Prices and Investment

with

Collateral and Default

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Abstract: This paper studies an OLG economy with three-period lived agents and two goods, one perishable and the other durable. The durable good serves as collateral for loans. We study the effect of an unanticipated income shock when the economy is in a steady state equilibrium, focusing on the consequence of the possibility of default on loans when the value of the collateral falls below the value of the debt it secures. This requires studying the steady state equilibria of the economy and their stability properties. We find that when the depreciation of the durable good is not too high and the durable good is desired, the Golden Rule steady state is (saddle-point) stable and show that there is an asymmetry between the effect of a positive and a negative income shock arising from default on collateralized loans.
1 Introduction

While borrowing and lending is perhaps the most basic form of mutually beneficial intertemporal exchange, the possibility of default by the borrower has always acted as a potential impediment to the smooth functioning of loan markets. One of the important ways of securing a loan is by posting collateral which is kept by the lender if the borrower fails to repay the debt. Traditionally durable goods—land, houses and other forms of property—have served as collateral and more recently the development of the repo and derivative markets have been based on the use of government bonds and other securitized assets as collateral for securing the loans.

Geanakoplos (1997) introduced collateral in a general equilibrium model with durable goods and potential default and the model was extended to an infinite horizon by Araujo et al (2002). However while it is possible to prove existence of an equilibrium, it is difficult to derive properties of an equilibrium in a stochastic model with durable goods serving as collateral, and even to numerically compute equilibrium in a model with only a few agents and a few goods (see Araujo et al (2012).

In this paper to retain the dynamic component of the equilibrium, we revert to the deterministic model in which a durable good is used as collateral—the mortgage market serving as the canonical example—and we adopt the method widely used in macroeconomics of studying the effect of an unanticipated income shock when the economy is in a steady state equilibrium. This method was used by Kiyotaki-Moore (1997) to argue that collateral constraints amplify income shocks in a model with production and infinitely lived capital (land). In this paper the focus is not so much on the magnitude of the response of the equilibrium variables to a shock but rather on the consequence of the default when the price of the collateral decreases, an effect which was not taken into account in Kiyotaki-Moore.

To model an agent’s need to borrow and lend to smooth consumption and permit the purchase of the durable good we use an overlapping generations model with three-period lived agents and two goods, one perishable and the other durable, the durable good being used as collateral for borrowing. The young and middle aged have an endowment of the consumption good while retired agents have no exogenous endowment. There is no exogenous supply of the durable good which must be produced from the consumption good with a constant returns technology. Loans must be guaranteed by collateral, the collateral constraint in equilibrium only being binding for young agents who cannot
borrow against their future income. Thus in our model the need to post collateral endogenously justifies the assumption that “Junior Can’t Borrow” (Constantinides-Donaldson-Mehra (2002). This leads to a surprisingly tractable model in which to study the price and production of the durable good.

To examine the consequence of a collateral borrowing constraint when the economy is subject to shocks, we assume that an unexpected shock occurs while the deterministic economy is in a steady state equilibrium. The shock is assumed to be small, so that the change in the endogenous variables can be studied using the linearized dynamics. If the perturbed economy has a unique equilibrium path which stays close to the steady state, then we can study the impulse response function—namely how the equilibrium reverts to the steady state after the initial shock. We are especially interested in the difference between a positive shock to the agents’ endowments which increases the value of the durable good, and a negative shock which lowers its price and triggers default on the collateralized loans. This gives an idea of what would happen in a richer model with incomplete markets in which agents take into account the possibility of shocks but do not have access to loan contracts contingent on these shocks.

Since the OLG model typically has several steady states, carrying out the above procedure requires an understanding of the (local) stability properties of the steady states of the economy, since an impulse response function can only be studied around a (locally saddle-point) stable steady state. We find that for reasonable values of the utility/endowment parameters the Golden Rule steady state (i.e the zero-interest rate steady state) is stable when the basic difference equation describing equilibrium is of low order as with log utilities, or saddle-point stable when it is of higher order as with CES utilities. Furthermore all other steady states when they exist are unstable. Thus because retired agents can get income from investing in the durable good which gives them an additional source of income over their savings on the financial markets, the economy has the stability characteristics of a “classical” economy rather than a “Samuelson” economy in the terminology of Gale (1973).

The paper is organized as follows: Section 2 presents the model. Section 3 studies an economy in which agents have log preferences and describes the equilibrium dynamics and the steady state equilibria which can be analytically derived. The effect of an unanticipated shock is also analyzed, showing the asymmetry between a positive and a negative shock. Section 4 generalizes the analysis to the case of CES preferences: in this case the dynamics is higher order and only the local properties
of the steady states can be studied. The Golden Rule steady state is shown to be saddle-point stable, which implies that there is a unique perturbed equilibrium which reverts to the steady state after an unanticipated shock. We study the impulse response functions of price and investment in the durable good which exhibit both the asymmetry between a positive and a negative shock, and a strong reaction (overshooting) effect on the date following the income shock.

2 Model

Consider an overlapping generations economy with two goods, one perishable and the other durable with depreciation rate $\delta > 0$ per period. Agents live for three periods as young, middle aged and retired: a new cohort of young agents of the same size enters at each date $t = 0, 1, \ldots$, while retired agents of the previous period exit. Thus at each date the three cohorts of young, middle and retired are of the same size. Every young agent enters with a lifetime endowment stream $(e^y, e^m, 0)$ consisting solely of the perishable good, and through trades on the markets obtains a lifetime consumption stream

$$(c, h) = (c^y, h^y, c^m, h^m, c^r, h^r)$$

where $c$ and $h$ denote the consumption of the perishable and durable good respectively, and the superscripts $(y, m, r)$ refer to the stages of the agent’s life. The preferences of every entering agent are represented by the same separable utility function

$$U(c, h) = u(c^y, h^y) + \beta u(c^m, h^m) + \beta^2 u(c^r, h^r) \quad (1)$$

where $u$ is increasing, concave and satisfies the Inada condition for both variables, and $0 < \beta \leq 1$ is the discount factor. Since the lifetime preferences and endowments of agents are always the same, and since the cohort sizes do not change we can focus on trades at each date between representative agents of each generation. At each date $t = 0, 1, \ldots$ there are markets for the two goods and a financial market for borrowing/lending. Let $(1, q_t)$ denote the spot prices of the two goods, the perishable good serving as the numeraire for the transactions of each period. Borrowing/lending takes place through competitive infinitely-lived intermediaries and we assume that there is no legal system for enforcing the payment of the debts, so that all borrowing must be guaranteed by collateral, an appropriate amount of durable good which can be seized if the debt is not repaid.

While the stock of the perishable good $(e^y + e^m)$ available each period is exogenously given, the stock of durable good can be altered by production. The durable good is produced using the
perishable good as input with a constant returns technology with a one-period lag, \( y_{t+1} = \alpha z_t \), \( \alpha > 0 \), where \( z_t \) is the input of the perishable good invested at date \( t \) and \( y_{t+1} \) is the output of durable at date \( t + 1 \). If \( r_t \) denotes the interest rate on the market for loans between dates \( t \) and \( t + 1 \) then no-arbitrage between investing one unit of the perishable good in the loan market or in production implies

\[
1 + r_t \geq \alpha q_{t+1} \tag{2}
\]

\( z_t \) being equal to zero if the inequality is strict. We restrict attention to sequences of prices satisfying (2) for studying the optimal choice of the representative agent since otherwise the maximization of utility would not have a solution. Note that the units of the perishable good are determined by the choice of the values for \((e^y, e^m)\) and the units of the durable good are determined by setting \( \alpha = 1 \): one unit of the durable is what can be produced with one unit of the perishable good.

Consider a young agent entering the economy at date \( t \). His lifetime consumption \((c, h)_t = (c_t^y, h_t^y, c_{t+1}^m, h_{t+1}^m, c_{t+2}^r, h_{t+2}^r)\) will be obtained through purchases on the spot markets at dates \( t, t + 1, t + 2 \) at prices \((1, q_t), (1, q_{t+1}), (1, q_{t+2})\) when he is young, middle aged and retired. These purchases will be financed by income obtained from his endowment \((e^y, e^m, 0)\), from the sale of the depreciated previously bought durable good \(((1 - \delta)h_t^y, (1 - \delta)h_{t+1}^m)\), from borrowing \( b_t^y, b_{t+1}^m \) on the loan market, and from investment \( z_{t+1}^m \) in the production of the durable good. The consumption stream and portfolio \((c, h)_t , b_t^y, b_{t+1}^m, z_{t+1}^m)\) of an agent entering at date \( t \) must satisfy the sequence of budget equations

\[
c_t^y + q_t h_t^y = e^y + b_t^y \tag{3}
\]
\[
c_{t+1}^m + q_{t+1} h_{t+1}^m + z_{t+1}^m = e^m - \min\{b_t^y(1 + r_t), q_{t+1}h_t^y(1 - \delta)\} + q_{t+1}h_t^y(1 - \delta) + b_{t+1}^m \tag{4}
\]
\[
c_{t+2}^r + q_{t+2} h_{t+2}^r = q_{t+2}(z_{t+1}^m + h_{t+1}^m(1 - \delta)) - b_{t+1}^m(1 + r_{t+1}) \tag{5}
\]

In addition the agent’s borrowing \((b_t^y, b_{t+1}^m)\) from the intermediaries in youth and middle age must satisfy the collateral constraints

\[
b_t^y(1 + r_t) \leq q_{t+1} h_t^y(1 - \delta) \tag{6}
\]
\[
b_{t+1}^m(1 + r_{t+1}) \leq q_{t+2}(z_{t+1}^m + h_{t+1}^m(1 - \delta)) \tag{7}
\]

and investment \( z_{t+1}^m \) must be non negative, \( z_{t+1}^m \geq 0 \).

(6) and (7) are similar to the collateral constraints used by Kiyotaki-Moore (1997) and the
subsequent literature\textsuperscript{1}: an agent can borrow up to the point where the reimbursement due next period is equal to the value of the collateral guaranteeing the loan. We consider settings where the collateral constraint of the young (6) always binds, while that of the middle-aged (7) never binds. For if (7) were binding then (5) would imply that the agent has no income to buy consumption once retired: thus (7), while included for consistency, can in fact be omitted.

Since we are interested in equilibria in which the borrowing constraint of the young is binding, we have not included an investment decision $z_y^t$ for the young: for in this case, even if given the choice, the agent would choose $z_y^t = 0$. In the perfect foresight deterministic case, the ‘min’ in the budget equation (4) can be omitted since the collateral constraint (6) ensures that the debt is paid, in which case (4) can be replaced by the budget equation

$$c_{m}^{t+1} + q_{t+1}h_{m}^{t+1} + z_{m}^{t+1} = e_{m}^{t} - b_{y}^{t}(1 + r_{t}) + q_{t+1}h_{y}^{t}(1 - \delta) + b_{m}^{t+1}$$ (4)

However when we consider an unanticipated income shock at some date, then the price of the durable differs from what was anticipated and the ‘min’ becomes relevant. In this case, at the date when the shock occurs, the middle-age budget constraint is of the form (4).

Finally note that the agent does not inherit an endowment of the durable good when young or middle aged: this implies that the durable good purchased by the agent when retired is not bequested. That is, we assume that when retired agents exit, their durable good exits with them. This assumption simplifies the model and avoids the presence of a bequest motive in the maximum problem of an agent.

**Simplified Maximum Problem.** The agent’s maximum problem consists in choosing a consumption stream and portfolio $\left((c, h)^t, b_y^t, b_m^{t+1}, z_m^{t+1}\right)$ which maximizes utility $U((c, h)^t)$ subject to the budget equations (2)-(4) and the collateral constraint (6). We analyze the deterministic problem where constraint (4) in middle age is given by (4\textsuperscript{'}). This problem has the interesting property that if the collateral constraint (6) is binding it decomposes into a simple one-period problem for the agent when young, independent of what happens later in life, and a two-period problem for the agent in middle age, choosing consumption for middle age and retirement. This arises from the fact that a collateral-constrained young agent can not borrow against future income. These properties can be seen from the first-order conditions which, in conjunction with the constraints

\textsuperscript{1}See in particular Kocherlakota (2000).
Proposition 1 characterize the solution to the agent's maximum problem. Let \((\lambda_t^y, \lambda_{t+1}^m, \lambda_{t+2}^r)\) denote the multipliers induced by (3), (4'), (5), and let \(\mu_t\) be the multiplier associated with (6).

\[
\begin{align*}
\text{(young)} & \quad c_t^y : \quad u_t^y = \lambda_t^y \quad \text{ (8)} \\
& \quad h_t^y : \quad u_h^y = \lambda_t^y q_t - (\mu_t + \lambda_{t+1}^m)q_{t+1}(1 - \delta) \quad \text{ (9)} \\
& \quad b_t^y : \quad \lambda_t^y = (\mu_t + \lambda_{t+1}^m)(1 + r_t), \quad (10) \\
\text{(middle-age)} & \quad c_{t+1}^m : \quad \beta u_c^m = \lambda_{t+1}^m \quad \text{ (11)} \\
& \quad h_{t+1}^m : \quad \beta u_h^m = \lambda_{t+1}^m q_{t+1} - \lambda_{t+2}^r q_{t+2}(1 - \delta) \quad \text{ (12)} \\
& \quad b_{t+1}^m : \quad \lambda_{t+1}^m = \lambda_{t+2}^r (1 + r_{t+1}) \quad \text{ (13)} \\
& \quad z_{t+1}^m : \quad \lambda_{t+1}^m \geq \lambda_{t+2}^r q_{t+2}, \quad \text{ if } z_{t+1}^m > 0 \quad \text{ (14)} \\
\text{(retired)} & \quad c_{t+2}^r : \quad \beta^2 u_c^r = \lambda_{t+2}^r \quad \text{ (15)} \\
& \quad h_{t+2}^r : \quad \beta^2 u_h^r = \lambda_{t+2}^r q_{t+2} \quad \text{ (16)}
\end{align*}
\]

where the partial derivatives are evaluated at the optimal decision, and expressions like \(u_c(c_t^y, h_t^y)\) have been abbreviated to \(u^y\) indicating the period of life and the partial differentiation.

**Proposition 1 (Decomposition of agent’s maximum problem)**

(a) If \((c, h)_t\) maximizes \(U((\tilde{c}, \tilde{h})_t)\) under the sequential budget constraints (3), (7), (5) and the collateral constraint (6) is binding \((\mu_t > 0)\) then

(i) \((c_t^y, h_t^y)\) maximizes \(u(\tilde{c}^y, \tilde{h}_t^y)\) under the constraint

\[
\tilde{c}^y + \left(q_t - q_{t+1}(1 - \delta) \right) \tilde{h}_t^y = e^y \quad \text{ (17)}
\]

(ii) \((c_{t+1}^m, h_{t+1}^m, c_{t+2}^r, h_{t+2}^r)\) maximizes \(u(\tilde{c}_{t+1}^m, \tilde{h}_{t+1}^m) + \beta u(\tilde{c}_{t+2}^r, \tilde{h}_{t+2}^r)\) under the present-value budget constraint

\[
\tilde{c}^m + \left(q_{t+1} - q_{t+2}(1 - \delta) \right) \tilde{h}_{t+1}^m + \left(1 + r_{t+1} \right) \frac{1}{1 + r_{t+1}} \left(c^r + q_{t+2}h^r \right) = e^m \quad \text{ (18)}
\]

(b) Conversely if \((c, h)_t\) is such that \((c_t^y, h_t^y)\) satisfies (i) and \((c_{t+1}^m, h_{t+1}^m, c_{t+2}^r, h_{t+2}^r)\) satisfies (ii) and if

\[
\begin{align*}
\begin{aligned}
u_c(c_t^y, h_t^y) & > \beta(1 + r_t)u_c(c_{t+1}^m, h_{t+1}^m) \\
1 + r_{t+1} & \geq q_{t+2}
\end{aligned}
\end{align*}
\]
then there exist a portfolio \((b_t^y, b_{t+1}^m, z_{t+1}^m)\) such that \(((c, h)_t, b_t^y, b_{t+1}^m, z_{t+1}^m)\) maximizes \(U\) under the constraints (3), (4'), (5) and (6).

**Proof:** (a) Substituting the first statement in (10) into (9) we see that the FOCs (8)-(10) imply that the FOCs of problem (i) hold. Since \(\mu_t > 0\), \(b_t^y = q_{t+1} h_t^y (1 - \delta) / (1 + r_t)\) so that the sequential constraint (3) implies that (17) holds. In the same way, when (13) is substituted in (12) then it is clear that the FOCs (11)-(16) imply that the FOCs for the problem in (ii) are satisfied. Since \(b_t^y (1 + r_t) = q_{t+1} h_t^y (1 - \delta)\) and since either \(z_{t+1}^m = 0\) or \(1 + r_{t+1} = q_{t+2}\), multiplying the sequential constraint (5) by \(1 + r_{t+1}\) and adding to (4') shows that the present-value constraint holds.

(b) Let \((c, h)_t\) be a solution of the problems in (i) and (ii) with budget constraints (17) and (18). We need to show that \((c, h)_t\) is also solution of the original problem of maximizing \(U((\tilde{c}, \tilde{h})_t)\) subject to the sequential budget constraints (3),(4'),(5), and the collateral constraint (6). Let \(\tilde{\lambda}_t\) denote the multiplier for (17) and \(\tilde{\lambda}_{t+1}\) the multiplier for (18). Then \((c, h)_t\) satisfies the FOCs for the problems (i) and (ii)

\[
\begin{align*}
\lambda^y_c &= \tilde{\lambda}_t, \\
\lambda^m_c &= \tilde{\lambda}_{t+1}, \\
\lambda^y_h &= \tilde{\lambda}_t \left( q_t - \frac{q_{t+1} (1 - \delta)}{1 + r_t} \right) \\
\lambda^m_h &= \tilde{\lambda}_{t+1} \left( q_{t+1} - \frac{q_{t+2} (1 - \delta)}{1 + r_{t+1}} \right) \\
\lambda^y_r &= \tilde{\lambda}_{t+1} \left( 1 + r_{t+1} \right), \\
\lambda^m_r &= \frac{\tilde{\lambda}_{t+1} q_{t+2}}{1 + r_{t+1}} \\
\end{align*}
\]

If \((\lambda_t, \lambda_{t+1}, \lambda_{t+2}) = (\tilde{\lambda}_t, \tilde{\lambda}_{t+1}, \tilde{\lambda}_{t+2})\) then (21)-(23) imply that the FOCs (8)-(16) are satisfied. Let \(b_t^y = c_t^y + q_t h_t^y - e^y_t\). Then (17) implies that \(b_t^y = \frac{q_{t+1} (1 - \delta)}{1 + r_t} \) so that the collateral constraint (6) and (3) hold. Let \(b_{t+1}^m - z_{t+1}^m = e_{t+1}^m - q_{t+1} h_{t+1}^m - e_{t+1}^m\). Since (6) holds with equality, (4) is satisfied. If \(1 + r_{t+1} > q_{t+2}\) let \(z_{t+1}^m = 0\), otherwise only the difference \(b_{t+1}^m - z_{t+1}^m\) is determined \((z_{t+1}^m\) being determined by the equilibrium conditions). Finally (18) implies

\[
b_{t+1}^m - z_{t+1}^m = \frac{1}{1 + r_{t+1}} (q_{t+2} h_{t+1}^m (1 - \delta) - c_{t+2}^r - q_{t+2} h_{t+2}^r)
\]

and since \(1 + r_{t+1} = q_{t+2}\) or \(z_{t+1}^m = 0\) then (5) is satisfied.

**Remark.** As shown in Proposition 1 the agent’s intertemporal maximum problem is simplified when the borrowing constraint binds. It decomposes into a problem in youth and a problem in
middle age/retirement. A young agent who is borrowing constrained can not satisfactorily solve the trade-off between consumption when young and consumption in later stages. Thus the agent simply spends as much he can given current income $e_y$ and the down payment that must be made on the durable good: in (17) all the agent has to pay to obtain one unit of the durable good is the down payment $q_t - \frac{q_{t+1}(1-\delta)}{1+r_{t+1}}$ which depends of its resale value, its depreciation, and the interest rate on the loan. Since in middle age the agent uses the resale value of the durable acquired in youth to pay for the debt, the agent’s second stage problem (ii) starts as a fresh two-period problem over middle age and retirement with income $(e_m, 0)$. The resale value of the durable acquired in middle age reduces its effective price to $q_{t+1} - \frac{q_{t+2}(1-\delta)}{1+r_{t+1}}$, while the durable purchased in retirement costs $q_{t+2}$ since there is no following period in which to sell it. Since in middle age the borrowing constraint does not bind, the sequential budget constraints in middle age and retirement reduce to a present-value constraint in period $t + 1$.

We are interested in studying equilibria and in particular steady-state equilibria of the economy. To this end we first define an equilibrium over $\mathbb{Z} = (-\infty, +\infty)$.

**Equilibrium.** An equilibrium is a sequence $((c, h)_t, q_t, r_t)_{t \in \mathbb{Z}}$ such that for each $t \in \mathbb{Z}$

(i) $(c, h)_t$ maximizes $U((c, h)_t)$ subject to (3)-(6)

(ii) $c^y_t + c^m_t + c^r_t + z^m_t = e_y + e^m$

(iii) $h^y_t + h^m_t + h^r_t = z^m_{t-1} + (1 - \delta)(h^y_{t-1} + h^m_{t-1})$

**Steady state equilibrium.** A steady state equilibrium is an equilibrium in which all the variables are constant in time

$$((c, h)_t, q_t, r_t) = ((c, h), q, r), \quad \forall t \in \mathbb{Z}$$

Note that in a steady state equilibrium $z^m > 0$ since the depreciated durable good must be replaced in each period: thus in a steady state $q = 1 + r$ must hold.

**Proposition 2.** (Golden Rule) If $((c, h), q, r)$ satisfies

1. $(\tilde{c}^y, \tilde{h}^y) \in \text{argmax} \left\{ u(\tilde{c}^y, \tilde{h}^y) \mid \tilde{c}^y + (q - (1 - \delta))\tilde{h}^y = e_y \right\}$

2. $(\tilde{c}^m, \tilde{h}^m, \tilde{c}^r, \tilde{h}^r) \in \text{argmax} \left\{ u(\tilde{c}^m, \tilde{h}^m) + \beta u(\tilde{c}^r, \tilde{h}^r) \mid \tilde{c}^m + (q - (1 - \delta))\tilde{h}^m + \tilde{c}^r + q\tilde{h}^r = e^m \right\}$

3. $u^y_c > \beta u^m_c$
4. \((q, r) = (1, 0)\)
then \((c, h, 1, 0)\) is the steady state equilibrium which we call the Golden Rule (GR) equilibrium.²

**Proof:** By Proposition 1 the optimality conditions for the consumption stream of a typical agent subject to the sequential budget and borrowing constraints are satisfied. Adding the budget constraints in (i) and (ii) gives

\[
c^y + c^m + c^r + (q - (1 - \delta))(h^y + h^m) + qh^r = e^y + e^m
\]

or, since \(q = 1\),

\[
c^y + c^m + c^r + h^y + h^m + h^r - (1 - \delta)(h^y + h^m) = c^y + c^m
\]

Let \(z^m = h^y + h^m + h^r - (1 - \delta)(h^y + h^m)\). Then the market clearing conditions for both the perishable good and the durable good are satisfied at all times, so that \((c, h, 1, 0)\) is a steady state equilibrium. \(\square\)

**Equilibrium Dynamics.** An equilibrium is a sequence of durable good prices and interest rates \((q_t, r_t)_{t \in \mathbb{Z}}\) such that at each date the markets for the consumption and the durable good clear. We are interested in equilibria close to the Golden Rule steady state in which the collateral constraint binds and there is positive investment to replace the depreciated durable good. In this case the market-clearing equations for the consumption and durable good can be summarized by the relation

\[
c^y_t + c^m_t + c^r_t + h^y_{t+1} + h^m_{t+1} + h^r_{t+1} - (1 - \delta)(h^y_t + h^m_t) = e^y + e^m, \quad t \in \mathbb{Z}
\] ²(24)

and the prices satisfy the relation \(1 + r_t = q_{t+1}\). Replacing the interest rate by its value in the budget sets of Proposition 1, the agents’ demand functions can be expressed as functions of the durable good prices \((q_t)_{t \in \mathbb{Z}}\). It then follows from Proposition 1 that \((c^y_t, h^y_t)\) depends on \(q_t\), while \((c^m_t, h^m_t)\) depend on \((q_t, q_{t+1})\) and \((c^r_t, h^r_t)\) on \((q_{t-1}, q_t)\). Since the indices at date \(t + 1\) move up 1, the dynamic equation (24) defines a relation between the prices \((q_{t-1}, q_t, q_{t+1}, q_{t+2})\) at four dates. Provided that we can solve this equation as \(q_{t+2} = f(q_{t+1}, q_t, q_{t-1})\), an equilibrium is described by a third-order difference equation.

²We use the terminology “Golden Rule” because of the property that \(r = 0\), i.e. the interest rate is equal to the rate of growth of the population, which is a characteristic property of the Golden Rule equilibrium in OLG economies without imperfections.
3 Log Utility

The equilibrium recursive equation is simplified when the agents have log preferences

\[ u(c, h) = \ln(c) + \gamma \ln(h) \] (25)

Since agents spend a fixed share of their income on each good, the dependence of the demand on \( q_{t-1} \) and \( q_{t+2} \) in (24) disappears, and the equilibrium equation has the simple form \( q_{t+1} = f(q_t) \). This leads to a straightforward study of the existence and stability of the steady states, and the change in price and investment caused by an endowment shock at date 0.

When the utility \( u \) in (1) is given by (25), the economy is characterized by two preference parameters \((\beta, \gamma)\), the depreciation rate \( \delta \), and the agents’ endowments of consumption good \((e^y, e^m)\). The following demands for young, medium and retired in (24) can then be deduced from Proposition 1:

\[
\begin{align*}
  c^y_t &= \frac{e^y}{1 + \gamma}, \\
  c^m_t &= \frac{e^m}{(1 + \gamma)(1 + \beta)}, \\
  c^r_t &= \frac{\beta e^m q_t}{(1 + \gamma)(1 + \beta)}, \\
  h^y_t &= \frac{\gamma e^y}{1 + \gamma q_t - (1 - \delta)}, \\
  h^m_t &= \frac{\gamma e^m}{(1 + \gamma)(1 + \beta) q_t - (1 - \delta)}, \\
  h^r_t &= \frac{\beta \gamma e^m}{(1 + \gamma)(1 + \beta)}.
\end{align*}
\] (26)

To ensure that the collateral constraint for the young is always binding we make the mild assumption that the endowment of the young is sufficiently scarce relative to their endowment in middle age:

\( \text{CC (binding collateral constraint)} \quad \frac{e^m}{e^y} \geq \beta(1 + \beta). \)

With log preferences the market clearing equation (24) becomes

\[
\frac{e^y + \frac{e^m}{1 + \beta}}{1 + \gamma} + \frac{\beta e^m q_t}{(1 + \gamma)(1 + \beta)} + \frac{\gamma}{1 + \gamma} \left( \frac{e^y + \frac{e^m}{1 + \beta}}{q_{t+1} - (1 - \delta)} + \frac{\beta e^m}{1 + \beta} - (1 - \delta) \frac{e^y + \frac{e^m}{1 + \beta}}{q_t - (1 - \delta)} \right) = e^y + e^m
\]

which after the change of variable

\[ x_t = q_t - (1 - \delta) \]

can be written as

\[ A x_t + \frac{B}{x_{t+1}} - \frac{(1 - \delta) B}{x_t} - (B + \delta A) = 0 \]

with

\[ A = \frac{\beta e^m}{1 + \beta}, \quad B = \gamma \left( e^y + \frac{e^m}{1 + \beta} \right). \]
Multiplying by \( x_t x_{t+1} \) (which introduces the fictitious solution \( x_t = x_{t+1} = 0 \)), the dynamic equation becomes

\[
x_{t+1} = \frac{B x_t}{-A x_t^2 + (B + \delta A)x_t + (1 - \delta)B}
\]

The steady state solutions are the solutions of the equation

\[
-A x^2 + (B + \delta A)x - \delta B = 0
\]

which has two positive roots

\[
x_1^* = \delta, \quad x_2^* = \frac{B}{A} = \frac{\gamma}{\beta} \left( 1 + \beta \right) e^y + e^m
\]

The first root corresponds to the Golden Rule since \( q = 1, r = 0 \), implies \( x = \delta \). To study which steady state is stable we write the difference equation (27) as \( x_{t+1} = f(x_t) \). Then \( f'(x) = \frac{B D}{Ax^2 + (1 - \delta)B} \), where \( D \) is the denominator in (27). Thus \( f'(0) = \frac{1}{1 - \delta} > 1 \) and \( f'(\delta) = (1 - \delta) + \frac{\delta^2}{x_2} \).

\( f'(\delta) < 1 \iff x_2 > \delta \) which is equivalent to

\[
\frac{\gamma}{\delta} > \frac{\beta e^m}{(1 + \beta) e^y + e^m}
\]

There are thus two possible cases which can arise: if (29) is satisfied the graph of \( f \) intersects the diagonal at \((0, \delta, x_2^*)\) in this order as shown in Figure 1(a), and if the inequality in (29) is reversed then the graph of \( f \) intersects the diagonal at \((0, x_2^*, \delta)\) (Figure 1(b)).

With log preferences, agents’ demand functions have the Gross Substitute property and, despite the fact that the durable good is produced, our economy has the properties that have been established for multigood exchange economies with Gross Substitute: there are exactly two steady states, the Golden Rule, (sometimes called the “nominal” steady state since it requires the presence of an ‘intermediary’ or of ‘money’ to be feasible) and the autarky, or “real” steady state corresponding to the price \( q_2 = (1 - \delta) + x_2^* \) for which it can be shown that \( b^y + b^m = 0 \), so that borrowing and lending can be achieved by middle-aged agents lending to the young and being reimbursed in their retirement (without the need for an intermediary).

The condition that the Golden Rule is stable, which corresponds to the case depicted in Figure 1(a), is that the parameters of the economy satisfy (29), i.e. the utility parameter \( \gamma \) must be sufficiently large, relative to the depreciation \( \delta \). This is the case on which we focus attention

\[\text{See Kehoe-Levine-Mas-Colell-Woodford (1991).}\]
Figure 1: Golden Rule stable in Figure 1(a), unstable in Figure 1(b).

since we are interested in economies where the durable good plays an important role—for example housing. At first sight it might seem that the economy falls into the Samuelson category of Gale (1973) since agents have no endowment in retirement. However in our economy retired agents get income from their purchase of the durable good and their investment in production in middle age. If (29) holds and the production of durable is non negligible, then retired agents are sufficiently rich for the economy to fall into the “classical” category where there is net aggregate borrowing \((b^y + b^m > 0)\) by the young and middle-aged at the Golden Rule, and the Golden Rule is stable.

Assuming that (29) holds we examine the effect of an unanticipated shock at the Golden Rule steady state equilibrium \(((c, h), 1, 0)\), in which a typical agent’s lifetime consumption stream and financial variables are given by

\[
\begin{align*}
  c^y &= \frac{e^y}{1 + \gamma}, \\
  h^y &= \frac{\gamma}{1 + \gamma} \frac{e^y}{\delta}, \\
  b^y &= \frac{\gamma(1 - \delta)e^y}{\delta(1 + \gamma)} \\
  c^m &= \frac{e^m}{(1 + \gamma)(1 + \beta)}, \\
  h^m &= \frac{\gamma}{(1 + \gamma)(1 + \beta)} \frac{e^m}{\delta}, \\
  b^m &= \frac{1}{1 + \gamma} \left( \gamma e^y + \frac{(\gamma - \beta\delta)e^m}{\delta(1 + \beta)} \right) \\
  z^m &= \frac{\gamma}{1 + \gamma} (e^y + e^m) \\
  c^r &= \frac{\beta e^m}{(1 + \gamma)(1 + \beta)}, \\
  h^r &= \frac{\beta \gamma e^m}{(1 + \gamma)(1 + \beta)}. 
\end{align*}
\]
Negative Shock. Suppose that there is a once and for all unanticipated shock to the agents’ endowments at date 0: the endowments at date 0 become

\[(e^y, e^m) \Rightarrow ((1 + \Delta)e^y, (1 + \Delta)e^m)\]

and revert to their standard values thereafter. Thus

\[(e^y_t, e^m_t) = (e^y, e^m), \quad t \neq 0\]

\[\Delta = \Delta e/e\] is the proportional change in the agents’ endowments (and in the total endowment \(e = e^y + e^m\)). As we shall see the effect on the equilibrium is different depending on whether \(\Delta < 0\) or \(\Delta > 0\), which we write as \(\Delta^–\) and \(\Delta^+\) respectively. We begin with the case \(\Delta^–\) and assume that \(q_0 < 1\) so that middle-aged agents default on their loans: as we will see this assumption is satisfied in the equilibrium we calculate.

Since the economy was following the steady state prior to date 0 when the shock occurs, the supply of houses at date 0 is

\[H_0 = (1 - \delta)(h^y + h^m) + z^m\] (31)

The demand of the young and the middle-aged has the same form as in (26), the incomes \((1 + \Delta^–)e^y\) and \((1 + \Delta^–)e^m\) replacing \(e^y\) and \(e^m\). The middle-aged “foreclose” rather than paying their debts and thus ‘start fresh’ at date 0. The demand of the retired does not have the same form as in (26) since their income is different from what they anticipated: the amount of durable good

\[a = (1 - \delta)h^m + z^m\]

which they inherit from their middle age sells for a lower price than anticipated. Since the interest charged at date \(-1\) is zero, their income is

\[I^r_0 = q_0a - b^m\] (32)

where we assume that the shock \(\Delta^–\) is such that \(q_0\), although less than 1, is sufficiently large for \(I^r_0\) to be positive, in which case the retired agents do not default on the loan \(b^m\). Their demand for the durable good is then \(h^r_0 = \frac{\gamma I^r_0}{(1 + \gamma)q_0}\) and the equilibrium equation on the durable good market is

\[\frac{\gamma}{1 + \gamma} \left[ \frac{(1 + \Delta^–)(e^y + e^m)}{q_0 - (1 - \delta)} + a - \frac{b^m}{q_0} \right] = H_0\]
Let \((\Delta q_0)^- = q_0 - 1\) denote the deviation of the durable good price a date 0 from the steady-state value 1 induced by the shock in the agents’ endowments.\(^4\) Assuming that \(\Delta^-\) is sufficiently small for a first-order approximation to be justified, the relation between \((\Delta q_0)^-\) and \(\Delta^-\) is obtained by replacing \(\frac{1}{q_0 - (1 - \delta)}\) by \(\frac{1}{\delta} \left(1 - \frac{(\Delta q_0)^-}{\delta}\right)\) and \(\frac{1}{q_0} = \frac{1}{1 + (\Delta q_0)^-}\) by \((1 - (\Delta q_0)^-)\). This leads to

\[
\left[-\left(\frac{1}{\delta}\right)^2 \left(e^y + \frac{e^m}{1 + \beta}\right) + b^m\right] (\Delta q_0)^- + \frac{1}{\delta} \left(e^y + \frac{e^m}{1 + \beta}\right) \Delta^- = 0 \tag{33}
\]

or

\[
(\Delta q_0)^- = \frac{\delta \Delta^-}{1 - \left(\frac{\delta^2 b^m}{e^y + \frac{e^m}{1 + \beta}}\right)} \tag{34}
\]

The denominator is positive in view of the expression for \(b^m\) in (30). Thus a decrease in the economy’s endowment leads to a fall in the price of the durable at date 0, the price elasticity being of the order of, but larger than, \(\delta\) since the denominator in (34) is less than 1.

The date 0 investment \(z_0^m\) can be deduced from the date 0 market clearing condition for the perishable good

\[
c_y^0 + c_y^m + c_r^0 + z_0^m = (1 + \Delta^-)(e^y + e^m) \tag{35}
\]

where the demand \(c_y^0\) and \(c_y^m\) have the form in (26) and \(c_r^0 = \frac{I_r^0}{1 + \gamma}\). Let \((\Delta z_0)^- = z_0^m - z^m\) denote the deviation of the date 0 investment from the steady-state value \(z^m\) in (30). Using first order approximations for the changes leads to

\[
(\Delta z_0)^- = n \Delta^- - m (\Delta q_0)^- \tag{36}
\]

where

\[
n = \left(1 - \frac{1}{1 + \gamma}\right) e^y + \left(1 - \frac{1}{(1 + \beta)(1 + \gamma)}\right) e^m, \quad m = \frac{a}{1 + \gamma}. \tag{37}
\]

\(n\) is is the net contribution to the input for investment of the young and the middle-aged in the steady state, namely the difference between their endowment and their demand for the consumption good. Since \(n > 0\) the first term is negative: the contribution of the young and the middle-aged to investment input decreases with the decrease in their endowment. The term \(m(\Delta q_0)^-\) comes from the demand of the retired agents. A fall in \(q\), which has mainly an income effect for the retired, leads to a fall in their demand for the consumption good, which increases the amount of the good

\(^4\)Since \(q = 1\) at the Golden Rule, \(\Delta q_0\) is also the percentage deviation from the steady state price.
available for investment. The first of the two terms in (36) dominates so that a negative shock reduces investment at date 0.⁵

**Positive Shock.** Now suppose there is a positive shock to the economy at date 0, so that agents’ endowments are increased by the percentage $\Delta^+$. We assume that the price $q_0$ which establishes itself on the durable good market is higher than 1 and this assumption will be justified below. Since the price of the durable exceeds its steady-state value, the middle-aged agents make a capital gain on the durable good inherited from their youth. Their income now comes from two sources: $q_0h^y(1-\delta) - b^y$ from the sale of houses inherited from youth and $e^m(1+\Delta^+)$ from the exogenous endowment. The equation for the equilibrium on the durable good market which determines the price $q_0$ thus becomes

\[
\frac{\gamma}{1 + \gamma} \left[ (1 + \Delta^+)\left(e^y + \frac{e^m}{1 + \beta}\right) + \frac{q_0h^y(1-\delta) - b^y}{1 + \beta} + a - \frac{b^m}{q_0} \right] = H_0
\]

where $H_0$ is the supply of durable good inherited from the previous period given by (31). Using the same approximations as in the derivation of (33) gives

\[
\left[ -\left(\frac{1}{\delta}\right) \left(e^y + \frac{e^m}{1 + \beta}\right) + \frac{(1-\delta)h^y}{1 + \beta} + \delta b^m \right] (\Delta q_0)^+ + \left(e^y + \frac{e^m}{1 + \beta}\right) \Delta^+ = 0
\]

or

\[
(\Delta q_0)^+ = \frac{\delta \Delta^+}{1 - \frac{\delta^2 b^m}{e^y + \frac{e^m}{1 + \beta}} - \frac{\delta(1-\delta)h^y}{e^y + \frac{e^m}{1 + \beta}}}
\]

In view of the expression for $b^m$ and $h^y$ in (30) the denominator is positive: the positive shock $\Delta^+$ leads to an increase in the price of durable good at date 0, the price elasticity being of the order of $\delta$. However since the denominator in (39) is smaller than in (34), the volatility of the price is greater with a positive shock $\Delta^+$ than with a negative shock $\Delta^-$. The difference comes from the asymmetric way in which the unanticipated capital gain (loss) affects the income of the middle-aged. With a positive shock the middle-aged agents get a boost to their income coming from the

---

⁵This can be seen by using the approximation $(\Delta q_0)^- \approx \delta \Delta^-$ which leads to

\[
(\Delta z_0)^- \approx \frac{1}{1 + \gamma} \left( \gamma \left(1 - \frac{\delta}{1 + \gamma}\right) e^y + \left(\beta(1 + \gamma) + \frac{\gamma(\gamma - \delta \beta)}{1 + \gamma}\right) \frac{e^m}{1 + \beta}\right) (\Delta q_0)^-
\]

which has the sign of $(\Delta q_0)^-$. 

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induced capital gain, while the capital loss from a negative shock is passed to the intermediaries through default, but is not felt in the income of the middle-aged agents.

The effect of a positive shock to investment can be deduced from the market clearing on the consumption good market

$$\frac{(1 + \Delta^+)(e^y + e^m)}{1 + \gamma} + q_0 h^y (1 - \delta) - b^y \frac{q_0 a - b^m}{1 + \gamma} + z_0^m = (1 + \Delta^+)(e^y + e^m)$$

If $$(\Delta z_0)^+$$ denotes the deviation investment from the steady state, this market clearing equation implies that

$$(\Delta z_0)^+ = n \Delta^+ - m^+ (\Delta q_0)^+$$

where $n$ is given by (37) and $m^+ = m + \frac{1 - \delta}{(1 + \beta)(1 + \gamma)} h^y$. Comparing the price changes in (34) and (39) implies

$$\frac{(\Delta q_0)^+}{\Delta^+} > \frac{(\Delta q_0)^-}{\Delta^-}$$

and it follows that

$$\frac{(\Delta z_0)^+}{\Delta^+} = n - m^+ (\Delta q_0)^+ \Delta^+ < n - m (\Delta q_0)^- \Delta^- = (\Delta z_0)^- \Delta^-$$

The log utility case is valuable because it leads to explicit closed-form solutions for the equilibrium path of prices and investment following a shock to agents’ endowments at date 0. Formula (34), (36), (39) and (40) show explicitly the asymmetry between the (first-order approximations of) the equilibria following a positive or a negative shock. Two effects are present in each case: a direct effect on the income of the young and middle-aged agents, and an indirect effect from the change in value of the inherited stock of durable good. While the capital gain or loss affects the retired agents in a symmetric way, it has an asymmetric effect on the middle-aged. With a negative shock the option to default cushions middle-aged agents from the capital loss, so that their income is higher than it would have been if they had to incur the loss. As a result the demand is higher than it otherwise would have been, implying a smaller decrease in the price of the durable and a smaller amount of consumption good available for investment. For a positive shock the wealth effect due to an increase in the price of the durable good is fully felt by the middle-aged agents, leading to an increase in demand for both the consumption and the durable good which is greater than the decrease in demand in the negative case, leading to a commensurately higher price of the consumption good.
So far we have studied only the contemporaneous date 0 response of equilibrium prices and investment to the shock at date 0. In the next section the analysis is extended to the entire perturbed equilibrium trajectory for all subsequent periods. To check the robustness of the qualitative response of the equilibrium to both positive and negative shocks at date 0, we extend the analysis to the larger class of CES preferences.

4 CES Utility and Local Dynamics

We extend the analysis to the case where a typical agent’s utility function \( u \) lies in the CES family

\[
    u(c, h) = \frac{1}{1-\frac{1}{\sigma}} \left( c^{1-\frac{1}{\sigma}} + \gamma h^{1-\frac{1}{\sigma}} \right), \quad \sigma \neq 1
\]

the limiting case where the elasticity of substitution is \( \sigma = 1 \) corresponding to the log case of Section 3: all other characteristics of the economy remain the same.

Assuming that the collateral constraint binds in equilibrium so that Proposition 1 can be applied, the demand function of a typical agent in the cohort entering at date \( t \) is given by

\[
    c^y_t = \frac{e^y_t}{I(q_t)}, \quad h^y_t = \frac{\gamma^\sigma}{(q_t - (1 - \delta))^\sigma} \frac{e^y_t}{I(q_t)}
\]

\[
    c^m_{t+1} = \frac{e^m_t}{J(q_{t+1}, q_{t+2})}, \quad h^m_{t+1} = \frac{\gamma^\sigma}{(q_{t+1} - (1 - \delta))^\sigma} \frac{e^m_t}{J(q_{t+1}, q_{t+2})}
\]

\[
    c^r_{t+2} = \frac{\beta^\sigma (q_{t+2})^\sigma e^m_t}{J(q_{t+1}, q_{t+2})}, \quad h^r_{t+2} = \frac{(\beta \gamma)^\sigma e^m_t}{J(q_{t+1}, q_{t+2})}
\]

where

\[
    I(q_t) = 1 + \gamma^\sigma (q_t - (1 - \delta))^{1-\sigma}, \quad J(q_{t+1}, q_{t+2}) = I(q_{t+1}) + \frac{\beta^\sigma}{(q_{t+2})^{1-\sigma}} + (\beta \gamma)^\sigma
\]

where we have used the relation \( 1 + r_{t+1} = q_{t+2} \) which holds in equilibria with positive investment and thus in the steady state equilibria. Of particular interest is the Golden Rule steady state with prices \( q_t = 1 \) for \( t \in \mathbb{Z} \). Evaluating the demands (41) at the prices \( q_t = 1 \) for \( t \in \mathbb{Z} \) gives the value \( (c, h) = (c^y, h^y, c^m, h^m, c^r, h^r) \) of the agents’ consumption at the Golden Rule, from which it is easy to deduce the financial variables \( (b^y, b^m, z^m) \). In order that the borrowing constraint be binding for the young agents at the GR steady state we must have \( u'(c^y) > \beta u'(c^m) \) which is equivalent to

\[
    \text{CC' (binding collateral constraint)} \quad \frac{e^m}{e^y} \geq \frac{\beta^\sigma \left( 1 + \beta^\sigma + \gamma^\sigma \delta^{1-\sigma} + (\beta \gamma)^\sigma \right)}{1 + \gamma^\sigma \delta^{1-\sigma}}
\]
CC' reduces to CC when $\sigma = 1$: as in Section 3 we restrict the analysis to parameter values satisfying this condition.

To study the dynamics of the equilibrium after date 0, we substitute the above demand functions into the combined market-clearing equation (24) for consumption and the durable good: this gives a single implicit equation which must be satisfied by the equilibrium prices $(q_{t-1}, q_t, q_{t+1}, q_{t+2})$ at each date. The dynamics is however better understood by using the durable good price and investment as the basic variables and retaining the two market clearing equations at each date for consumption and the durable good. An equilibrium is then described by the pair of equations for all $t$

$$F(q_{t-1}, q_t, q_{t+1}, z_{t-1}, z_t) = 0$$
$$G(q_{t-1}, q_t, q_{t+1}, z_{t-1}, z_t) = 0$$

(43)

with appropriate boundary conditions, where $(F, G)$ are given by

$$F = c^y(q_t) + c^m(q_t, q_{t+1}) + c^r(q_{t-1}, q_t) + z_t - (e^y + e^m)$$
$$G = h^y(q_t) + h^m(q_t, q_{t+1}) + h^r(q_{t-1}, q_t) - z_{t-1} - (1 - \delta) \left( h^y_{t-1}(q_{t-1}) + h^m_{t-1}(q_{t-1}, q_t) \right)$$

the superscript ‘$m$’ on $z^m_t$ being omitted to simplify notation. A steady state $(q, z)$ is a solution to the pair of equations

$$F(q, q, q, z, z) = 0, \quad G(q, q, q, z, z) = 0$$

The first equation can be viewed as a steady state “supply of investment” $z$ as a function of $q$ and the second as a “demand for investment” $z$ to replace the durable good as a function of $q$. By Proposition 2 the Golden Rule price $q = 1$ along with the Golden rule investment $z^m$ is always a solution of these equations. Analyzing the pair of equations for different parameter values shows that the property found for the log case that there are two steady states, the Golden Rule and a real steady state (with no intermediation), carries over to the CES family when the elasticity of substitution is above a critical value $\sigma \geq \sigma^*$, while for low elasticities $\sigma < \sigma^*$ the real steady state disappears and the Golden Rule is the only steady state. $\sigma^*$ depends on the other parameters $(\beta, \gamma, \delta, e^y, e^m)$ and is less than one since the properties for $\sigma \geq \sigma^*$ are those for the log case.

Linearizing (43) around the Golden Rule steady state $(1, z^m)$ leads to the local dynamics

$$\begin{bmatrix} dq_{t+1} \\ dz_t \\ dq_t \end{bmatrix} = \Gamma \begin{bmatrix} dq_t \\ dz_{t-1} \\ dq_{t-1} \end{bmatrix}$$

(45)
with
\[
\Gamma = \begin{bmatrix}
-M^{-1}N \\
1 & 0 & 0
\end{bmatrix}, \quad M = \begin{bmatrix}
F_{q_{t+1}} & F_{z_t} \\
G_{q_{t+1}} & G_{z_t}
\end{bmatrix}, \quad N = \begin{bmatrix}
F_{q_t} & F_{z_{t-1}} & F_{q_{t-1}} \\
G_{q_t} & G_{z_{t-1}} & G_{q_{t-1}}
\end{bmatrix}
\]

where \((dq, dz_t)\) denote displacements from the steady state values \((1, z^m)\) and the partial derivatives in the matrices \(M\) and \(N\) are evaluated at the steady state. The dependence of \(F\) and \(G\) on investment implies \(F_{z_t} = 1, G_{z_t} = 0\) so that \(M\) is triangular. In view of (41) \(\sigma \neq 1\) implies \(G_{q_{t+1}} = \frac{\partial h}{\partial q_{t+1}} \neq 0\) so that \(M\) is invertible.

Since
\[
\Gamma = \begin{bmatrix}
-F_{q_{t+1}} + \frac{F_{q_{t+1}}}{G_{q_{t+1}}}G_{q_t} & \frac{1}{G_{q_{t+1}}} & -\frac{G_{q_{t-1}}}{G_{q_{t+1}}} \\
-F_{q_{t+1}} + \frac{F_{q_{t+1}}}{G_{q_{t+1}}}G_{q_t} & -\frac{F_{q_{t+1}}}{G_{q_{t+1}}} + \frac{F_{q_{t-1}}}{G_{q_{t+1}}}G_{q_{t+1}} & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

the characteristic polynomial whose zeros are the eigenvalues of \(\Gamma\) is given by
\[
G_{q_{t+1}}\lambda^3 + (G_{q_t} + F_{q_{t+1}})\lambda^2 + (G_{q_{t-1}} + F_{q_t})\lambda + F_{q_{t-1}} = 0 \quad (46)
\]

Since the local dynamics (45) has two predetermined variables \((dz_t, dq_t)\) and one forward variable \(dq_{t+1}\), in order that the system, when shocked at date 0, reverts to the steady state on a unique trajectory (46) must have two roots inside the unit circle and one outside.

It follows from Rouche’s Theorem (see e.g. Ahlfors (1979)) that if a polynomial \(a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 = 0\) is such that
\[
|a_k| > |a_n| + \cdots + |a_{k-1}| + |a_{k+1}| + \cdots + |a_0|
\]
i.e. if the coefficient of the term of order \(k\) exceeds the sum of the other coefficients, then the polynomial has \(k\) roots inside the unit circle and \(n - k\) outside. Thus a sufficient condition that there be two roots of the polynomial (46) inside the unit circle is
\[
|G_{q_t} + F_{q_{t+1}}| > |G_{q_{t+1}}| + |G_{q_{t-1}} + F_{q_t}| + |F_{q_{t-1}}| \quad (47)
\]
where the partial derivatives are evaluated at the Golden Rule \(q_{t-1} = q_t = q_{t+1} = 1\). Note that in the case where \(\sigma = 1\) (the log case), \(F_{q_{t-1}} = F_{q_{t+1}} = G_{q_{t+1}} = 0\) so that (46) reduces to \(G_{q_t}\lambda + G_{q_{t-1}} + F_{q_t} = 0\) and the condition that the Golden Rule is stable is
\[
|\lambda| = \frac{|G_{q_{t-1}} + F_{q_t}|}{|G_{q_t}|} < 1 \iff |G_{q_t}| > |G_{q_{t-1}} + F_{q_t}| \quad (48)
\]
which is equivalent to the stability condition (29). Thus (47) is a generalisation to the CES family of the stability condition (48) when \( \sigma = 1 \). It can also be expressed as a condition on the parameters \( (\beta, \delta, \gamma, \sigma, e^y, e^m) \) which is however much less tractable than the expression (29) to which it reduces when \( \sigma = 1 \). When \( \sigma < 1 \) it is satisfied for most values of the parameters except when \( \gamma \) is close to zero; when \( \sigma > 1 \) it requires that \( \gamma/\delta \) be sufficiently large, expressing the requirement that the durable good is sufficiently desirable and durable. The reason why (47) holds for most parameter values when \( \sigma < 1 \) is that the dominant term in (47) is the direct price effect \( G_q \) of a change in the current price of the durable on the (current) excess demand for this good: all other terms except \( G_q \) are indirect substitution effects which are small when \( \sigma < 1 \). \( G_q \) is essentially the direct effect of a change in price of the durable at date \( t-1 \) on the contemporaneous demand of the young and middle aged, but it has a coefficient \( 1 - \delta \) making it smaller (in absolute value) than \( G_q \).

Assuming that the parameters are such that (47) is satisfied, we show how to derive the deviations in the prices \( (dq_0, dq_1, dq_2) \) and investment \( (dz_0, dz_1) \) from the steady state and their subsequent deviations, following a once and for all shock to agents’ endowments at date 0.

**Negative shock.** Suppose that the economy has been following the GR steady state and that at date 0 the agents’ endowments are unexpectedly shocked to \( (e^y_0, e^m_0) = (1 + \Delta^-)(e^y, e^m) \); after date 0 the endowments return to the values \( (e^y, e^m) \). The excess demand function at date 0 can be written as

\[
F_0(e'^y_0, e'^m_0, q_0, z_0, q_1) = c'^y(e'^y_0, q_0) + c'^m(e'^m_0, q_0, q_1) + c'_n(q_0) + z_0 - (e'^y_0 + e'^m_0)
\]

\[
G_0(e'^y_0, e'^m_0, q_0, q_1) = h'^y(e'^y_0, q_0) + h'^m(e'^m_0, q_0, q_1) + h'_n(q_0) - H_0
\]

where

- \( H_0 \) is the supply of durable good inherited from date \(-1\), given by (31)
- the variables in the demand functions are those which differ from steady state values: \( (e'^y_0, e'^m_0) \) for the endowments, \( (q_0, z_0, q_1) \) for prices and investment
- the subscript ‘\( n \)’ (for negative shock) has been added to the functions which differ from the form given in (41). Since middle-aged agents default on their loans and begin ‘fresh’ at date 0 the only demand functions with a subscript ‘\( n \)’ are those of the retired agents at date 0.

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whose income is not what was anticipated at date $-1$. Their demand functions are

$$c^r_n(q_0) = \frac{I_0^r}{1 + \gamma q_0^{-1-\sigma}}, \quad h^r_n(q_0) = \frac{\gamma^\sigma I_0^r}{q_0^\sigma (1 + \gamma q_0^{-1-\sigma})}$$

(49)

where $I_0^r$ is given by (32).

At date 1 the effect of the date 0 shock to endowments still leads to excess demand functions different from $F$ and $G$, since the inherited stock of the durable and the demand of the retired agents depends on the shocked endowments ($e_y^0, e_m^0$).

$$F_1(e_y^0, e_m^0, q_0, q_1, z_1, q_2) = c^y(q_1) + c^m(q_1, q_2) + c^r(e_m^0, q_0, q_1) + z_1 - (e^y + e^m)$$

$$G_1(e_y^0, e_m^0, q_0, z_0, q_1, z_1, q_2) = h^y(q_1) + h^m(q_1, q_2) + h^r(e_m^0, q_0, q_1) - z_0 - (1-\delta)(h^y(e_y^0, q_0) + h^m(e_m^0, q_0, q_1))$$

From date 2 on the excess demand functions are given by $F$ and $G$ defined by (44) and the dynamics are described by (43) with initial conditions $(q_2, z_1, q_1)$. To determine these initial conditions we have four market clearing equations $F_0 = 0, G_0 = 0, F_1 = 0, G_1 = 0$ in five unknowns $(q_0, z_0, q_1, z_1, q_2)$. To obtain a determinate solution we assume that the date 0 shock to endowments is sufficiently small so that the deviation of the above variables from their steady-state values can be studied using the linear approximations to the market-clearing equations. In order that the variables return to their steady-state values after date 2 following the dynamics (45) we require that the initial conditions $(dq_2, dz_1, dq_1)$ lie in the stable subspace $\langle V_2, V_3 \rangle$ spanned by the eigenvectors corresponding to the eigenvalues inside the unit circle. Writing this condition

$$(dq_2, dz_1, dq_1) = \nu_2 V^2 + \nu_3 V^3$$

(50)

adds three equations and two unknowns ($\nu_2, \nu_3$) so that we now have seven equations in the seven unknowns

$$\xi = (dq_0, dz_0, dq_1, dz_1, dq_2, \nu_2, \nu_3)$$

The system of equations for calculating the changes in price and investment at dates 0 and 1 when the change in price $dq_2$ is compatible with convergence back to the steady state is

$$A\xi = -B[e^y, e^m] \Delta^-$$

(51)
Positive shock. As in Section 3 let $\Delta^+$ denote the percentage increase in agents’ endowments at date 0. The analysis is similar to the negative-shock case except that with a positive shock the demand of the middle-aged agents at date 0 and the retired agents at date 1 (who are the same agents) have a different form since their income consists not only of their endowment income $e^m_0$ but also of the capital gain on the durable good inherited from youth. The demand functions of these agents are thus based on the income

$$I^m_0 = e^m_0 + q_0(1-\delta) h^y - b^y$$

and are given by

$$c^m_p(e^m_0, q_0, q_1) = \frac{I^m_0}{J(q_0, q_1)}$$

$$h^m_p(e^m_0, q_0, q_1) = \frac{\gamma^s}{(q_0 - (1-\delta))^s} \frac{I^m_0}{J(q_0, q_1)}$$

$$c^r_p(e^m_0, q_0, q_1) = \frac{\beta^s(q_1)^s I^m_0}{J(q_0, q_1)}$$

$$h^r_p(e^m_0, q_0, q_1) = \frac{(\beta^s)^r I^m_0}{J(q_0, q_1)}$$

The excess demand functions $F0$ and $G0$ of the negative case are replaced by

$$\widetilde{F0}(e^y_0, e^m_0, q_0, z_0, q_1) = e^y(e^y_0, q_0) + c^m_p(e^m_0, q_0, q_1) + c^r_p(q_0) + z_0 - (e^y_0 + e^m_0)$$

$$\widetilde{G0}(e^y_0, e^m_0, q_0, q_1) = h^y(e^y_0, q_0) + h^m_p(e^m_0, q_0, q_1) + h^r_p(q_0) - H_0$$

and the functions $F1$ and $G1$ are replaced by

$$\widetilde{F1}(e^y_0, e^m_0, q_0, q_1, z_1, q_2) = e^y(q_1) + c^m(q_1, q_2) + c^r_p(e^m_0, q_0, q_1) + z_1 - (e^y + e^m)$$
\[\tilde{G}_1(e^y_0, e^m_0, q_0, z_0, q_1, z_1, q_2) = h^y(q_1) + h^m(q_1, q_2) + h^r_p(e^m_0, q_0, q_1) - z_0 - (1 - \delta)(h^y(e^y_0, q_0) + h^m(e^m_0, q_0, q_1))\]

where the demand functions which differ from those in (41) have a subscript \(p\) (positive shock).

The demand functions \((e^r_p, h^r_p)\) of the retired agents at date 0 are the same as those in (49); the demand functions which now differ importantly from those in the negative case are those of the middle aged: they no longer default, pay off their loans, and with their increased income increase their demand both for the consumption and the durable good.

Proceeding as in the negative-shock case we use the market clearing equations \(\tilde{F}_0 = 0, \tilde{F}_1 = 0, \tilde{G}_0 = 0, \tilde{G}_1 = 0\) and the three subspace equations (50) to determine \(\xi = (dq_0, dz_0, dq_1, dz_1, dq_2, \nu_2, \nu_3)\) by the system of linear equations

\[\tilde{A}\xi = -\tilde{B}[e^y, e^m]\Delta^+\]

where \(\tilde{A}\) and \(\tilde{B}\) are obtained from \(A\) and \(B\) by replacing the excess demand functions \((F_0, F_1, G_0, G_1)\) by their ‘tilde’ versions \(\tilde{F}_0, \tilde{F}_1, \tilde{G}_0, \tilde{G}_1\). From date 2 on the deviations \((dq_t, dz_t)\) are given by the local dynamics (45).

To study the response of the prices and investment to an anticipated shock to the agents’ endowments at date 0 we need to choose reference values for the parameters \((e^y, e^m, \beta, \gamma, \delta, \sigma)\). As in our earlier paper (Geanakoplos-Magill-Quinzii (2004)) we have taken the economic life of an agent to last for three periods, young, middle age and retirement. If childhood is included as the ‘non economic’ part of an agent’s life, and if the life span is 80 years then each period corresponds to 20 years. What matters for the agents’ endowments is not their magnitude but the relative magnitude \(e^y/e^m\). To reflect the fact that middle-aged agents are more productive we set \((e^y, e^m) = (2, 5)\). We choose \(\beta = 0.7\) corresponding to an annual discount rate of 2%. We think of the durable good as housing and choose \(\delta = 0.3\), implying that after 20 years 1/3 of a house needs to be replaced to maintain its original condition. If a young agent with CES preferences spends a proportion \(\pi\) of his income on the durable good and \(1 - \pi\) on the consumption good at the Golden Rule, then \(\delta \left(\frac{\gamma}{\delta}\right)^\sigma = \frac{\pi}{1 - \pi}\). To express the condition that the durable good is ‘desirable’ for the agent we assume that \(\pi \geq 1/4\) so that

\[\delta \left(\frac{\gamma}{\delta}\right)^\sigma \geq \frac{1/4}{3/4} = \frac{1}{3}\]

(53)

Since the elasticity of substitution affects the relative sizes of the effects on prices and investment, we present two cases, \(\sigma = 1/3\) and \(\sigma = 3\). For \(\sigma = 1/3\) the inequality (53) implies \(\gamma \geq 0.41\) and for \(\sigma = 3\) it implies \(\gamma \geq 0.31\): we choose \(\gamma = 0.5\) which is compatible with both cases.
Figures 2 and 3 show the deviations of the durable good prices \( dq_t \) and investment \( dz_t \) from the GR steady state following unanticipated negative and positive shocks of -10% and +10% to the agents’ endowments at date 0, for the reference values of the parameters and an elasticity of substitution of \( \sigma = \frac{1}{3} \). Figures 4 and 5 show the impulse response functions for the same values of the parameters and \( \sigma = 3 \). The curves \( IRF_q \) and \( IRF_z \)—the impulse response functions to the date 0 shock—have a superscript – and + when they represent the response of prices and investment to a negative and positive shock respectively. The dotted curve \( symIRF^+ \) is the symmetric image of \( IRF^+ \) with respect to the steady-state value (it graphs the values of \( q - dq^+_0 \) in Figure 2 and \( z^m - dz^+_0 \) in Figure 3) which shows the asymmetry between the response to the negative and positive shock. The price at date 0 responds more strongly to a positive than a negative shock \( (symIRF^+_q \) below \( IRF^-_q \)) while the investment responds more strongly \( (IRF^-_z \) below \( symIRF^+_z \)). Because the decrease in investment at date 0 when a negative shock occurs creates a low supply of the durable good at date 1 the price rebounds strongly, overshooting the steady state. Conversely the increase in investment following a positive shock depresses the price of the durable at date 1 below the steady state price. However since investment responds less strongly to a positive than a negative shock the fall in price is relatively less pronounced than the rise in the negative case \( (IRF^-_q \) stays below \( symIRF^+_q \) at date 1). As could be predicted, the response of the durable good price to an income shock is much larger when the elasticity of substitution is small \( (\sigma = 1/3) \) than when it is large \( (\sigma = 3) \): in the latter case the response of investment is somewhat larger since the demand for the consumption good varies more with the change in income.

5 Conclusion

This paper draws on the OLG model with a durable good produced from an all-purpose good to study an equilibrium in which the durable good serves as collateral for loans. Agents are assumed to be opportunistic and to default on their loans as soon as the value of the collateral falls below the value of the debt it secures. This assumption is a relatively good approximation of the mortgage market in the US where either by law or de facto mortgage loans are non-recourse loans: a lender can foreclose a house if the mortgage is not paid but is not allowed, or finds it too costly, to seize other income from a defaulting borrower. The model however ignores some aspects of the mortgage market: in practice the credit rating of a borrower falls if the borrower defaults and the depreciation
Figure 2: Impulse response function of the durable good price in the case of a negative shock ($IRF_q^-$), of a positive shock ($IRF_q^+$), and the symmetric image of the latter with respect to the steady state line ($symIRF_q^+$) for $\sigma = 1/3$.

Figure 3: Impulse response function of investment in the case of a negative shock ($IRF_z^-$), of a positive shock ($IRF_z^+$), and the symmetric image of the latter with respect to the steady state line ($symIRF_z^+$) for $\sigma = 1/3$. 
Figure 4: Impulse response function of the durable good price in the case of a negative shock ($\text{IRF}_\eta^-$), of a positive shock ($\text{IRF}_\eta^+$), and the symmetric image of the latter with respect to the steady state line ($\text{symIRF}_\eta^+$) for $\sigma = 3$.

Figure 5: Impulse response function of investment in the case of a negative shock ($\text{IRF}_z^-$), of a positive shock ($\text{IRF}_z^+$), and the symmetric image of the latter with respect to the steady state line ($\text{symIRF}_z^+$) for $\sigma = 3$. 
on a foreclosed house is substantially higher than on a non-distressed house.

We used the unanticipated-shock approach of macro to study how the equilibrium of demand and supply on the consumption and durable good markets responds to an initial income shock. Summarizing the economic life of an agent by three periods means that we have privileged the transparency and tractability of the model rather than its realism. Making the model more amenable to calibration by decreasing the length of the period and by incorporating more attributes of the mortgage market is left for future research.

6 References


