The Distribution of Money Balances and the Non-Neutrality of Money

Aleksander Berentsen  Gabriele Camera  Christopher J. Waller
University of Basel  Purdue University  University of Notre Dame

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Abstract

We construct a model where agents hold money for transactions purposes, and trade on a sequence of spot markets during a period. Agents choose spending strategies and cash holdings taking into account expected cash flows. Because buying and selling opportunities are idiosyncratic, intra-period heterogeneity in money holdings emerges. Due to this distribution, money transfers may increase output and welfare, in the short-run.
1 Introduction

In this paper we present a monetary framework in which increases in the money growth rate are detrimental to economic activity yet a one-time monetary injection, while neutral in the long-run, can have beneficial effects in the short-run on output and welfare.

The notion of long-run money neutrality but short-run non-neutrality dates back to Hume and has given rise to a large body of theoretical and empirical research (see Lucas, 1996). It is clear from this research that the answer one obtains regarding the neutrality of money hinges on the types of trading imperfections in the economy and the manner in which the money supply changes. Typical market frictions are Keynesian sticky prices and wages, imperfect information [Lucas (1972)], age heterogeneity [Samuelson (1958)], limited participation in financial markets [Lucas (1990), Fuerst (1992)] or segmented markets [Alvarez, Atkinson and Kehoe (2002)]. Typical monetary policy experiments include changes in the money stock, the money growth rate or the variance of the money growth rate. We also know from this literature that it matters whether the change is anticipated or unanticipated and whether it is done with lump-sum transfers of money or proportional transfers.

However, a drawback of many of these models is that they take a ‘shortcut’ by assuming money is necessary for trade without explaining why. While a shortcut may reduce the complexity of the model, it raises doubts as to whether the results obtained regarding the neutrality of money are driven by the shortcut itself or reflect a fundamental property of money. Rather than debate the validity of shortcuts, Wallace (1998) forcefully argued for constructing models that explain why money is necessary and then proceed to study how monetary changes affect the economy. There is now a growing literature in monetary economics that follows this strategy. We contribute to this literature by building a model emphasizing the medium of exchange role of money, in the tradition of Kiyotaki and Wright (1989,1993), to address the issue of long run versus short run neutrality of money.

Our model combines aspects of the monetary economies in Lagos and Wright
(2002) and Aruoba, Waller and Wright (2003). Infinitely lived agents trade on a sequence of competitive spot markets within a period and face idiosyncratic production and consumption shocks in some of these markets. There is anonymous trade, imperfect record-keeping and no contract enforcement. These frictions, combined with the absence of double coincidence of wants, are sufficient to make money necessary for trade.

The randomness of shocks gives rise to intra-period heterogeneity in money holdings; buyers deplete their money balances while sellers acquire them. This intra-period distribution of money holdings is the key innovation of the model and it drives the results we obtain regarding short-run non-neutrality and welfare. To make the money distribution analytically tractable, we assume that everyone can produce and consume in the last market. This allows agents to adjust their money holdings in such a way that the distribution of money holdings becomes degenerate in the last market under appropriate conditions.

With regard to money neutrality, we show that, in accordance with most monetary models, there is long-run neutrality from changes in the money stock and the optimal steady-state money growth rate follows the Friedman rule. However, the model is capable of generating short-run non-neutralities. We find that one-time injections of money, taking place in the middle of the trading sequence, can cause output and welfare to increase despite the resulting jump in the price level. In particular, for essentially all injection schemes we consider, announced or unannounced, aggregate production is positively affected if the steady-state money growth rate is sufficiently low. For sufficiently high money growth rates, production is unaffected by the injection but welfare may still increase.

We show that equilibrium spending patterns depend on whether steady-state money growth is low or high. In the simple case of three sequential markets, buyers spend only a fraction of their money in the first market so that they have some money to spend in the second market if need be. In short, they hold precautionary money balances. In the second market, poor buyers spend any remaining balances, while rich buyers do so only if money growth is sufficiently high. A monetary injection
relaxes the cash constraint of poor buyers and allows them to consume more on the margin. When money growth is low, rich buyers are not cash constrained and so they do not change their consumption decisions. As a result, aggregate production and welfare increase. When money growth is high, both rich and poor buyers are cash constrained but there is a redistribution of spending from rich agents to the poor agents who value consumption more on the margin. Thus, even though total demand for goods is unaffected, the redistribution of purchasing power from rich to poor buyers improves welfare on the margin. This result holds in all cases considered except when injections are announced and equally distributed to all agents. We demonstrate that this result generalizes to the case of \( n \) sequential markets and non-competitive pricing.

## 2 The Environment

The basic environment is that of Lagos and Wright (2002) modified as in Aruoba, Waller and Wright (2003). Time is discrete and in each period a \([0, 1]\) continuum of infinitely-lived agents trade on \( n = 3 \) Walrasian markets, that open and close sequentially. Only one market, denoted by \( j = 1, 2, 3 \), is open at any one time.

One perishable good is produced and consumed by all agents. Before entering the first two markets an agent receives one of two equally probable consumption/production shocks. He may want to consume or produce but not both. As a result, there is an equal number of consumers and producers in each market. Agents get utility \( u(q) \) from consuming \( q > 0 \) in the first two markets, where \( u(0) = 0 \), \( u'(q) > 0 \), \( u'(0) = \infty \), \( u'(\infty) = 0 \) and \( u''(q) \leq 0 \). In the last market all agents consume and produce, getting utility \( U(q) \) from \( q \) consumption, with \( U'(q) > 0 \), \( U'(0) = \infty \), \( U'(\infty) = 0 \) and \( U''(q) < 0 \).\(^1\) Production of \( q \) output generates disutility \( q \). The discount factor across dates is \( \beta \in (0, 1) \).

\(^1\)The difference in preferences over the good sold in the last market is a technical device we use to obtain a degenerate distribution of money holdings, at the beginning of a period.
trading, no record-keeping and no enforcement of contracts. The law of motion of the
money stock is $M_t = \mu M_{t-1}$. In $t$, agents receive a perfectly anticipated lump-sum
transfer of money $(\mu - 1)M_t$, while trading in the last market.

2.1 Sequential Market Trades in a Stationary Equilibrium

In period $t$, let $p_{j,t}$ be the nominal price in market $j$, and $\phi_t = 1/p_{3,t}$ be the real price
in the last market. We study equilibria where end-of-period real money balances are
time-invariant

$$\phi_t M_t \mu = \phi_{t-1} M_{t-1} \mu \Rightarrow \phi_t M_t = \phi_{t-1} M_{t-1}. \quad (1)$$

We refer to it as a stationary equilibrium. For this reason we omit the time subscript
when understood, and study a representative period working backwards from last to
first market, within the period.

Let $V_j(m_j)$ denote the expected value from trading in market $j$ with $m_j$ money.
Let $q_{j,b}$ and $q_{j,s}$ respectively denote the quantities bought or sold by an agent trading
in market $j$. We let $q_j^*$ be the solution to $U'(q_j) = 1$ and $q^*$ the solution to $u'(q_j) = 1$
for $j = 1,2$.

2.1.1 The last market

In the last market agents can produce and consume. They choose how much to buy,
$q_{3,b}$, how much to sell, $q_{3,s}$, and how much money to take into the next period, $m_{1,+1}$.
As a result, the representative agent’s program is

$$V_3(m_3) = \max_{q_{3,b}, q_{3,s}, m_{1,+1}} [U(q_{3,b}) - q_{3,s} + \beta V_1(m_{1,+1})]$$

s.t. $q_{3,b} + \phi m_{1,+1} = q_{3,s} + \phi m_3 + \phi(\mu - 1)M$

where $m_{1,+1}$ is the money taken into period $t + 1$. Substituting for $q_{3,s}$ yields

$$V_3(m_3) = \phi m_3 + \phi(\mu - 1)M + \max [U(q_{3,b}) - q_{3,b} - \phi m_{1,+1} + \beta V_1(m_{1,+1})] \quad (2)$$

where $(q_{3,b}, m_{1,+1})$ are choice variables, hence the conditions for maximization are

$$U'(q_{3,b}) = 1$$

$$-\phi + \beta V_1'(m_{1,+1}) \leq 0 \quad (= \text{ if } m_{1,+1} > 0) \quad (3)$$
so that $\beta V'_1(m_1) = \phi$ in a stationary monetary equilibrium, since $m_1 > 0$.

There are two key results. First, trades are always efficient in the last market, since $q_{3b} = q^*_3$ always and for every agent. Second, and most importantly, the distribution of beginning-of-period money holdings is degenerate. This is because $m_{1,+1}$ is chosen independently of $m_3$. The reason is $V_3(m_3)$ is linear, so the equilibrium marginal value of money in the last market is independent of the agent’s holdings, i.e.

$$V'_3(m_3) = \phi.$$  \hspace{1cm} (4)

It follows that in equilibrium everyone exits the last market with identical money holdings, regardless of how much money they brought into the last market. Those who bring excessive money into the last market, spend some on goods, while those with too little money sell output. Thus, everyone starts a period with identical money balances.\footnote{Conditions need to be imposed to ensure $q_{s3} \geq 0$. See later.} This feature of the Lagos and Wright model makes the distribution of money degenerate at the beginning of market one.\footnote{Thus, we avoid analytical intractabilities and the need to solve numerically for the distribution of money as done, for example, in Molico (1998) or İmrohorğlu (1992). See Shi (1997) for an alternative way to obtain a degenerate distribution.}

### 2.1.2 The second market

An agent who has $m_2$ money balances at the opening of the second market has expected lifetime utility

$$V_2(m_2) = \frac{1}{2} [u(q_{2b}) + V_3(m_2 - d_{2b})] + \frac{1}{2} [-q_{2s} + V_3(m_2 + d_{2s})]. \hspace{1cm} (5)$$

Here $d_{2b} = p_2 q_{2b}$ is the amount of money spent when buying $q_{2b}$ goods, and $d_{2s} = p_2 q_{2s}$ is the money received when selling $q_{2s}$ goods.

The agent chooses quantities to buy and sell, taking the price $p_2$ as given. Specifically, as a seller, the agent chooses $q_{2s}$ to maximize $-q_{2s} + V_3(m_2 + p_2 q_{2s})$. This yields the first-order condition

$$p_2 V'_3(m_2 + p_2 q_{2s}) = 1 \Rightarrow p_2 = p_3 = \frac{1}{\phi}.$$  \hspace{1cm} (6)
where we have used (4). That is prices in the last two markets must be equal and are pinned down by the agent’s value of money in the last market.

The intuition is that the seller can acquire a unit of money in the second or the third market and will do so at the lowest cost. Since sellers have linear production costs, if \( p_2 > p_3 \) it is cheaper to acquire money in the second market and vice versa if \( p_2 < p_3 \). At price \( p_2 = p_3 \) sellers are indifferent as to which market they sell in to acquire money. This also implies that they are willing to supply all that is demanded, so the supply curve in the second market is flat.

As a buyer, the agent chooses \( q_{2b} \) to maximize his expected utility \( u(q_{2b}) + V_3(m_2 - p_2 q_{2b}) \), given his cash constraint \( p_2 q_{2b} \leq m_2 \). Letting \( \lambda_2 \) be the multiplier on the cash constraint, the conditions for maximization are

\[
\begin{align*}
u'(q_{2b}) &= p_2 V'_3(m_2 - p_2 q_{2b}) + p_2 \lambda_2 \\
\lambda_2 (m_2 - p_2 q_{2b}) &= 0 \quad \text{and} \quad \lambda_2 \geq 0
\end{align*}
\]

Using (6) this reduces to

\[
\begin{align*}
u'(q_{2b}) &= 1 + \lambda_2 / \phi \\
\lambda_2 (m_2 - q_{2b} / \phi) &= 0 \quad \text{and} \quad \lambda_2 \geq 0
\end{align*}
\]

We can state the following

**Lemma 1.** Let \( m^* = q^*/\phi \). In equilibrium, if

(i) \( m_2 < m^* \) then \( \lambda_2 > 0 \), \( q_{2b} = \phi m_2 < q^* \) and \( V_2(m_2) \) is concave.

(ii) \( m_2 \geq m^* \) then \( \lambda_2 = 0 \), \( q_{2b} = q^* \leq \phi m_2 \) and \( V_2(m_2) \) is linear.

The key implication is that trades in the second market will be inefficient, \( q_{2b} < q^* \), if the buyer is cash constrained, \( m_2 < m^* \). They will not be only if the buyer has \( m_2 \geq m^* \).

As shown in the appendix, if the constraint is binding,

\[
V'_2(m_2) = \frac{u'(q_{2b}) + 1}{2p_2} > \phi
\]

whereas if it is not binding

\[
V'_2(m_2) = \phi.
\]
Intuitively, if the cash constraint does not bind then agents do not spend all of their money and its marginal value is given by its value from carrying it into the last market, \( \phi \). If the constraint binds, the marginal value of money is the expected value of money and has two components. With probability one half the agent spends the unit of money, buying \( \frac{1}{p_2} \) goods whose marginal utility is \( u'(q_{2b}) \). With probability one half, he does not spend it and values it at the prevailing market price \( \frac{1}{p_2} \). Since spending a marginal unit of money is optimal, then its marginal value must be greater than the value of simply holding onto it so \( V'_2(m_2) > \phi \).

### 2.1.3 The first market

An agent starting a period with \( m_1 \) money has expected lifetime utility

\[
V_1(m_1) = \frac{1}{2} [u(q_{1b}) + V_2(m_1 - d_{1b})] + \frac{1}{2} [-q_{1s} + V_2(m_1 + d_{1s})]
\]  

(10)

where \( d_{1s} = p_1 q_{1s} \) and \( d_{1b} = p_1 q_{1b} \) are, respectively, the amounts of money received as a seller and spent as a buyer. As a seller, the agent chooses \( q_{1s} \) to maximize his expected profit \( -q_{1s} + V_2(m_1 + p_1 q_{1s}) \), taking the price \( p_1 \) as given. This yields the first-order condition

\[
p_1 V'_2(m_1 + p_1 q_{1s}) = 1.
\]  

(11)

Production takes place until the marginal value of money \( V'_2(m_1 + p_1 q_{1s}) \), equals its real price \( 1/p_1 \). This money can be used to buy consumption in markets that open later.

As a buyer the agents chooses \( q_{1b} \) to maximize \( u(q_{1b}) + V_2(m_1 - p_1 q_{1b}) \) subject to the cash constraint \( p_1 q_{1b} \leq m_1 \). The conditions for maximization are

\[
u'(q_{1b}) = p_1 V'_2(m_1 - p_1 q_{1b}) + p_1 \lambda_1
\]

\[
\lambda_1 (m_1 - p_1 q_{1b}) = 0 \quad \text{and} \quad \lambda_1 \geq 0
\]

(12)

where \( \lambda_1 \) is the multiplier on the cash constraint.

We can state the following

**Lemma 2.** In equilibrium \( \lambda_1 = 0, q_{1b} = q_{1s} = q_1 \leq q^*, q_1 < m_1/p_1, \) and \( V_1(m_1) \) is concave if \( q_1 < q^* \).
The main implication of Lemma 2 is that agents never spend all of their money in the first market, \( q_1 < m_1/p_1 \). The marginal value of consuming even a little bit in the second market is very high for every agent, should a consumption opportunity arise. Consequently, for precautionary reasons, agents always want to carry some cash into the second market.

We also have that
\[
V_1'(m_1) = \frac{1}{p_1} \left[ \frac{u'(q_1) + 1}{2} \right].
\] (13)

Notice that the marginal value of money in the first market is given by an expression similar to (8). The difference is that \( 1/p_2 \) is equal to \( \phi \) whereas \( 1/p_1 \) may or may not be. As will be shown, this depends on the steady state money growth rate, \( \mu \).

\section{Equilibria}

It should be now clear that the idiosyncratic consumption and production shocks generate intra-period heterogeneity in money balances. As we demonstrate later, the existence of a non-degenerate distribution of money holdings is what opens the door to possible beneficial effects of money creation in the short-run.

More precisely, every agent enters a period with \( m_1 = M \) money (see Figure 1). Since agents are divided equally between buyers and sellers in markets 1 and 2, when the second market opens half of the agents will be ‘poor’, holding \( M - p_1q_1 \) money, and half will be ‘rich’, with \( M + p_1q_1 \) money. Agents will once more be divided into sellers and buyers. Since in equilibrium rich and poor buyers spend different amounts, and the sellers’ supply is flat, we allocate second market production by assuming that each seller produces for a randomly chosen buyer, and receives that buyer’s cash expenditure.\(^4\) Therefore, when market 3 opens the support of the distribution of \( m_1 \)...

\(^4\)If we had increasing marginal cost of production, then we would not have this problem. However, it would greatly complicate the analysis without changing the basic results.
money will have six mass points. However, all agents leave market 3 with the same money holdings.

Figure 1 - Equilibrium Heterogeneity

From what we have learned so far, consumption may differ across buyers only in the second market, due to heterogeneity in money holdings. Thus, let \((q_2^p, q_2^r)\) and \((\lambda_2^p, \lambda_2^r)\) denote consumption and cash constraint multipliers of, respectively, poor and rich buyers in market 2. If, by a small abuse in notation, we let \(q_2 = (q_2^p, q_2^r)\), \(\lambda_2 = (\lambda_2^p, \lambda_2^r)\), and let \(\bar{m}_j\) denote the vector of possible money holdings at the opening of market \(j\) we can state the following

**Definition.** A stationary monetary equilibrium is a time-invariant list \(\{p_j, q_j, \bar{m}_j\}_{j=1}^3\) and \(\{\lambda_1, \lambda_2]\) that satisfy (1)-(3), (5)-(7), and (10)-(12).

We can state our main proposition regarding the existence and uniqueness of equilibria as functions of the money growth rate.

**Proposition 1** A stationary monetary equilibrium exists if and only if \(\mu \geq \beta\).

For \(\mu \in (\beta, \bar{\mu}]\) a unique monetary equilibrium exists such that \(\lambda_2^p > 0\) and \(\lambda_2^r = 0\). For \(\mu > \bar{\mu}\), if an equilibrium exists it must be the case that \(\lambda_2^p > 0\) and \(\lambda_2^r > 0\).
The key implication of the proposition is that there are two possible monetary equilibria. In both equilibria, poor buyers are always cash constrained in the second market so $q^*_2 < q^*$. Rich buyers will be unconstrained if the money growth rate is low so they do not spend all of their cash and buy $q^*_2 = q^*$. If the money growth rate is too high they are also constrained and spend their remaining money balances in market 2 and buy $q^*_2 < q^*$. In what follows we characterize each equilibria.

3.1 Characterization of equilibria

Consider the case when the money growth rate is low. Here, rich buyers do not spend all of their money in the second market. We show in the proof of Proposition 1 that in this case prices are identical across markets within a period, $p_j = 1/\phi \forall j = 1, 2, 3$. The reason is that sellers in the first market know they will not spend all of their acquired money balances in the second market. Consequently, the remainder will be spent in the last market where its value is $\phi$ per unit. Thus, the value of acquiring a unit of money in the first market must be $\phi$ as well. So $p_1 = p_3 = 1/\phi$ but we also know $p_2 = p_3$.

A key policy implication of the model, as in Shi (1997) and Lagos-Wright (2002), is that money is neutral but not super-neutral and the Friedman rule sustains the most efficient trades. To see this note that in equilibrium the inflation rate is $\mu$ and the unique equilibrium value of $q_1$ satisfies (see proof of Proposition 1)

$$\frac{\mu - \beta}{\beta} = \frac{u'(q_1) - 1}{2}.$$ 

It is clear from this expression that changes in the money growth rate affect the equilibrium value of $q_1$. Thus, monetary policy can have real effects in this model. For $\mu > \beta$, $q_1 < q^*$ so trades are inefficient in the first market. As $\mu \to \beta$, $u'(q_1) \to 1$ and $q_1 \to q^*$ which also implies $q_2 \to q^*$ for both rich and poor buyers. Since $\mu = \beta$ corresponds to the Friedman rule in our framework, it generates the social optimum.

This equilibrium only exists for sufficiently low values of $\mu$. The reason is that real balances must be sufficiently high that rich buyers prefer to hold onto excess cash rather than spend it. Clearly, for sufficiently high inflation rates rich agents will
not want to do so. Consequently, this equilibrium can be sustained only under ‘low inflation’ i.e., small values of $\mu$.

To determine production in the last market, recall that $q_{3s} = q_{3b} - \phi m_3 - \phi(\mu - 1)M + \phi m_{1,+1}$, while in equilibrium $q_{3b} = q^*_3$ and $m_{1,+1} = \mu M$. Hence,

$$q_{3s} = q^*_3 - \phi m_3 + \phi M.$$  

Thus, in the last market, the richest agents produce very little, while the poorest agents produce more than they consume to increase their cash holdings. Using (37) and $m_3$ for the richest agent (see Figure 1) it is easy to show that $q_{3s} > 0$ as long as $q^*_3 \geq 2q^*$.

Now consider a money growth rate that is sufficiently high. Here both poor and rich buyers are cash constrained in the second market. As a result, no trade is efficient. An interesting feature of this equilibrium is that prices are no longer identical across markets. In particular $p_1 < p_2 = p_3$. The intuition is the following. Since both types of buyers are cash constrained in the second market, they would like to bring a bit more cash into the second market from the first market. Since sellers in the first market are the rich buyers in the second market, they now have an incentive to acquire a bit more money in the first market to spend in the second. They do so by selling more on the margin. This pushes down the price in the first market relative to the second and third markets.

To summarize our findings, two types of stationary monetary equilibria exist – a low and a high money growth equilibrium. In both cases, buyers do not spend their entire money balances in the first market, to self-insure against possible consumption opportunities in the second market. In the second market, however, poor buyers always spend any remaining balances, while rich buyers do so only in the high money growth equilibrium.

Money growth not only affects intertemporal nominal prices but also intra-period prices across markets. Nominal prices are constant across all markets (within a period) if money growth is low, but are lower in the first, relative to the second and last markets, if money growth is too high.
Thus, although our model has similarities with typical cash in advance models, it differs from them along two important dimensions: trade patterns and velocity of money. Unlike standard cash-in-advance models, some of the agents in our model acquire more money than they want to spend. Furthermore, the velocity of money in our model depends on the equilibrium money growth rate.

4 Monetary Injections and Welfare

Given that some buyers (possibly all) are cash constrained in the second market, a natural question is whether a one-time injection of money can be beneficial. For this reason we now consider a one-time, small and perfectly observable monetary injection that takes place after the first market closes but before the second market opens. In doing so we consider symmetric and asymmetric transfers that are: 1) unannounced or 2) announced before the first market opens. We do so for both the high and low money growth equilibrium.

Specifically, the poor receive a transfer \( \tau > 0 \), and the rich receive \( x\tau \) where \(-1 \leq x \leq 1\). This allows us to consider, for example, symmetric transfers \( x = 1 \), transfers only to the poor \( x = 0 \) and pure redistributions of money from rich to poor \( x = -1 \). The money supply changes as follows:

\[
M_{t+1} = \mu M_t + \frac{(1 + x) \tau}{2}.
\]

To ensure that we are in the low money growth equilibrium, we need to set \( \mu \) close to \( \beta \). For the high money growth equilibrium \( \mu \) has to be ‘large’. Since money is neutral in the long run, \( \phi M \) is constant, \( \phi \) adjusts instantaneously after the monetary injection takes place in the second market. Recalling that \( M = m_1 \)

\[
\frac{\partial \phi}{\partial \tau} = -\frac{(1 + x) \phi}{2m_1} \quad \text{and} \quad \frac{\partial p_2}{\partial \tau} = \frac{(1 + x) p_2}{2m_1}.
\]

If the transfer increases the stock of money, \(-1 < x \), the goods price of money falls in the last market and nominal prices rise in the second market. However, if the transfer is purely redistributive, \( x = -1 \), then there are no price effects. In this case
the money stock available in the last market is unaffected, so both prices do not change.

If the transfer is announced right before the first market opens, welfare is measured by

\[ W = \frac{1}{2} \left[ u(q_1) + V_2(m_1 - p_1q_1 + \tau) \right] + \frac{1}{2} \left[ -q_1 + V_2(m_1 + p_1q_1 + x\tau) \right] \]  

(15)

Differentiate (15) with respect to \( \tau \) and evaluate at \( \tau = 0 \) to get

\[ \frac{\partial W}{\partial \tau} = \frac{1}{2} \left[ u'(q_1) \frac{\partial q_1}{\partial \tau} + \frac{\partial V_2(m_1 - p_1q_1)}{\partial \tau} \right] + \frac{1}{2} \left[ -\frac{\partial q_1}{\partial \tau} + \frac{\partial V_2(m_1 + p_1q_1)}{\partial \tau} \right] \]  

(16)

From this expression it is clear that announced transfers affect agents’ decisions to buy and sell in the first market, even though the transfer does not occur until the second market. If the transfer is unannounced, then \( \frac{\partial q_1}{\partial \tau} = 0 \) so the welfare effects are captured completely by changes in \( V_2(m_2) \).

Monetary injections affect individual agents through two channels. First, they modify agents’ consumption and production decisions in the first and second markets, a substitution effect. Second, injections affect the agents’ real money balances, a wealth effect. We can state the following

**Proposition 2** Aggregate wealth effects are zero. Injections affect welfare only by modifying the quantities produced and consumed by agents with different cash holdings.

The basic reason for this is that \( V_3(m_3) \) is linear in \( m_3 \). Any price effects or changes in monetary expenditures simply increase one agent’s nominal wealth while decreasing the wealth of someone else by the same amount and these changes are valued equally due to linearity of \( V_3(m_3) \). Consequently, these changes cancel out in the aggregate.

We can now state our main welfare result

**Proposition 3** Transfers are welfare increasing except in one case – when the transfer is announced and \( x = 1 \). Welfare increases more when the transfers are unannounced, and the optimal transfer scheme in all cases is a pure redistribution of money from rich to poor.
Since there are no aggregate wealth effects, the key effect of the transfer is to loosen the liquidity constraints of at least some of the buyers. At the same time, the transfer causes prices to increase which tightens the constraints. In all but one case, the transfer more than compensates poor buyers for the higher prices. As a result they can consume more on the margin. For any value of $\mu > \beta$, the quantities traded are inefficient. Thus the marginal utility for poor buyers exceeds the marginal cost incurred by sellers. Hence, this additional consumption is welfare improving. Consumption of the rich buyers is unaffected in the low money growth equilibrium because they are not liquidity constrained. In the high money growth equilibrium, they are constrained but the price increase dominates the value of the transfer so their consumption falls on the margin. Due to concavity of the utility function, however, the marginal utility loss of the rich buyer is smaller than the marginal utility gain of the poor buyer. Thus, average welfare increases.\footnote{This effect on welfare is similar to that found, for example, in Green and Zhou (2003), Eden (1994), or Levine (1991).}

The only exception to this welfare result is when symmetric transfers are announced. Here, prices increase instantly and identically across all markets, which completely offset the transfer in terms of purchasing power. This is in contrast to the other cases where prices do not increase as much or do not increase equally across markets.

It is not surprising that unannounced transfers have bigger welfare effects because prices in the first market are unaffected. What is interesting is that welfare can go up even if the transfer is announced and prices in the first market increase before transfers take place. It is also interesting that transfers can have positive welfare effects in the high money growth equilibrium. This is surprising because this is the equilibrium with ample money creation, via lump-sum injections in the last market, yet monetary transfers in the second market are still welfare enhancing. One would be hard pressed to say ex ante that this is obvious in any sense.

With regards to production the following holds:
Proposition 4  Consider aggregate production in the second market. In the low money growth equilibrium, it increases when the transfer is announced if \( x < 1 \) and is constant if \( x = 1 \). When the transfer is not announced, it increases for all \( x \). In the high money growth equilibrium, it is totally unaffected by transfers.

The intuition is that in the low money growth equilibrium, poor buyers buy more with the transfers while rich buyers continue buying \( q^* \). Thus aggregate production increases in the second market. In the high money growth equilibrium, as explained above, the poor buyers are able to buy more while the rich buyers are able to buy less. These changes average to zero.

Finally, we end this section with a description of the change of the aggregate production in the first market. Clearly, production can be affected only when the transfer is announced.

Proposition 5  Consider aggregate production in the first market. When the transfer is announced and \( x < 1 \), it increases in the low and high money growth equilibria. It is unaffected if \( x = 1 \).

Parts of the welfare gains arise because consumption of agents with low marginal utilities (the rich) is reduced and given to agents with high marginal utilities (the poor agents). But note that there is more going on here because aggregate production across the period is increasing in all but one case when \( x < 1 \). The only case where aggregate production is constant across the period is when the injection is unannounced in the high money growth equilibrium. In this case the welfare gains are purely due to the shift in consumption from agents with low marginal utilities to agents with high marginal utilities. In all other cases, part of the welfare gains comes from the fact that there is a temporal increase in aggregate production and consumption.\(^6\)

\(^6\)The increase in output due to the loosening of cash constraints of some agents is similar in spirit to that of Sheinkman and Weiss (1986) or Wallace (1997)
5 Extensions

In this section we discuss three extensions. First, we generalize the model to the case of \( n > 3 \) markets. Second, we consider bargaining instead of competitive pricing. Third, we consider the case when all money transfers take place at the beginning of the second market.\(^7\)

5.1 More markets

If \( n > 3 \) markets open sequentially during a period, the model has to be adapted as follows. In the last market, \( n \), all agents can produce and consume. In markets \( j < n \) agents receive production and consumption shocks as seen for markets 1 and 2 in the previous sections.

Agents’ decisions in the last market are not affected by assuming \( n > 3 \) instead of \( n = 3 \). While money holdings will be more heterogeneous, the optimization problem is identical to the one studied earlier. Now consider any market \( j < n \). An agent entering market \( j \) with \( m_j \) money has expected lifetime utility

\[
V_j(m_j) = \frac{1}{2} [u(q_{jb}) + V_{j+1}(m_j - p_j q_{jb})] + \frac{1}{2} [-q_{js} + V_{j+1}(m_j + p_j q_{js})] \tag{17}
\]

Using a procedure similar to those in earlier sections, it is easy to verify that seller’s maximization requires

\[
p_j V'_{j+1}(m_j + p_j q_{js}) = 1. \tag{18}
\]

The maximization conditions for buyer’s optimization are

\[
u'(q_{jb}) = p_j V'_{j+1}(m_j - p_j q_{jb}) + p_j \lambda_j
\]

\[
\lambda_j(m_j - p_j q_{jb}) = 0 \quad \text{and} \quad \lambda_j \geq 0 \tag{19}
\]

where \( \lambda_j \) is the multiplier on the liquidity constraint.

Differentiating (17), recalling that \( \frac{\partial q_{js}}{\partial d_{js}} = \frac{\partial q_{jb}}{\partial d_{js}} = \frac{1}{p_j} \), and using (18)-(19) yields:

\[
V'_j(m_j) = \frac{1}{p_j} \left[ \frac{u'(q_{jb}) + 1}{2} \right]. \tag{20}
\]

\(^7\)So far we have assumed that only one-time money injections take place at the beginning of the second market, while repeated injections take place in the last market.
As before, the marginal value of money in market $j$ is the expected value of money, in that market.

5.1.1 Low money growth equilibrium

In market $j$ the distribution of money holdings will have $2^{j-1}$ elements. Consequently, in market $n-1$ buyers will differ in their money holdings and, much as in the previous sections, different types of equilibria will arise depending on which buyers are constrained. In fact, there can be $2^{n-1}$ different possible equilibria that are differentiated by the money growth rates that can sustain them.

In the following we focus on the equilibrium where in market $n-1$ only the poorest buyers’ are cash constrained. To relate this to the prior sections, we call this the low money growth equilibrium. Notice that in markets $j < n-1$ no agent is cash constrained, since the Inada condition make consumption in market $n-1$ so valuable that no agent wants to spend all his balances before reaching this market. Therefore, prices must be the same across markets.

To see this, note that because sellers have linear production cost, then (18) implies $p_{n-1} = p_n = 1/\phi$. Since rich buyers do not spend all their money in market $n-1$, $V'_{n-1}(m_{n-1}) = \frac{1}{p_{n-1}} = \phi$ for all but the poorest buyers. Then consider market $n-2$. Here the first-order conditions of rich sellers is $V'_{n-2}(m_{n-2} + p_{n-2}q_{n-2,s}) = \frac{1}{p_{n-2}}$, which implies $p_{n-2} = p_{n-1} = p_n = 1/\phi$. By proceeding in this way, one can show that $p_j = \frac{1}{\phi}$, $\forall j$.

If we consider markets $j < n-1$, then (20) holds for all $j$. In a steady state $\mu \phi = \phi_{-1}$ and $\beta V'_1(m_1) = \phi$; thus (20) for $j = 1$ implies

$$\frac{\mu - \beta}{\beta} = \frac{u'(q_{1b}) - 1}{2} \quad (21)$$

As seen before, $q_{1b} \leq q^*$ if $\mu \geq \beta$. Since consumption in the first market is independent of $n$, we can pin down the entire consumption sequence of agents who end up buying in every market. Their period budget constraint allows us to determine the equilibrium price sequence.
To see this, consider the first-order condition of a buyer in market $j$:

$$u'(q_{jb}) = p_j V'_{j+1}(m_{j+1}).$$  \hspace{1cm} (22)

Then, (20) and (22) imply

$$u'(q_{jb}) = \frac{u'(q_{j+1b}) + 1}{2} \quad \text{for} \quad j < n - 1.$$  \hspace{1cm} (23)

Thus, (21) gives us $q_{1b}$, and by using (23) recursively we get $q_{jb}$ for $j \leq n - 1$. Since the period budget constraint of the agent that always buys is $\sum_{j=1}^{n-1} p_j q_{jb} = m_1$, and $p_j = \frac{1}{\phi}$, we get

$$\phi m_1 = \sum_{j=1}^{n-1} q_{jb}.$$ \hspace{1cm} (24)

We now compare the low-money growth equilibrium consumption and price sequences across economies that differ only in the number of markets open in a period, $n$.

**Proposition 6** In the low money growth equilibrium prices, in every market, decrease in $n$. Consumption of the poorest buyer increases in $n$.

The proof is immediate. Since $q_{1b}$ is independent of $m_1$ and $n$, then (23) implies $q_j$ is also independent of $n$ and $m_1$. But then $\phi$ must grow in $n$ because $\sum_{j=1}^{n-1} q_{jb}$ is increasing in $n$ (see (24)).

As an example, consider the iso-elastic utility function $u(q) = q^{\alpha}/\alpha$ with $0 < \alpha < 1$. Then, $q_{1b} = \left(\frac{\beta}{2\mu-\beta}\right)^{\frac{1}{\alpha-1}}$ and (23) implies

$$q_{jb} = \left(\frac{2^j \left[ \frac{1}{\alpha-\beta} \right] + 1}{\alpha-\beta} + 1 \right)^{\frac{1}{\alpha}} \quad \text{for} \quad j \leq n - 1.$$  

Thus $q_{jb}$ is decreasing in $j$ and converges to zero when $j \to \infty$. Moreover, $q_{jb}$ falls in $\alpha$, since the surplus generated by each trading pair falls. This is because if $\alpha$ increases marginal utility approaches the constant disutility of production, at any consumption level.

\footnote{The buyer’s cash constraint is not binding in markets $j < n - 1$.}
The ratio of consumption between markets is
\[
\frac{q_{jb}}{q_{j+1b}} = \left( \frac{2^{j+1} \left[ \frac{\mu - \beta}{\beta} \right] + 1}{2^{j} \left[ \frac{\mu - \beta}{\beta} \right] + 1} \right)^{\frac{1}{1-\alpha}}
\]
increasing in \( \alpha \) because of consumption smoothing across markets. The larger is \( \alpha \), the less important is consumption smoothing across markets so agents want to shift consumption towards the first market. The ratio is also increasing in \( j \) because the closer one gets to the last market the more eager one is to spend all the money. The ratio \( \frac{q_{jb}}{q_{j+1b}} \) also increases in \( \mu \) because higher rates of money growth make it more costly to keep unspent cash.

Finally, from (24)
\[
\phi m_1 = \sum_{j=1}^{n-1} \left( \frac{1}{2^j \left[ \frac{\mu - \beta}{\beta} \right] + 1} \right)^{\frac{1}{1-\alpha}}.
\]
Therefore, real money balances at the beginning of a period rise in \( n \) and fall in \( \alpha \). The intuition is that as the number of markets increases, buyers can consume more often, in a given period. If \( \alpha \) increases, less surplus is created in any given trade, therefore the value of money falls.

For existence of this equilibrium, we need the richest sellers coming into the \( n^{th} \) market to be able to unravel their money holdings so that the distribution of money holdings is degenerate again. The simplest way to do this is to assume that in every market \( j < n \), the richest sellers (those who never buy) are assigned to sell to the poorest buyers (those who buy in every period). This does not imply that sellers know this is who they are selling to – it is simply a method for allocating equilibrium output across sellers. As a result, entering market \( n \) the richest sellers will be holding \( 2M \) in money balances – their own plus those of the poorest buyers who by construction spend all of their cash in markets 1 through \( n - 1 \). Thus, for the richest sellers to produce \( q_{ns} > 0 \), imposing \( m_1 = M \), we need
\[
q^*_n > 2\phi M = 2 \sum_{j=1}^{n-1} \left( \frac{1}{2^j \left[ \frac{\mu - \beta}{\beta} \right] + 1} \right)^{\frac{1}{1-\alpha}}.
\]
With this utility function, as $\mu \to \beta$, $q_j \to q^* = 1$ for all $j$, implying that $q_n^* > 2(n-1)$ is sufficient.

5.2 Bargaining

We now go back to $n = 3$ markets and consider terms of trade that, in the first two markets, are determined by take-it-or-leave-it offers from buyers to sellers. We assume in the first two markets each buyer is matched to a seller, so that all agents trade bilaterally. The last market is as before.

5.2.1 The second market

Let capital letters denote market variables, i.e. variables that are not controlled by the agent under consideration. The expected lifetime utility of entering the second market holding $m_2$ is

$$V_2 (m_2) = \frac{1}{2} [u (q_{2b}) + V_3 (m_2 - d_{2b})] + \frac{1}{2} [-Q_2s + V_3 (m_2 + D_{2s})] \quad (25)$$

where $d_{2b}$ is the amount of money spent as a buyer, and $D_{2s}$ represent money receipts as a seller.

In a match, a buyer solves the following problem

$$\max_{q_{2b}, d_{2b}} [u (q_{2b}) + V_3 (m_2 - d_{2b})]$$

s.t $-q_{2b} + V_3 (M_2 + d_{2b}) = V_3 (M_2)$ and $d_{2b} \leq m_2$

where $M_2$ denotes the seller’s money holdings. Since $q_{2b}$ is pinned down by the take-it-or-leave-it offer constrain, the first-order condition for $d_{2b}$ is

$$u'(q_{2b}) V_3' (M_2 + d_{2b}) = V_3' (m_2 - d_{2b}) + \lambda_2$$

Using (4), this reduces to

$$u'(q_{2b}) = \frac{\lambda_2}{\phi}. \quad (26)$$

If the constraint is slack, trade is efficient. If the constraint is binding, the buyer spends all his money getting $q_{2b} = m_2/p_2$, implying $q_{2b} < q^*$. One can prove a statement identical to Lemma 1, and that, in particular (8) and (9) hold in this case.
The first market

The expected lifetime utility of an agent who enters the first market holding \( m_1 \) is

\[
V_1(m_1) = \frac{1}{2} [u(q_{1b}) + V_2(m_1 - d_{1b})] + \frac{1}{2} [-Q_{1s} + V_2(m_1 + D_{1s})]. \tag{27}
\]

As a buyer, the agent program is

\[
\max_{q_{1b},d_{1b}} \quad [u(q_{1b}) + V_2(m_1 - d_{1b})] \\
\text{s.t.} \quad -q_{1b} + V_2(M_1 + d_{1b}) = V_2(M_1) \quad \text{and} \quad d_{1b} \leq m_1
\]

yielding the first-order condition

\[
u'(q_{1b}) V_2'(M_1 + d_{1b}) = V_2'(m_1 - d_{1b}) + \lambda_1. \tag{28}\]

By the same reasoning used earlier on, in the first market the buyer’s constraint is never binding, i.e. \( \lambda_1 = 0 \). Then

\[
u'(q_{1b}) = \frac{V_2'(m_1 - d_{1b})}{V_2'(M_1 + d_{1b})} \tag{29}\]

which implies \( q_{1b} < q^* \), by concavity of \( V_2(m_2) \).

Differentiating (27) we get (see appendix):

\[
V_1'(m_1) = \frac{u'(q_{1b}) V_2'(M_1 + d_{1b})}{2} + \frac{V_2'(m_1)}{2}. \tag{30}\]

Equation (30) determines the expected marginal value of entering market 1 with an additional unit of money. It has two components. With probability one half the agent is a buyer who can acquire \( V_2'(M_1 + d_{1b}) \) extra goods whose marginal utility is \( u'(q_{1b}) \). Note that the bargaining protocol adopted implies this additional consumption depends on the seller’s money holdings, pinning down his willingness to sell. Second, with probability one half the agent is a seller, in which case the value of the additional unit of money depends on his own money holdings, \( V_2'(m_1) \). Therefore the key difference relative to the competitive pricing case, is that now the agent faces two ‘prices’ in market 1, depending on whether he is a buyer or a seller.

Finally, using (29),

\[
V_1'(m_1) = \frac{V_2'(m_1 - d_{1b}) + V_2'(m_1)}{2}. \tag{31}\]
Intuitively, (31) can be interpreted as an arbitrage condition. Since there is no discounting across markets, during a period, the value of an additional unit of money in market 1 must equal its average marginal value in the second market given that the agent spends \(d_1\) units with probability one half and nothing otherwise.

Note that (31) has an interesting implication. Since \(0 < d_{1b} < m_1 \forall m_1 > 0\), the curvature of the value function in the first market is strictly smaller at \(m_1\) than the curvature of the value function in the second market at \(m_1\). Since this is true for any \(m_1\), the value of holding a given amount of money is higher in the first market than in the second market.

### 5.3 Repeated asymmetric transfers

We now relax the assumption that money growth occurs via transfers taking place while the third (last) market is open. Rather, the process of money creation takes place via regular transfers occurring right before the second opens, and that are fully announced. Assume \(M_{t+1} = \mu M_t\). The additional money is distributed as follows, before the second market opens, in each period. Each rich agent gets the transfer \(\tau^r_t = (\mu - 1)xM_t\) and each poor agent gets the transfer \(\tau^p_t = (\mu - 1)(2 - x)M_t\) where \(x \in [0, 1]\).

Consider a low-money growth steady state where prices increase at small but constant rate \(\mu\), so that beginning of period real money balances are constant. Here the rich buyers are never cash constrained. However, poor buyers are cash constrained in the second market. Their problem is

\[
\max_{q_{2b}, d_{2b}} \left[ u(q_{2b}) + V_3(m_2 - d_{2b} + \tau^p_t) \right]
\]

s.t.

\[ p_2q_{2b} = d_{2b} \text{ and } d_{2b} \leq m_2 + \tau^p_t \]

or

\[
\max_{q_{2b}} \left[ u(q_{2b}) + V_3(m_2 - p_2q_{2b} + \tau^p_t) \right] + \lambda_2 (m_2 + \tau^p_t - p_2q_{2b})
\]

Differentiate with respect to \(x\) to get

\[
u'(q_{2b}) \frac{dq_{2b}}{dx} - (\phi + \lambda_2) \left[ \frac{dp_2}{dx} q_{2b} + p_2 \frac{dq_{2b}}{dx} + (\mu - 1)M_t \right]
\]
From the first order condition \( u'(q_{2b}) - (\phi + \lambda_2) p_2 = 0 \) we get

\[-(\phi + \lambda_2) \left( \frac{dp_2}{dx} q_{2b} + \gamma M_t \right) < 0\]

because \( \frac{dp_2}{dx} = 0 \) (equilibrium prices are independent of how money is distributed).

Therefore, poor buyers are better off when \( x \) falls. In a similar way one can show that rich buyers are worse off when \( x \) falls. If we take a standard welfare measure that aggregates symmetrically the agents utilities, then the concavity of preferences implies that welfare rises as \( x \) falls.

We stress that given that there is a money creation policy in place, it is better to transfer new money entirely to agents with high marginal utilities (poor agents). However, this is only a second best policy. The optimal policy is still deflationary, at rate \( \beta - 1 \).

6 Conclusion

We have presented a framework in which a monetary expansion, while neutral in the long-run, can have beneficial effects in the short-run. The objective is to take a first step in complementing the large literature of the effects of money creation, by building on a recent research program that emphasizes the medium of exchange role of money.

The key feature of our model is that agents trade on a sequence of competitive markets while being subject to idiosyncratic shocks. For this reason, there is equilibrium heterogeneity in money balances, so that one-time monetary transfers can be used to redistribute purchasing power from rich to poor. Since this allows consumption to be redirected to those who most value it, societal welfare is positively affected. We are also able to show how the equilibrium money growth rates affected spending patterns and how one time injections can increase output. Interestingly this happens even if the transfers are announced, as long as they are not symmetric. In the long run, however, money creation is not welfare improving. Rather, the Friedman rule is the optimum. Finally, our main result on the neutrality of money is that unannounced injections expand output when inflation is low but has no quantitative effects.
when inflation is sufficiently high. This is somewhat comforting in that it suggests policymakers will only get positive expansions from surprise monetary injections if they are ’well-behaved’ in general by keeping inflation low. Otherwise, no gains in output will be realized.
References


Appendix

Proof of Lemma 1

If the constraint is not binding then \( \lambda_2 = 0 \). Using (7) then \( u'(q_{2b}) = 1 \). Here trades are efficient. The buyer spends \( q^*/\phi \) money, and we let \( m^* = q^*/\phi \) denote money holdings such that the cash constraint does not bind. If the constraint is binding, \( \lambda_2 > 0 \), then (7) implies

\[
u'(q_{2b}) = 1 + \frac{\lambda_2}{\phi} \quad \text{and} \quad p_2q_{2b} = m_2\]

(32)

Here trades are inefficient. The buyer spends all his money, \( p_2q_{2b} = m_2 < m^* \), and consumes \( q_{2b} = m_2/p_2 < q^* \).

To examine concavity of \( V_2 \) differentiate (5) with respect to \( m_2 \) to get

\[
V'_2(m_2) = \frac{1}{2} \left[ u'(q_{2b}) \frac{\partial q_{2b}}{\partial m_2} + V'_2(m_2 - d_{2b}) \left( 1 - \frac{\partial d_{2b}}{\partial m_2} \right) \right] + \frac{1}{2} \left[ \frac{\partial q_{2s}}{\partial m_2} + V'_2(m_2 + d_{2s}) \left( 1 + \frac{\partial d_{2s}}{\partial m_2} \right) \right]
\]

Since \( q_{2s} = \frac{d_{2s}}{p_2} \) and \( q_{2b} = \frac{d_{2b}}{p_2} \), then \( \frac{\partial q_{2s}}{\partial m_2} = \frac{\partial q_{2b}}{\partial m_2} = \frac{1}{p_2} = \phi \). Moreover, \( V'_2(m_2 - d_{2b}) = \phi \) and \( \frac{\partial d_{2b}}{\partial m_2} = 0 \) because \( q_{2s} \) is independent of the seller’s money holdings. Hence,

\[
V'_2(m_2) = \frac{\phi}{2} \left[ u'(q_{2b}) \frac{\partial d_{2b}}{\partial m_2} + \left( 1 - \frac{\partial d_{2b}}{\partial m_2} \right) \right] + \frac{\phi}{2}
\]

If \( \lambda_2 = 0 \), then relaxing the buyer’s cash constrain does not induce him to spend more. Thus, \( \frac{\partial d_{2b}}{\partial m_2} = 0 \) and \( V'_2(m_2) = \phi \), so \( V_2(m_2) \) is linear in \( m_2 \) for \( m \geq m^* \).

If \( \lambda_2 > 0 \), then \( \frac{\partial d_{2b}}{\partial m_2} = 1 \) hence

\[
V'_2(m_2) = \phi \left[ \frac{u'(q_{2b}) + 1}{2} \right] > \phi
\]

since \( u'(q_{2b}) > 1 \). Note that \( V''_2(m_2) < 0 \) because \( \frac{\partial q_{2b}}{\partial m_2} > 0 \), so that \( V_2(m_2) \) is concave \( \forall m_2 < m^* \).

Proof of Lemma 2

First prove that \( \lambda_1 = 0 \) always. Suppose \( \lambda_1 > 0 \). Then \( m_2 = 0 \). From (32), then \( q_{2b} = 0 \) and \( u'(0) = 1 + \lambda_2/\phi \), which is not possible since \( u'(0) = \infty \). Thus \( \lambda_1 = 0 \), in which case (11)-(12) yield

\[
u'(q_{1b}) = \frac{V'_2(m_1 - p_1q_{1b})}{V'_2(m_1 + p_1q_{1s})}.
\]

(33)
If \( m_1 - p_1 q_{1b} < m^* \) then \( V_2(m_1 - p_1 q_{1b}) \) is concave, hence \( u'(q_{1b}) > 1 \) and \( q_{1b} < q^* \). If \( m_1 - p_1 q_{1b} \geq m^* \) then both numerator and denominator are linear, hence \( u'(q_{1b}) = 1 \) and \( q_{1b} = q^* \). Hence, \( q_{1b} \leq q^* \).

Differentiating (10) with respect to \( m_1 \)

\[
V'_1(m_1) = \frac{1}{2} \left[ u'(q_{1b}) \frac{\partial d_{1b}}{\partial m_1} + V_2'(m_1 - d_{1b}) \left( 1 - \frac{\partial d_{1b}}{\partial m_1} \right) \right] + \frac{1}{2} \left[ - \frac{\partial q_{1s}}{\partial d_{1s}} \frac{\partial d_{1s}}{\partial m_1} + V_2'(m_1 + d_{1s}) \left( 1 + \frac{\partial d_{1s}}{\partial m_1} \right) \right]
\]

Since \( q_{1s} = \frac{d_{1s}}{p_1} \) and \( q_{1b} = \frac{d_{1b}}{p_1} \), then \( \frac{\partial q_{1s}}{\partial d_{1s}} = \frac{\partial q_{1b}}{\partial d_{1b}} = \frac{1}{p_1} \), hence the second line collapses to \( \frac{1}{2p_1} \). Using (11) then:

\[
V'_1(m_1) = \frac{1}{2} \left[ u'(q_{1b}) \frac{1}{p_1} \frac{\partial d_{1b}}{\partial m_1} + V_2'(m_1 - d_{1b}) \left( 1 - \frac{\partial d_{1b}}{\partial m_1} \right) \right] + \frac{1}{2p_1}.
\]

Using (12) then:

\[
V'_1(m_1) = \frac{1}{2} \left[ u'(q_{1b}) \frac{1}{p_1} \frac{\partial d_{1b}}{\partial m_1} + u'(q_{1b}) \frac{1}{p_1} \left( 1 - \frac{\partial d_{1b}}{\partial m_1} \right) \right] + \frac{1}{2p_1} = \frac{u'(q_{1b}) + 1}{2p_1}.
\]

Thus, if \( q_{1b} < q^* \) then \( V'_1(m_1) \) is concave. Note that we did not have to solve for \( \frac{\partial d_{1b}}{\partial m_1} \). However, it is straightforward to show that for \( \lambda_1 = 0, 0 < \frac{\partial d_{1b}}{\partial m_1} < 1 \). In short, if agents take an additional unit of money into the first market, they expect to spend only part of it, since they are not cash constrained. Since everyone enters the first market with identical money balances, and there is an identical number of buyers and sellers, in equilibrium, \( q_{1b} = q_{1s} = q_1 \) hence \( d_{1b} = d_{1s} = d_1 \).

**Proof of Proposition 1**

Suppose that \( \lambda_1^p = \lambda_2^r = 0 \). Then using (9), (12) and (13) yields

\[
u'(q_1) = \phi p_1 = 1 \Rightarrow q_1 = q^* \Rightarrow V'_1(m_1) = \phi\]

This condition says that if agents take a unit of money into the first market but do not intend to spend it in either the first or second markets, then the value of this extra unit of money is the goods it buys in the last market. Substituting this expression into (3), and backdating it, gives

\[-\phi_{-1} + \beta \phi \leq 0.\] (34)
In a steady state equilibrium, $\phi M = \phi_{-1} M_{-1}$, implying $\phi_{-1} = \phi \mu$. Using this in (34) yields $-\phi (\mu - \beta) \leq 0$. For $\mu > \beta$ this expression is negative implying $m_{1,+1} = 0$ which cannot be an equilibrium. For $\mu < \beta$, then $-\phi (\mu - \beta) > 0$ implying agents want to hold an infinite amount of money, which cannot be an equilibrium. For $\mu = \beta$, there is an infinity of monetary equilibria, one for each value of $\phi$.

Suppose $\lambda_{2}^{r} > \lambda_{2}^{p} = 0$. From (7) this is not feasible since $m_{2}$ is larger for rich agents.

Suppose $\lambda_{2}^{p} > \lambda_{2}^{r} = 0$. To derive $p_{1}$ use (3) and (11), noting that $u'(q_{1b}) > 1$. Then $V'(m_{1} + p_{1} q_{1}) = \phi \Rightarrow p_{1} = 1/\phi$, hence $p_{1} = p_{2} = p_{3} = 1/\phi$. Using this, (13) and (3) we get

$$\phi_{-1} = \beta \phi \left[ \frac{u'(q_{1}) + 1}{2} \right].$$

In a steady state equilibrium $\phi M = \phi_{-1} M_{-1} \Rightarrow \mu \phi = \phi_{-1}$. Thus,

$$\frac{\mu - \beta}{\beta} = \frac{u'(q_{1}) - 1}{2}.$$  \hspace{1cm} (35)

Because of strict concavity of $u(q)$ there is a unique value $q_{1}$ that solves (35), and for $\mu > \beta$, $q_{1} < q^{*}$. As $\mu \to \beta$, $u'(q_{1}) \to 1$ and $q_{1} \to q^{*}$. Note that $q_{2b}^{p} = q^{*}$, since rich buyers are unconstrained, and $q_{3b} = q_{3}^{*}$ from (3). In equilibrium, $m_{1} = M$. Since a poor buyer’s balances are $m_{2} = M - p_{1} q_{1}$, and he is cash constrained, then $p_{2} q_{2b}^{p} = M - p_{1} q_{1}$. Since $p_{1} = p_{2} = 1/\phi$, then $q_{2b}^{p} = \phi M - q_{1}$. Using this expression jointly with (8), (12) and $\lambda_{2}^{p} > 0$ implies

$$u'(q_{1}) = \frac{u'(\phi M - q_{1}) + 1}{2}.$$ \hspace{1cm} (36)

Thus, given $q_{1}$, there is a unique steady level of real balances that is decreasing in $\mu$ and satisfies $q_{1} < \phi M < q^{*} + q_{1}$. In the conjectured equilibrium, $\phi M$ must satisfy

$$q^{*} - q_{1} \leq \phi M \leq q^{*} + q_{1}.$$ \hspace{1cm} (37)

To see this, note that a poor buyer’s period budget constraint satisfies $p_{1} q_{1} + p_{2} q_{2b}^{p} = M$, while $p_{1} = p_{2} = 1/\phi$. Hence, $\phi M = q_{1} + q_{2b}^{p} \leq q^{*} + q_{1}$ because $q_{2b}^{p} \leq q^{*}$. For a rich buyer, however, $p_{2} q^{*} \leq M + p_{1} q_{1}$ because he is not constrained. Hence, $q^{*} - q_{1} \leq \phi M$. 29
Differentiating (36) yields
\[
\frac{\partial \phi M}{\partial q_1} = 1 + \frac{2u''(q_1)}{u''(\phi M - q_1)} > 1
\]
Real money balances increase more than one for one with increases in \(q_1\). As \(\mu \rightarrow \beta\), \(q_1 \rightarrow q^*\) and so \(q_{2b}^o = \phi M - q_1 \rightarrow q^*\) which implies that \(\phi M \rightarrow 2q^*\). In this equilibrium, under the Friedman rule, the beginning-of-period real balances of an agent are exactly what is needed to buy \(q^*\) in both the first and second market. Since \(q_1\) falls in \(\mu\), then (36) shows that real balances decline as well but at a faster rate. This implies that there exists a \(\bar{\mu} > \beta\) such that, \(q^* - q_1 > \phi M\) for \(\mu > \bar{\mu}\).

Suppose \(\lambda_2^o > 0, \lambda_2^r > 0\). Then \(q_{2b}^o = \phi m_1 + \phi p_1 q_1\), so (8) and (11) imply
\[
\frac{1}{p_1} = \phi \left[ \left( \frac{u'(\phi m_1 + \phi p_1 q_1) + 1}{2} \right) \right] \tag{38}
\]
hence \(p_1 < p_2 = 1/\phi\). Substituting this expression into (13) gives
\[
V_1'(m_1) = \phi \left( \frac{u'(q_1) + 1}{2} \right) \left( \frac{u'(\phi m_1 + \phi p_1 q_1) + 1}{2} \right)
\]
Note that \(V_1'(m_1)\) is greater now relative to the low money growth case. In equilibrium, \(m_1 = M\) so that \(q_{2b}^o = \phi M - \phi p_1 q_1\), and \(q_{2b}^r = \phi M + \phi p_1 q_1 < q^*\). From (3), in a stationary monetary equilibrium
\[
\frac{\mu}{\beta} = \left( \frac{u'(q_1) + 1}{2} \right) \left( \frac{u'(\phi M + \phi p_1 q_1) + 1}{2} \right). \tag{39}
\]
The right-hand side of (39) is a decreasing function of \(q_1\) and is greater than one since \(\phi M + q_1 < q^*\) under the conjectured equilibrium. Hence, (39) cannot be satisfied unless \(\mu\) is sufficiently high. Since \(\lambda_1 = 0\) expressions (8) and (33) imply
\[
u'(q_1) = \frac{u'(\phi M - \phi p_1 q_1) + 1}{u'(\phi M + \phi p_1 q_1) + 1}. \tag{40}
\]
For sufficiently large values of \(\mu\), (38)-(40) allow us to find \(q_1, p_1\) and \(\phi\).

**Proof of Proposition 2**

We first determine how a change in \(\tau\) affects \(V_3(m_3)\). From (2)
\[
\frac{\partial V_3(m_3)}{\partial \tau} = \frac{\partial \phi}{\partial \tau} m_3 + \frac{\partial m_3}{\partial \tau} \phi
\]
We next determine how $\tau$ affects $V_2(m_2)$ for poor and rich agents:

$$V_2(m_1 - d_1 + \tau) = \frac{1}{2} \left[ u(q_{2b}^p) + V_3(m_1 - d_1 + \tau - d_{2b}^p) \right]$$
$$+ \frac{1}{2} \left[ -q_{2s}^p + V_3(m_1 - d_1 + \tau + d_{2s}^p) \right]$$

$$V_2(m_1 + d_1 + x\tau) = \frac{1}{2} \left[ u(q_{2b}^r) + V_3(m_1 + d_1 + x\tau - d_{2b}^p) \right]$$
$$+ \frac{1}{2} \left[ -q_{2s}^r + V_3(m_1 + d_1 + x\tau + d_{2s}^p) \right]$$

Since we consider small transfers, differentiate $V_2(m_2)$ with respect to $\tau$ and evaluate at $\tau = 0$. Recalling that $V_3'(m_3) = \phi \forall m_3$:

$$\frac{\partial V_2(m_1 - d_1)}{\partial \tau} = \frac{1}{2} \left[ u'(q_{2b}^p) \frac{\partial q_{2b}^p}{\partial \tau} + \frac{\partial \phi}{\partial \tau} (m_1 - d_1 - d_{2b}^p) + \phi(1 - \frac{\partial d_1}{\partial \tau} + \frac{\partial d_{2b}^p}{\partial \tau}) \right]$$
$$- \frac{\partial q_{2s}^p}{\partial \tau} \left( m_1 - d_1 + d_{2s}^p + \phi(1 - \frac{\partial d_1}{\partial \tau} + \frac{\partial d_{2s}^p}{\partial \tau}) \right)$$

$$\frac{\partial V_2(m_1 + d_1)}{\partial \tau} = \frac{1}{2} \left[ u'(q_{2b}^r) \frac{\partial q_{2b}^r}{\partial \tau} + \frac{\partial \phi}{\partial \tau} (m_1 + d_1 - d_{2b}^p) + \phi(x + \frac{\partial d_1}{\partial \tau} - \frac{\partial d_{2b}^p}{\partial \tau}) \right]$$
$$- \frac{\partial q_{2s}^r}{\partial \tau} \left( m_1 + d_1 + d_{2s}^p + \phi(x + \frac{\partial d_1}{\partial \tau} + \frac{\partial d_{2s}^p}{\partial \tau}) \right)$$

If the injection is not announced, then $\frac{\partial q_{2s}}{\partial \tau} = 0$. Simplifying the expressions above:

$$\frac{\partial V_2(m_1 - d_1)}{\partial \tau} = \frac{1}{2} \left[ u'(q_{2b}^p) \frac{\partial q_{2b}^p}{\partial \tau} - \frac{\partial q_{2s}^p}{\partial \tau} \right] + \frac{1}{2} \left[ \frac{\partial \phi}{\partial \tau} (d_{2s}^p - d_{2b}^p) + \left( \frac{\partial \phi}{\partial \tau} - \frac{\partial \phi}{\partial \tau} \right) \phi \right]$$
$$+ \frac{\partial \phi}{\partial \tau} (m_1 - d_1) + \left( 1 - \frac{\partial d_1}{\partial \tau} \right) \phi$$

$$\frac{\partial V_2(m_1 + d_1)}{\partial \tau} = \frac{1}{2} \left[ u'(q_{2b}^r) \frac{\partial q_{2b}^r}{\partial \tau} - \frac{\partial q_{2s}^r}{\partial \tau} \right] + \frac{1}{2} \left[ \frac{\partial \phi}{\partial \tau} (d_{2s}^r - d_{2b}^r) + \left( \frac{\partial \phi}{\partial \tau} - \frac{\partial \phi}{\partial \tau} \right) \phi \right]$$
$$+ \frac{\partial \phi}{\partial \tau} (m_1 + d_1) + \left( x + \frac{\partial d_1}{\partial \tau} \right) \phi$$

Then,

$$\frac{\partial V_2(m_1 - d_1)}{\partial \tau} + \frac{\partial V_2(m_1 + d_1)}{\partial \tau} = \frac{1}{2} \left[ u'(q_{2b}^p) \frac{\partial q_{2b}^p}{\partial \tau} - \frac{\partial q_{2s}^p}{\partial \tau} + u'(q_{2b}^r) \frac{\partial q_{2b}^r}{\partial \tau} - \frac{\partial q_{2s}^r}{\partial \tau} \right]$$
$$+ 2 \frac{\partial \phi}{\partial \tau} m_1 + (1 + x) \phi$$

because $\frac{\partial \phi}{\partial \tau} (d_{2s}^p - d_{2b}^p + d_{2s}^r - d_{2b}^r) = 0$, i.e. payments received by the sellers must equal payments made by buyers, and $\left( \frac{\partial \phi}{\partial \tau} - \frac{\partial \phi}{\partial \tau} + \frac{\partial \phi}{\partial \tau} - \frac{\partial \phi}{\partial \tau} \right) \phi = 0$, i.e. additional payments received by the sellers must equal additional payments made by buyers. Then, (14) implies $2 \frac{\partial \phi}{\partial \tau} m_1 + (1 + x) \phi = 0$. Hence,

$$\frac{\partial V_2(m_1 - d_1)}{\partial \tau} + \frac{\partial V_2(m_1 + d_1)}{\partial \tau} = \frac{1}{2} \left[ u'(q_{2b}^p) \frac{\partial q_{2b}^p}{\partial \tau} - \frac{\partial q_{2s}^p}{\partial \tau} + u'(q_{2b}^r) \frac{\partial q_{2b}^r}{\partial \tau} - \frac{\partial q_{2s}^r}{\partial \tau} \right]$$

Consequently, the welfare effect (16) of an injection satisfies

$$\frac{\partial W}{\partial \tau} = \frac{1}{2} \left[ u'(q_1) - 1 \right] \frac{\partial q_1}{\partial \tau} + \frac{1}{4} \left[ u'(q_{2b}^p) \frac{\partial q_{2b}^p}{\partial \tau} - \frac{\partial q_{2s}^p}{\partial \tau} + u'(q_{2b}^r) \frac{\partial q_{2b}^r}{\partial \tau} - \frac{\partial q_{2s}^r}{\partial \tau} \right]$$

(41)
where $\frac{\partial q}{\partial \tau} = 0$ if the transfer is not announced. Since the only effects on welfare are from changes in quantities, not changes in prices or money balances, there is no aggregate wealth effect.

**Proof of Propositions 3 through 5**

The proof of Propositions 3 through 5 involve four Claims: the low and high money growth equilibria with and without announcement.

We first evaluate the welfare effect (41) in the low money growth equilibrium when the injections are not announced, Claim 1, and announced, Claim 2.

**Claim 1** *In the low money growth equilibrium with no announcement, an injection is strictly welfare increasing and increases aggregate production in the second market if*

$$x < \frac{m + d_1}{m - d_1}. \quad (42)$$

*Moreover, $\frac{\partial W}{\partial \tau}$ is strictly decreasing in $x$ and maximized at $x = -1$.***

**Proof of Claim 1**: In the low money growth equilibrium, only poor buyers are constrained, i.e. $p_2 q_2^p = m_1 - d_1 + \tau$. Totally differentiate this equation and evaluate at $\tau = 0$ to get

$$\frac{\partial q_2^p}{\partial \tau} = \frac{\phi}{2} \frac{m_1 (1 - x) + d_1 (1 + x)}{m_1} > 0. \quad (43)$$

Rich buyers are not constrained so $\frac{\partial q_2^r}{\partial \tau} = 0$. Thus, aggregate production in the second market changes by $\frac{1}{4} \frac{\partial q_2^p}{\partial \tau} + \frac{1}{4} \frac{\partial q_2^r}{\partial \tau} = \frac{1}{8} \frac{m_1 (1 - x) + d_1 (1 + x)}{m_1}$, strictly positive if (42) holds.

Each seller is equally likely to meet a poor or a rich buyer. Consequently, the expected additional production for a seller is

$$\frac{\partial q_2^p}{\partial \tau} = \frac{\phi}{2} \frac{m_1 (1 - x) + d_1 (1 + x)}{m_1}.$$

With no announcement $\frac{\partial \psi}{\partial \tau} = 0$ and so (41) reduces to

$$\frac{\partial W}{\partial \tau} = \frac{1}{4} \frac{m_1 (1 - x) + d_1 (1 + x) \phi}{2} \left( u' \left( q_2^p \right) - 1 \right) > 0.$$
$u'(q_{2b}^p) + 1 = 2u'(q_1)$. Hence

$$\frac{\partial W}{\partial \tau} = \frac{m_1 (1-x) + d_1 (1+x) \phi}{2 m_1} \left( u'(q_1) - 1 \right) > 0. \quad (44)$$

Since $u'(q_1) > 1$ when $\mu > \beta$, $\frac{\partial W}{\partial \tau} > 0$ if $x < \frac{m_1 + d_1}{m_1 - d_1}$.

Finally, note that the injection can be welfare increasing even when more money is given to the rich than to the poor, i.e. $1 < x < \frac{m_1 + d_1}{m_1 - d_1}$. However, reducing $x$ increases the beneficial effect of the injection and this effect is maximal with the purely redistributive scheme $x = -1$.

**Claim 2** In the low money growth equilibrium with announcement, an injection is strictly welfare increasing and increases aggregate production in the second and first market if $x < 1$. If $x = 1$, there is no welfare effect and aggregate production in the second and first market is unchanged. Moreover, $\frac{\partial W}{\partial \tau}$ is strictly decreasing in $x$ and maximized at $x = -1$.

**Proof of Claim 2:** For a poor buyer we have $p_2 q_{2b}^p = m_1 - d_1 + \tau$. Totally differentiate this equation and evaluate at $\tau = 0$ to get

$$\frac{\partial p_2}{\partial \tau} q_{2b}^p + \frac{\partial q_{2b}^p}{\partial \tau} p_2 = \frac{(1+x)p_2}{2m_1} q_{2b}^p + \frac{\partial p_2}{\partial \tau} p_2 = 1 - \frac{d_1}{\partial \tau}.$$  

Because $p_2 q_{2b}^p = m_1 - d_1$ when $\tau = 0$:

$$\frac{\partial q_{2b}^p}{\partial \tau} = \phi \left( \frac{m_1 (1-x) + d_1 (1+x)}{2m_1} - \frac{\partial d_1}{\partial \tau} \right). \quad (45)$$

Then, $p_1 q_1 = d_1$, $q_1 \frac{\partial q_1}{\partial \tau} + p_1 \frac{\partial q_1}{\partial \tau} = \frac{\partial d_1}{\partial \tau}$, and $\frac{\partial q_1}{\partial \tau} = \frac{(1+x)m_1}{2m_1}$ (recall $p_1 = p_2$) imply that

$$\frac{\partial q_{2b}^p}{\partial \tau} = \phi \left( \frac{1-x}{2} - p_1 \frac{\partial q_1}{\partial \tau} \right). \quad (46)$$

Rich buyers are not constrained, hence $\frac{\partial q_{2b}^p}{\partial \tau} = 0$. This implies aggregate production in the second market changes by $\frac{1}{4} \frac{\partial q_{2b}^p}{\partial \tau} + \frac{1}{4} \frac{\partial q_{2b}^p}{\partial \tau} = \phi \left( \frac{1-x}{2} - p_1 \frac{\partial q_1}{\partial \tau} \right)$. Since sellers meet both types of buyers with equal probability, $\frac{\partial q_{2b}^p}{\partial \tau} = \frac{\partial q_{2b}^p}{\partial \tau} = \phi \left( \frac{1-x}{2} - p_1 \frac{\partial q_1}{\partial \tau} \right)$.

With announcement (41) becomes

$$\frac{\partial W}{\partial \tau} = \frac{1}{2} \left[ \left( u'(q_1) - 1 \right) \frac{\partial q_1}{\partial \tau} + \frac{1}{2} \left( u'(q_{2b}^p) - 1 \right) \phi \left( \frac{1-x}{2} - p_1 \frac{\partial q_1}{\partial \tau} \right) \right].$$

In the proof of Claim 1 we have shown that $u'(q_{2b}^p) + 1 = 2u'(q_1)$. Using this relation and rearranging yields

$$\frac{\partial W}{\partial \tau} = \left( \frac{1-x}{2} \right) \phi \left( \frac{2}{2} (u'(q_1) - 1) \right)$$
Since \( u'(q_1) > 1 \) when \( \mu > \beta, \frac{\partial W}{\partial \tau} > 0 \) if \( x < 1 \).

To see that aggregate production is increasing in the second market if \( x < 1 \), note that:
\[
\frac{\partial q_1}{\partial \tau} = \frac{u''(q_{2b})}{2u'(q_1)} \frac{\partial q_{2b}^p}{\partial \tau}
\]

Then, (46) simplifies to
\[
\frac{\partial q_{2b}^p}{\partial \tau} = \phi \left( \frac{1 - x}{2} \right) \left( \frac{2u''(q_1) + u''(q_{2b})}{2u'(q_1)} \right) > 0.
\]

Aggregate production in the first market is increasing if \( x < 1 \) because \( \frac{\partial q_1}{\partial \tau} = \frac{u''(q_{2b})}{u'(q_1)} \frac{\partial q_{2b}^p}{\partial \tau} > 0 \).

If we compare Claims 1 and 2 we find the following: First, in both cases \( \frac{\partial W}{\partial \tau} \) is strictly decreasing in \( x \). Second, aggregate production in the second market is increasing in the low money growth equilibrium if \( x < 1 \). If \( x = 1 \), it is increasing only if the injection is not announced. Finally, the welfare effect with no announcement dominates the welfare effect without if \( x > -1 \), because \( \frac{m_1(1-x) + d_1(1+x)}{m_1} > \frac{(1-x)}{2} \) if \( x > -1 \).

We next evaluate the welfare effect (41) in the high money growth equilibrium when the injections are not announced, Claim 3, and announced, Claim 4.

**Claim 3** In the high money growth equilibrium with no announcement, an injection is strictly welfare increasing if \( x < \frac{m_1+d_1}{m_1-d_1} \). Moreover, \( \frac{\partial W}{\partial \tau} \) is strictly decreasing in \( x \) and maximized at \( x = -1 \). Finally, for any \( x \), the injection does not affect aggregate production in the second market.

**Proof of Claim 3**: In the high money growth equilibrium, all buyers are constrained. For poor buyers from (43) we have
\[
\frac{\partial q_{2b}^p}{\partial \tau} = \phi \frac{m_1(1-x) + d_1(1+x)}{2m_1} > 0
\]
For a rich buyer we have \( p_2q_{2b}^r = m_1 + d_1 + x \tau \). Hence,
\[
\frac{\partial q_{2b}^p}{\partial \tau} + \frac{\partial q_{2b}^r}{\partial \tau} p_2 = \frac{(1+x)p_2}{2m_1} q_{2b}^r + \frac{\partial q_{2b}^r}{\partial \tau} p_2 = x.
\]
Because \( p_2q_{2b}^r = m_1 + d_1 \) when \( \tau = 0 \):
\[
\frac{\partial q_{2b}^r}{\partial \tau} = -\phi \frac{m_1(1-x) + d_1(1+x)}{2m_1}.
\]
Note that $\frac{\partial q_{2b}^p}{\partial \tau} + \frac{\partial q_{2b}^r}{\partial \tau} = 0$. Thus, because each seller meets with equal probability a poor or a rich buyer, the expected additional production for a seller is zero. With no announcement $\frac{\partial V}{\partial \tau} = 0$ and so (41) reduces to

$$\frac{\partial V}{\partial \tau} = \frac{1}{4} \left[ u' \left( q_{2b}^p \right) \frac{\partial q_{2b}^p}{\partial \tau} + u' \left( q_{2b}^r \right) \frac{\partial q_{2b}^r}{\partial \tau} \right]$$

$$= \frac{1}{4} \left[ (u' \left( q_{2b}^p \right) - u' \left( q_{2b}^r \right)) \phi \frac{m_1 (1-x) + d_1 (1+x)}{2m_1} \right]$$

We know that in this equilibrium $u' \left( q_{2b}^p \right) + 1 = \frac{2u'(q_1)p_2}{p_1}$ and $u' \left( q_{2b}^r \right) + 1 = \frac{2p_2}{p_1}$. Hence

$$\frac{\partial V}{\partial \tau} = \frac{m_1 (1-x) + d_1 (1+x)}{4p_1m_1} (u'(q_1) - 1) > 0.$$ 

Since $u'(q_1) > 1$ when $\mu > \beta$, $\frac{\partial V}{\partial \tau} > 0$ if $x < \frac{m_1 + d_1}{m_1 - d_1}$.

**Claim 4** In the high money growth equilibrium with announcement, an injection is strictly welfare increasing if $x < 1$. Moreover, $\frac{\partial V}{\partial \tau}$ is strictly decreasing in $x$ and maximized at $x = -1$. Finally, for all $x$ an injection does not affect aggregate production in the second market.

**Proof of Claim 4:** From (45), for a poor buyer we have

$$\frac{\partial q_{2b}^p}{\partial \tau} = \phi \left( \frac{m_1 (1-x) + d_1 (1+x)}{2m_1} - \frac{\partial d_1}{\partial \tau} \right) = \phi \left( \frac{1-x}{2} - \frac{p_1 \partial q_1}{\partial \tau} \right) \quad (47)$$

For rich buyers, who are also constrained, we have $p_2 q_{2b}^r = m_1 + d_1 + x \tau$. Hence,

$$\frac{\partial q_{2b}^r}{\partial \tau} = \frac{(1+x)p_2 q_{2b}^r}{2m_1} + \frac{\partial q_{2b}^r}{\partial \tau} p_2 = x + \frac{\partial d_1}{\partial \tau}.$$ 

Because $p_2 q_{2b}^r = m_1 + d_1$ when $\tau = 0$: 

$$\frac{\partial q_{2b}^r}{\partial \tau} = -\phi \left( \frac{m_1 (1-x) + d_1 (1+x)}{2m_1} - \frac{\partial d_1}{\partial \tau} \right).$$

Note that $\frac{\partial q_{2b}^p}{\partial \tau} + \frac{\partial q_{2b}^r}{\partial \tau} = 0$, which implies that aggregate production in the second market is constant. Also, because each seller meets equally likely a poor or a rich buyer, the expected extra production for a seller is 0. With announcement (41) is

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \left( u' \left( q_1 \right) - 1 \right) \frac{\partial q_1}{\partial \tau} + \frac{1}{4} \left[ (u' \left( q_{2b}^p \right) - u' \left( q_{2b}^r \right)) \frac{\partial q_{2b}^p}{\partial \tau} \right]$$

In this equilibrium $u' \left( q_{2b}^p \right) + 1 = \frac{2u'(q_1)p_2}{p_1}$ and $u' \left( q_{2b}^r \right) + 1 = \frac{2p_2}{p_1}$. Using these relations and rearranging yields

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \left( u' \left( q_1 \right) - 1 \right) \left[ \frac{\partial q_1}{\partial \tau} + \frac{p_2}{p_1} \frac{\partial q_{2b}^p}{\partial \tau} \right]$$

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Then, \((47)\) implies
\[
\frac{\partial W}{\partial \tau} = \frac{1}{2p_1} \left( u'(q_1) - 1 \right) \left( \frac{1-x}{2} \right)
\]
Since \(u'(q_1) > 1\) when \(\mu > \beta\), \(\frac{\partial W}{\partial \tau} > 0\) if \(x < 1\).

Aggregate production in the first market is increasing. To see this, note that
\[
u'(q_{2b}) + 1 = \frac{2u'(q_1)p_2}{p_1}
\]
and \(u''(q_{2b}) + 1 = \frac{\partial q_{2b}}{\partial \tau}\) imply
\[
u''(q_{2b}) \frac{\partial q_{2b}}{\partial \tau} = u''(q_{2b}) u'(q_1) \frac{\partial q_{2b}}{\partial \tau} + (u'(q_{2b}) + 1) u''(q_1) \frac{\partial q_{2b}}{\partial \tau}
\]
Then, \(\frac{\partial q_{2b}}{\partial \tau} + \frac{\partial q_{2b}}{\partial \tau} = 0\) implies
\[
[u''(q_{2b}) + u''(q_{2b}) u'(q_1)] \frac{\partial q_{2b}}{\partial \tau} = (u'(q_{2b}) + 1) u''(q_1) \frac{\partial q_{2b}}{\partial \tau}
\]
Hence, \(\frac{\partial q_{2b}}{\partial \tau}\) and \(\frac{\partial q_{2b}}{\partial \tau}\) must have the same sign. Both must be strictly positive since \(\frac{\partial W}{\partial \tau} > 0\) if \(x < 1\). Hence, in the high money growth equilibrium when the transfer is announced and \(x < 1\) aggregate production is strictly increasing in the first market.

If we compare Claims 3 and 4 we find the following. In both cases \(\frac{\partial W}{\partial \tau}\) is strictly decreasing in \(x\). Second, aggregate production in the second market is constant in both cases. Finally, the welfare effect with no announcement is the largest if \(x > -1\) because \(m_1(1-x) +d_1(1+x)\) is strictly increasing in \(x\).

**Technical derivations for the case of Bargaining**

Expression (25) implies
\[
V_2'\left( m_2 \right) = \frac{1}{2} \left[ u'(q_{2b}) \frac{\partial q_{2b}}{\partial m_2} \frac{\partial d_{2b}}{\partial m_2} + V_3'\left( m_2 - d_{2b} \right) \left( 1 - \frac{\partial d_{2b}}{\partial m_2} \right) \right.
\]
\[-\frac{\partial Q_{2b}}{\partial D_{2s}} \frac{\partial D_{2s}}{\partial m_2} + \left. V_3'\left( m_2 + D_{2s} \right) \left( 1 + \frac{\partial D_{2s}}{\partial m_2} \right) \right]\]
Note that \(V_2'\left( m_2 + D_{2s} \right) = \phi\) and that if the agent is a seller, then \(-Q_{2s} + V_3\left( m_2 + D_{2s} \right) = V_3\left( m_2 \right)\), which implies \(\frac{\partial Q_{2s}}{\partial D_{2s}} = \phi\). Consequently,
\[
V_2'\left( m_2 \right) = \frac{1}{2} \left[ u'(q_{2b}) \frac{\partial q_{2b}}{\partial d_{2b}} \frac{\partial d_{2b}}{\partial m_2} + V_3'\left( m_2 - d_{2b} \right) \left( 1 - \frac{\partial d_{2b}}{\partial m_2} \right) \right] + \frac{\phi}{2}
\]
If the agent is a buyer, then $-q_{2b} + V_3 (M_2 + d_{2b}) = V_3 (M_2)$, implying $\frac{\partial q_{2b}}{\partial m_{2b}} = \phi$.

Hence, since $V_3' (m_2 - d_{2b}) = \phi$

$$V_2' (m_2) = \frac{\phi}{2} \left[ u' (q_{2b}) - \frac{\lambda_2}{\phi} \left( 1 - \frac{\partial d_{2b}}{\partial m_2} \right) \right] + \frac{\phi}{2}.$$  

Recalling that if $\lambda_2 > 0$ then $\frac{\partial d_{2b}}{\partial m_2} = 1$ implies

$$V_2' (m_2) = \phi \left[ \frac{u' (q_{2b}) + 1}{2} \right]$$

for $\lambda_2 \geq 0$. If $\lambda_2 = 0$, then $q_{2b} = q^*$ and $V_2' (m_2) = \phi$. If $\lambda_2 > 0$, then $q_{2b} < q^*$ which implies that (8). Since $u' (q_{2b}) > 1$. For $m < m^*$, $\frac{\partial q_{2b}}{\partial m_2} > 0$ so $V_2'' (m_2) < 0$. Thus, $V_2(m_2)$ is concave in $m_2$.

Using (27) we get

$$V_1' (m_1) = \frac{1}{2} \left[ u' (q_{1b}) \frac{\partial q_{1b}}{\partial m_1} \frac{\partial d_{1b}}{\partial m_1} + V_2' (m_1 - d_{1b}) \left( 1 - \frac{\partial d_{1b}}{\partial m_1} \right) \right] + \frac{1}{2} \left[ -\frac{\partial Q_{1s}}{\partial m_1} \frac{\partial D_{1s}}{\partial m_1} + V_2' (m_1 + D_{1s}) \left( 1 + \frac{\partial D_{1s}}{\partial m_1} \right) \right]$$

If the agent is a seller, $-Q_{1s} + V_2' (m_1 + D_{1s}) = V_2 (m_1)$, which implies $-\frac{\partial Q_{1s}}{\partial D_{1s}} \frac{\partial D_{1s}}{\partial m_1} + V_2' (m_1 + D_{1s}) \left( 1 + \frac{\partial D_{1s}}{\partial m_1} \right) = V_2 (m_1)$. Consequently, the second line of the expression above is $\frac{1}{2} V_2' (m_1)$. If the agent is a buyer, $-q_{1b} + V_2 (M_1 + d_{1b}) = V_2 (M_1)$, which implies $\frac{\partial q_{1b}}{\partial d_{1b}} = V_2' (M_1 + d_{1b})$. As a result

$$V_1' (m_1) = \frac{1}{2} \left[ u' (q_{1b}) \frac{\partial q_{1b}}{\partial m_1} \frac{\partial d_{1b}}{\partial m_1} + V_2' (m_1 - d_{1b}) \left( 1 - \frac{\partial d_{1b}}{\partial m_1} \right) \right] + \frac{1}{2} V_2' (m_1).$$

Using (28) we get

$$V_1' (m_1) = \frac{1}{2} \left[ u' (q_{1b}) \frac{\partial q_{1b}}{\partial m_1} \frac{\partial d_{1b}}{\partial m_1} - \lambda_1 \left( 1 - \frac{\partial d_{1b}}{\partial m_1} \right) \right] + \frac{1}{2} V_2' (m_1).$$

If the constraint on money binds, then $\lambda_1 > 0$ and $\frac{\partial d_{1b}}{\partial m_1} = 1$. Otherwise $\lambda_1 = 0$. 

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