

Early and Late Human Capital Investments, Borrowing Constraints, and the Family* (Preliminary and Incomplete)

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1 Introduction

This paper investigates the importance of family borrowing constraints in determining human capital investments in children at early and late ages. While a number of papers have recently examined the effects of credit constraints on college-going behavior (see Carniero and Heckman (2002) for a summary of the empirical literature, and Caucutt and Kumar (2003) for a calibrated theoretical approach), very little attention has been paid to the role of borrowing constraints when children are younger.¹ Yet, it seems possible that constraints at early ages play a more important role in determining investment decisions for a number of reasons. First, most empirical studies indicate that early investments in children produce high long-term payoffs (see Karoly et al. (1998) or Blau and Currie (2005) and references therein). Not only are children able to learn quickly when they are young, but early learning begets later learning as emphasized by Cunha (2004), Cunha and Heckman (2004), and Cunha, et al (2005). Second, empirical studies suggest

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¹Only recently, have economists (e.g. Restuccia and Urrutia 2004, Cunha 2004, and Cunha and Heckman 2004) begun to consider multiple investment periods at young ages. Yet, none of these studies examines the role of early vs. late borrowing constraints. Restuccia and Urrutia (2004) abstract from financial asset accumulation, while Cunha (2004) and Cunha and Heckman (2004) shut down late borrowing altogether and do not focus on early borrowing constraints.

that family income has a greater impact on child outcomes at earlier ages (e.g. Duncan and Brooks-Gunn (1997), Levy and Duncan (1999)). Third, parents age with their children. And, as both children and parents age, parental resources tend to increase with the accumulation of human capital and the associated rise in earnings. Fourth, despite generous government student loan programs for college-age students and their families in the U.S. and other developed countries, governments have not traditionally offered loans to parents with young children to help finance earlier human capital investments.

While the direct costs of public elementary and secondary education are fully subsidized, a good education through high school is not free. In many U.S. communities, parents must choose between sending their child to poor public schools and paying for their child to attend better private schools. Alternatively, parents may choose between high-cost communities with good public schools and low-cost neighborhoods with poor ones. Other investments in young children can also be costly. Preschool programs in the U.S. can cost as much as attendance at a top university. While the government does not fully neglect poor preschool-age children, the quality of publicly provided pre-school programs (e.g. Head Start) is far below what it could be (Zigler 1994, Blau and Currie, 2005). Expenditures on computers and books also add up. Finally, parental time is an important, yet costly, input that poor parents may be unable to afford.

We provide a number of empirical tests for family credit constraints when children are teenagers or younger. We distinguish between *intergenerational* borrowing constraints, which would prevent parents from borrowing against their children's future earnings, and *intragenerational* borrowing constraints, which would prevent individuals from borrowing against their own future income. Intuitively, testing for the former amounts to estimating whether the present value of parental lifetime income affects child success, while testing for the latter amounts to estimating whether the timing of family income matters. We focus on the latter, estimating the effects of family income at different ages of the child on the math and reading achievement of those children at ages 5-14. We interpret evidence that the timing of income matters as evidence that *intragenerational* borrowing constraints distort investment decisions. In particular, we would expect early income to matter more if families with young children are constrained from borrowing against

their future earnings, thereby causing them to pass on highly productive investment options.²

We provide three tests for *intragenerational* constraints using data from Children of the National Longitudinal Survey of Youth (NLSY). First, we estimate the effects of past income on math and reading scores on the Peabody Individual Achievement Tests (PIAT), to see whether income earned when a child is younger has larger effects on test scores (at any given age) than does income earned at later ages. Empirically, we find statistically different (though quantitatively small) effects of income earned at different ages. Second, we test whether past family income has a greater effect on test scores than does future family income. Even when we limit our sample to individuals with smooth earnings trajectories (a proxy for the level of uncertainty), we find that past income matters more than future income. Finally, we test whether the slope of a family's income profile affects child test scores after conditioning on the present value of income discounted over many years. Again, our results are consistent with borrowing constraints in that the slope is negatively related to test scores. Taken altogether, these results suggest that borrowing constraints inhibit early childhood investments within some families and that early deficiencies are not fully compensated for with later investment once income levels have risen. In short, early borrowing constraints have long-term consequences for children.

The rest of this paper develops a theory of the family that incorporates the dynamic nature of investment in children, as well as the dynamic nature of borrowing constraints faced by parents and college-age youth. Our theory recognizes that later investments build on earlier investments, that early childhood investments are made by young parents at the beginning of their careers, and that opportunities for borrowing may differ substantially over the life-cycle of an individual.

We assume that people live through six stages in their lives: childhood, young adulthood, young parenthood, old parenthood, post-parenthood, and retirement. Young parents make early investments in their children and provide them with consumption. These parents also make their own consumption choices and borrow or save to intertemporally allocate resources. Constraints on their borrowing may limit consumption and investments in young children. Once children

²If investments are perfectly substitutable over time, then the timing of income may not matter even if a family is borrowing constrained. Strictly speaking, our tests offer a joint test against perfect substitutability and borrowing constraints.

become young adults, they make investments in themselves (e.g. college), using their own earnings, transfers from their (now older) parents, and student loans to cover their own costs and consumption. Again, choices may be constrained by imperfect credit markets. Older parents must decide how much to transfer to their college-age children and how much to borrow or save for their own current and future consumption. Once a child leaves the home to establish his own family, parents continue to work, save, and consume until retirement. This cycle repeats itself, as young adults grow into parenthood.

The human capital production process, which translates a child's ability and early and late human capital investments into final skill levels, is of central importance.³ We assume that each child has a learning ability that is correlated with his parent's ability. Total human capital acquired at the end of the two investment periods is increasing in learning ability and the investments made in each period. Ability and investments are assumed to be complements so that an extra unit of investment for a child with higher ability results in a greater increase in human capital.⁴ We make no assumptions about the complementarity or substitutability of investments across periods; although, Cunha, et al (2005) argue that empirical evidence on the returns to early and late investments is most consistent with complementarity. An important contribution of this research will be the use of different empirical strategies and calibration of our model to gain a better understanding of the interaction of investments over time (i.e. the extent of complementarity) and the role borrowing constraints at different ages.⁵

This is a novel and complex problem for several reasons. First, we include an early and late childhood investment period along with potential borrowing constraints at each of these stages. This extends previous work by Becker and Tomes (1976), Keane and Wolpin (2001), Aiyagari, Greenwood, and Seshadri (2002), and Caucutt and Kumar (2003), who consider a single investment period. Second, we allow for intergenerational linkages by assuming Barro-Becker preferences. This breaks from the standard micro literature on human capital production,

³For simplicity, we abstract from post-school human capital investment in life stages 3-5.

⁴Alternatively, ability may reflect the "parenting" ability of parents to turn investments in their children into human capital. We do not distinguish between these two interpretations.

⁵Restuccia and Urrutia (2004) assume a Cobb-Douglas production function for early and late investments, thereby imposing an elasticity of substitution equal to one.

which typically neglects the family altogether or assumes parental transfers are exogenous (e.g. Ben-Porath, 1967, Cameron and Heckman 1998, Keane and Wolpin, 2001). And, third, we assume that parents can pass financial assets and human capital on to their children. This breaks from the work by Caucutt and Kumar (2003) and Restuccia and Urrutia (2004), which assumes that parents can only transfer human capital to their children and skews family decisions towards investment in human capital rather than financial assets.⁶ In their models, parents with low ability children who would benefit little from investments in human capital cannot transfer their children assets instead. Further, both Caucutt and Kumar (2003) and Restuccia and Urrutia (2004) assume that young adults are either successful or unsuccessful in college, and that young adults with low ability are unlikely to be successful. It is, therefore, difficult for parents of low ability children to transfer anything at all to their children.

In order to evaluate the role of many simplifying assumptions made in previous work on human capital and borrowing constraints, we nest a series of simple models (that have been used in most previous studies on this topic) within our more general intergenerational model. We first consider a static problem from the child's perspective, and assume that parental transfers are exogenous. By including two childhood investment periods, we expand on work by Becker and Tomes (1986), Keane and Wolpin (2001), Aiyagari, Greenwood, and Seshadri (2002), and Caucutt and Kumar (2003), who consider a single investment period (and, implicitly, that children are exogenously different at college-going ages). Due to forward-looking behavior and the interaction of early and late investment decisions, a change in borrowing limits at the college age affects early childhood investments in addition to college-age investments. If investment across periods is sufficiently complementary, relaxing borrowing constraints at later ages will increase investment at all ages. Failure to incorporate effects on early investment may, therefore, understate the full impact of a college student loan or subsidy policy. When investments at different ages are highly substitutable, individuals may simply respond to a policy that relaxes college-age borrowing constraints by

⁶Our work is most closely related to that of Cunha (2004) and Cunha and Heckman (2004); however, they do not focus on the role of borrowing constraints at different ages. By assuming individuals only live for two periods, with parents and children overlapping for two, they are unable to explore the role of borrowing constraints during late childhood. They assume that parents cannot borrow against their children's income and do not allow older children to borrow.

shifting some early investment to later ages. In this case, the policy will still increase final human capital levels, but by less than estimates that do not consider changes in early investment. This very simple model highlights importance of considering dynamic investment decisions and the role of complementarity in investment over time, even if one is only interested in understanding final human capital or schooling outcomes.

We next endogenize parental transfers by assuming that parents receive utility from transferring resources to their children. This utility function may represent “warm glow” preferences over the amount a parent transfers to his child, or more interestingly, it may reflect parental altruism (as in Barro-Becker preferences) for the child’s utility. In the latter case, the parent’s and the child’s problem become interdependent: parents’ choices depend on their children’s ability and investment choices, and children’s choices depend on the transfer functions of their parents. Endogenizing parental transfers generalizes the work of Keane and Wolpin (2001) who assume that transfers are independent of policy changes. We explore how parents increase or decrease their transfers at different ages in response to changes in borrowing constraints and other policy changes. Furthermore, we study the total effects of policy changes on investment decisions. In general, we find that changes in borrowing constraints faced by college-age children affect investment decisions (when parental transfers are endogenous) in much the same way as when the parental transfer function is taken as exogenous. In some cases, constrained parents may ‘capture’ some of the benefits from additional lending provided to their college-age children (by transferring them less at all ages), but they do not take it all. In other cases, parents may transfer less to the child when he is old but more when he is young. The complementarity or substitutability of investment over time plays a key role in determining how both parents and children react to changes in college lending limits.

It is also possible to examine the effects of loosening parental borrowing constraints on transfer and investment decisions. Relaxing borrowing constraints on older parents with older children unambiguously raises college-age investments. Early investment may increase or decrease depending on the complementarity of investments (with investment rising when complementarity is sufficiently strong). Relaxing the borrowing constraint faced by young parents tends to encourage

transfers and investment at early ages, but it has two (sometimes opposing) effects on investment at later ages. First, changes in early investment alter the return to later investments – raising them under sufficient complementarity. Second, an increase in borrowing raises the level of debt carried over into old parenthood, which tends to reduce the amount of transfers a parent wants to provide at that stage. By ignoring asset accumulation entirely, Caucutt and Kumar (2003) and Restuccia and Urrutia (2004) cannot examine these issues.

Our most general model assumes that children grow up to become parents who have children of their own, making the problem fully dynamic and intergenerational. Assuming Barro-Becker preferences, parents will make the same investment decisions as their children would make conditional on the level of transfers. Thus, it is possible to think of the entire problem from the point of view of parents. Because it is difficult to analytically derive results regarding the interaction of borrowing constraints and investment decisions in this general model, we do so computationally with an empirically founded model calibrated to relationships between parental ability, education, and income and children’s early and late outcomes.

Our quantitative analysis requires a serious calibration. We consider five values of final human capital that correspond to different levels of schooling attainment. These human capital levels and the distribution of earnings shocks (we incorporate uncertainty in earnings for our quantitative analysis) are determined from U.S. data on earnings by education and age. Parameters determining the human capital production technology and the levels of borrowing allowed at each stage of life are calibrated to best fit the relationship between parental earnings at both early and late ages of their children and the final schooling outcomes of their children.

Following the analysis of Lochner and Monge (2004), we allow for the fact that borrowing limits may differ across schooling choices in response to the fact that an individual’s ability to re-pay his loans may depend on his schooling attainment. In particular, borrowing limits depend on the amount a person will be able to re-pay under the worst-case scenario (i.e. the person receives the worst possible earnings shocks in all future periods). Because this amount is increasing in schooling, our borrowing limits are also increasing in schooling. This is consistent with the structure of the federal student loan system, which links borrowing limits to actual schooling

expenditures. It is also likely to accurately reflect private lending opportunities. See Lochner and Monge (2005) for a more detailed discussion of these issues.

We use our model to analyze a number of important issues in education. First, we study how constraints on borrowing for young parents limit early childhood investments and, more importantly, how those early deficiencies manifest themselves later on. Can these deficiencies be compensated for with the aid of generous college student loan programs or subsidies and, if so, at what cost? This analysis has implications for university affirmative action policies and more general initiatives for disadvantaged youth when they reach college-going ages, since the success of these initiatives depends on the extent to which they can adequately redress the lack of early human capital investments.

Second, we examine the role of government lending and subsidy policies on the dynastic outcomes of families. Children receiving inadequate human capital investments eventually become parents with few skills or resources and are, therefore, likely to invest little in their children. How do the impacts of government education policies grow or change with each passing generation? These questions are addressed in Restuccia and Urrutia (2004), Caucutt and Kumar (2003), Hanushek, Leung, and Yilmaz (2004), and Aiyagari, Greenwood, and Seshadri (2002), but no one has done so in an intergenerational model with asset accumulation and a multi-period childhood.

Finally, we explore the quantitative importance of various simplifying assumptions that have been made in previous empirical studies. For example, Keane and Wolpin (2001) empirically study the importance of borrowing constraints and parental transfers on the intergenerational correlation in educational attainment, but they abstract from investments prior to age sixteen and assume an exogenous parental transfer function. In this environment, they conclude that loosening borrowing constraints at the college age would have virtually no effect on college attendance and, hence, no effect on the intergenerational correlation in educational attainment. Cameron and Heckman (1998, 2001) and Cameron and Taber (2004) similarly argue that credit constraints at the college age are empirically unimportant – holding earlier skill levels and investments constant. By quantitatively exploring the effects of relaxing these assumptions, our results can serve as a guide for future theoretical and empirical work on human capital. To the extent that the

responses in early investment or parental transfers do not alter the implications of policies that relax borrowing constraints or subsidize tuition, we can be more confident in the results from empirical studies that abstract from these features of the decision problem. However, to the extent that either of these channels have important effects on our results, future work should take them into account.

This paper proceeds as follows. In Section 2, we use data from the NLSY to empirically study whether families face borrowing constraints when their children are young. Section 3 describes an intergenerational model of lifecycle human capital investment with borrowing constraints. We theoretically discuss the role of parental transfers and credit constraints at different ages on human capital levels and eventual labor market earnings. In doing so, we explore various assumptions about the link between parents and children. In Section 4, we calibrate a numerical version of the model, and in Section 5, we examine the quantitative importance of a number of government loan and subsidy policies. The effects of policy on both a single generation as well as a succession of generations are examined. Section 6 concludes.

2 Empirical Evidence on the Existence of Family Borrowing Constraints

While there is a growing literature on the importance of borrowing constraints among college students, less is known about constraints faced by families with younger children. Indirect evidence suggests that families may, indeed, be constrained. First, randomized studies of early intervention and preschool programs for disadvantaged children have estimated large benefit-cost ratios, suggesting that many poor families are not making investments in their children even though those investments more than pay for themselves in the long-run. Second, recent studies have shown that increases in family income lead to increases in the test scores of adolescents and young children (e.g. Blau 1999, Duncan and Brooks-Gunn 1997, Levy and Duncan 1999, and Dahl and Lochner, 2005). In this section, we build on the second set of evidence based on child achievement test scores, providing more direct empirical tests for whether families are borrowing constrained.

It is important to distinguish between two types of constraints: an *intergenerational* borrowing constraint and an *intra-generational* borrowing constraint.⁷ First, parents may be intergenerationally constrained from borrowing against their children's future income. That is, parents may not be able to pass on debts to their children. This is more like a generational budget constraint than a borrowing constraint, but it does imply that family income may affect child achievement. Importantly, this type of constraint, if binding, suggests that the present discounted value of lifetime family income will be an important determinant of child achievement. However, if parents can save and borrow against their own future earnings, the timing of their income should not be important. A second, stronger intra-generational constraint more closely matches the standard idea of a borrowing constraint. Parents may be unable to borrow against their own future earnings in order to smooth consumption or to make investments in their children at young ages. If parents are borrowing constrained in this way, the timing of their income will be important. Namely, constrained parents who earn more of their income at later ages will tend to invest less in their children when they are young than will parents who earn more of their income early on. This should be true even after conditioning on the discounted present value of lifetime family income.

The first type of constraint is, in theory, easy to test. Do children from wealthier families perform better than children from poorer families? In practice, this type of test is difficult to implement, since innate abilities of children may be correlated with those of their parents. Additionally, rich parents and poor parents may be different in many unobservable ways that have little to do with income. This problem has plagued past work on credit constraints and college-going behavior as discussed in Carniero and Heckman (2002). Testing for the second type of constraint relies on examining how the timing of income matters conditional on the discounted value of lifetime income. This amounts to comparing the test scores of children in families with the same total lifetime earnings but with different income profiles. If families are unaffected by borrowing constraints, children from families who earn a larger share of their income early on should perform equally well as children from families who earn more of their income late

⁷Below, we discuss these constraints and their implications in greater detail within our theoretical model of family human capital investment decisions.

in their careers. If borrowing constraints are binding, the first set of children should perform better than the second set. There is less concern about a correlation between parent and child abilities in this case, since lifetime income is held constant.⁸ We, therefore, focus on tests of intra-generational borrowing constraints, however, many of our results also speak to the importance of intergenerational constraints (assuming we adequately control for parental ability).

Income earned earlier should be worth more in a present value sense. That is, income earned in one period is worth $1 + r$ times income earned in the next period (in the absence of credit constraints), where r is the annual interest rate. It is important to account for this when examining the effects of income timing. Therefore, we discount income earned by the family so that it is measured in present value terms as of the birth year of the child. (We use a discount rate of $r = 0.05$; although, other reasonable rates yield similar conclusions.) After making this adjustment, the absence of intra-generational borrowing constraints suggests that the timing of income should be irrelevant conditional on the discounted present value of lifetime family income.

We examine three types of evidence on the importance of income timing. First, we estimate the effects of past income on child achievement test scores to see whether income earned at earlier ages of the child has a different effect on test scores than does income earned at later ages. Second, we estimate whether future income has the same effect on test scores as does past and current income. Third, we test whether the slope of a family's income profile significantly affects test scores conditional on the discounted present value of lifetime family income over a long time span. The first test compares the effects of income earned at different points in the past; the second compares the effects of past income with future income; and the final test examines the role of income timing over the past, present, and future. As we next show, these tests point to the existence of intra-generational borrowing constraints (as well as intergenerational constraints): family income earned at earlier ages has significantly larger effects on child test scores than income earned at later ages.

⁸One might expect more able parents to earn more of their income later, as they invest more in their human capital early on. Then, if more able parents have more able children (a problem in the test of the intergenerational constraint), families earning more of their income at later ages should have more innately able children. This suggests that, in the absence of credit constraints, children from families earning their income earlier should perform worse, on average – the opposite of the prediction based on credit constraints. Thus, a positive intergenerational correlation in ability may make it difficult to find evidence of intra-generational borrowing constraints.

2.1 Data from the National Longitudinal Survey of Youth

We use data from the Children of the NLSY and the main NLSY sample of mothers. These data are ideal for studying the effects of family income on children for several reasons. First, we can link children to their mothers, and second, we can follow families over time. Additionally, the NLSY contains achievement test scores for children and good measures of family income, parental education, and mother's ability. Finally, the NLSY oversamples minority and poor white families, which provides a larger sample of families more likely to face borrowing constraints. We use data drawn from more than 7,000 interviewed children born to over 3,500 interviewed mothers.

The NLSY collects a rich set of variables for both children and mothers repeatedly over time. For children, biannual measures of family background and cognitive and behavioral assessments are available from 1986 to 2000. Detailed longitudinal demographic, educational, and labor market information for the mothers is available annually from 1979 through 1994 and biannually thereafter. Equally important, family income measures are available for the same period. Hence, for children born after 1979, we compile an income history for almost every year since birth (except for non-responses, of course). Repeated achievement test scores for children are also available every other year since 1988.

We use Peabody Individual Achievement Tests (PIAT) in math and reading for children ages five to fourteen. The assessments measure ability in mathematics, oral reading ability, and the ability to derive meaning from printed words.⁹ To simplify our exposition, we focus on a combined math and reading achievement test score, which places a weight of 50% on the math score and 25% on each of reading comprehension and reading recognition. To make these scores more easily interpretable, we create standardized test scores with a mean of zero and standard deviation of one.¹⁰

We restrict our sample to children ages 5-14 who take at least one PIAT test after 1988 and for

⁹From 1988 to 2000, the tests were administered biannually to children five years of age and older. Starting in 1994, the tests were given only to children who had not reached their 15th birthday by the end of the calendar year. Around two percent of children took the PIAT tests after their 15th birthday before this rule was put in place.

¹⁰We first normalize each individual math or reading test by subtracting off the mean score for the random sample of test takers and dividing by the estimated standard deviation. We then weight them to calculate a combined math and reading score. Finally, we re-normalize this score to make it mean zero with a standard deviation of one.

whom we can calculate a valid family income measure. Children are scheduled to take the PIAT tests biannually, so that the maximum number of repeated test score measurements is five. For our sample of individuals, the fraction of children taking the PIAT tests as scheduled is relatively high. Children in our sample completed the math and reading recognition tests as scheduled 91% of the time, and the reading recognition test 75% of the time. The number of children taking the PIAT tests in any given year varies from a low of 2,073 (for reading recognition in 2000) to a high of 3,703 (for math in 1993).

2.2 Empirical Results

We first estimate the effects of family income (measured in 10,000's of year 1979 dollars) earned at all earlier ages on current PIAT scores, controlling for a variety of individual and family background characteristics as well as average discounted family income measured over the life of the child (through the final survey date).¹¹

To simplify the discussion suppose family income at age j ($I_{i,j}$) affects child test scores at age $k > j$ ($T_{i,k}$), conditional on average lifetime income and family background, in the following way:

$$\frac{\partial T_{i,k}}{\partial I_{i,j}} = \alpha_1 + \alpha_2 j + \alpha_3 k.$$

In this case, α_2 tells us how income earned at different ages affects subsequent child test scores, while α_3 tells us at what ages test scores respond most to changes in past income. These effects can be estimated based on the following regression:

$$T_{i,a} = X_{i,a}\gamma_x + \bar{I}_i\gamma_I + \alpha_0 I_{i,a} + \alpha_1 \left(\sum_{j=0}^a I_{i,j} \right) + \alpha_2 \left(\sum_{j=0}^a j I_{i,j} \right) + \alpha_3 \left(\sum_{j=0}^a a I_{i,j} \right) + \epsilon_{i,a},$$

where $X_{i,a}$ represents background characteristics for child i at age a and \bar{I}_i represents average family income from the child's birth through the final period of observation. The inclusion of $I_{i,a}$ allows for an additional temporary effect of contemporaneous income on test scores, so $\frac{\partial T_{i,a}}{\partial I_{i,a}} = \alpha_0 + \alpha_1 + (\alpha_2 + \alpha_3)a$. Table 1 reports estimates of this regression equation. Column

¹¹In computing average 'lifetime income', we calculate a weighted average of all observed family income measures since the birth of a child where the weights for observed income at age j are given by $\frac{(1+r)^{-j}}{\sum_k (1+r)^{-k}}$ (the denominator sums over all periods for which an income measure is observed).

1 does not control for any background measures or average lifetime family income. Column 2 controls for a set of *basic background characteristics*, including mother's education, scores on the Armed Forces Qualifying Test¹² (AFQT), and family background (foreign born, rural residence at age 14, living with both parents at age 14), highest grade completed by grandparents, and year indicators. Column 3 controls for *additional family background characteristics*, including the number of adults and children in the household, current marital status of the mother, age and education of the spouse/father, and whether or not the child's mother is living with her father and mother. Column 4 drops current income from the regression in column 3, since it's coefficient is no longer statistically significant after accounting for family background. Columns 5 and 6 reflect results from the most general specifications that include average lifetime family income and it's interaction with the child's current age. Focusing on the estimates of α_2 , which are all negative, we see that income earned at later ages has a smaller effect on subsequent child achievement scores than does income earned at earlier ages. These estimates are statistically significant at the 0.05 level in both columns 4 and 6. A coefficient of -0.001 (consistent with most specifications) suggests that a \$10,000 increase in family income at age 1 would increase test scores at age 14 by about 0.01 standard deviation more than would the same increase in income at age 11. More generally, the estimates suggest that past income has a positive effect on test scores at ages when the math and reading tests were administered (recall that tests were not administered to children before age 5). The final specification suggests that lifetime income has little effect (or a negative effect) on test scores over ages 5-10, but has growing effects as a child ages. That is, children from wealthier families (as measured by lifetime family income) perform better and better over time relative to children from less fortunate families.

The finding that income at earlier ages of the child has larger effects than income earned at later ages is consistent with binding intra-generational borrowing constraints; though, it is difficult to quantify the severity of constraints. To the extent that controls for family background and mother's ability account for differences in innate ability of children and other factors that affect child development, the results are also consistent with important intergenerational borrowing

¹²This test was administered to most respondents in the NLSY and is widely used as a measure of cognitive ability.

constraints

The permanent income hypothesis does not distinguish between income earned in the past or the future. If individuals are reasonably certain about their future income prospects and unconstrained by intra-generational borrowing constraints, income earned in the past, present, and future should all affect child test scores equally. Because the NLSY offers panel data over a long time period, it is possible to observe family income measured both before and after some tests taken by children. Table 2 reports estimates from regressions of child test scores on the present value of past and current income as well as the present value of all future income. Columns 2 and 3 control for additional family background variables as described earlier. All of these estimates suggest that past and current income have a significantly greater effect on test scores than does future income. In the final column, the coefficient estimate for future income is not statistically different from zero, while the effect of past income is significantly positive at 0.06. Again, this result is consistent with both intergenerational and intra-generational borrowing constraints.

An obvious concern with this approach is uncertainty about future earnings. If individuals are completely uncertain about future income, actual realizations of that income process should have no effect on current decisions or outcomes even if individuals are not borrowing constrained. However, since future income primarily represents income earned over the next 1-5 years, it seems unlikely that it is all that uncertain for most families. To more formally examine the role of uncertainty, we ask whether the results of Table 2 hold for families with fairly predictable income profiles. To measure the predictability of income, we estimate family-specific log income regressions on age and age-squared using all income observations after the child's birth. We can then compute two potential measures of uncertainty (or variability, at least): (i) R^2 statistics which measure the fraction of the variance in log income that can be explained by age and age-squared alone, and (ii) the square root of the mean squared error (RMSE) from the regression. We separate individuals according to these measures of ability and separately run regressions of child test scores on past/current and future family income. If the insignificant coefficient on future income in Table 2 is due primarily to uncertainty in future earnings, we expect that future income should have similar effects to past/current income among those with predictable earnings

profiles (i.e. a high R^2 statistic or a low RMSE). This is not the case as shown in Table 3. Using either measure of predictability, we find that past/current income has a significantly greater effect than future income among those with highly predictable income profiles. A combined measure which takes only those individuals with an R^2 above 0.75 and those within the lowest quartile of RMSE, we find a more dramatic difference in coefficients on past/current and future income than we observed in Table 2. While predictability in an *ex post* sense (as implied by our measures) does not necessarily imply a high degree of predictability in an *ex ante* sense (i.e. low uncertainty), these results are at least consistent with a more important role for constraints rather than uncertainty.

Finally, we examine whether the slope of a family’s income profile affects child test scores after controlling for the discounted present value of lifetime income. These results are presented in Table 4. Consistent with credit constraints, we generally find negative effects of the slope on children; although, the estimates are not always statistically significant.¹³ Furthermore, average lifetime income also has an important and significant effect (that grows with age according to estimates in columns 4-6). Thus, these estimates are consistent with both intra- and intergenerational borrowing constraints.

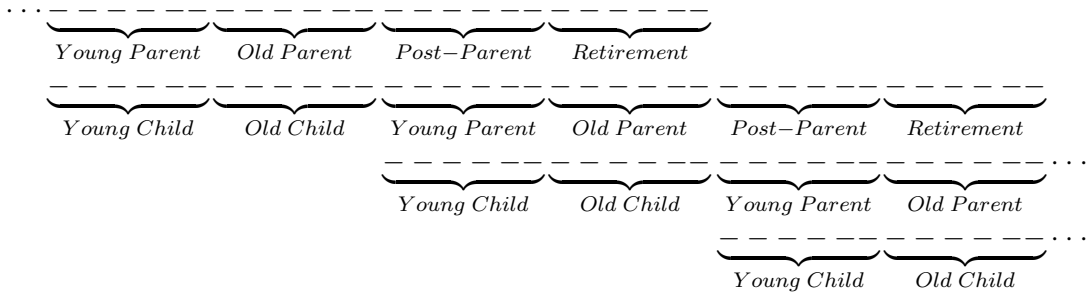
While all of these ‘tests’ for intragenerational borrowing constraints are subject to different criticisms, we view the combination of all three sets of results as convincing evidence that some families with young children are constrained. At the very least, this evidence suggests that a better understanding of the role credit constraints play in determining family investments in children is warranted.

¹³Estimates of the coefficient on the slope of income profiles is likely to be biased upwards for two reasons. First, since we only use income over a limited number of years in computing average lifetime income, this number will tend to be too low for those with a steeper slope relative to those with a flatter income profile. Because average lifetime has a positive effect on children, this mis-measurement will tend to bias estimates of the coefficient on the slope of income profiles upward. However, this is unlikely to have much affect on our estimates, since income far into the future is heavily discounted. A second potential source of bias would arise if unobserved differences across families are related to the slope of family income profiles. If family income profiles are rising because parents are accumulating human capital, then those parents with the steepest earnings profiles will tend to be accumulating the most human capital. If those parents also tend to invest more in their children or if their investments are more productive, this will produce upwardly biased estimates of the coefficient on the slope of the income profile.

3 An Intergenerational Lifecycle Model of Human Capital Investment and Borrowing Constraints

People live through six stages in their life: young and old childhood, young and old parenthood, post-parenthood, and retirement. These stages each last roughly 12 years. The lifecycle of different generations in a dynasty can be described as follows:

Diagram 1: Generations of a Dynasty



We assume that all transfers from parents to children take place during the first two phases of a child's life. To avoid solving the decision problem when three generations of a family co-exist, we abstract from transfers during post-parenthood and retirement.¹⁴

Let subscripts $j \in \{1, 2, 3, 4, 5, 6\}$ represent the six stages of the lifecycle, and denote a child's ability to learn by θ . Investment in children during stages 1 and 2 are given by i_1 and i_2 respectively. Together, these investments yield human capital as of young parenthood equal to

$$h_3 = h_{min} + f(i_1, i_2, \theta) \quad (1)$$

for a child of ability θ . This reflects the fact that individuals possess a minimum amount of human capital, $h_{min} \geq 0$, before any human capital investments are made. The human capital production function $f(\cdot)$ is increasing in all of its arguments and concave in both investment arguments. We further assume that f_{13} and f_{23} are non-negative, so that ability and investments

¹⁴Uncertainty of future earnings and ability of future grandchildren, along with binding borrowing constraints could cause post-parents and retired individuals to want to transfer resources to their children when they are young parents and old parents. Absent these issues, a parent should be indifferent between transferring resources when he is an old parent and later in life.

are complements. To guarantee appropriate second order conditions hold in the decision problems described below, we assume the following:

Assumption 1. $f_{12} \geq \max \left\{ f_{22} \left(\frac{f_1}{f_2} \right), f_{11} \left(\frac{f_2}{f_1} \right) \right\}$ and $f_{12}^2 \leq f_{11} f_{22}$.

In our computational analysis, we employ a modified-CES human capital production function of the form

$$f(i_1, i_2, \theta) = \theta(ai_1^b + (1 - a)i_2^b)^{c/b}, \quad (2)$$

where $a \in (0, 1)$, $b < 1$, and $c \in (0, 1)$; however, our theoretical analysis does not rely on any particular functional form.¹⁵ In the absence of any credit constraints, it is necessary to impose decreasing returns to scale (i.e. $c < 1$), otherwise individuals will want to invest an infinite amount.¹⁶ While our model includes borrowing constraints such that this is never a problem, it seems likely that some families are unconstrained even though they do not make infinite investments in their children. We, therefore, focus on the case of decreasing returns to scale and $c < 1$.

Because we are mainly concerned with the interaction between investments made during childhood, we assume that human capital exogenously increases after young parenthood:

$$h_4 = \Gamma_4 h_3 \quad \text{and} \quad h_5 = \Gamma_5 h_4. \quad (3)$$

Given human capital levels and a wage per unit of skill, w , labor market earnings are easily defined. First, we assume that older children earn $W_2 = \phi w \bar{h}$, where \bar{h} is level of human capital all older children are assumed to have, and $\phi \in [0, 1]$ represents the fraction of second stage years an individual can potentially work. For adults, earnings are given by

$$W(h_j) = wh_j, \text{ for } j \in \{3, 4, 5\}. \quad (4)$$

If individuals choose to save, they earn a gross return of R on savings. This interest rate also applies to any borrowing. Assets saved in period j are given by A_{j+1} , and total borrowing

¹⁵Assumption 1 holds for this production function.

¹⁶This is easily understood, once one recognizes that unconstrained families will simply choose investment to maximize discounted lifetime income net of discounted costs. With constant (increasing returns) to scale, if any combination of first and second period investment produces a positive net value, then doubling those investments will double (more than double) the net value.

(negative A_{j+1}) may be limited by a restriction on debt carried over to the next period, $-L_j$. Note that in the last period, retirement, individuals live off of savings and do not work.

3.1 The Child's Problem Given Exogenous Parental Transfers

We first discuss the child's problem conditional on parental transfers. Assume that young children cannot borrow and, therefore, depend on parental transfers for consumption and investment. (We implicitly assume that parents do not transfer so much that children want to save.) Old children can borrow as long as their net debt does not exceed their debt limit, L_2 . For now, parental transfers (y_1 and y_2) are assumed to be pre-determined and independent of the child's choices.

Individuals choose investment (i_j) and consumption (c_j) each period to maximize their lifetime utility. For a given wage rate, w , discount factor $\beta \in (0, 1)$, utility function $u(c)$ (an increasing and concave function), and value function for a young parent with human capital h_3 and assets A_3 , $\tilde{V}(h_3, A_3)$ (assume for now that $\tilde{V}(\cdot)$ is increasing and concave in both its arguments—we discuss this in more detail below), the decision problem facing a child of ability θ receiving parental transfers y_1 and y_2 is

$$U(y_1, y_2, \theta) = \max_{c_1, c_2, i_1, i_2} u(c_1) + \beta u(c_2) + \beta^2 \tilde{V}(h_3, A_3) \quad (5)$$

subject to

$$\begin{aligned} y_1 &= i_1 + c_1, \\ A_3 &= y_2 + W_2 - i_2 - c_2, \\ A_3 &\geq -L_2, \\ c_1 &\geq 0, \quad c_2 \geq 0, \\ i_1 &\geq 0, \quad i_2 \geq 0, \end{aligned}$$

and the human capital accumulation equation (1). This specification assumes that young children cannot work, but that older children earn W_2 on their own, as described earlier.

If there are no future borrowing constraints, then the value function after the two investment periods, $\tilde{V}(\cdot)$, can be more simply written as

$$\tilde{V}(h_3, A_3) = v(RA_3 + wh_3\chi), \quad (6)$$

where, $\chi = 1 + R^{-1}\Gamma_1 + R^{-2}\Gamma_1\Gamma_2$, $v' > 0$, and $v'' < 0$. In this case, future welfare is determined entirely by the amount of total lifetime wealth (including current assets and the value of human capital in the future) carried into period three.¹⁷

The complementarity of investment across periods plays an important role in determining individual responses to borrowing constraints. In particular, the following complementarity condition is important for a number of results:

Condition 1. $\frac{f_{12}}{f_1 f_2} > -\frac{v''(\cdot)}{v'(\cdot)} w\chi$.

It is useful to consider this condition with our CES production function, Equation (2), and with a CIES value function $v(x) = \eta \frac{x^{1-\sigma}}{1-\sigma}$. In this case, Condition 1 cannot hold if $c \leq b$, but this only rules out very strong substitution between early and late investments, since $c > 0$. For the more relevant case, $c > b$ and Condition 1 simplifies to

$$\underbrace{\frac{1}{1-b}}_{\text{elast. of sub.}} < \underbrace{\frac{1}{\sigma}}_{\text{CIES}} \cdot \underbrace{\left(1 + \frac{RA_3}{w\chi h_3}\right)}_{1 - \frac{\text{debt}}{\text{income}}} \cdot \left(\frac{h_3}{h_3 - h_{min}}\right) \cdot \left(\frac{c-b}{c(1-b)}\right).$$

As the elasticity of substitution between early and late investments decreases (i.e. investments become more complementary) or the consumption intertemporal elasticity of substitution increases (i.e. individuals become less concerned about maintaining smooth consumption profiles), this inequality is more likely to hold. For $b \leq 0$ (complementarity at least as strong as implied by a Cobb-Douglas production function) and $L_2 = 0$ (i.e. no borrowing allowed for older children), this condition will be satisfied whenever the intertemporal elasticity of substitution for consumption is greater than the elasticity of substitution between early and late investments. Generally

¹⁷If these children will have children of their own and they care about their children as we describe in the next subsection, this simplification still holds because of the implicit assumption embodied in equation (1) that parental human capital does not affect the productivity of children's human capital. However, to the extent that ability embodied in θ is correlated across generations, the value function will also depend on θ .

speaking, when individual preferences for smooth consumption are strong, Condition 1 requires strong complementarity between early and late investments.

Now, consider the case where individuals face binding constraints on their borrowing when they are older children, but that they will be unconstrained thereafter. What role does the debt constraint play in both early and later investments in skill?

Proposition 1. *Assume that borrowing constraints bind during old childhood (i.e. $A_3 = -L_2$), but there are no future borrowing constraints. Then,*

- (i) $\frac{\partial i_2}{\partial L_2} \in (0, 1)$;
- (ii) if Condition 1 holds, then $\frac{\partial i_1}{\partial L_2} > 0$;
- (iii) $\frac{\partial h_3}{\partial L_2} > 0$.

Proof: See appendix.

If an individual is borrowing constrained as an old child and this constraint is relaxed, then he will respond by investing more in the second period. However, this increase in investment will be less than the increase in borrowing, since he will consume some of the extra borrowing. If early and late human capital investments are complementary enough, he will also increase his investment during early childhood. This is the sense in which constraints at college-going ages can impact earlier investment decisions. Note that if early and late human capital investments are not complementary enough, first period investment can either rise or fall. Assuming the production function is given by Equation (2), if investments are perfect substitutes, then first period investment falls. Regardless of whether early investment rises or falls, individuals acquire more total human capital, h_3 , when the borrowing constraint on older children is relaxed.

We assume that young children cannot borrow or save, which precludes investigation of how investments react to changes in borrowing limits when a child is young. We can consider the effects of shifting income across periods via changes in the exogenous parental transfers. Suppose a young child is borrowing constrained, and his parental transfers while young increase at the expense of his parental transfers in the next period. Such a shift can have interesting consequences. If investment across periods is substitutable enough, investment in period one rises, while investment in period

two falls. However, if investment across periods is complementary, it is possible that shifting resources forward could cause investment to decline in both periods, and the accumulated human capital to fall.

Let y represent an increase in period one parental transfers, $y_1 + y$, which are gained at the expense of period two parental transfers, $y_2 - Ry$.

Proposition 2. *Assume that a young child would like to borrow, and that borrowing constraints bind during late childhood (i.e. $A_3 = -L_2$), but there are no future borrowing constraints. Then,*

- (i) *if Condition 1 does not hold, $\frac{\partial i_1}{\partial y} > 0$ and $\frac{\partial i_2}{\partial y} < 0$;*
- (ii) *if $\frac{\partial i_1}{\partial y} < 0$, then $\frac{\partial i_2}{\partial y} < 0$.*

Proof: See appendix.

To understand the forces at work in case (ii), consider a simplified version of the problem. Imagine a Leontief production function, $f(i_1, i_2) = \min(i_1, \frac{i_2}{\alpha})$, where investment in the first period must be matched by an α scaled investment in the second period in order to have any impact. If a person is optimizing, he chooses investment so that the marginal benefit is equal to the marginal cost. Suppose income rises in the first period at the expense of income in the second period. If we hold investment fixed, the marginal benefit of investment does not change, however the marginal cost of investment does change. The marginal cost of investment is the marginal utility of consumption in the first period plus the marginal utility of consumption in the second period weighted by α . Consumption rises in period one and falls in period two, causing marginal utility in the first period to fall, and in the second period to rise. If the increase in second period marginal utility, weighted by α , is greater than the decrease in first period marginal utility, then investment falls. What is happening is that the marginal cost of investment increases with the shift in income. Note that, because the person is borrowing constrained, if $\alpha \leq 1$ and the third derivative of the utility function is positive, then the decrease in marginal cost in period one will outweigh the increase in marginal cost in period two, and investment will rise. If $\alpha > 1$, it is possible that the increase in marginal cost in period two is greater than the decrease in period one, which implies a decline in investment. With the production function given by Equation

(2) and the CIES utility function that we use in the computation, the issues are a bit more subtle, but it is still possible to generate a situation where investment falls in both periods.¹⁸ The key is that investment in the second period needs to be costly, a is small, and investments need to be complementary, but not too complementary. The reason investments cannot be too complementary is that in the limit the CES production function converges to the Leontief with $\alpha = 1$. So as investments get more complementary, the role of a diminishes, and the fact that the young child is borrowing constrained implies that the marginal cost of investment falls.

These results assume that parental transfers, y_1 and y_2 , are exogenous. Next, we introduce parental transfer decisions to determine how optimal transfers respond to changes in borrowing limits. One might expect that parents respond to expanded borrowing opportunities for their children by reducing their own transfers. If so, we are interested in to what extent the responses of parents undo the effects of relaxing borrowing limits on older children.

3.2 Parental Transfer Decisions

We now discuss the parent's decision problem assuming that parents receive utility from transferring resources to their children. Denote the value of transferring y_1 and y_2 to a child of ability θ by $\tilde{U}(y_1, y_2, \theta)$ (where $\tilde{U}(\cdot)$ is increasing and concave in both y_1 and y_2). Parents choose their own consumption, savings/borrowing, and transfers to their children subject to any constraints on their borrowing. As of young parenthood, individuals with assets A_3 , human capital h_3 , and a child with ability θ solve the following problem:

$$V(h_3, A_3, \theta) = \max_{c_3, c_4, c_5, y_1, y_2} u(c_3) + \beta u(c_4) + \beta^2 u(c_5) + \beta^3 u(c_6) + \tilde{U}(y_1, y_2, \theta) \quad (7)$$

¹⁸Using the parameters calibrated in Section 4.1, first period investment rises.

subject to

$$\begin{aligned}
A_4 &= RA_3 + W(h_3) - c_3 - y_1, \\
A_4 &\geq -L_3, \\
A_5 &= RA_4 + W(h_4) - c_4 - y_2, \\
A_5 &\geq -L_4, \\
A_6 &= RA_5 + W(h_5) - c_5, \\
c_6 &= RA_6, \\
y_1 \geq 0 \quad y_2 \geq 0 \quad c_3 \geq 0, \\
c_4 \geq 0 \quad c_5 \geq 0 \quad c_6 \geq 0,
\end{aligned}$$

human capital growth given in Equation (3) and the earnings Equation (4). The function $\tilde{U}(\cdot)$ may represent “warm glow” preferences over the amount of transfers a parent gives to a child, in which case parental transfer decisions are not necessarily linked to the investment decisions of their children. More interestingly, $\tilde{U}(\cdot)$ may reflect parental altruism (as in Becker-Barro preferences) for the child’s utility over those transfers, so $\tilde{U}(\cdot) = \rho U(\cdot)$ where $U(\cdot)$ is defined in Equation (5). In this case, the parent’s and the child’s problem clearly become interdependent. Parents’ choices depend on their children’s ability and investment choices, and children’s choices depend on the transfer functions of their parents.¹⁹

Suppose that a parent cares about the utility his child derives from parental transfers, and that the child is borrowing constrained. We are interested in how investment across periods reacts

¹⁹Whether the parents determine the level of transfers and investments, or merely the level of transfers and allow the child to determine the level of investments is important. Brown, Mazzocco, Scholz, and Seshadri (2005) point out that in this situation strategic behavior can arise. They consider children and parents who overlap for two periods. The first period includes an investment in human capital, while the second does not. Parents can tie first period transfers to educational investment. They show that because the child has incentive to under invest in period one to extract more from the parent in period two, the parent responds by tying transfers to educational spending. In our environment, we consider an even stronger version of tied transfers. Parents earmark a certain amount for educational investment, which the child cannot reduce or increase. The reason for the restriction against increasing first period investment is that if the educational investments are complementary across periods (Condition 1), the child has incentive to over invest in Period one to extract greater transfers in period two. In Brown, Mazzocco, Scholz, and Seshadri (2005) first period investment is substitutable with second period transfers so this effect does not arise. If we assume this stronger version of tied transfers, in essence we are assuming the parent makes all of the decisions in each period.

to changes in the severity of the borrowing constraint when parents are free to adjust transfers in response.

Proposition 3. *Assume that $\tilde{U}(\cdot) = \rho U(\cdot)$, the old child is borrowing constrained (i.e. $A_3 = -L_2$) while a child, but not as an adult. Then,*

- (i) $\frac{\partial i_2}{\partial L_2} \in (0, 1)$;
- (ii) if Condition 1 holds, then $\frac{\partial i_1}{\partial L_2} > 0$;
- (iii) $\frac{\partial h_3}{\partial L_2} > 0$;
- (iv) $0 > \frac{\partial y_2}{\partial L_2} > \frac{\partial i_2}{\partial L_2} - 1$;
- (v) if $\frac{\partial i_1}{\partial L_2} > 0$, then $\frac{\partial i_1}{\partial L_2} > \frac{\partial y_1}{\partial L_2} > 0$, else $\frac{\partial i_1}{\partial L_2} \leq \frac{\partial y_1}{\partial L_2} \leq 0$.

Proof: See appendix.

Results (i)-(iii) are the same as those of Proposition 1. As in the case with exogenous transfers, relaxing the borrowing constraint on older children increases late investment, increases early investment if early and late investment are complementary enough, and increases overall human capital production.²⁰ However, parents do adjust their transfers. Result (iv) shows that they reduce transfers to older children as one might expect, but they do not fully offset the benefits of a greater borrowing capacity, so the older child's consumption rises. Interestingly, result (v) shows that parents may increase transfers to their children at young ages to help finance increases in early investment; although they do not fully defray the costs of any additional investments. If investments are sufficiently substitutable over time so that children respond to relaxed borrowing constraints by shifting investment from the earlier to the later period, parents will reduce transfers to their children at all ages.

Because we now model the parent's decision, we can also consider the effects of relaxing their borrowing constraints. We begin with the constraint on young parents. If a young constrained parent can borrow more, he will. When the young parent borrows more, this worsens his debt

²⁰As when transfers were exogenous, using Equation (2) as the production function, and assuming investments are perfect substitutes implies that early investment falls when the borrowing constraint on the older child is relaxed.

position as an old parent. As a result, relaxing the borrowing constraint on young parents reduces resources available for transfers to older children.

Proposition 4. *Assume that $\tilde{U}(\cdot) = \rho U(\cdot)$, the old child is borrowing constrained (i.e. $A_3 = -L_2$) while a child, but not as an adult. Then, if the young parent is borrowing constrained,*

- (i) *if Condition 1 does not hold, then $\frac{\partial i_1}{\partial L_3} > 0$ and $\frac{\partial i_2}{\partial L_3} < 0$;*
- (ii) *if $\frac{\partial i_1}{\partial L_3} < 0$, then $\frac{\partial i_2}{\partial L_3} < 0$;*
- (iii) *if $\frac{\partial i_1}{\partial L_3} > 0$, then $\frac{\partial y_1}{\partial L_3} > 0$;*
- (iv) *if $\frac{\partial i_2}{\partial L_3} < 0$, then $\frac{\partial y_2}{\partial L_3} < 0$.*

Proof: See appendix.

Proposition 4 is the endogenous transfer analog to Proposition 2. Even when we allow parental transfers to be chosen optimally, it is still possible for investment to fall in both periods as a result of the shift of resources from the second to the first period. Again, if investments are substitutable enough, the child responds to the change by investing more in period one and less in period two, which helps alleviate his first period constraint by shifting resources forward. We can show that parental transfers rise in the first period if investment rises in the first period, and fall in the second period if investment falls in the second period.

Lastly, we look at the effects of loosening the borrowing constraint on older parents.

Proposition 5. *Assume that $\tilde{U}(\cdot) = \rho U(\cdot)$, the old child is borrowing constrained (i.e. $A_3 = -L_2$) while a child, but not as an adult. Then, if the parent is borrowing constrained when old,*

- (i) $\frac{\partial i_2}{\partial L_4} \in (0, 1)$;
- (ii) *if Condition 1 holds, then $\frac{\partial i_1}{\partial L_4} > 0$, else $\frac{\partial i_1}{\partial L_4} < 0$;*
- (iii) $\frac{\partial h_3}{\partial L_4} > 0$;
- (iv) $\frac{\partial i_2}{\partial L_4} < \frac{\partial y_2}{\partial L_4} < 1$;
- (v) *if $\frac{\partial i_1}{\partial L_4} > 0$ then $\frac{\partial i_1}{\partial L_4} > \frac{\partial y_1}{\partial L_4} > 0$, else $\frac{\partial i_1}{\partial L_4} \leq \frac{\partial y_1}{\partial L_4} \leq 0$.*

Proof: See appendix.

This is a straightforward exercise. With changes in L_2 , the child is affected by his own future debt position, and with changes in L_3 , the child is affected by his parent's future debt position. Here, the link between parent and child is severed after this period, so an increase in L_4 gives the older child an income transfer with no strings attached. As a result, when the old parent can borrow more, he does, and passes some of the extra borrowing onto his old child. The old child takes this additional income and increases his investment and consumption in period two. If investments are complementary enough, period one investment increases, otherwise they fall. Regardless of the change in first period investment, human capital increases. The parent responds to increases (decreases) in first period investment by increasing (decreasing) his first period transfer, although by less than the change in investment.

3.3 Embedding the Child's Problem into the Parent's Problem

If we embed the child's problem (5) into the parent's problem (7), assuming $\tilde{U}(\cdot) = \rho U(\cdot)$, and assume that the child's continuation utility, $\tilde{V}(h_3, A_3)$, is given by the parent's value function (7) then the problem becomes fully dynamic and intergenerational. Parents transfer resources to their children, who grow up, become parents of their own children, and transfer resources to them in an analogous fashion.

We believe that it is reasonable to assume young parents make investment decisions for their young children. When old children make investment decisions, we assume that it is their last period of financial interaction with their parents, so that there is no room for strategic behavior. Therefore, it is possible to re-write the entire family problem from the point of view of parents. We assume that the ability of a child depends on that of the parent following a simple Markov process. Once a child is born, the parent's ability becomes irrelevant. However, the child's ability will affect parental decisions, because it affects the child's ability to accumulate human capital, and it affects the future ability of the grandchild. Therefore, the value function for a young parent with a young child will depend on his child's ability in addition to his own human capital and assets. Letting prime superscripts denote the child's variables, the problem facing a young parent

with a young child is given by:

$$V_3(h_3, A_3, \theta') = \max_{c_3, c'_1, i'_1} \{u(c_3) + \rho u(c'_1) + \beta V_4(h_4, A_4, i'_1, \theta')\}$$

subject to

$$\begin{aligned} A_4 &= RA_3 + W(h_3) - c_3 - i'_1 - c'_1, \\ A_4 &\geq -L_3, \\ c'_1 \geq 0 \quad i'_1 \geq 0 \quad c_3 \geq 0, \end{aligned}$$

human capital equations (3), and the earnings equation (4). Since young children are not allowed to borrow on their own, the only constraint on borrowing represents that imposed on young parents.

The problem facing an old parent with an old child is given by:

$$V_4(h_4, A_4, i'_1, \theta') = \max_{c_4, c'_2, i'_2, A_5} \{u(c_4) + \beta V_5(h_5, A_5) + \rho[u(c'_2) + \beta E_{\theta''}(V_3(h'_3, A'_3, \theta''|\theta'))]\}$$

subject to

$$\begin{aligned} A'_3 + A_5 &= RA_4 + W(h_4) + W_2 - c_4 - c'_2 - i'_2, \\ A'_3 + c'_2 + i'_2 - W_2 &\geq 0, \\ A_5 &\geq -L_4, \\ A'_3 &\geq -L_2, \\ c'_2 \geq 0 \quad i'_2 \geq 0 \quad c_4 \geq 0, \end{aligned}$$

human capital equations (1) and (3), and the earnings equation (4). The second constraint ensures that parental transfers are non-negative. That is, parents cannot borrow against their offspring's earnings, leaving debt for them to re-pay. At this stage, both the old parent and the old child face constraints on their borrowing as shown in the third and fourth constraints. The expectation of

V_3 is taken over the ability level of the future grandchild conditional on the ability of the child, θ' .

The problem facing a post-parent with no child at home is standard:

$$V_5(h_5, A_5) = \max_{A_6} \{u(RA_5 + W(h_5) - A_6) + \beta u(RA_6)\}.$$

This can easily be solved analytically (given a specific utility function) and incorporated into the old parent's problem. After solving the parent's problems computationally, it is straightforward to back out the value functions for children.

4 An Empirically Based Quantitative Analysis

In our computational analysis, we assume that there are a finite number of early investment, human capital, and ability levels, but a continuum of asset levels. We modify the intergenerational problem above in three ways: we (1) introduce earnings shocks, (2) allow for human capital-specific growth rates, and (3) allow borrowing constraints to depend on human capital levels.

To account for the variation in earnings within education classes, we introduce period j -specific earnings shocks, ϵ_j for young and old parents so

$$W(h_j, \epsilon_j) = wh_j + \epsilon_j, \quad \text{for } j = 3, 4.$$

These shocks are distributed such that earnings are always non-negative, as we discuss further below.

To better account for empirical differences in growth rates in earnings by education, we allow growth rates in human capital to depend on current human capitals. Since human capital after period three is determined by h_3 , we represent growth rates from period three to four by $\Gamma_4(h_3)$ and from period four to five by $\Gamma_5(h_3)$ (generalizing Γ_4 and Γ_5). Since we will consider n discrete h_3 types, this implies there may be up to n different growth rates each period.

Finally, we allow borrowing constraints to depend on the future human capital of an individual to account for the possibility that more educated persons are able to borrow more. This is both

theoretically and empirically attractive for reasons discussed in Lochner and Monge (2004).²¹ Following Ayagari (199?), we assume that borrowing limits depend on the lowest possible discounted value of future earnings, since that determines the amount a person can credibly commit to re-pay under any circumstances. Since earnings after childhood are fully determined by h_3 , denote these constraints by $L_j(h_3)$. Letting $\underline{\epsilon}_j = \min\{\epsilon_j\}$ represent the lowest possible earnings shock in period j , we assume that

$$\begin{aligned} L_2(h_3) &= -\gamma[R^{-1}(wh_3 + \underline{\epsilon}_3) + R^{-2}(w\Gamma_4(h_3)h_3 + \underline{\epsilon}_4) + R^{-3}w\Gamma_4(h_3)\Gamma_5(h_3)h_3], \\ L_3(h_3) &= -\gamma[R^{-1}(\Gamma_4(h_3)h_3 + \underline{\epsilon}_4) + R^{-2}w\Gamma_4(h_3)\Gamma_5(h_3)h_3], \\ L_4(h_3) &= -\gamma R^{-1}w\Gamma_4(h_3)\Gamma_5(h_3)h_3, \end{aligned}$$

where $\gamma \in [0, 1]$. Intuitively, the parameter γ reflects the efficiency of credit markets, since γ near zero implies that no borrowing is allowed while γ near one implies that individuals can borrow fully against guaranteed future earnings. It is important to note, however, that constraints may be severe even when $\gamma = 1$.²²

These three modifications are easily incorporated into the intergenerational problem of the previous section without generating any important computational difficulties. From this point on, we discuss this modified version of the intergenerational problem

We assume a CES human capital production function, as in Equation (2), and a CIES utility function, given by

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma \geq 0.$$

4.1 Calibration

We use data from the March Current Population Survey (CPS), National Longitudinal Survey of Youth 1979 Cohort (NLSY79), and Children of the NLSY79 to calibrate our model. The six model periods are mapped into ages 1-12, 13-24, 25-36, 37-48, 49-60, and 61-72. We examine

²¹Lochner and Monge (2004) argue that more skilled individuals can commit to re-pay higher debts, so that rational lenders will offer them more credit. Furthermore, the federal student loan system allows students to borrow more if they spend more on their education.

²²Of course, γ could vary across stages of the lifecycle. We do not pursue this extension, because we do not expect to be able to separately calibrate three different γ values given our data. However, in our policy exercises, we allow for stage dependent γ .

5 schooling groups, each representing a different level of h_3 : high school dropouts (less than 12 years of completed schooling), high school graduates (exactly 12 years of completed schooling), some college (13-15 years of completed schooling), college graduates (exactly 16 years of completed schooling), and post-graduates (17 or more years of completed schooling). An annual interest rate of $r = 0.05$ is assumed throughout, so $R = (1 + r)^{12} = 1.7959$. We assume $\beta = R^{-1}$. All earnings are in 1979 dollars (deflated by the CPI-U). We normalize $w = 1$, so human capital is measured in 1979 dollars per hour. We choose the preference parameter governing the intertemporal elasticity of substitution for consumption, σ , to be 2.

Second Period Earnings

We use the random sample of male high school dropouts over ages 16-24 in the NLSY79 to estimate earnings during the second period of life, $w\phi\bar{h}$. This reflects the potential earnings of all persons if they do not invest more than the minimum—any reduction in earnings during this period is part of schooling investment (i.e. foregone earnings). Since we do not model labor supply decisions, we use wage measures rather than actual earnings. Afterwards, these measures can be multiplied by any number of hours to represent annual or period earnings. We compute the average present value of real wages over ages 16-24 in 1979 dollars discounted forward or backward to age 18 (roughly the midpoint of the second model period). We assume earnings are zero from ages 13-15 and multiply the 16-24 average wage by 0.75 to arrive at our estimate of second period earnings: $w\phi\bar{h} = 3.2506$

Period Three Human Capital Levels, Wage Rates, and the Distribution of Earnings Shocks

We use the random sample of men ages 25-36 in the NLSY79 to determine post-school human capital levels, h_3 , and the distribution of earnings shocks.²³ We assume that h_{min} is the minimum human capital level, that associated with high school dropouts. We specifically estimate the

²³Since survey respondents are ages 14-21 at the beginning of the NLSY79 sample period (1979) and we observe them annually through 1994 and bi-annually after then until 2000, these ages fall entirely within the sample period for nearly all respondents.

distribution for ϵ_3 and assume that the distribution of later shocks, ϵ_4 , is the same.²⁴ Since we do not model labor supply decisions, we use wage measures rather than earnings to determine human capital levels. Afterwards, these measures can be multiplied by any number of hours to represent annual or period earnings. Since our model aggregates ages 25-36, we prefer a measure based on the present value of wages over all ages within that period.²⁵ We compute the average present value of real wages over ages 25-36 in dollars as of age 30.

Notice that the empirically observed wage for individual i in model period 3 is given by

$$wage_{i,3} = wh_3 + \epsilon_{i,3}.$$

Distinguishing between the distribution of the shocks and the h_3 levels for each schooling type is non-trivial, since $\epsilon_{i,3}$ has an unknown non-negative mean. We first regress wages on five education dummies (reflecting the five education groups described above). This yields estimates of average wages for each education group S :

$$\widehat{wage}_3(S) = wh_3(S) + E(\epsilon_3).$$

These values, along with the fraction of the population in each schooling category, are reported in Table 5. This represents the wage distribution in period 3 that will guide our calibration.

Table 5: Average Present Value of Wages over ages 25-36 by Education

Education	Avg. PV Wage	Fraction of Pop.
HS Dropout	5.1258	0.1410
HS Graduate	6.2861	0.4295
Some College	6.9533	0.1963
College Graduate	8.7227	0.1440
Post-Graduate	9.5493	0.0892

From these values alone, it is impossible to separate h_3 from $E(\epsilon_{i,3})$. We must examine higher moments of the residual distribution to identify the mean, μ_d , and standard deviation, σ_d , which

²⁴In principle, one could use a similar approach for model period 4 as is used here for period 3; however, NLSY79 respondents do not have observations over most of the period 4 ages.

²⁵Simply using the distribution of wages at any single age would suffice if shocks were not correlated across time. While we do not model serial correlation in wage shocks across model periods, it seems appropriate to account for serial correlation within model periods in calculating the variability of the shocks.

can then be used to separately identify the values of h_3 . Notice that the population is almost evenly split between those that never attended college and those that did.

Residuals from the wage regression yield $\tilde{\varepsilon}_{i,3} = \varepsilon_{i,3} - E(\varepsilon_{i,3})$ for each person. Given our assumption of log normality for $\varepsilon_{i,3}$, we can identify the mean, μ_d , and variance, σ_d^2 , of the underlying normal distribution for $\log(\varepsilon_{i,3})$ using the following central moment conditions:

$$\begin{aligned} E(\tilde{\varepsilon}_{i,3}^2) &= e^{(\sigma_d^2+2\mu_d)} (e^{\sigma_d^2} - 1) \\ E(\tilde{\varepsilon}_{i,3}^3) &= e^{(1.5\sigma_d^2+3\mu_d)} (e^{\sigma_d^2} - 1)^2 (e^{\sigma_d^2} + 2). \end{aligned}$$

With the sample analogs $\frac{1}{n} \sum_{i=1}^n \tilde{\varepsilon}_{i,3}^2 = 11.3911$ and $\frac{1}{n} \sum_{i=1}^n \tilde{\varepsilon}_{i,3}^3 = 115.8334$, we have two equations and need to solve for two unknown parameters, μ_d and σ_d . Finally, given these values of μ_d and σ_d , we can subtract $E(\varepsilon_{i,3}) = e^{(\mu_d+0.5\sigma_d^2)}$ from each $\widehat{wage}_3(S)$ to obtain estimates of $h_3(S)$ for each schooling category. These estimates are given in Table 6.

Table 6: Calibrated Human Capital and Wage Shock Distribution Parameters

Parameter	Value
<i>h</i> ₃ by Education:	
HS Dropout	1.0114
HS Graduate	2.1717
Some College	2.8389
College Graduate	4.6083
Post-Graduate	5.4349
Shock Distribution:	
μ_d	1.1572
σ_d	0.7173

It is important to recognize that these h_3 values reflect the lowest possible wage for each schooling group in period three; average wages are given by $h_3 + E(\varepsilon_3)$, where $E(\varepsilon_3) = 4.1144$.

Post-School Wage Growth

We use 2001-03 March CPS data on men ages 25-60 who worked at least five weeks last year and at least five hours per week last week to compute the growth rates in human capital over model periods 3 to 4, $\Gamma_4(S)$, and periods 4 to 5, $\Gamma_5(S)$. First, we compute the average present

discounted value of earnings as of ages 30, 42, and 54 for age categories 25-36, 37-48, and 49-60, respectively. This yields estimated growth rates in average wages from model periods 3 to 4, $\gamma_4(S)$, and from periods 4 to 5, $\gamma_5(S)$, for all schooling groups. Since these growth rates reflect growth in wages rather than human capital, they do not directly correspond to the parameters of interest ($\Gamma_4(S)$ and $\Gamma_5(S)$). We must take account of the fact that

$$\begin{aligned}\widehat{wage}_3(S) &= wh_3(S) + E(\varepsilon_3), \\ \widehat{wage}_4(S) &= wh_4(S) + E(\varepsilon_4), \\ \widehat{wage}_5(S) &= wh_5(S).\end{aligned}$$

We, therefore, compute growth rates in human capital by schooling category using estimates of $\gamma_4(S)$, $\gamma_5(S)$, $E(\varepsilon_3)$, $E(\varepsilon_4)$, and $h_3(S)$ as follows:

$$\begin{aligned}\Gamma_4(S) &\equiv \frac{h_4(S)}{h_3(S)} = \gamma_4(S) + (\gamma_4(S) - 1) \frac{E(\varepsilon_3)}{h_3(S)} \\ \Gamma_5(S) &\equiv \frac{h_5(S)}{h_4(S)} = \gamma_5(S) \left(1 + \frac{E(\varepsilon_3)}{\Gamma_4 h_3(S)} \right).\end{aligned}$$

Estimated rates of growth in wages and human capital are reported in Table 7.

Table 7: Rates of Growth in Wages and Human Capital

Education Category	Wage Growth Rate		Human Capital Growth Rate	
	$\gamma_4(S)$	$\gamma_5(S)$	$\Gamma_4(S)$	$\Gamma_5(S)$
HS Dropout	1.1163	1.0573	1.5893	3.7502
HS Graduate	1.1778	1.0599	1.5147	2.3170
Some College	1.2607	1.0425	1.6384	1.9885
College Graduate	1.3415	0.9861	1.6464	1.5373
Post-Graduate	1.3662	1.0057	1.6434	1.4823

Not surprisingly, growth rates in wages tend to be much larger between periods 3 and 4 than between periods 4 and 5. Periods 3 and 4 include a non-negative wage shock, but period 5 does not. Because the mean value for these shocks is rather large, higher growth in earnings between periods 3 and 4 does not always translate into higher values of $\Gamma_4(S)$ than $\Gamma_5(S)$.

Remaining Parameters

A number of remaining parameters need to be calibrated. In particular, we need to determine parameters of the human capital production function, a , b , and c , ability levels, θ , along with its transition matrix, π , parental altruism towards their children, ρ , and the debt constraint parameter, γ . These parameters are found by simulating the model in steady state to best fit a number of moments in the NLSY79 data. We fit to the education distribution found in Table 5 and to the education distribution conditional on mother's education and income terciles while a young parent and while an old parent.

4.2 Benchmark

5 Policy Analysis

6 Conclusion

7 Appendix

Proof of Proposition 3: Rewrite Problem (5) by substituting in Equations (6) and (1), and the two budget constraints. Because we are assuming this person is constrained as an old child, let $A_3 = -L_2$.

$$U(y_1, y_2, \theta) = \max_{i_1, i_2} u(y_1 - i_1) + \beta u(y_2 + W_2 - i_2 + L_2) + \beta^2 v(-RL_2 + w\chi(h_{min} + f(i_1, i_2, \theta))). \quad (8)$$

The first order conditions for i_1 and i_2 of this problem are given by:

$$-u'(y_1 - i_1) + \beta^2 v' w \chi f_1 = 0, \quad (9)$$

$$-u'(y_2 + W_2 - i_2 + L_2) + \beta v' w \chi f_2 = 0, \quad (10)$$

respectively.

Using Equation (9), implicitly differentiate i_1 with respect to L_2 and i_2 :

$$\frac{\partial i_1^*}{\partial L_2} = \frac{\overbrace{R\beta^2 w \chi v'' f_1}^{-}}{\underbrace{u''(c_1) + \beta^2 w \chi v' f_{11} + \beta^2 w^2 \chi^2 v'' f_1^2}_{-}} > 0, \quad (11)$$

$$\frac{\partial i_1^*}{\partial i_2} = \frac{\overbrace{-\beta^2 w \chi (v' f_{12} + w \chi v'' f_1 f_2)}^{\text{Condition 1}}}{\underbrace{u''(c_1) + \beta^2 w \chi v' f_{11} + \beta^2 w^2 \chi^2 v'' f_1^2}_{-}} \quad (12)$$

where the latter is positive if Condition 1 holds, and negative otherwise. The implicit derivatives are labeled with an asterisk to highlight that they are not the complete partial derivative since they do not take into account Equation (10)

Using Equation (10), implicitly differentiate i_2 with respect to L_2 , realizing that $i_1(i_2, L_2)$, and substituting in the above implicit derivatives:

$$\frac{\partial i_2}{\partial L_2} = \frac{u''(c_2) + \beta w \chi (R v'' f_2 - \frac{\partial i_1^*}{\partial L_2} (v' f_{12} + w \chi v'' f_1 f_2))}{u''(c_2) + \beta w \chi (v' f_{22} + w \chi v'' f_2^2 + \frac{\partial i_1^*}{\partial i_2} (v' f_{12} + w \chi v'' f_1 f_2))} \quad (13)$$

It can be shown using second order conditions (Assumption 1) that the denominator of this expression is negative. First, multiply the denominator through by the denominator in Equation (12), and cancel terms. Note that this causes the sign of the original denominator to change, so the task is to show that the following expression is positive:

$$u''(c_2) [u''(c_1) + \beta^2 w \chi (v' f_{11} + w \chi v'' f_1^2)] + u''(c_1) \beta w \chi (v' f_{22} + w \chi v'' f_2^2) + \beta^3 w^2 \chi^2 v'^2 f_{11} f_{22} + \beta^3 w^3 \chi^3 v' v'' f_1^2 f_{22} + \beta^3 w^3 \chi^3 v' v'' f_2^2 f_{11} - \beta^3 w^2 \chi^2 v'^2 f_{12}^2 - 2\beta^3 w^3 \chi^3 v' v'' f_1 f_2 f_{12} > 0.$$

Note that the expressions on the first line are positive. It suffices to show that the expressions on the second line are positive. Collecting terms we need to show:

$$\underbrace{\beta^3 w^2 \chi^2 v'^2 [f_{11} f_{22} - f_{12}^2]}_{+} \underbrace{+ \beta^3 w^3 \chi^3 v' v'' f_1 [f_1 f_{22} - f_2 f_{12}]}_{+} \underbrace{+ \beta^3 w^3 \chi^3 v' v'' f_2 [f_2 f_{11} - f_1 f_{12}]}_{+} > 0.$$

This follows from the second order conditions in Assumption 1.

(i.) To prove the first part of Proposition 1 part (i.), that $\frac{\partial i_2}{\partial L_2} > 0$, the numerator of Equation (13) must be negative. Let the denominator of Equations (12) and (11) be denoted by $D = u''(c_1) + \beta^2 w \chi v' f_{11} + \beta^2 w^2 \chi^2 v'' f_1^2 < 0$. Multiply Equation (13) by D/D .

$$\frac{\partial i_2}{\partial L_2} = \frac{u''(c_2)D + R\beta w \chi v'' f_2 D - R\beta^3 w^2 \chi^2 v'' f_1 (v' f_{12} + w \chi v'' f_1 f_2)}{u''(c_2)D + \beta w \chi v' f_{22} D + \beta w^2 \chi^2 v'' f_2^2 D - \beta^3 w^2 \chi^2 (v' f_{12} + w \chi v'' f_1 f_2)^2} \quad (14)$$

Note that denominator is now positive, so for the entire expression to be positive, the numerator must be positive. The first term of the numerator is clearly positive, it is enough to show the remaining terms are positive:

$$-R\beta w\chi v''(\beta^2 w\chi f_1(v' f_{12} + w\chi v'' f_1 f_2) - f_2 D) > 0.$$

Since $-R\beta w\chi v'' > 0$, show:

$$\beta^2 w\chi f_1(v' f_{12} + w\chi v'' f_1 f_2) - f_2(u''(c_1) + \beta^2 w\chi(v' f_{11} + w\chi v'' f_1^2)) > 0.$$

Multiply out, and cancel terms. Note $-f_2 u''(c_1) > 0$. It suffices to show:

$$\beta^2 w\chi v' \underbrace{(f_1 f_{12} - f_{11} f_2)}_{+Assumption1} > 0.$$

Using second order conditions given in Assumption 1, this is true. Therefore:

$$\frac{\partial i_2}{\partial L_2} > 0.$$

This is a strict inequality as long as u is strictly concave, because the first piece of the first term of the numerator of Equation (14) is then strictly positive.

To prove the second part of Proposition 1 part (i.), that $\frac{\partial i_2}{\partial L_2} < 1$, the numerator of Equation (14) must be less than the denominator of Equation (14). Note that it has been shown that both terms are positive. Multiply out both the numerator and the denominator, cancel out the terms that contain $u''(c_2)$, and divide through by $\beta w\chi$:

$$\begin{aligned} & Rv'' f_2 D - R\beta^2 w\chi v' v'' f_1 f_{12} - R\beta^2 w^2 \chi^2 v''^2 f_1^2 f_2 < \\ & v' f_{22} D + w\chi v'' f_2^2 D - \beta^2 w\chi v'^2 f_{12}^2 - \beta^2 w^3 \chi^3 v''^2 f_1^2 f_2^2 - 2\beta^2 w^2 \chi^2 v' v'' f_1 f_2 f_{12}. \end{aligned}$$

Substitute in for D and cancel terms:

$$\begin{aligned} & Ru''(c_1)v'' f_2 + R\beta^2 w\chi v' v'' f_2 f_{11} - R\beta^2 w\chi v' v'' f_1 f_{12} < u''(c_1)v' f_{22} + \\ & \beta^2 w\chi v'^2 f_{11} f_{22} + \beta^2 w^2 \chi^2 v' v'' f_1^2 f_{22} + u''(c_1)w\chi v'' f_2^2 + \beta^2 w^2 \chi^2 v' v'' f_2^2 f_{11} - \beta^2 w\chi v'^2 f_{12}^2 - 2\beta^2 w^2 \chi^2 v' v'' f_1 f_2 f_{12}. \end{aligned}$$

Collect terms:

$$0 < \underbrace{u''(c_1)}_{-} \underbrace{(v'f_{22} + v''f_2(w\chi f_2 - R))}_{-} + \beta^2 w\chi v'^2 \underbrace{(f_{11}f_{22} - f_{12}^2)}_{+} + \beta^2 w\chi v'v''(w\chi f_1^2 f_{22} + w\chi f_2^2 f_{11} - 2w\chi f_1 f_2 f_{12} - Rf_{11}f_2 + Rf_1 f_{12}).$$

Because the old child is constrained, the marginal benefit from investing in the second period ($w\chi f_2$) is greater than the marginal cost (R). Therefore, the first term on the right hand side is positive. The second term on the right hand side is positive because of second order conditions given in Assumption 1. It remains to be shown that the third piece on the right hand side is positive:

$$\beta^2 w\chi v'v''(w\chi f_1^2 f_{22} + w\chi f_2^2 f_{11} - 2w\chi f_1 f_2 f_{12} - Rf_{11}f_2 + Rf_1 f_{12}) > 0,$$

or that:

$$w\chi f_1^2 f_{22} + w\chi f_2^2 f_{11} - 2w\chi f_1 f_2 f_{12} - Rf_{11}f_2 + Rf_1 f_{12} < 0.$$

Collecting terms:

$$w\chi f_1 \underbrace{(f_1 f_{22} - f_2 f_{12})}_{-Assumption1} + \underbrace{(f_2 f_{11} - f_1 f_{12})}_{-Assumption1} \underbrace{(w\chi f_2 - R)}_{+} < 0.$$

The first term is negative because of second order conditions given in Assumption 1. The second term is negative because of those same second order conditions and the fact that the old child is constrained, and $w\chi f_2 > R$.

As long as $u''(c_2) < 0$, $v' > 0$, and $f_{22} < 0$, the numerator of Equation (14) is *strictly* less than the denominator, and hence $\frac{\partial i_2}{\partial L_2} < 1$.

(ii.) The second part of Proposition 1 states that if Condition 1 holds, then $\frac{\partial i_1}{\partial L_2} > 0$. To show this, note:

$$\frac{\partial i_1}{\partial L_2} = \underbrace{\frac{\partial i_1^*}{\partial L_2}}_{+} + \underbrace{\frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_2}}_{+}.$$

It has been shown that $\frac{\partial i_1^*}{\partial L_2}$ and $\frac{\partial i_2}{\partial L_2}$ are positive, and that if Condition 1 holds, $\frac{\partial i_1^*}{\partial i_2} > 0$. Therefore, when Condition 1 holds, $\frac{\partial i_1}{\partial L_2} > 0$.

iii. The third part of Proposition 1 states that $\frac{\partial h_3}{\partial L_2} > 0$. To show this, note:

$$\frac{\partial h_3}{\partial L_2} = f_1 \frac{\partial i_1}{\partial L_2} + f_2 \frac{\partial i_2}{\partial L_2},$$

which can be rewritten as:

$$\frac{\partial h_3}{\partial L_2} = f_1 \frac{\partial i_1^*}{\partial L_2} + f_1 \frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_2} + f_2 \frac{\partial i_2}{\partial L_2}.$$

Collect terms, substitute in for $\frac{\partial i_1^*}{\partial L_2}$ and $\frac{\partial i_1^*}{\partial i_2}$, and show:

$$\frac{R\beta^2 w \chi v'' f_1^2}{D} + \frac{\partial i_2}{\partial L_2} \left[f_2 + \frac{-\beta^2 w \chi v' f_1 f_{12} - \beta^2 w^2 \chi^2 v'' f_1^2 f_2}{D} \right] > 0.$$

Multiply through by $-D$, collect and cancel terms, and show:

$$\underbrace{-R\beta^2 w \chi v'' f_1^2}_{+} + \frac{\partial i_2}{\partial L_2} \left[\underbrace{-u''(c_1) f_2}_{+} + \beta^2 w \chi v' \underbrace{(f_1 f_{12} - f_2 f_{11})}_{+ \text{Assumption 1}} \right] > 0.$$

As indicated by the plus signs on the above equation, each individual term is positive. The last term is positive because of second order conditions given in Assumption 1.

Q.E.D.

Proof of Proposition 2: Rewrite Problem (8) by substituting in $y_1 + y$ for y_1 and $y_2 - Ry$ for y_2 :

$$U(y_1, y_2, \theta) = \max_{i_1, i_2} u(y_1 + y - i_1) + \beta u(y_2 - Ry + W_2 - i_2 + L_2) + \beta^2 v(-RL_2 + w\chi(h_{min} + f(i_1, i_2, \theta))). \quad (15)$$

The first order conditions for i_1 and i_2 of this problem are given by:

$$-u'(y_1 + y - i_1) + \beta^2 v' w \chi f_1 = 0, \quad (16)$$

$$-u'(y_2 - Ry + W_2 - i_2 + L_2) + \beta v' w \chi f_2 = 0, \quad (17)$$

respectively.

Using Equation (16), implicitly differentiate i_1 with respect to y (note that the implicit derivative of i_1 with respect to i_2 is given by Equation (12)):

$$\frac{\partial i_1^*}{\partial y} = \frac{\overbrace{u''(c_1)}^{-}}{\underbrace{u''(c_1) + \beta^2 w \chi v' f_{11} + \beta^2 w^2 \chi^2 v'' f_1^2}_{-}} > 0. \quad (18)$$

Using Equation (17), implicitly differentiate i_2 with respect to y , realizing that $i_1(i_2, y)$, and substituting in the above implicit derivatives:

$$\frac{\partial i_2}{\partial y} = \frac{-u''(c_2)R - \beta w \chi \frac{\partial i_1^*}{\partial y} (v' f_{12} + w \chi v'' f_1 f_2)}{u''(c_2) + \beta w \chi (v' f_{22} + w \chi v'' f_2^2) + \frac{\partial i_1^*}{\partial i_2} (v' f_{12} + w \chi v'' f_1 f_2)}. \quad (19)$$

Note that the denominator of Equation (19) was shown to be negative in the proof of Proposition 1.

(i.) To prove the first part of Proposition 2, that if Condition 1 does not hold, $\frac{\partial i_1}{\partial y} > 0$ and $\frac{\partial i_2}{\partial y} < 0$, note that if Condition 1 does not hold, the numerator of Equation (19) is positive. Therefore, $\frac{\partial i_2}{\partial y} < 0$. Next, consider the following expression for $\frac{\partial i_1}{\partial y}$:

$$\frac{\partial i_1}{\partial y} = \underbrace{\frac{\partial i_1^*}{\partial y}}_{+} + \frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial y}.$$

If Condition 1 does not hold, $\frac{\partial i_1^*}{\partial i_2} < 0$ and $\frac{\partial i_2}{\partial y} < 0$, hence their product is positive, and $\frac{\partial i_1}{\partial y} > 0$.

(ii.) To prove the second part of Proposition 2, that if $\frac{\partial i_1}{\partial y} < 0$, then $\frac{\partial i_2}{\partial y} < 0$, first note that if $\frac{\partial i_1}{\partial y} < 0$ it must be the case that $\frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial y} < 0$. If Condition 1 does not hold, this product must be positive, hence Condition 1 holds. If Condition 1 holds, then $\frac{\partial i_1^*}{\partial i_2} > 0$. If the product $\frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial y} < 0$, that means that $\frac{\partial i_2}{\partial y} < 0$.

Q.E.D.

Proof of Proposition 3: Rewrite Problem (7) by substituting in Equations (6) and (1), and the budget constraints facing the parent and the child. Because we are assuming this person is constrained as an old child, let the child's asset choice be: $A_3 = -L_2$. The remaining A_3 in the

problem is the asset level of the young parent who is not assumed to be constrained.

$$\max_{i_1, i_2, y_1, y_2} \rho u(y_1 - i_1) + \rho\beta u(y_2 + W_2 - i_2 + L_2) + \rho\beta^2 v(-RL_2 + w\chi(h_{min} + f(i_1, i_2, \theta))) + u(RA_3 + W(h_3) - y_1 - A_4) + \beta u(RA_4 + W(h_4) - y_2 - A_5) + \beta^2 u(RA_5 + W(h_5) - A_6) + \beta^3 u(RA_6).$$

The first order conditions for i_1 , i_2 , y_1 , and y_2 of this problem are given by:

$$-u'(y_1 - i_1) + \beta^2 v' w \chi f_1 = 0, \quad (20)$$

$$-u'(y_2 + W_2 - i_2 + L_2) + \beta v' w \chi f_2 = 0, \quad (21)$$

$$-u'(RA_3 + W(h_3) - y_1 - A_4) + \rho u'(y_1 - i_1) = 0, \quad (22)$$

$$-u'(RA_4 + W(h_4) - y_2 - A_5) + \rho u'(y_2 + W_2 - i_2 + L_2) = 0, \quad (23)$$

respectively.

Using Equation (23), implicitly differentiate y_2 with respect to L_2 and i_2 :

$$-1 < \frac{\partial y_2^*}{\partial L_2} = \frac{\overbrace{-\rho u''(c_2)}^+}{\underbrace{u''(c_4) + \rho u''(c_2)}_-} < 0, \quad (24)$$

$$0 < \frac{\partial y_2^*}{\partial i_2} = \frac{\overbrace{\rho u''(c_2)}^-}{\underbrace{u''(c_4) + \rho u''(c_2)}_-} < 1. \quad (25)$$

The implicit derivatives are again labeled with an asterisk to highlight that they are not the complete partial derivative since they do not take into account Equations (20-22).

Using Equation (22), implicitly differentiate y_1 with respect to i_1 :

$$0 < \frac{\partial y_1^*}{\partial i_1} = \frac{\overbrace{\rho u''(c_1)}^-}{\underbrace{u''(c_3) + \rho u''(c_1)}_-} < 1. \quad (26)$$

Using Equation (20), implicitly differentiate i_1 with respect to L_2 and i_2 , realizing that $y_1(i_1)$, and substituting in the above implicit derivatives:

$$\frac{\partial i_1^*}{\partial L_2} = \frac{\overbrace{R\beta^2 w \chi v'' f_1}^-}{\underbrace{u''(c_1) \left[1 - \frac{\partial y_1^*}{\partial i_1} \right] + \beta^2 w \chi v' f_{11} + \beta^2 w^2 \chi^2 v'' f_1^2}_-} > 0, \quad (27)$$

$$\frac{\partial i_1^*}{\partial i_2} = \frac{\overbrace{-\beta^2 w \chi (v' f_{12} + w \chi v'' f_1 f_2)}^{\text{Condition 1}}}{\underbrace{u''(c_1) \left[1 - \frac{\partial y_1^*}{\partial i_1} \right] + \beta^2 w \chi v' f_{11} + \beta^2 w^2 \chi^2 v'' f_1^2}_{-}}, \quad (28)$$

where the latter is positive if Condition 1 holds, and negative otherwise.

Using Equation (21), implicitly differentiate i_2 with respect to L_2 , realizing that $y_2(i_2, L_2)$ and $i_1(i_2, L_2)$, and substituting in the above implicit derivatives:

$$\frac{\partial i_2}{\partial L_2} = \frac{u''(c_2) \left[\frac{\partial y_2^*}{\partial L_2} + 1 \right] + \beta w \chi \left[R v'' f_2 - \frac{\partial i_1^*}{\partial L_2} (v' f_{12} + w \chi v'' f_1 f_2) \right]}{u''(c_2) \left[1 - \frac{\partial y_2^*}{\partial i_2} \right] + \beta w \chi \left[v' f_{22} + w \chi v'' f_2^2 + \frac{\partial i_1^*}{\partial i_2} (v' f_{12} + w \chi v'' f_1 f_2) \right]}. \quad (29)$$

It can be shown using second order conditions (Assumption 1) that the denominator of this expression is negative. First, note that the first expression in this denominator is negative. It suffices to show the remaining term in this denominator is negative. Multiply the remaining term through by the denominator in Equation (28), and cancel terms. Note that this causes the sign of the original denominator to change, so the task is to show that the following expression is positive:

$$\beta w \chi (v' f_{22} + w \chi v'' f_2^2) \frac{u''(c_1) u''(c_3)}{u''(c_3) + \rho u''(c_1)} + \beta^3 w^2 \chi^2 v'^2 f_{11} f_{22} + \beta^3 w^3 \chi^3 v' v'' f_1^2 f_{22} + \beta^3 w^3 \chi^3 v' v'' f_2^2 f_{11} - \beta^3 w^2 \chi^2 v'^2 f_{12}^2 - 2\beta^3 w^3 \chi^3 v' v'' f_1 f_2 f_{12} > 0.$$

Note that the expression on the first line is positive. It is enough to show that the expressions on the second line are positive. Collecting terms we need to show:

$$\underbrace{\beta^3 w^2 \chi^2 v'^2 [f_{11} f_{22} - f_{12}^2]}_{+} + \underbrace{\beta^3 w^3 \chi^3 v' v'' f_1 [f_1 f_{22} - f_2 f_{12}]}_{+} + \underbrace{\beta^3 w^3 \chi^3 v' v'' f_2 [f_2 f_{11} - f_1 f_{12}]}_{+} > 0.$$

This follows from the second order conditions in Assumption 1.

(i.) To prove the first part of Proposition 3 part (i.), that $\frac{\partial i_2}{\partial L_2} > 0$, the numerator of Equation (29) must be negative. Substituting in implicit derivatives yields:

$$\frac{u''(c_1) u''(c_2) u''(c_3) u''(c_4)}{u''(c_3) + \rho u''(c_1)} + u''(c_2) u''(c_4) \beta^2 w \chi v' f_{11} + u''(c_2) u''(c_4) \beta^2 w^2 \chi^2 v'' f_1^2 - \underbrace{u''(c_4) R \beta^3 w^2 \chi^2 v' v'' f_1 f_{12}}_A - \underbrace{\rho u''(c_2) R \beta^3 w^2 \chi^2 v' v'' f_1 f_{12}}_B + \frac{u''(c_1) u''(c_3) u''(c_4)}{u''(c_3) + \rho u''(c_1)} R \beta w \chi v'' f_2 + \underbrace{u''(c_4) R \beta^3 w^2 \chi^2 v'' v' f_2 f_{11}}_C +$$

$$\rho \frac{u''(c_1)u''(c_2)u''(c_3)}{u''(c_3) + \rho u''(c_1)} R\beta w \chi v'' f_2 + \underbrace{\rho u''(c_2) R\beta^3 w^2 \chi^2 v' v'' f_2 f_{11}}_D.$$

First, note that all the terms that are not labeled in the above equation are negative. So it suffices to show that $A + B + C + D < 0$.

$$A + B + C + D = u''(c_4) R\beta^3 w^2 \chi^2 v' v'' \underbrace{[f_2 f_{11} - f_1 f_{12}]}_{-Assumption1} + \rho u''(c_2) R\beta^3 w^2 \chi^2 v' v'' \underbrace{[f_2 f_{11} - f_1 f_{12}]}_{-Assumption1} < 0.$$

(ii.) The second part of Proposition 3 states that if Condition 1 holds, then $\frac{\partial i_1}{\partial L_2} > 0$. To show this, note:

$$\frac{\partial i_1}{\partial L_2} = \underbrace{\frac{\partial i_1^*}{\partial L_2}}_+ + \frac{\partial i_1^*}{\partial i_2} \underbrace{\frac{\partial i_2}{\partial L_2}}_+.$$

It has been shown that $\frac{\partial i_1^*}{\partial L_2}$ and $\frac{\partial i_2}{\partial L_2}$ are positive, and that if Condition 1 holds, $\frac{\partial i_1^*}{\partial i_2} > 0$.

Therefore, when Condition 1 holds, $\frac{\partial i_1}{\partial L_2} > 0$.

iii. The third part of Proposition 3 states that $\frac{\partial h_3}{\partial L_2} > 0$. To show this, note:

$$\frac{\partial h_3}{\partial L_2} = f_1 \frac{\partial i_1}{\partial L_2} + f_2 \frac{\partial i_2}{\partial L_2},$$

which can be rewritten as:

$$\frac{\partial h_3}{\partial L_2} = f_1 \frac{\partial i_1^*}{\partial L_2} + f_1 \frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_2} + f_2 \frac{\partial i_2}{\partial L_2} = f_1 \frac{\partial i_1^*}{\partial L_2} + \frac{\partial i_2}{\partial L_2} \left[f_2 + f_1 \frac{\partial i_1^*}{\partial i_2} \right].$$

Because $f_1 \frac{\partial i_1^*}{\partial L_2} > 0$, and $\frac{\partial i_2}{\partial L_2} > 0$, it is sufficient to show:

$$f_2 + f_1 \frac{\partial i_1^*}{\partial i_2} > 0.$$

Substitute in $\frac{\partial i_1^*}{\partial i_2}$, cancel terms, and it is left to show:

$$-u''(c_1) \left[1 - \frac{\partial y_1}{\partial i_1} \right] f_2 - \beta^2 w \chi v' f_2 f_{11} + \beta^2 w \chi v' f_1 f_{12} > 0.$$

Note that the first term of this expression is positive. Combine the second two terms, and show:

$$\beta^2 w \chi v' \underbrace{[f_1 f_{12} - f_2 f_{11}]}_{+Assumption1} > 0,$$

which follows from a second order condition given in Assumption 1.

iv. The fourth part of Proposition 3 states that $0 > \frac{\partial y_2}{\partial L_2} > \frac{\partial i_2}{\partial L_2} - 1$. To show this, note:

$$\frac{\partial y_2}{\partial L_2} = \frac{\partial y_2^*}{\partial L_2} + \frac{\partial y_2^*}{\partial i_2} \frac{\partial i_2}{\partial L_2}. \quad (30)$$

From Equations (24) and (25) it is clear that:

$$\frac{\partial y_2^*}{\partial L_2} = -\frac{\partial y_2^*}{\partial i_2}.$$

Substitute this into Equation (30):

$$\frac{\partial y_2}{\partial L_2} = \frac{\partial y_2^*}{\partial i_2} \left[\frac{\partial i_2}{\partial L_2} - 1 \right].$$

Because $\frac{\partial y_2^*}{\partial i_2} > 0$, and $0 < \frac{\partial i_2}{\partial L_2} < 1$, it follows that $\frac{\partial y_2}{\partial L_2} < 0$. And since $\frac{\partial y_2^*}{\partial i_2} < 1$, then

$$\frac{\partial y_2}{\partial L_2} > \frac{\partial i_2}{\partial L_2} - 1.$$

v. The fifth part of Proposition 3 states that if $\frac{\partial i_1}{\partial L_2} > 0$, then $\frac{\partial i_1}{\partial L_2} > \frac{\partial y_1}{\partial L_2} > 0$, else

$\frac{\partial i_1}{\partial L_2} \leq \frac{\partial y_1}{\partial L_2} \leq 0$. To show this, note:

$$\frac{\partial y_1}{\partial L_2} = \frac{\partial y_1^*}{\partial i_1} \frac{\partial i_1}{\partial L_2}. \quad (31)$$

Since $\frac{\partial y_1^*}{\partial i_1} > 0$, the sign of $\frac{\partial y_1}{\partial L_2}$ is the same as the sign of $\frac{\partial i_1}{\partial L_2}$. And since $\frac{\partial y_1^*}{\partial i_1} < 1$, the magnitude of $\frac{\partial y_1}{\partial L_2}$ is less than that of $\frac{\partial i_1}{\partial L_2}$.

Q.E.D.

Proof of Proposition 4: Begin by substituting in $A_4 = -L_3$ in Equations (22) and (23).

Implicitly differentiate y_2 with respect to L_3 in Equation (23):

$$\frac{\partial y_2^*}{\partial L_3} = \frac{\overbrace{-Ru''(c_4)}^{+}}{\underbrace{u''(c_4) + \rho u''(c_2)}_{-}} < 0. \quad (32)$$

Implicitly differentiate y_1 with respect to L_3 in Equation (22):

$$0 < \frac{\partial y_1^*}{\partial L_3} = \frac{\overbrace{u''(c_3)}^{-}}{\underbrace{u''(c_3) + \rho u''(c_1)}_{-}} < 1. \quad (33)$$

Next implicitly differentiate i_1 with respect to L_3 in Equation (20), recognizing that $y_1(i_1, L_3)$ and substituting in the appropriate implicit derivatives:

$$\frac{\partial i_1^*}{\partial L_3} = \frac{\overbrace{u''(c_1) \frac{\partial y_1^*}{\partial L_3}}}{\underbrace{u''(c_1) \left[1 - \frac{\partial y_1^*}{\partial i_1} \right] + \beta^2 w \chi v' f_{11} + \beta^2 w^2 \chi^2 v'' f_1^2}} > 0, \quad (34)$$

Lastly, implicitly differentiate i_2 with respect to L_3 in Equation (21), recognizing that $y_2(i_2, L_3)$ and $i_1(i_2, L_3)$ and substituting in the appropriate implicit derivatives:

$$\frac{\partial i_2}{\partial L_3} = \frac{u''(c_2) \frac{\partial y_2^*}{\partial L_3} - \beta w \chi \frac{\partial i_1^*}{\partial L_3} (v' f_{12} + w \chi v'' f_1 f_2)}{u''(c_2) \left[1 - \frac{\partial y_2^*}{\partial i_2} \right] + \beta w \chi \left[v' f_{22} + w \chi v'' f_2^2 + \frac{\partial i_1^*}{\partial i_2} (v' f_{12} + w \chi v'' f_1 f_2) \right]}. \quad (35)$$

i. The first part of Proposition 4 states that if Condition 1 does not hold, then $\frac{\partial i_1}{\partial L_3} > 0$ and $\frac{\partial i_2}{\partial L_3} < 0$. The denominator of Equation (35) was shown to be negative in the proof of Proposition 3. The first piece in the numerator of Equation (35) is positive. The second piece is positive only if Condition 1 does not hold. If both pieces are positive than we know the numerator of Equation (35) is positive, and $\frac{\partial i_2}{\partial L_3} < 0$. Next, note that:

$$\frac{\partial i_1}{\partial L_3} = \underbrace{\frac{\partial i_1^*}{\partial L_3}}_+ + \frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_3}. \quad (36)$$

This expression is positive if the product $\frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_3}$ is positive. If Condition 1 does not hold, then $\frac{\partial i_1^*}{\partial i_2} < 0$ and $\frac{\partial i_2}{\partial L_3} < 0$, implying that $\frac{\partial i_1}{\partial L_3} > 0$.

ii. The second part of Proposition 4 states that if $\frac{\partial i_1}{\partial L_3} < 0$, then $\frac{\partial i_2}{\partial L_3} < 0$. To show this, first note that if $\frac{\partial i_1}{\partial L_3} < 0$, from Equation (36) it must be the case that $\frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_3} < 0$. The only way for this to possibly be true is if Condition 1 holds, which implies that $\frac{\partial i_1^*}{\partial i_2} > 0$. In order for $\frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_3} < 0$, it must be the case that $\frac{\partial i_2}{\partial L_3} < 0$.

iii. The third part of Proposition 4 states that if $\frac{\partial i_1}{\partial L_3} > 0$, then $\frac{\partial y_1}{\partial L_3} > 0$. To demonstrate this, note that:

$$\frac{\partial y_1}{\partial L_3} = \underbrace{\frac{\partial y_1^*}{\partial L_3}}_+ + \underbrace{\frac{\partial y_1^*}{\partial i_1} \frac{\partial i_1}{\partial L_3}}_+.$$

If $\frac{\partial i_1}{\partial L_3} > 0$, then $\frac{\partial y_1}{\partial L_3} > 0$.

iv. The fourth part of Proposition 4 states that if $\frac{\partial i_2}{\partial L_3} < 0$, then $\frac{\partial y_2}{\partial L_3} < 0$. To demonstrate this, note that:

$$\frac{\partial y_2}{\partial L_3} = \underbrace{\frac{\partial y_2^*}{\partial L_3}}_{-} + \underbrace{\frac{\partial y_2^*}{\partial i_2} \frac{\partial i_2}{\partial L_3}}_{+}.$$

If $\frac{\partial i_2}{\partial L_3} < 0$, then $\frac{\partial y_2}{\partial L_3} < 0$.

Q.E.D.

Proof of Proposition 5:

i. The first part of Proposition 5 states that

$$0 < \frac{\partial i_2}{\partial L_4} < 1.$$

To show this, begin by substituting in $A_5 = -L_4$ in Equation (23) and, implicitly differentiate y_2 with respect to L_4 :

$$0 < \frac{\partial y_2^*}{\partial L_4} = \frac{\overbrace{u''(c_4)}}{u''(c_4) + \rho u''(c_2)} < 1. \quad (37)$$

Then, using Equation (21), implicitly differentiate i_2 with respect to L_4 , realizing that $y_2(i_2, L_4)$ and $i_1(i_2)$, and substituting in the above implicit derivatives:

$$\frac{\partial i_2}{\partial L_4} = \frac{\overbrace{u''(c_2) \frac{\partial y_2^*}{\partial L_4}}}{\underbrace{u''(c_2) \left[1 - \frac{\partial y_2^*}{\partial i_2} \right] + \beta w \chi \left[v' f_{22} + w \chi v'' f_2^2 + \frac{\partial i_1^*}{\partial i_2} (v' f_{12} + w \chi v'' f_1 f_2) \right]}} > 0. \quad (38)$$

Note, that the denominator of this expression was demonstrated to be negative in the proof of Proposition 3. In that proof it was shown that the second piece of the denominator is negative. Since, $\frac{\partial y_2^*}{\partial L_4} = 1 - \frac{\partial y_2^*}{\partial i_2}$, the numerator of Equation (38) is smaller in magnitude than the denominator of Equation (38). Therefore, $\frac{\partial i_2}{\partial L_4} < 1$.

ii. The second part of Proposition 5 states that if Condition 1 holds, then $\frac{\partial i_1}{\partial L_4} > 0$, else $\frac{\partial i_1}{\partial L_4} < 0$.

To show this, note that:

$$\frac{\partial i_1}{\partial L_4} = \frac{\partial i_1^*}{\partial i_2} \underbrace{\frac{\partial i_2}{\partial L_4}}_+$$

Since $\frac{\partial i_1^*}{\partial i_2}$ is positive if Condition 1 holds, and negative otherwise, $\frac{\partial i_1}{\partial L_4}$ is positive if Condition 1 holds, and negative otherwise.

iii. The third part of Proposition 5 states that $\frac{\partial h_3}{\partial L_4} > 0$. Note that:

$$\frac{\partial h_3}{\partial L_4} = f_1 \frac{\partial i_1}{\partial L_4} + f_2 \frac{\partial i_2}{\partial L_4} = f_1 \frac{\partial i_1^*}{\partial i_2} \frac{\partial i_2}{\partial L_4} + f_2 \frac{\partial i_2}{\partial L_4} = \frac{\partial i_2}{\partial L_4} \left[f_1 \frac{\partial i_1^*}{\partial i_2} + f_2 \right].$$

Since $\frac{\partial i_2}{\partial L_4} > 0$, and it was shown in the proof of Proposition 3 part (iii) that $f_1 \frac{\partial i_1^*}{\partial i_2} + f_2 > 0$,

$\frac{\partial h_3}{\partial L_4} > 0$.

iv. The fourth part of Proposition 5 states that $\frac{\partial i_2}{\partial L_4} < \frac{\partial y_2}{\partial L_4} < 1$. Combining the following two equations:

$$\frac{\partial y_2}{\partial L_4} = \frac{\partial y_2^*}{\partial L_4} + \frac{\partial y_2^*}{\partial i_2} \frac{\partial i_2}{\partial L_4}, \quad (39)$$

and

$$\frac{\partial y_2^*}{\partial i_2} = 1 - \frac{\partial y_2^*}{\partial L_4},$$

yields:

$$\frac{\partial y_2}{\partial L_4} = \underbrace{\frac{\partial i_2}{\partial L_4}}_+ + \underbrace{\frac{\partial y_2^*}{\partial L_4}}_+ \underbrace{\left[1 - \frac{\partial i_2}{\partial L_4} \right]}_+.$$

Since $1 - \frac{\partial i_2}{\partial L_4} > 0$ and $\frac{\partial y_2^*}{\partial L_4} > 0$, clearly, $\frac{\partial i_2}{\partial L_4} < \frac{\partial y_2}{\partial L_4}$. To show that $\frac{\partial y_2}{\partial L_4} < 1$, substitute in for $\frac{\partial y_2^*}{\partial L_4}$ and $\frac{\partial y_2^*}{\partial i_2}$ in Equation (39):

$$\frac{\partial y_2}{\partial L_4} = \frac{u''(c_4) + \frac{\partial i_2}{\partial L_4} \rho u''(c_2)}{u''(c_4) + \rho u''(c_2)}.$$

Since $0 < \frac{\partial i_2}{\partial L_4} < 1$, the numerator of this expression is smaller in magnitude than the denominator, and hence $\frac{\partial y_2}{\partial L_4} < 1$.

v. The fifth part of Proposition 5 states that if $\frac{\partial i_1}{\partial L_4} > 0$ then $\frac{\partial i_1}{\partial L_4} > \frac{\partial y_1}{\partial L_4} > 0$, else $\frac{\partial i_1}{\partial L_4} \leq \frac{\partial y_1}{\partial L_4} \leq 0$.

Note that:

$$\frac{\partial y_1}{\partial L_4} = \frac{\partial y_1^*}{\partial i_1} \underbrace{\frac{\partial i_1}{\partial L_4}}_+$$

Since, $\frac{\partial y_1^*}{\partial i_1} > 0$, $\frac{\partial y_1}{\partial L_4}$ has the same sign as $\frac{\partial i_1}{\partial L_4}$. And because $\frac{\partial y_1^*}{\partial i_1} < 1$, the magnitude of $\frac{\partial y_1}{\partial L_4}$ is less than the magnitude of $\frac{\partial i_1}{\partial L_4}$.

Q.E.D.