Informative Cheap Talk Equilibria as Fixed Points*

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Abstract: We introduce a new fixed point method to analyze cheap talk games, in the tradition of Crawford and Sobel (1982). We illustrate it in a class of one-dimensional games, where the sender's bias may depend on the state of the world, and which contains Crawford and Sobel's model as a special case. The method yields new results on the structure of the equilibrium set. For games in which the sender has an *outward bias*, i.e. the sender is more extreme than the receiver whenever the state of the world is extreme, we prove that for any positive integer k, there is an equilibrium with exactly k pools, and at least one equilibrium with an infinite number of pools. We discuss the extent to which the fixed point method can be used to analyze other cheap talk signalling problems.

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1. Introduction

A stream of research examines how a privately informed agent, the "sender," can influence a decision maker, the "receiver," by supplying relevant unverifiable information. To influence the decision, all the sender can do is talk. Talking is free of costs, in the sense that messages do not enter the payoff of the players. This problem of cheap talk signalling is interesting when the sender and the receiver do not have the same preferences, i.e. when the sender is "biased."

The model of Crawford and Sobel (1982) is important in this literature. It is one of the first models to address the issue, and has served as a building block for most of the work in the area. We now have at hand an entire family of cheap talk signalling models that either enrich, build on, or apply to more specific settings, the model of Crawford and Sobel. In this paper, we introduce a new method to analyze models in this family. The key idea of the method is to look at cheap talk equilibria as the fixed points of a certain mapping. We thus label it the "fixed point method."

The method can be used to analyze a large class of cheap talk signalling games. In particular, it can help to analyze models that have raised technical difficulties, such as models where actions and types have more than one dimension. The method also leads to a new natural way to address the problem of selecting among the many equilibria that typically arise in cheap talk signalling games.

In this paper, we show how the fixed-point method works for a model in one dimension, which contains Crawford and Sobel's as a special case. Ours is more general, in that we allow the direction of the sender's bias to be either left or right, depending on the state of the world. In contrast, these authors require the sender's bias to be consistently in the same direction, across all states of the world. Thus, our model can be applied to a larger set of situations.

In this unidimensional context, using our method has three advantages. First, it works exactly the same way, for games that satisfy Crawford and Sobel's consistent bias direction restriction, and for those that do not. Second, it yields a richer description of the structure of the equilibrium set, even for games that do satisfy all the assumptions of Crawford and Sobel. Third, the method requires few regularity assumptions, and some of our results hold even when the receiver's decision rule is not continuous in its information. All of these improvements are dividends of the fixed-point method.

In general, cheap talk signalling games can be described by a set of receiver's possible actions, a set of types that represent the sender's private information, a set of preferences for the sender, indexed by his type, and a decision rule under uncertainty for the receiver. An equilibrium outcome can be described by a partition of the sender type space in pools, and a list of actions indexed by the pools in the partition, satisfying two conditions. An interpretation is that sender types in a same pool send the same information to the receiver, therefore they induce the same action, but types in different pools send different information, therefore they induce different actions. The first condition is that the action associated with a certain pool must be the decision prescribed by the receiver's rule when all he knows is that the type is in this pool. In other words, the receiver transforms the information he receives into actions in a way that is consistent with his decision rule. The second condition is that all types in any pool must like the action they induce at least as much as any other action in the list. This condition simply says that the sender types pool in an incentive-compatible fashion.

The fixed point method. We can map each pool partition into another pool partition in the following manner. For each pool in the initial partition, consider the action prescribed by the receiver's decision rule when all he knows is that the type is in this pool. This defines a list of actions. Next, sort sender types according to which action in the list they like the best. This yields a new pool partition of the type space. The equilibria of the game are exactly the fixed points of the mapping we just defined. Therefore, studying the equilibrium set amounts to study the set of fixed points of this mapping.[1]

A larger class of one dimensional models. The model we consider is more general that Crawford and Sobel's, in that we allow the direction of the sender's

[1] In unpublished work, Dimitrakas and Sarafidis (2006) use a version of the "fixed point method" outlined here, to study a variant of Crawford and Sobel's model. Their results and ours were obtained independently. For a discussion of their work, see the last section.

bias to be either left or right, depending on the state of the world. These authors require the sender's bias to be strictly in the same direction, across all states of the world. For example, all sender types could have a strict upward bias compared to the receiver. Or they could all have a strict downward bias. While this restriction is appropriate in many situations, it excludes a large class of problems. For example, the sender could have an outward bias. In this case, his preferred action is lower than the receiver's when his type is low, and higher than the receiver's when his type is high. He could also have an inward bias. In this case, his preferred action is higher than the receiver's when his type is low, and lower than the receiver's when his type is high. Our model contains upward, downward, outward and inward biases as special cases. More generally, we allow the direction of the bias to depend on the sender's type. We now provide examples of situations, where the sender has an outward or inward bias.

Outward and inward bias: examples. In our first example, the receiver is the government, and the sender is an expert, hired by the government to advise it on a one dimensional policy reform from a current status quo a^* to a new policy a. The expert's type represents the policy the expert believes the government should take. The government trusts the expert to indicate the direction of the change, i.e. whether a should be greater or lesser than a^*. The government takes into account factors that the expert will tend to ignore, such as the greater risks of facing popular resistance incurred when carrying out large changes. Thus, the government is more conservative than the expert, in the sense that it is reluctant to implement large policy changes. To fix ideas, let the type t be distributed in $[-1, 1]$, and the preferred policy of the government under complete information be $R(t) = a^* + \frac{t}{2}$. Instead, the expert would like the government to implement $S(t) = a^* + t$. In this example, the sender has an outward bias, since $S(-1) < R(-1) < R(1) < S(1)$.

In our second example, the receiver is a legislature with two members, and the sender is an expert, hired to advise it on a one dimensional policy a. The expert reports to the legislature, which then collectively choose the policy a. Specifically, the chosen policy is the outcome of a bargaining game among the two members of the legislature. To fix ideas, let the type t be any real number in $[0, 1]$, and let $S(t) = t$

be the policy the expert would like the government to implement. Let the preferred policy of one of member 1 under complete information be $R_1(t) = -\frac{3}{4} + \frac{3t}{2}$, and let the preferred policy of member 2 under complete information be $R_2(t) = \frac{1}{4} + \frac{3t}{2}$. Let the outcome of the legislative bargaining under complete information be $R(t) = \frac{R_1(t)+R_2(t)}{2} = -\frac{1}{4} + \frac{3t}{2}$. Here, the expert has an upward bias, with respect to member 1, and a downward bias, with respect to member 2. Indeed, we have for all $t \in [0,1]$, $R_1(t) < S(t) < R_2(t)$. But when comparing the expert, and the legislature's rule $R(t)$, the sender has an inward bias, since $R(-1) < S(-1) < S(1) < R(1)$.

Other examples can be found in the literature. Stein (1989) uses a unidimensional model, where the sender is a central bank, and the receiver is a financial market. The equilibrium of this market determines an exchange rate. The central bank has a target exchange rate for today, but the market expect a reversal of the policy tomorrow. As a result, it is less reactive than the central bank would like it to be. Thus, the central bank has an outward bias, compared to the market. Melumad and Shibano (1991) also study cheap talk signalling, among other mechanisms, without Crawford and Sobel's restriction. Their main focus is on comparing equilibria with one and two pools, from the point of view of the expected utility of the sender and the receiver. In both cases, the authors restrict attention to the special case where the preferences of the sender are quadratic, and the decision rule of the receiver is linear. Our analysis applies to a much larger set of situations, as it does not rely on these assumptions.

A classic result for unidimensional cheap talk signalling, which holds also in our model, is that equilibrium pools must be intervals. Crawford and Sobel prove that, when the sender's bias is strictly upward (or strictly downward), the set of integers such that there are equilibria of size k, i.e. with exactly k intervals, is of the form $\{1, \ldots, K\}$ and there are no equilibria of infinite size. In contrast, we prove that when the bias is outward, there are equilibria of any finite size and at least one of infinite size. If we interpret the maximal equilibrium size as a measure of the sender's influence on the receiver, our results suggest that a sender with an outward bias enjoys greater influence on the receiver than a sender with an strictly upward bias. Regardless of the form of the bias, the following holds. Either the set of equilibrium sizes is of the form $\{1, \ldots, K\}$, like in the strictly upward bias case, or it is $\mathbb{N} \cup \{\infty\}$, like in the outward bias case. In other words, if there is an equilibrium of size $k > 1$, then there exists an equilibrium of size $k-1$. We also show that the latter is "nested" into the former, in the sense that the boundary points of the size k equilibrium define bounds within which a size $k-1$ equilibrium necessarily exists.

We also obtain other new results on the structure of the set of equilibria of a given size. When the sender has an outward bias, the set of equilibria of a given size $k \geq 2$ is nonempty and has a lattice structure. In particular, it has a minimal element and a maximal element. Under the assumption that the highest sender type has an upward bias (this includes upward the upward and outward cases), the set of equilibria of a given size $k \geq 2$ may or may not be empty. If it is nonempty, this set is an upper-semilattice. In particular, it has a maximal element. We then provide further results on this maximal equilibrium of size k. First, we provide a simple algorithm that converges monotonically to this equilibrium. We then provide comparative statics results on this equilibrium. Crawford and Sobel (1982) proved some results of this type. Ours are stronger, in that we do not assume the unicity of the equilibrium of size k to obtain them.

As we pointed out, the fixed-point method yields both a more precise description of the equilibrium set, and for a broader class of models, than Crawford and Sobel's work. However, our main contribution here is the introduction of a fixed point method in the context of cheap talk signalling. The method can be used to address other questions in the cheap talk signalling literature. Fixed point methods are pervasive in many areas of economic theory. We show that they are a powerful tool to analyze cheap talk signalling models as well.

The rest of the paper is organized as follows. Section 2 lays out the model. Section 3 studies the set of possible equilibrium sizes in general. Section 4 introduces a taxonomy of sender's biases, and specializes the results of section 3 to certain special cases. Section 5 provides further results on the structure of the equilibrium set. The case where the receiver maximizes a von Neuman Morgenstern utility function is studied in section 6. This section includes a comparative statics analysis. In section 7, we study the important uniform-quadratic case. Section 8 discusses the technical aspect of this paper and its articulation with other works. In section 9, we discuss the extent to which the fixed-point method can be used to address other cheap talk signalling problems.

2. The model

There are two players, the sender and the receiver. Only the sender has payoff-relevant private information, his type. The sender observes his type, and sends a message to the receiver. The receiver then reads this message, and takes an action. Talking is "cheap", in the sense that messages do not directly affect payoffs.

Let $T := [0, 1]$ be the sender's set of types, with typical element t. Let $A \subseteq \mathbb{R}$ be a nonempty set of receiver's possible actions, with typical element a. A preference over A is a binary relation that is reflexive, transitive and complete. The sender has a preference relation $\succeq_t$ over A, which depend on his type t. For all $a, b \in A$, the proposition $a \succeq_t b$ means that the sender of type t weakly prefers action a to action b. The corresponding strict preference and indifference relations are denoted by $\succ_t$ and $\simeq_t$. Let $\succeq$ denote the family of preferences $\{\succeq_t\}_{t\in[0,1]}$.

A *pool* is a nonempty subset of T and represents a piece of information that the sender might provide to the receiver. Let $\mathcal{T}$ be the collection of all pools. The receiver takes decisions on the basis of the information he receives from the sender. His behavior is modeled by a reaction function $R : \mathcal{T} \to A$. A sender strategy is described by a partition Π of T in pools. A typical pool I in the partition Π is a set of sender types that send identical signals or messages. The encoding of information, i.e. what messages are sent by each of the pools, is irrelevant.

For any strategy Π, the *outcome for* Π is a function $T \to A$ that maps each sender type to the action it induces, under the strategy Π and the reaction R. For any pool I, let $1_I : T \to \{0, 1\}$ be the characteristic function of I. Then for all $t \in T$, the outcome for Π equals the sum $\sum_{I\in\Pi} R(I)1_I(t)$. Two partitions Π and Π' are *equivalent* if they induce the same outcome. An *equilibrium strategy* for $(R, \succeq)$ is a partition Π such that for all $I, I' \in \Pi$, for all $t \in I$, we have $R(I) \succeq_t R(I')$. Clearly, a strategy that is equivalent to an equilibrium strategy for $(R, \succ)$ is itself an equilibrium strategy $(R, \succeq)$. An *equilibrium outcome* for $(R, \succeq)$, is an outcome that is induced by some equilibrium strategy for $(R, \succeq)$.[2]

[2]Our setup differs from Crawford and Sobel's (1982) in various ways. Our aim is to underline the structure we study in the rest of the paper. Thus, we avoid introducing objects whose role in this structure is inessential, such as messages, beliefs of the receiver, preferences of the receiver, or a utility representation of the preferences of the sender. In many respects, our model is more general

We now introduce assumptions on the sender's preferences. A preference $\succeq_t$ is **single-peaked** if it has a unique preferred action $S(t) \in A$ (its peak) and, among any two distinct actions on the same side of the peak, the one closest the peak is preferred. More precisely, there is an action $S(t) \in A$ such that for all $a, b \in A$ satisfying either $a < b \leq S(t)$ or $S(t) \leq b < a$, we have $b \succ_t a$. We say that the family $\succeq$ is single-peaked if, for all $t \in T$, the preference $\succeq_t$ is single-peaked. Next, we assume that the sender's preference unambiguously shifts in favor of higher actions, as his type increases. The family $\succeq$ is **single-crossing** if, for all $s, t \in T$ such that $s < t$, and for all $a, b \in A$ such that $a < b$, we have $b \succeq_s a \Rightarrow b \succ_t a$. If the family $\succeq$ is *single-peaked* and *single-crossing*, then the function $S(\cdot)$ is nondecreasing on T. Last, we impose regularity on the way the sender's preferences change, as the type varies. A strict preference relation should not be reversed by an infinitesimal change of the type. We say that the family $\succeq$ is **type-continuous** if, for all $a, b \in A$, the set $\{t \in T : a \succeq_t b\}$ is closed.

We now turn attention to the receiver and introduce two assumptions on his reaction function. First, we require that if the receiver disregards some information when taking his action, then suppressing this information should not affect his decision. The reaction R is **consistent** if, for all family of disjoint pools $\mathcal{T}^* \subset \mathcal{T}$ and all $a \in A$ satisfying (for all $I \in \mathcal{T}$, $R(I) = a$), we have $R(\cup_{I \in \mathcal{T}^*} I) = a$. Second, we impose regularity on the way the receiver reacts to interval pools. The reaction R is **robust** if the receiver reaction to an interval pool that is not a singleton does not depend on whether the endpoints of the interval are included in the pool, i.e. for all $s, t \in T$ such that $s < t$, we have $R([s, t]) = R(]s, t[) = R(]s, t]) = R([s, t[)$. Abusing notations, for all $s, t \in T$ such that $s \leq t$, let $R(s, t) := R([s, t])$.

In the remainder of the section, we present some of the basic implications of some

than the classic one. For example, the receiver need not be a single decision maker, but could be the outcome of a game played by a group of players who receive the same information from the sender. The receiver need not be a Subjective Expected Utility maximizer, nor Bayesian. Our equilibrium concept is the equivalent of the Nash-Bayesian Equilibrium concept in the classic framework, known to be as strong as the Perfect-Bayesian concept in that setting. Our set of sender strategies is as rich as the set of pure sender strategies in the classic model. We do rule out mixed strategies, but nondegenerate mixed strategies would not be used in equilibrium in the classic setting anyway, so this restriction is not binding. Overall, our approach nests the classic framework as a special case.

of these assumptions.

LEMMA 1: *Let R be consistent. Let γ be the outcome for some strategy Π. Then γ is also the outcome for the partition Π' in level curves of γ. Thus Π' is equivalent to Π.*

Proof. Let $I' \in \Pi'$, and let $a \in A$ such that $I' = \{t \in T : \gamma(t) = a\}$. Let Π^* be the (possibly infinite) sub-collection of Π consisting of sets I that have a nonempty intersection with I'. For all $I \in \Pi^*$, we have $R(I) = a$, which implies $I \subseteq I'$. Therefore I' equals the union of the members of Π^*. Since R is *consistent*, then $R(I') = a$, the desired conclusion.■

As a consequence, if R is consistent, then any partition whose pools are the level curves of some equilibrium outcome for $(R, \succeq)$ is itself an equilibrium partition for $(R, \succeq)$. An *interval partition* is a partition whose pools are all intervals in T (some of them possibly singletons). Lemma 2 provides sufficient conditions on $(R, \succeq)$ under which any equilibrium partition for $(R, \succeq)$ is equivalent to an interval partition.

LEMMA 2: *Let R be consistent and let $\succeq$ be single-crossing. Then any equilibrium partition Π for $(R, \succeq)$ is equivalent to an interval partition.*

Proof. Let Π be an equilibrium partition for $(R, \succeq)$, and let Π' be the partition whose pools are the level curves of the outcome for Π. By Lemma 1, Π and Π' are equivalent. By Theorem 2.8.1 in Topkis (1998), since $\succeq$ is *single-crossing*, then all pools in the partition Π' are intervals.■

We now introduce partial orders on vectors and sets of on vectors and monotonicity notions for correspondences. Let m, n be arbitrary positive integers. For any two nonempty subsets $X, Y \subseteq \mathbb{R}^m$, let $X \leq Y$ if, for all $x \in X$, and all $y \in Y$, we have $x \leq y$. Similarly, let $X < Y$ if, for all $x \in X$, and all $y \in Y$, we have $x < y$. Let $X \subseteq \mathbb{R}^m$ and $Z \subseteq \mathbb{R}^n$. Let $G : X \twoheadrightarrow Z$ be a correspondence such that $G(x)$ is nonempty for all $x \in X$. We say that G is *nondecreasing* if, for all $x \leq y \in X$, we have $G(x) \leq G(y)$. We say that G is *increasing*[3] if for all $x < y \in X$, we have $G(x) < G(y)$.

[3]An equivalent definition is that G is nondecreasing (increasing) if all selections from G are nondecreasing (increasing) functions, in the usual sense.

We now introduce an *indifference correspondence* (for the sender), which will play an important role in our results. Let τ be the correspondence such that, for all $a, b \in A$ satisfying $a < b$, $a \succeq_0 b$ and $b \succeq_1 a$, we have $\tau(a,b) := \{t \in [0,1] : a \simeq_t b\}$, and for all $a \in A$ satisfying $S(0) \leq a \leq S(1))$, we have $\tau(a,a) := \{t \in [0,1] : S(t) = a\}$. Let

$$D_\tau := \{(a,b) \in A^2 : (a < b \text{ and } a \succeq_0 b \text{ and } b \succeq_1 a) \text{ or } (S(0) \leq a = b \leq S(1))\}.$$

LEMMA 3: *Let $\succ$ be single-peaked, single-crossing and type-continuous.*[4] *Then τ is nonempty valued and increasing on D_τ.*

Proof. Since $\succeq$ is *single-crossing* and *type-continuous*, then $a \succeq_0 b$ and $b \succeq_1 a$ imply that the set $\{t \in [0,1] : a \simeq_t b\}$ is a singleton. Similarly, since $\succeq$ is *single-peaked, single-crossing*, and *type-continuous*, the inequalities $S(0) \leq a \leq S(1)$ imply that the set $\{t \in [0,1] : S(t) = a\}$ is a nonempty closed interval. Therefore $\tau(a,b)$ is a nonempty closed interval for all $a \leq b$, and is a singleton when $a < b$. Thus τ is well-defined on D_τ.

We now prove that τ is increasing on D_τ. Let $(a,b) \in D_\tau$ and $(c,d) \in D_\tau$, such that $(a,b) \leq (c,d)$ and $(a,b) \neq (c,d)$. Let $s \in \tau(a,b)$ and $t \in \tau(c,d)$. We will prove that $s < t$. We distinguish three cases. *Case 1:* $c < d$. Since $\succeq_s$ is *single-peaked*, then $c \succ_s d$. Since $c \simeq_t d$, and $\succeq$ is *single-crossing*, then $s < t$. *Case 2:* $a < b$. Since $\succeq_t$ is *single-peaked*, then $b \succ_t a$. Since $b \simeq_s a$, and $\succeq$ is *single-crossing*, then $s < t$. *Case 3:* $a = b < c = d$. Since $S(s) = a$ and $S(t) = c$ and $S(\cdot)$ is nondecreasing, therefore $s < t$, the desired conclusion.■

3. General results

A problem $(R, \succeq)$ is **admissible** if the sender preferences $\succeq$ are *single-peaked, type-continuous* and *single-crossing*; the receiver reaction R is *consistent* and *robust*; and the function $(s,t) \mapsto R(s,t)$ is *increasing*. In the remainder of the paper, we restrict attention to admissible problems. For examples and applications, see Sections 6 and

[4]Neither Lemma 2 nor Lemma 3 would hold, if we were to replace our definition of single-crossing with the standard strict single-crossing condition that, for all $a, b \in A$ such that $a < b$, we have $b \succeq_s a \Rightarrow b \succeq_t a$ and $b \succ_s a \Rightarrow b \succ_t a$.

7. In this section, we characterize the set of equilibrium partitions for any admissible problem. We first examine the structure of the set of equilibria with finitely many intervals. We then turn our attention to equilibria with infinitely many intervals.

3.1. *Informative equilibria with finitely many intervals*

For all partition Π, let the *size of* Π be the number of distinct (possibly singleton) pools in the partition. An important class of interval equilibrium partitions are the ones that have a finite size. It is well known that any cheap talk game has an equilibrium of size one, the trivial partition with only one interval $[0,1]$, the babbling equilibrium. Our goal is to describe the structure of the equilibria of size greater than one, the informative equilibria. Any partition of size $\kappa \geq 2$ can be represented by a vector $x \in T^{\kappa+1}$ such that $0 = x_0 \leq \ldots \leq x_\kappa = 1$. Let X_κ be the set of such vectors. For each $l = 1, \ldots, \kappa - 1$, the type x_l is the boundary between the l-th and the $(l+1)$-th intervals of the partition, ranked in increasing order.[5] Also, let W_κ be the set of vectors $x \in T^{\kappa+1}$ such that $0 \leq x_0 \leq \ldots \leq x_\kappa \leq 1$.

We are now ready to introduce the size κ equilibrium correspondence. For each $\kappa \geq 2$, let $\theta^\kappa(\cdot)$ be the correspondence that maps each vector x from a subset of W_κ, to a set of vectors $\theta^\kappa(x) := \theta_0(x) \times \ldots \times \theta_\kappa(x) \subseteq T^{\kappa+1}$, where $\theta_0(x) := \{x_0\}$, $\theta_\kappa(x) := \{x_\kappa\}$, and for all $l = 1, \ldots, \kappa - 1$, we have

$$\theta_l(x) := \tau(R(x_{l-1}, x_l), R(x_l, x_{l+1})).$$

Let $D_2 := \{x \in W_2 : (R(x_0, x_1), R(x_1, x_2)) \in D_\tau\}$. The domain on which $\theta^\kappa(\cdot)$ is nonempty-valued is the set $D_\kappa \subseteq W_\kappa$ of vectors x such that $(x_0, x_1, x_2) \in D_2$ and $(x_{\kappa-2}, x_{\kappa-1}, x_\kappa) \in D_2$. Indeed, these two relations, together with the inequalities $x_0 \leq \ldots \leq x_\kappa$, and the monotonicity of $R(\cdot)$ ensure that, for all $l = 1, \ldots, \kappa - 1$, we have $(x_{l-1}, x_l, x_{l+1}) \in D_2$, i.e. $\theta_l(x)$ is well-defined on D_κ, so that $\theta^\kappa : D_\kappa \twoheadrightarrow T^{\kappa+1}$. Notice that, in general, the set D_κ could be empty.

[5]Each size κ partition admits a unique representation $x \in X_\kappa$. For example, the interval strategy $\{[0, 1/3],]1/3, 1/2[, [1/2, 1]\}$ is represented only by $x = (0, 1/3, 1/2, 1)$. However, not all vectors in X_κ represent a partition. For example the vector $(0, 0, 0, 1) \in X_3$ does not represent any partition. A necessary and sufficient condition for a vector x to represent a partition is that for all $l \in \{1, \ldots, \kappa - 1\}$, we de not have $x_{l-1} = x_l = x_{l+1}$.

For all $x \in D_\kappa$, we have $\{0\} \leq \theta_1(x) \leq \ldots \leq \theta_{\kappa-1}(x) \leq \{1\}$. In addition, for all $x \in D_\kappa \cap X_\kappa$, we also have $\{0\} = \theta_0(x) \leq \theta_1(x)$ and $\theta_{\kappa-1}(x) \leq \theta_\kappa(x) = \{1\}$, so that for all $x \in D_\kappa \cap X_\kappa$, we have $\theta^\kappa(x) \subseteq X_\kappa$. We will now show that the vectors $x \in X_\kappa$ that represent an interval equilibrium partition of size κ are the fixed-points of the correspondence $\theta^\kappa(\cdot)$ in the set $D_\kappa \cap X_\kappa$.

Lemma 4: *Let $\kappa \geq 2$. Let $x \in X_\kappa$. Then, the vector x defines an interval equilibrium partition of size κ if and only if $x \in D_\kappa$ and $x \in \theta^\kappa(x)$. When this is the case and $\kappa > 2$, we have $x_1 < \ldots < x_{\kappa-1}$.*

Proof. Since $\succeq$ is *type-continuous*, by the definition of an equilibrium strategy, and by Lemma 2, if x represents an equilibrium of size κ, then $x \in D_\kappa$ and $x \in \theta^\kappa(x)$. We will now prove that the converse also holds.

Let $\kappa > 2$ and let $x \in D_\kappa \cap X_\kappa$ be a fixed-point of $\theta^\kappa(\cdot)$. To alleviate notations, for all relevant indices l, let $S_l := S(x_l)$, let $a_l := R(x_{l-1}, x_l)$, and let $I_l :=]x_{l-1}, x_l[$. We will prove that $S_1 < \ldots < S_{\kappa-1}$. Let $H := \{h \in \{1, \ldots, \kappa-2\} : S_h < S_{h+1}\}$. Since $x \in X_\kappa$ and $\succeq$ is *single-crossing,* we have $S_0 \leq S_1 \leq \ldots \leq S_\kappa$ and $S_0 < S_\kappa$. Therefore $H \neq \emptyset$. Let $h \in H$. Then in particular $x_h < x_{h+1}$. Suppose that $h > 1$. Since $R(\cdot)$ is *increasing*, we have $a_h < a_{h+1}$. Since x is a fixed-point, we have $a_{h-1} \simeq_{x_{h-1}} a_h$ and $a_h \simeq_{x_h} a_{h+1}$. Since $\succeq_{x_{h-1}}$ and $\succeq_{x_h}$ are *single-peaked*, then $S_{h-1} \leq a_h < S_h < a_{h+1}$. In particular, $h-1 \in H$. By induction, we obtain that $1, \ldots, h \in H$. By an identical reasoning, we obtain that $h, \ldots, \kappa-2 \in H$, which proves the claim.

In particular, if $\kappa \geq 2$, then x represents a partition of size κ. It only remains to prove that this partition is an equilibrium for $(R, \succeq)$. For all $l = 1, \ldots, \kappa-1$, we have $a_l \simeq_{x_l} a_{l+1}$. Since $\succeq$ is *single-crossing*, this further implies that for all $t \in I_1 \cup \ldots \cup I_l$, we have $a_l \succeq_t a_{l+1}$, and that for all $t \in I_{l+1} \cup \ldots \cup I_\kappa$, we have $a_{l+1} \succeq_t a_l$. Thus for all $h, l \in \{0, \ldots, \kappa\}$, all $t \in I_h$, we have $a_h \succeq_t a_l$. Thus the $(I_1, \ldots, I_\kappa)$ form an equilibrium partition for $(R, \succeq)$.■

A set of positive integers is **connected to 1** if it is either $\mathbb{N}$ or of the form $\{1, \ldots, K\}$. We are now ready to state our first main result.

Theorem 1: *The set of integers κ such that there are equilibria of size κ is connected to 1.*

The proof of Theorem 1 rests on Lemmas 4, 5, 6 and 7. For all positive integer m, and all vectors $x, z \in \mathbb{R}^m$, we let $[x, z] := \{y \in \mathbb{R}^m : x \leq y \leq z\}$. Sets of this form are called **closed intervals**.

LEMMA 5: *For all $\kappa \geq 2$, the correspondence $\theta^\kappa(\cdot)$ is increasing on D_κ. For all $x \in D_\kappa$, the set $\theta^\kappa(x)$ is a closed interval.*

Proof. Since $R(\cdot)$ is *increasing* on $\{(s,t) \in [0,1]^2 : s \leq t\}$, and (by Lemma 3) τ is increasing on D_τ, therefore $\theta^\kappa(\cdot)$ is increasing on D_κ. Since $R(\cdot)$ is a function and $\tau(a,b)$ is a closed interval for all $(a,b) \in D_\tau$, thus $\theta^\kappa(x)$ is a closed interval for all $x \in D_\kappa$. ■

To state the next result, we need the following definitions. A subset $L \subseteq \mathbb{R}^m$ is a **lattice** if, for each nonempty subset $H \subseteq L$, the set $\{x \in L : \{x\} \leq H\}$ is nonempty and has a greatest element in L, the infimum of H in L, denoted by $\inf_L[H]$; and the set $\{x \in L : \{x\} \geq H\}$ is nonempty and has a least element in L, the supremum of H in L, denoted by $\sup_L[H]$. In particular, a nonempty lattice L has a least element and a greatest element.[6]

The set $T^{\kappa+1}$ (for each $\kappa \geq 0$) is a lattice that plays a central role in this paper. Its least element is $(0, \ldots, 0)$, and its greatest element is $(1, \ldots, 1)$. For each nonempty subset $H \subseteq T^{\kappa+1}$, let $\inf[H]$ and $\sup[H]$ (without subscript) be the infimum and the supremum of H in $T^{\kappa+1}$. For each $l = 0, \ldots, \kappa$, the l-th coordinate of $\inf[H]$ is the infimum of the image of H by the projection on the l-th coordinate. Similarly, the l-th coordinate of $\sup[H]$ is the supremum of the image of H by the projection on the l-th coordinate.

A subset $L \subseteq T^{\kappa+1}$ (for some $\kappa \geq 2$) is a **sublattice** of $T^{\kappa+1}$ if, for each nonempty $H \subseteq L$, we have $\inf[H] \in L$ and $\sup[H] \in L$. For example, the sets W_κ and X_κ are both sublattices of $T_{\kappa+1}$. The least element of W_κ is $(0, \ldots, 0)$ and its greatest element is $(1, \ldots, 1)$. The least element of X_κ is $0_\kappa := (0, \ldots, 0, 1)$ and its greatest element is $1_\kappa := (0, 1, \ldots, 1)$. Furthermore, for any $\kappa \geq 0$, and all vectors $x, z \in T^{\kappa+1}$, the closed

[6]The objects we define here as a lattice, a sublattice, an upper-semilattice (Section 5) and an upper-subsemilattice (Section 6) are commonly called a *complete* lattice, a *subcomplete* sublattice, a *complete* upper-semilattice and a *subcomplete* upper-subsemilattice, e.g. in Topkis (1998). Since we only consider (sub)complete objets, we omit the reference to this property throughout.

interval $[x, z]$ is a sublattice of $T^{\kappa+1}$. In particular, for all $\kappa \geq 2$, and all $x \in D_\kappa$, the set $\theta^\kappa(x)$ is a sublattice. Finally, if L and L' are sublattices of $T^{\kappa+1}$, then $L \cap L'$ is a sublattice of $T^{\kappa+1}$.

Lemma 6: *Let $\kappa \geq 2$. Suppose that there is a nonempty $L \subseteq D_\kappa$, that is a sublattice of $T^{\kappa+1}$ such that for all $x \in L$, we have $\theta^\kappa(x) \subseteq L$. Then the set of fixed-points of $\theta^\kappa(\cdot)$ in L is a nonempty lattice.*

Proof. By Lemma 5, $\theta^\kappa(\cdot)$ is increasing. For all $x \in D_\kappa$, the set $\theta^\kappa(x)$ is a sublattice of $T^{\kappa+1}$. The result then follows from Zhou's (1994) extension of Tarski's fixed-point theorem to correspondences. Note that $\theta^\kappa(\cdot)$ satisfies a stronger monotonicity condition than the one required for Zhou's result.■

For any vector $x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$, let $x_{-j} := (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$.

Lemma 7: *Let $\kappa \geq 3$. Suppose that $x \in D_\kappa \cap X_\kappa$ is a fixed-point of $\theta^\kappa(\cdot)$. Then the set $L := \left[x_{-(\kappa-1)}, x_{-1}\right]$ is a subset of $D_{\kappa-1} \cap X_{\kappa-1}$, it is a nonempty sublattice of $T^{\kappa+1}$, and for all $y \in L$, we have $\theta^{\kappa-1}(y) \subseteq L$. Finally, the correspondence $\theta^{\kappa-1}(\cdot)$ admits a fixed-point in L.*

Proof. Let us first verify that $L \subseteq D_{\kappa-1} \cap X_{\kappa-1}$. Since $x \in D_\kappa$, then the relations $(x_0, x_1, x_2) \in D_2$ and $(x_{\kappa-2}, x_{\kappa-1}, x_\kappa) \in D_2$ hold. Since $x_0 \leq \ldots \leq x_\kappa$ and by monotonicity of $R(\cdot)$, then $(x_0, x_2, x_3) \in D_2$ and $(x_{\kappa-3}, x_{\kappa-2}, x_\kappa) \in D_2$, therefore $L \subseteq D_{\kappa-1}$, and clearly $L \subseteq X_{\kappa-1}$.

Second, L is a nonempty sublattice of $T^{\kappa+1}$. Third, we show that $\theta^{\kappa-1}(x_{-1}) \leq x_{-1}$. Since $x \in \theta^\kappa(x)$, then $x_{-0} \in \theta^{\kappa-1}(x_{-0})$. We have $x_{-1} \leq x_{-0}$. Since $\theta^{\kappa-1}(\cdot)$ is increasing, then $\theta^{\kappa-1}(x_{-1}) \leq \theta^{\kappa-1}(x_{-0})$. Therefore $\theta^{\kappa-1}(x_{-1}) \leq x_{-0}$. But since the first coordinate of $\theta^{\kappa-1}(x_{-1})$ is $\{x_0\}$, and x_{-1} only differs from x_{-0} by its first coordinate, which precisely equals 0, therefore $\theta^{\kappa-1}(x_{-1}) \leq x_{-1}$. Fourth, by an identical reasoning, we can prove that $x_{-(\kappa-1)} \leq \theta^{\kappa-1}(x_{-(\kappa-1)})$. From these last two inequalities and since $\theta^{\kappa-1}(\cdot)$ is increasing, we conclude that for all $y \in L$, we have $\theta^{\kappa-1}(y) \subseteq L$, the desired conclusion. Lemma 6 ensures then that $\theta^{\kappa-1}(\cdot)$ has a fixed-point in L.■

Proof of Theorem 1. It is immediate, from Lemmas 4 and 7.■

3.2. Equilibria with infinitely many intervals

In this section, we turn our attention to infinite size equilibria. Under certain continuity assumptions, we show that exactly one of the following alternatives is true. Either there are finite size equilibria of any positive integer size and there is at least one equilibrium of infinite size, or the set of finite equilibrium sizes is a bounded set connected to 1, and there are no equilibria of infinite size. The following lemma is useful.

LEMMA 8: *The function $S(\cdot)$ is continuous in t.*

Proof. Let $t \in [0,1]$ and $\varepsilon > 0$. Let $a := S(t)$. Since $\succeq$ is *type-continuous*, the set $O := \{s \in [0,1] : a \succ_s a - \varepsilon \text{ and } a \succ_s a + \varepsilon\}$ is open in $[0,1]$. Therefore it is a neighborhood of t in $[0,1]$. For all $s \in O$, since $\succeq_s$ is *single-peaked*, then $S(s) \in (a - \varepsilon, a + \varepsilon)$. Therefore $S(\cdot)$ is continuous at t.■

For our next result, we impose additional regularity conditions on the receiver's reaction and the sender's preferences. On the receiver side, we require that the receiver's reaction be continuous in the information he receives. More precisely, we say that the receiver reaction $R(\cdot)$ is **continuous** if the mapping $(s,t) \mapsto R(s,t)$ is continuous in the usual sense. On the sender side, we require that a strict preference relation should not be reversed by infinitesimal changes of the alternatives. For each $t \in T$, we say that the preference $\succeq_t$ is **action-continuous**, if the set $\{(a,b) \in A : a \succeq_t b\}$ is closed. We say that $\succeq$ is action-continuous if, for all $t \in T$, the preference $\succeq_t$ is action-continuous. We are ready to state our second main result.

THEOREM 2: *Let $R(\cdot)$ be continuous, and let $\succeq$ be action-continuous. Then, the set of integers κ such that there are equilibria of size κ is $\mathbb{N}$ if and only if there is at least one equilibrium of infinite size.*

Proof. The continuity of $R(\cdot)$ and the action-continuity of $\succeq$ are needed only to prove the *only if* implication. We first prove the *only if* implication. (Claims 1 to 6). Let Π^κ be a sequence of equilibria such that for all $\kappa = 1, 2, \ldots$, the equilibrium Π^κ is of size κ. For all $\kappa \geq 0$, let $i_\kappa : [0,1] \to [0,1]$ and $s_\kappa : [0,1] \to [0,1]$ such that for all $t \in [0,1]$, the real numbers $i_\kappa(t)$ and $s_\kappa(t)$ are respectively the infimum and the supremum of the pool containing t in the partition Π^κ. Clearly, these functions are both nondecreasing and satisfy for all κ and all $t \in [0,1]$, the inequalities $i_\kappa(t) \leq t \leq s_\kappa(t)$.

<u>Claim 1:</u> There is a subsequence $\{n\}$ and (unique) nondecreasing functions $i(\cdot)$ and $s(\cdot)$ such that $i_n(\cdot)$ converges to $i(\cdot)$ and $s_n(\cdot)$ converges to $s(\cdot)$. Moreover, for all $t \in [0,1]$, we have $i(t) \leq t \leq s(t)$.

<u>Proof:</u> The functions $i_\kappa(\cdot)$ and $s_\kappa(\cdot)$ are all nondecreasing and uniformly bounded on $[0,1]$. Helly's Selection Theorem guarantees that a sequence of nondecreasing uniformly bounded functions on $[0,1]$, has a subsequence which converges to a nondecreasing function. Let $\{m\}$ denote a sequence and let $i : [0,1] \rightarrow [0,1]$ be a nondecreasing function such that $i_m(\cdot)$ converges to $i(\cdot)$. Next, let $\{n\}$ be a subsequence from $\{m\}$ and let $s : [0,1] \rightarrow [0,1]$ be a nondecreasing function such that $s_n(\cdot)$ converges to $s(\cdot)$. The last inequalities are obvious.$\|$

Let Π^* be a (possibly infinite) partition of $[0,1]$ into level curves of $i(\cdot) + s(\cdot)$. Since $i(\cdot) + s(\cdot)$ is nondecreasing, each pool in Π^* is an interval, possibly a singleton.

<u>Claim 2:</u> The functions $i(\cdot)$ and $s(\cdot)$ are constant on any pool of Π^*.

<u>Proof:</u> Let t, t' be in the same pool of the partition Π^*. Then $i(t)+s(t) = i(t')+s(t')$ holds. Since both $i(\cdot)$ and $s(\cdot)$ are nondecreasing, the equality implies that $i(t) = i(t')$ and $s(t) = s(t')$. $\|$

For all $t \in [0,1]$, let $I(t)$ be the interval that contains t in the partition Π^*.

<u>Claim 3:</u> For all $t \in [0,1]$, we have $\inf[I(t)] = i(t)$ and $\sup[I(t)] = s(t)$.

<u>Proof:</u> By Claim 2, for all $t \in [0,1]$ and all $t' \in I(t)$, we have $i(t) = i(t')$. Since $i(t') \leq t'$, we obtain $i(t) \leq t'$, for all $t' \in I(t)$. Therefore $i(t) \leq \inf[I(t)]$ for all $t \in [0,1]$. An identical reasoning proves $\sup[I(t)] \leq s(t)$ for all $t \in [0,1]$. Thus for all $t \in [0,1]$, we have $i(t) \leq \inf[I(t)] \leq t \leq \sup[I(t)] \leq s(t)$. For any type t satisfying $i(t) = s(t)$, all these inequalities hold as equalities and there is nothing more to prove.

For the other types, we still need to prove that $]i(t), s(t)[\subseteq I(t)$. Let then t be such that $i(t) \neq s(t)$. Let t' be such that $i(t) < t' < s(t)$. Let u and v be types such that $i(t) < u < t'$ and $t' < v < s(t)$. Since $\lim_{n\infty} i_n(t) = i(t)$ and $\lim_{n\infty} s_n(t) = s(t)$, there is a positive integer n^* such that for all $n \geq n^*$, we have $i_n(t) \leq u$ and $s_n(t) \geq v$. Thus for all $n \geq n^*$, we have $i_n(t') = i_n(t)$ and $s_n(t') = s_n(t)$. Taking the limit as n goes to infinity, we obtain $i(t') = i(t)$ and $s(t') = s(t)$. Therefore $t' \in I(t)$, for all

$t' \in]i(t), s(t)[$. Therefore $]i(t), s(t)[\subseteq I^*(t)$. This and the inequalities we obtained in the last paragraph yield the desired conclusion.‖

Claim 4: Π^* is an equilibrium.

Proof: For all $t \in [0,1]$, we have $\lim_{n\infty} R(i_n(t), s_n(t)) = R(i(t), s(t))$, by *continuity* of $R(\cdot)$. Since Π^n is an equilibrium, for all $t, t' \in [0,1]$, we have $R(i_n(t), s_n(t)) \succeq_t R(i_n(t'), s_n(t'))$. This relation, the *continuity* of $R(\cdot)$ and the *action-continuity* of $\succeq_t$ imply that for all $t, t' \in [0,1]$, we have $R(i(t), s(t)) \succeq_t R(i(t'), s(t'))$. Therefore Π^* is an equilibrium.‖

It only remains to show that Π^* has an infinity of intervals.

Claim 5: There is $t^* \in [0,1]$ such that $i(t^*) = t^* = s(t^*)$. The partition Π^* has an infinity of intervals.

Proof: For all $n \geq 2$, there exists $u_n, v_n \in [0,1]$ such that $s_n(v_n) - i_n(u_n) \leq 2/(n-1)$ and $s_n(u_n) = i_n(v_n)$. Let $\{q\}$ be a subsequence such that $s_q(u_q)$ converges to $t^* \in [0,1]$. Then the sequences $i_q(u_q)$ and $s_q(v_q)$ both converge to t^*. Since $R(\cdot)$ is *continuous*, we have $\lim_{q\infty} R(i_q(u_q), s_q(u_q)) = R(t^*, t^*)$. By Lemma 8, the function $S(t)$ is continuous and thus $\lim_{q\infty} S(s_q(u_q)) = S(t^*)$. For all q, by *single-peakedness* of the preference $\succeq_{s_q(u_q)}$, we have

$$R(i_q(u_q), s_q(u_q)) \leq S(s_q(u_q)) \leq R(i_q(v_q), s_q(v_q)).$$

In the limit where q goes to infinity, we obtain $R(t^*, t^*) = S(t^*)$. Since Π^q is an equilibrium, we have

$$R(i_q(t^*), s_q(t^*)) \succeq_{t^*} R(i_q(u_q), s_q(v_q)).$$

By *continuity* of $R(\cdot)$ and *action-continuity* of $\succeq_{t^*}$, we can take the limit as q goes to infinity, which yields $R(i(t^*), s(t^*)) \succeq_{t^*} R(t^*, t^*)$. Since $R(t^*, t^*) = S(t^*)$, we obtain $R(i(t^*), s(t^*)) = R(t^*, t^*)$. Since $R(\cdot)$ is *increasing* and $i(t^*) \leq t^* \leq s(t^*)$, then either we have $i(t^*) = t^* = s(t^*)$ or we have $i(t^*) < t^* < s(t^*)$. Suppose, by contradiction, that the second case holds, i.e. the inequalities are strict. Let u be a type such that $i(t^*) < u < t^*$ and let v be a type such that $t^* < v < s(t^*)$. Then there is a positive integer q° such that for all $q \geq q^\circ$, we have $u_q \in]u, v[$, $i_q(t^*) < u$ and $s_q(t^*) > v$. Thus for all $q \geq q^\circ$, we have $i_q(u_q) = i_q(t^*) < u$ and $s_q(u_q) = s_q(t^*) > v$. Thus for all

$q \geq q^\circ$, we have $s_q(v_q) - i_q(u_q) > v - u > 0$, which contradicts that $s_q(v_q) - i_q(u_q)$ converges to 0. Therefore $i(t^*) = t^* = s(t^*)$.

For all $t < t^*$, we have $i(t) < t^*$ and $s(t) \leq t^*$. Since $R(\cdot)$ is *increasing*, we have $R(i(t), s(t)) < R(t^*, t^*) = S(t^*)$. Therefore $S(t^*) \succ_{t^*} R(i(t), s(t))$ and thus $s(t) < t^*$. Therefore, if $t^* > 0$, the partition Π^* has infinitely many intervals in a left-neighborhood of t^*. Similarly, if $t^* < 1$, the partition Π^* has infinitely many intervals in a right-neighborhood of t^*. $\|$

We now prove the *if* implication.

Claim 6: If there is an equilibrium of infinite size, then for all $\kappa \geq 2$, there are two vectors $y, z \in D_\kappa \cap X_\kappa$, such that for all $x \in [y, z]$ we have $\theta^\kappa(x) \subseteq [y, z]$.

Proof: If $R(0,0) = S(0)$, let $y := 0_\kappa$. If $R(0,0) \neq S(0)$, then there are $t_1 < \ldots < t_{\kappa-1}$ in T, such that $i(t_1) = 0$ and for all $h = 1, \ldots, \kappa - 2$, we have $i(t_{h+1}) = s(t_h)$. Let $y := (0, s(t_1), \ldots, s(t_{\kappa-1}), 1)$. Similarly, if $R(1,1) = S(1)$, let $z := 1_\kappa$. If $R(1,1) \neq S(1)$, then there are $t'_1 < \ldots < t'_{\kappa-1}$ in T, such that $s(t'_{\kappa-1}) = 1$ and for all $h = 2, \ldots, \kappa - 1$, we have $s(t'_{h-1}) = i(t'_h)$. Let $z := \left(0, i(t'_1), \ldots, i(t'_{\kappa-1}), 1\right)$. It is then easy to verify that y and z satisfy the desired condition. $\|$

Claim 6 and Lemma 7 imply that $\theta^\kappa(\cdot)$ has a fixed point in $[y, z]$. By Lemma 4, this vector represents an equilibrium of size κ. This ends the proof of the Theorem.■

4. A taxonomy of biases

We introduce here a taxonomy of admissible problems, according to the nature of the bias of the sender versus the receiver. We then refine the results of Section 3 within some of these categories. Abusing notations, let $R(t) := R(t, t)$. This is the reaction of the receiver when he knows that the type of the sender is t. One important case occurs when one of the functions $R(t)$ or $S(t)$ dominates the other by at least some positive constant.

The sender has a **strictly upward bias** if there is $\varepsilon > 0$ such that either for all $t \in [0, 1]$, we have $R(t) + \varepsilon < S(t)$, and it has a **strictly downward bias** if or for all $t \in [0, 1]$, we have $S(t) < R(t) - \varepsilon$. The sender has a **strictly consistent bias** if it has a strict bias, either upward or downward.

Crawford and Sobel's main result is obtained under assumptions that imply that the sender has a strictly consistent bias.[7] The following result generalizes Theorem 1 in Crawford and Sobel (1982) to problems where $R(\cdot)$ is not necessarily continuous, and the sender has a strictly consistent bias.

THEOREM 3: *Let $(R, \succ)$ be such that the sender has a strictly consistent bias. Then the set of integers κ such that there are equilibria of size κ is connected to* 1 *and bounded, and no equilibrium has an infinite size.*

Proof. When the sender has a *strictly consistent bias*, there is $\varepsilon > 0$ such that if u and v are actions induced in equilibrium, they satisfy $|u - v| > \varepsilon$ (see Crawford and Sobel 1982, Lemma 1, for a detailed proof). Therefore the set of actions induced in equilibrium is finite. Let κ be a positive integer such that there is an equilibrium of size κ. Consider one such equilibrium and let a_1 and a_κ be the most extreme actions induced in equilibrium. Then $\epsilon(\kappa - 1) \leq a_\kappa - a_1 \leq R(1) - R(0)$. Therefore $\kappa \leq (R(1) - R(0))/\epsilon + 1$. The Theorem is an immediate consequence of this fact and Theorems 1 and 2.■

Another important case occurs when the locus of the sender's preferred actions contains the locus of the receiver's optimal actions. In other worlds, the sender is weakly *more* responsive to the state of the world than the receiver in extreme situations. This condition is incompatible with a *strictly consistent bias.*

Outward bias. The sender has an outward bias if $[R(0), R(1)] \subseteq [S(0), S(1)]$.

For all $\kappa \geq 2$, let X_κ^* be the set of vectors $x \in X_\kappa$ such that x defines an interval equilibrium partition of size κ.

THEOREM 4: *Let $(R, \succeq)$ satisfy outward bias. Then, for all $\kappa \geq 2$, the set X_κ^* is a nonempty lattice. If, in addition, the function $R(s,t)$ is continuous in (s,t) and $\succeq$ is action-continuous, then there is at least one equilibrium of infinite size.*

Proof. Let $\kappa \geq 2$. Under outward bias, we have $D_\kappa = W_\kappa$, so that $D_\kappa \cap X_\kappa = X_\kappa$. This set is a sublattice of $T^{\kappa+1}$. Moreover, for all $x \in X_\kappa$, we have $\theta^\kappa(x) \subseteq X_\kappa$.

[7] In their main result, Theorem 1, Crawford and Sobel (1982) assume that for all $t \in [0,1]$, we have $R(t) \neq S(t)$. Under type-continuity of $\succeq$ and continuity of $R(\cdot)$ (both of them are implied by their assumptions), this condition is equivalent to strictly consistent bias, as these authors show in their Lemma 1.

Therefore $L := X_\kappa$ satisfies the conditions of Lemma 6. The first claim in Theorem 4 then follows from Lemma 6. The second claim follows from Lemma 6 and Theorem 2.■

For completeness, we say that the sender has an *inward bias* when the locus of sender's preferred actions is strictly included in the locus of receiver's optimal actions. In other worlds, the sender is strictly *less* responsive to the state of the world than the receiver in extreme situations, i.e. $[S(0), S(1)] \subsetneq [R(0), R(1)]$. We do not have a more precise result than Theorem 1 for this case. We study an example in Section 7.

The three conditions of *strictly consistent bias, outward bias* and *inward bias* are mutually exclusive. But there are admissible problems that do not belong to any of the three cases. Such problems are such that $S(0) - R(0)$ and $S(1) - R(1)$ have strictly the same sign but the graphs of $R(\cdot)$ and $S(\cdot)$ are not bounded away from each other (e.g. they cross).

We end this section with a result that demonstrates the robustness of Theorem 3. A type $t \in T$ is an *agreement type* if the reaction of the receiver when he knows this type coincides with the preferred action of the sender of this type, i.e. $S(t) = R(t)$ holds.

THEOREM 5: *Suppose that the set of agreement types is at most countable. Then the size of any equilibrium is at most countable.*

Proof. Let T^* be the set of types $t \in T \setminus \{0, 1\}$ that satisfy the following two conditions. (i) The type t is not an agreement type. (ii) The mapping $s \mapsto R(s, s)$ is continuous at t. Since this mapping is increasing, it has at most countably many discontinuities. Therefore, the set $T \setminus T^*$ is at most countable. Therefore, the set T^* is nonempty. It is also clear that the set T^* is open. Therefore, there is an at most countable collection $\{]\underline{t}_k, \bar{t}_k[\}_{k \in K}$ of non-empty open intervals that partition the set T^*. Let k be an arbitrary index in K and let n be an arbitrary integer such that $n \geq 3$. Let $I_{k,n} := [\underline{t}_k + \frac{\bar{t}_k - \underline{t}_k}{n}, \bar{t}_k - \frac{\bar{t}_k - \underline{t}_k}{n}]$. Since $S(\cdot)$ and $R(\cdot)$ are continuous, then the function $t \mapsto S(t) - R(t)$ has a constant sign on $I_{k,n}$ and is bounded away from 0. Consider an arbitrary equilibrium strategy for the problem $(R, \succeq)$, and consider the set of actions induced in this equilibrium. We will show that this set is at most countable. Using the same argument as in Theorem 3, the set of actions induced by types in $I_{k,n}$ is finite. Since this is true for all $n \geq 3$, and $]\underline{t}_k, \bar{t}_k[= \bigcup_{n \geq 3} I_{k,n}$, then the set of actions induced by types in $]\underline{t}_k, \bar{t}_k[$ is at most countable. If follows that the set of actions induced by types in T^* is at most countable. Since $T \setminus T^*$ is at most countable, the set of actions induced by types in $T \setminus T^*$ is at most countable. Overall, the set of actions induced in this equilibrium is at most countable.■

5. On equilibria with the same number of intervals

We present here additional results on the structure of the set of equilibria with a given number κ of intervals, for a class of problems that includes both the *strictly upward bias* and the *outward bias* cases. The sender has an **upward bias at 1** if $R(1) \leq S(1)$. A symmetric situation also of interest occurs when the sender has a **downward bias at 0**, i.e. if $S(0) \leq R(0)$. Symmetric results can be obtained in this case, so we will restrict attention to situations where the sender has an upward bias at 1.

To state the next Lemma, we need the following definitions. A subset $L \subseteq \mathbb{R}^m$ is an **upper-semilattice** if, for any nonempty subset $H \subseteq L$, the set $\{x \in L : \{x\} \geq H\}$ is nonempty and has a least element, the supremum of H in L, denoted by $\sup_L[H]$. In particular, a nonempty upper-semilattice L has a greatest element. A subset $L \subseteq T^{\kappa+1}$ (for some $\kappa \geq 2$) is an **upper-subsemilattice** of $T^{\kappa+1}$ if, for each nonempty $H \subseteq L$, we have $\sup[H] \in L$.

Lemma 9: *Let $(R, \succeq)$ be such that the sender has an upward bias at 1. Let $\kappa > 1$ be an integer such that $D_\kappa \neq \emptyset$. Then the set $D_\kappa \cap X_\kappa$ is an upper-subsemilattice of $T^{\kappa+1}$. If, in addition, we have $X^*_\kappa \neq \emptyset$, then the set X^*_κ is an upper-semilattice. In particular, it has a greatest element.*

Proof. Since the sender has an upward bias at 1, then for all $x \in D_\kappa \cap X_\kappa$, we have $[x, 1_\kappa] \subseteq D_\kappa \cap X_\kappa$. Let $H \subseteq D_\kappa \cap X_\kappa$ be nonempty, and let $x' \in H$. Clearly $\sup[H] \in [x', 1_\kappa]$. Therefore $\sup[H] \in D_\kappa \cap X_\kappa$. Therefore, $D_\kappa \cap X_\kappa$ is an upper-subsemilattice of $T^{\kappa+1}$.

Suppose next that $X^*_\kappa \neq \emptyset$. Let Y be an arbitrary nonempty set of fixed-points of $\theta^\kappa(\cdot)$, i.e. $Y \subseteq X^*_\kappa$. Let $\hat{y} := \sup[Y]$. Since $D_\kappa \cap X_\kappa$ is an upper-subsemilattice of $T^{\kappa+1}$, then $\hat{y} \in D_\kappa \cap X_\kappa$. Since all elements in Y are fixed-points of $\theta^\kappa(\cdot)$, then for all $y \in Y$, we have $y \leq \sup[\theta^\kappa(y)] \leq \sup[\theta^\kappa(\hat{y})]$. Therefore $\hat{y} \leq \sup[\theta^\kappa(\hat{y})]$. Let $U := [\hat{y}, 1_\kappa] \subseteq D_\kappa \cap X_\kappa$. For all $u \in U$, we have $\hat{y} \leq \sup[\theta^\kappa(\hat{y})] \leq \sup[\theta^\kappa(u)] \leq 1_\kappa$. Thus for all $u \in U$, we have $\theta^\kappa(x) \cap U \neq \emptyset$. Let $Z(x) := \theta^\kappa(x) \cap U$. Consider the correspondence $Z : U \twoheadrightarrow U$. The set U is a closed interval, therefore it is a nonempty lattice. For all $x \in U$, the set $Z(x)$ is also a closed interval included in U, therefore it is a nonempty sublattice of U. Since Z is increasing, we can apply Zhou's (1994) extension of Tarski's fixed-point theorem to correspondences. Therefore the set of fixed-points of Z in U is a nonempty lattice. Let $\overline{y}$ be the least fixed-point of Z in U. The vector $\overline{y}$ has the following properties. *i)* It is a fixed-point of θ^κ in $D_\kappa \cap X_\kappa$, i.e. $\overline{y} \in X^*_\kappa$. *ii)* Since $\overline{y} \in U$, then $\overline{y}$ is an upper-bound of Y. *iii)* Any upper-bound u of Y in X^*_κ is a fixed-point of Z in U, and therefore $\overline{y} \leq u$. Therefore $\overline{y}$ is the supremum of Y in X^*_κ, the desired conclusion.■

Under the conditions of Lemma 9, the set of vectors that represent equilibria with κ intervals has a greatest element, whenever this set is nonempty. Let the **greatest equilibrium with κ intervals** be the equilibrium represented by the greatest element of X^*_κ. The following result shows that the greatest equilibrium with κ intervals is nested within the greatest equilibrium with $\kappa + 1$ intervals, whenever the latter exists.

Lemma 10: *Let $\kappa \geq 3$. Suppose that the sender has an upward bias at 1. Suppose that $X^*_{\kappa+1} \neq \emptyset$ (and therefore also $X^*_\kappa \neq \emptyset$). Let $\overline{x}$ be the greatest element in X^*_κ, and let $\overline{y}$ be the greatest element in $X^*_{\kappa+1}$. Then $\overline{y}_{-(\kappa-1)} \leq \overline{x} \leq \overline{y}_{-1}$.*

Proof. First, Lemma 7 ensures that there exists some $x \in X^*_\kappa$ such that $\overline{y}_{-(\kappa-1)} \leq x \leq \overline{y}_{-1}$. Since $x \leq \overline{x}$, it follows that $\overline{y}_{-(\kappa-1)} \leq \overline{x}$. Second, let $y^* \in X^*_{\kappa+1}$, let $y^\circ := (0, \overline{x}_0, \ldots, \overline{x}_\kappa) \in X_{\kappa+1}$, and let $L := [y^*, 1_\kappa] \cap [y^\circ, 1_\kappa] \cap X_{\kappa+1}$. Then $L \subseteq D_\kappa \cap X_\kappa$. This set is such that for all $y \in L$, we have $\theta^{\kappa+1}(y) \in L$, and it is a nonempty lattice. Therefore it contains a fixed point of $\theta^{\kappa+1}(\cdot)$. Thus, there exists $y \in X^*_{\kappa+1}$ such that $\overline{x} \leq y_{-1}$. Since $y \leq \overline{y}$, we then have $\overline{x} \leq \overline{y}_{-1}$.■

We now present a comparative statics result on the greatest equilibrium with κ intervals, for two distinct admissible problems where the sender has an upward bias at 1.

Corollary 1: *Let* $(R_1, \succeq^1)$ *and* $(R_2, \succeq^2)$ *be two admissible problems. Let* $\kappa \geq 2$. *Suppose that the sender has an upward bias at 1 in both of these problems. Suppose that for all* $x \in D_\kappa \cap X_\kappa$, *we have* $\inf[\theta_1^\kappa(x)] \leq \inf[\theta_2^\kappa(x)]$ *and* $\sup[\theta_1^\kappa(x)] \leq \sup[\theta_2^\kappa(x)]$. *Suppose further that problem 1 has an equilibrium of size* κ. *Then problem 2 also has an equilibrium of size* κ. *Let* $\overline{x}^1$ *and* $\overline{x}^2$ *be the respective greatest such equilibria for problem 1 and 2. Then* $\overline{x}^1 \leq \overline{x}^2$. *If, in addition, for all* $x \in D_\kappa \cap X_\kappa$, *we have* $\theta_1^\kappa(x) < \theta_2^\kappa(x)$, *then* $\overline{x}^1 < \overline{x}^2$.

Proof. This result follows directly from Lemma 9 in this paper, and Theorem 2.5.2 by Topkis (1998), which extends Milgrom and Roberts' (1994) Theorem 3 to correspondences. ■

In practice, the following conditions on the primitives $(R_1, \succeq^1)$ and $(R_2, \succeq^2)$ imply that $\theta_1^\kappa(\cdot) \leq \theta_2^\kappa(\cdot)$, which is stronger than the joint inequalities $\inf[\theta_1^\kappa(x)] \leq \inf[\theta_2^\kappa(x)]$ and $\sup[\theta_1^\kappa(x)] \leq \sup[\theta_2^\kappa(x)]$.

- *Sender 2 is more leftist than Sender 1; receivers are identical.*
  For all $t \in [0,1]$, all $a < b \in A$, we have $[a \succeq_t^1 b] \Rightarrow [a \succeq_t^2 b]$.

- *Receiver 2 is more rightist than Receiver 1; senders are identical.*
  For all $s \leq t \in [0,1]$, we have $R_1(s,t) \leq R_2(s,t)$.

Corollary 1 plays an important role in Section 6.2. There, we will show that comparative statics results on welfare due to Crawford and Sobel (1982) hold under broader conditions than what they assume.

We obtained the existence of a greatest equilibrium with κ intervals, as the greatest element of the set of fixed points of the correspondence $\theta^\kappa(\cdot)$. It is easy to show that this equilibrium is also the greatest fixed point of the function $\mu : x \mapsto \sup[\theta^\kappa(x)]$ in $D_\kappa \cap X_\kappa$ (as an application of Corollary 1, for example). In our next result, we provide an algorithm that converges to the greatest size κ equilibrium. The algorithm can be used to compute this equilibrium numerically. Let $\{x^n\}$ be the sequence of vectors in $D_\kappa \cap X_\kappa$ such that $x^0 = 1_\kappa$ and, for all $n \geq 0$, we have $x^{n+1} = \mu(x^n)$.

Theorem 6: *Let* $R(\cdot)$ *be continuous, and let* $\succeq$ *be action-continuous. Let* $\kappa \geq 2$. *If the sender has an upward bias at 1, and* $X_\kappa^* \neq \emptyset$, *then* $\{x^n\}$ *converges to the greatest element of* X_κ^*.

Proof. By Lemma 9, the set X_κ^* has a greatest element $\overline{x}$. Since the sender has an upward bias at 1, we have $x^1 \leq x^0$. Since the function $\mu(\cdot)$ is increasing, this implies that the sequence $\{x^n\}$ is nonincreasing. Since $\{x^n\}$ is bounded below by $\overline{x}$, therefore it converges to a limit $x^\infty \in [\overline{x}, 1_\kappa] \subseteq D_\kappa \cap X_\kappa$. Moreover, for all n, we have $x^\infty \leq x^n$. Thus, for all n, we have $\mu(x^\infty) \leq \mu(x^n)$. Since the sequence $\mu(x^n)$ converges to x^∞, therefore $\mu(x^\infty) \leq x^\infty$. Let l be an arbitrary index in $\{1, \ldots, \kappa - 1\}$, and let $\mu_l(\cdot)$ be the l-th coordinate of the function $\mu(\cdot)$. We will prove that $x_l' \leq \mu_l(x^\infty)$. Let n be an arbitrary integer, and let $x_0^n, \ldots, x_\kappa^n$ be the coordinates of the vector x^n. Since $x^\infty \leq \mu(x^n)$, thus $R(x_{l-1}^n, x_l^n) \succeq_{x_l^\infty} R(x_l^n, x_{l+1}^n)$. Since $R(\cdot)$ is *continuous,* then the sequence $\{R(x_{l-1}^n, x_l^n)\}$ converges to $R(x_{l-1}^\infty, x_l^\infty)$. Similarly, the sequence $\{R(x_l^n, x_{l+1}^n)\}$ converges to $R(x_l^\infty, x_{l+1}^\infty)$. Since the preference $\succeq_{x_l^\infty}$ is *action-continuous*, then $R(x_{l-1}^\infty, x_l^\infty) \succeq_{x_l^\infty} R(x_l^\infty, x_{l+1}^\infty)$, i.e. $x_l^\infty \leq \mu_l(x^\infty)$. Since this relation holds for all $l = 1, \ldots, \kappa - 1$, then $x^\infty \leq \mu(x^\infty)$. Since we also had $\mu(x^\infty) \leq x^\infty$, then $\mu(x^\infty) = x^\infty$, i.e. $x^\infty \in X_\kappa^*$. Since $\overline{x} \leq x^\infty$, therefore $x^\infty = \overline{x}$, the desired conclusion.■

6. Welfare comparisons for the receiver

In this section, we compare the receiver's welfare across equilibria, and as the preferences of the sender vary. We suppose that the receiver has preferences over actions that admit a Subjective Expected Utility representation. More precisely, we suppose that there is a utility function $U^r : A \times T \to \mathbb{R}$ and a positive and continuous density $f : T \to \mathbb{R}$ such that, the receiver's reaction $R(\cdot)$ satisfies, for all $\underline{t}, \overline{t}$ such that $0 \leq \underline{t} < \overline{t} \leq 1$,

$$R(\underline{t}, \overline{t}) := \arg\max_a [\int_{\underline{t}}^{\overline{t}} U^r(a, t) f(t) \, dt],$$

and for all $t \in [0, 1]$,

$$R(t, t) := \arg\max_a [U^r(a, t)].$$

The continuity of $f(\cdot)$ ensures that R is robust. Moreover, we maintain the assumptions that R is consistent and increasing. Finally, we suppose that the preferences represented by the utility function $U^r(\cdot)$ are *single-peaked*, *single-crossing*, and that $U^r(\cdot)$ is continuously differentiable in (a, t). This last assumption implies in particular that the preferences represented by the utility function $U^r(\cdot)$ are type-continuous and action-continuous.

For all $\kappa \geq 2$, and all $y \in X_\kappa$, let $E(y)$ be the indirect expected utility of the receiver, when his information when choosing an action is a partition in interval pools of $[0, 1]$ represented by a vector y, to which he responds according to R. We have

$$E(y) = \sum_{h=0}^{\kappa-1} \int_{y_h}^{y_{h+1}} U^r(R(y_h, y_{h+1}), s) f(s) ds.$$

In subsection 6.1, we establish that the function $E(y)$ is nondecreasing in y within a certain region of X_κ. Using this property, we then derive comparative statics results on the receiver's welfare in subsection 6.2.

6.1. Monotonicity of the indirect expected utility

We now show that the function $E(y)$ is nondecreasing in y within a certain region of X_κ, that we describe next. To define this region, we need to consider the problem $(R, \succeq)$, where the sender has preferences $\succeq$ represented by the utility $U^r(\cdot)$. Thus the sender and the receiver have the same preferences. In particular, we have $S(t) = R(t)$ for all $t \in [0, 1]$, and the function S is increasing. Clearly, the problem $(R, \succeq)$ is admissible. Since $S(t)$ is increasing, then the sender indifference correspondence τ is a function. Moreover, since $S(0) = R(0, 0)$ and $S(1) = R(1, 1)$, then its domain $\mathcal{D}_\tau$ is the entire set $\{(a, b) \in A^2 : a \leq b\}$.

For each $\kappa \geq 2$, consider the following construction. Since τ is a function, then so is the size κ equilibrium correspondence $\theta^\kappa(\cdot)$ for this problem. Since $\mathcal{D}_\tau = \{(a, b) \in A^2 : a \leq b\}$, then $D_\kappa = W_\kappa$. For all $t \in T$, we have $\tau(R(t, t), R(t, t)) = t$. By this equality, and since $R(\cdot)$ and τ are increasing, then for all $x \in W_\kappa$, we have $\theta^\kappa(x) \in W_\kappa$. We thus have a function $\theta^\kappa : W_\kappa \to W_\kappa$. Let $Z_\kappa := \{z \in W_\kappa : z \leq \theta^\kappa(z)\}$. The following result plays an important role in the analysis.

LEMMA 11: *Let $\kappa \geq 2$. Then Z_κ is an upper-subsemilattice of $T^{\kappa+1}$.*

Proof. Let Z be an arbitrary nonempty subset of Z_κ, and let z be an arbitrary element of Z. Since $\theta^\kappa(\cdot)$ is nondecreasing, then $\theta^\kappa(z) \leq \theta^\kappa(\sup[Z])$. Since $z \in Z_\kappa$, then $z \leq \theta^\kappa(z)$. Therefore $z \leq \theta^\kappa(\sup[Z])$. Since this holds for all $z \in Z$, therefore $\sup[Z] \leq \theta^\kappa(\sup[Z])$. Therefore $\sup[Z] \in Z_\kappa$, the desired conclusion.■

The following condition ensures that the set Z_κ is connected in a particular way.

Condition (N) let $x, x' \in W_\kappa$ satisfying $\theta^\kappa(x) = x$ and $\theta^\kappa(x') = x'$. Then if $x_0 = x'_0$ and $x_1 < x'_1$, then $x_h < x'_h$, for all $h = 2, \ldots, \kappa$.

Crawford and Sobel (1982) introduce a similar, but substantially stronger condition (M). Condition (N) only restricts the receiver's preferences, while condition (M) is a joint restriction on the preferences of the receiver and the sender. Unlike condition (M), condition (N) does not imply that there is at most one equilibrium of any given size κ.

LEMMA 12: *Let $\kappa \geq 2$, and let (N) hold. Let $x, x' \in W_\kappa$ satisfy $\theta^\kappa(x) = x$, $\theta^\kappa(x') \geq x'$ and $(x_0, x_\kappa) = (x'_0, x'_\kappa)$. Then $x' \leq x$.*

Proof. By Lemma 11, the set $Z := \{z \in Z_\kappa : z_0 = x_0 \text{ and } z_\kappa = x_\kappa\}$ is an upper-subsemilattice of $T^{\kappa+1}$. Let x^* be the greatest element of Z. Let $x^{**} := \theta^\kappa(x^*)$. Let us prove that $x^{**} \in Z$. We already know that $x^{**} \in W_\kappa$. Since $x^* \in Z_\kappa$, then $x^* \leq x^{**}$. Since $\theta^\kappa(\cdot)$ is increasing, we have $x^{**} \leq \theta^\kappa(x^{**})$. Therefore $x^{**} \in Z_\kappa$. Since $x_0^{**} = x_0^* = x_0$ and $x_0^{**} = x_0^* = x_0$, it follows that $x^{**} \in Z$. Since x^* is the greatest element of Z, then $x^{**} \leq x^*$. Thus $x^{**} = x^*$. By condition (N), we have $x = x^*$. Since $x' \in Z$, this implies $x' \leq x$.■

LEMMA 13: *Let $\kappa \geq 2$, and let (N) hold. Let $y' \leq y'' \in Z_\kappa$. For all $t \in [0,1]$, let $g(t) := (1-t)y' + ty''$. Let $y : [0,1] \to Z_\kappa$, such that for all $t \in [0,1]$, we have $y(t) := \sup(Z_\kappa \cap [0_\kappa, g(t)])$. Then the path $y(t)$ satisfies $y(0) = y'$ and $y(1) = y''$, and is increasing and continuous.*

Proof. By Lemma 11, the set Z_κ is an upper-subsemilattice of $T^{\kappa+1}$. Therefore the set $Z_\kappa \cap [0_\kappa, g(t)]$ is also an upper-subsemilattice of $T^{\kappa+1}$. Moreover $y' \in Z_\kappa \cap [0_\kappa, g(t)]$, therefore this set is nonempty. It follows that for all $t \in [0,1]$, we have $y(t) \in Z_\kappa \cap [0_\kappa, g(t)]$. For all $t \leq t' \in [0,1]$, we have $Z_\kappa \cap [0_\kappa, g(t)] \subseteq Z_\kappa \cap [0_\kappa, g(t')]$. Therefore $y(t)$ is nondecreasing.

It only remains to prove that $y(t)$ is continuous everywhere on $[0,1]$. Since $y(t)$ is nondecreasing, then for all $t \in]0,1]$, the limit $y(t^-) := \lim_{s \to t^-} y(s)$ exists, and we have $y(t^-) \leq y(t)$. Similarly, for all $t \in [0,1[$, the limit $y(t^+) := \lim_{s \to t^+} y(s)$ exist, and we have $y(t) \leq y(t^+)$. By continuity of $\theta^\kappa(\cdot)$, we have $y(t^-), y(t^+) \in Z_\kappa \cap [0_\kappa, g(t)]$. Since $y(t)$ is the greatest element of this set, then in fact $y(t) = y(t^+)$, for all $t \in [0, 1[$.

To prove that $y(t)$ is continuous everywhere on $[0, 1]$, it only remains to establish that $y(t^-) = y(t)$ also holds. Suppose, by contradiction, that this is not true, so that $y(t^-) < y(t)$. Then there are indices k, l such that $0 < k \leq l < \kappa$ and satisfying $y_{k-1}(t^-) = y_{k-1}(t)$, $y_{l+1}(t^-) = y_{l+1}(t)$, and for all $h \in \{k, \ldots, l\}$, we have $y_h(t^-) < y_h(t)$. Let h be an arbitrary index such that $k \leq h \leq l$. Since $y_h(t) \leq g_h(t)$, therefore we also have $y_h(t^-) < g_h(t)$. For all $\epsilon > 0$ small enough, we have $y_h(t-\epsilon) < g_h(t-\epsilon)$. The only other constraint that restricts $y_h(t-\epsilon)$ must then bind. Therefore $\theta_h^\kappa(y(t-\epsilon)) = y_h(t-\epsilon)$. By continuity of $\theta_h^\kappa(\cdot)$, it follows that $\theta_h^\kappa(y(t^-)) = y_h(t^-)$ holds, for all h such that $k \leq h \leq l$. Let $x := (y_{k-1}(t^-), \ldots, y_{l+1}(t^-))$ and $x' := (y_{k-1}(t), \ldots, y_{l+1}(t))$. We have $\theta^{l-k+2}(x) = x$ and $\theta^{l-k+2}(x') \geq x'$. By Lemma 12, we conclude that $x' \leq x$, a contradiction.■

LEMMA 14: *Let $\kappa \geq 2$, and let (N) hold. Then $E(y)$ is increasing on $Z_\kappa \cap X_\kappa$.*

Proof. Let $y' \leq y'' \in Z_\kappa$. By Lemma 12, the following object exists. Let $y(t)$ be a continuous increasing path such that $y(0) = y'$ and $y(1) = y''$ and $y(t) \in Z_\kappa$. For all $t \in [0, 1]$, let $W(t) := E(y(t))$. We will show that $W(0) \leq W(1)$. For all $t \in [0, 1)$, let

$$Dy(t) := \liminf_{h \to 0^+} \frac{y(t+h) - y(t)}{h}, \text{ and } DW(t) := \liminf_{h \to 0^+} \frac{W(t+h) - W(t)}{h}.$$

Since $E(y)$ is everywhere continuously differentiable, and $y(t)$ is continuous, we have

$$DW(t) = \sum_{k=1}^{\kappa-1} \frac{dE}{dy_k}(y(t)) Dy_k(t).$$

By the envelope theorem,

$$\frac{dE}{dy_k}(y(t)) = \left[U^r(R(y_{h-1}, y_h), y_h) - U^r(R(y_h, y_{h+1}), y_h)\right] f(y_h).$$

Since $y(t) \in Z_\kappa$, then $\frac{dE}{dy_k}(y(t)) \geq 0$. Since $y(t)$ is nondecreasing, then $Dy_k(t) \geq 0$. Therefore we obtain $DW(t) \geq 0$ for all $t \in [0, 1)$. Since $y(t)$ is continuous on $[0, 1]$, then $W(0) \leq W(1)$, the desired conclusion.■

6.2. Welfare comparisons

The next results are consequences of the previous lemma. They apply to situations where the sender has a particular form of strictly upward bias. Given two preferences $\succeq$ and $\succeq'$, we say that the preference $\succeq'$ has a **pairwise strictly upward bias** with respect to $\succeq$, if for all $t \in [0,1]$, all two actions $a < b \in A$, we have $b \succeq_t a \Rightarrow b \succ'_t a$.

THEOREM 7: *Let $\kappa \geq 2$, and let condition (N) hold. Suppose that the sender has a pairwise strictly upward bias with respect to the receiver. Let y' and y'' be two equilibria of size κ such that $y' \leq y''$. Then the receiver's expected payoff is greater at y'' than at y'.*

Proof. We have $y', y'' \in Z_\kappa \cap X_\kappa$. The Theorem then follows from Lemma 14.■

THEOREM 8: *Let $\kappa \geq 1$, and let (N) hold. Suppose further that the sender has a pairwise strictly upward bias with respect to the receiver. Let $\overline{x}$ represent the greatest equilibrium of size κ and let $\overline{y}$ represent the greatest equilibrium of size $\kappa + 1$. The receiver's expected payoff is then greater at $\overline{y}$ than it is at $\overline{x}$.*

Proof. Let $z \in X_{\kappa+1}$ such that $z := (0, \overline{x})$. We have $\overline{y}, z \in Z_{\kappa+1} \cap X_{\kappa+1}$. By Lemma 10, we have $z < \overline{y}$. The Theorem then follows from Lemma 14.■

Our last result compares the receiver's indirect utility at the greatest equilibrium of a given size κ, when informed by two different senders. The result shows that if sender 2 has a strictly pairwise upward bias with respect to sender 1, and sender 1 has a pairwise strictly upward bias with respect to the receiver, then the receiver's indirect utility is higher when informed by sender 1, than when informed by sender 2.

THEOREM 9: *Consider two sender preferences $\succeq^1$ and $\succeq^2$. Let $\kappa \geq 2$, and let condition (N) hold. Suppose that $\succeq^2$ has a pairwise strictly upward bias with respect to $\succeq^1$, and that $\succeq^1$ has a pairwise strictly upward bias with respect to U^r. Then the receiver's expected payoff at the greatest equilibrium of size κ is higher when informed by the sender $\succeq^1$ than when informed by the sender $\succeq^2$.*

Proof. Let y' and y'' represent the greatest equilibrium of size κ against $\succeq^1$, and against $\succeq^2$. By Corollary 1, we have $y'' < y'$. We also have $y', y'' \in Z_\kappa \cap X_\kappa$. The Theorem then follows from Lemma 14.[8]■

7. The uniform-quadratic example

We now consider the special case where the prior distribution is uniform and utilities are quadratic. Let $f(\cdot)$ be the uniform distribution over $T = [0,1]$. Let $d > 0$. Let

$$U^r(a,t) = -(a-t)^2 \text{ and } U^s(a,t) = -(a-b-dt)^2.$$

Straightforward calculations yield $R(s,t) = \frac{s+t}{2}$. It is immediate that the conditions listed at the beginning of this section are satisfied, so that the problem is admissible. Table 1 shows that nature of the sender's bias for different values of the parameters b and d.

	$b+d \leq 1$	$b+d \geq 1$
$b \leq 0$	Downward	Outward
$b \geq 0$	Inward	Upward

Table 1: Nature of the sender's bias for different values of b and d.

Crawford and Sobel (1982) studied in detail the case where $b > 0$ and $d = 1$ as an example of *strictly upward bias* and gave an explicit solution. We give here an explicit solution for all values of the parameters, using their difference equation method.

In equilibrium, a cutoff type x_h must be indifferent between inducing the receiver's reaction to information the interval $[x_{h-1}, x_h]$ and the receiver's reaction to the infor-

[8]We constructed a continuous path between the equilibrium of the first game and the equilibrium of the second game. Along the path, the indirect utility of the receiver decreases. An alternative strategy would be to consider a continuous path $\succeq^v$ from the preference $\succeq^1$ to the preference $\succeq^2$, indexed by $v \in [1,2]$, and such that for all $v < v'$, the preference $\succeq^{v'}$ has a strictly pairwise upward bias with respect to $\succeq^v$. We can then consider the greatest equilibrium of size κ for each of the games $(R, \succeq^v)$, which defines a path $v \mapsto \overline{x}^v$ in $Z_\kappa \cap X_\kappa$. By Corollary 1, the path $\overline{x}^v$ is decreasing. If the path $\overline{x}^v$ is also continuous, then by Lemma 14, the indirect utility of the receiver is decreasing along the path, and the conclusion of Theorem 9 holds. Therefore, to obtain this welfare comparison, it suffices to prove the continuity of the path $\overline{x}^v$, which may or may not hold, independently of whether condition (N) is satisfied. Crawford and Sobel's condition (M) implies this continuity.

mation $[x_h, x_{h+1}]$. This implies the arbitrage condition

$$b + dx_h - \frac{x_{h-1} + x_h}{2} = \frac{x_h + x_{h+1}}{2} - (b + dx_h),$$

which can be rewritten as

$$(A_h) \qquad x_{h+1} + (2 - 4d)\, x_h + x_{h-1} - 4b = 0.$$

The vector $x = (x_0, \ldots, x_\kappa)$ represents an equilibrium of size κ, if and only if it is nondecreasing $x_0 \leq \ldots \leq x_\kappa$, solves the system $A_1, \ldots, A_{\kappa-1}$ and satisfies the boundary conditions $x_0 = 0$ and $x_\kappa = 1$ (problem A). We now solve problem A for all values of the parameters.

The discriminant of the equation

$$(*) \qquad \omega^2 + (2 - 4d)\, \omega + 1 = 0.$$

is $16d\,(d-1)$. It is null if and only if $d = 1$, positive if and only if $d > 1$ and negative if and only if $d < 1$.

Case 1: $d = 1$.

Crawford and Sobel (1982) show that in this case, a vector $(x_0, \ldots, x_\kappa)$ is a solution of A if and only if it is nondecreasing, and for all $h = 0, \ldots, \kappa$, we have

$$x_h = 2bh^2 + (\frac{1 - 2b\kappa^2}{\kappa})h.$$

The vector defined by the formula above is nondecreasing if and only if

$$\kappa \leq \left\lfloor \frac{1 + \sqrt{1 + 2/\left|b\right|}}{2} \right\rfloor.$$

Therefore there is exactly one equilibrium of size κ, for each positive integer κ satisfying this last inequality (i.e. for a bounded set of integers connected to 1), and it is described by the vector x defined above.

Case 2: $d > 1$.

Let $\lambda < \theta$ be the solutions of $(*)$. We have $0 < \lambda < 1 < \theta$. Let $x^* := \frac{b}{d-1}$. A vector $(x_0, \ldots, x_\kappa)$ is a solution of A if and only if it is nondecreasing, and for all $h = 0, \ldots, \kappa$, we have

$$x_h = x^* + a_\kappa \lambda^h + b_\kappa \theta^h. \tag{1}$$

The boundary conditions $x_0 = 1$ and $x_\kappa = 1$ determine the constants

$$a_\kappa = -\frac{1 + x^*(\theta^\kappa - 1)}{\theta^\kappa - \lambda^\kappa} \text{ and } b_\kappa = \frac{1 - x^*(1 - \lambda^\kappa)}{\theta^\kappa - \lambda^\kappa}. \tag{2}$$

We now examine under what conditions the vector x is nondecreasing, i.e. defines an equilibrium with κ intervals. We distinguish three cases.

Outward bias: $0 \leq x^* \leq 1$. In this case, the vector x defined by the formula above is nondecreasing, for all $\kappa \in \mathbb{N}$, since $a_\kappa < 0$ and $b_\kappa > 0$. Therefore there is a unique equilibrium with κ intervals, for all $\kappa \in \mathbb{N}$, and it is described by the formula above. There is also a unique equilibrium with an infinity of intervals. It is described by the sequence $\{x_h^\infty\}_{h \in \mathbb{Z}}$ such that $x_0^\infty := x^*$ and for all $h > 1$, we have $x_h^\infty = x^*(1 - \lambda^{h-1})$ and $x_{-h}^\infty = x^* + (1 - x^*)\,\theta^{-h+1}$.

Strong downward bias: $x^* < 0$. For all $\kappa > 0$, we have $b_\kappa > 0$. A necessary and sufficient condition for x to be nondecreasing is that

$$\frac{a_\kappa}{b_\kappa} \leq \frac{\theta - 1}{1 - \lambda}$$

i.e.

$$b_\kappa \geq -\frac{(1 - \lambda)x^*}{\theta - \lambda}.$$

This inequality is compatible with (2) only within a bounded set of positive integers that is connected to 1. For all κ in this set, there is a unique equilibrium with κ intervals. It is defined by (1).

Strong upward bias: $x^* > 1$. For all $\kappa > 0$, we have $a_\kappa < 0$. A necessary and sufficient condition for x to be nondecreasing is that

$$\frac{b_\kappa}{a_\kappa} \leq \frac{1-\lambda}{\theta-1}$$

i.e.

$$a_\kappa \leq -\frac{(\theta-1)x^*}{\theta-\lambda}.$$

This inequality is compatible with (2) only within a bounded set of positive integers that is connected to 1. For all κ in this set, there is a unique equilibrium with κ intervals. It is defined by (1).

Case 3: $d < 1$.

Let $x^* := \frac{b}{1-d}$. If $0 \leq x^* \leq 1$, the sender has an inward bias. Otherwise he has either a *strictly upward bias*, or a *strictly downward bias*. Let $z = e^{\pm i\rho}$ be the complex solutions of $(*)$. A solution x for problem A satisfies

$$x_h = x^* + A_\kappa \sin(\rho h + \varphi_\kappa).$$

The constants φ_κ and A_κ are jointly determined by the boundary conditions

$$\begin{aligned} x^* + A_\kappa \sin(\varphi_\kappa) &= 0 \\ x^* + A_\kappa \sin(\rho\kappa + \varphi_\kappa) &= 1. \end{aligned}$$

The vector x is a nondecreasing solution if and only if φ_κ satisfies

$$\frac{\sin(\rho\kappa + \varphi_\kappa)}{\sin(\varphi_\kappa)} = -\frac{1-x^*}{x^*} \text{ and } \varphi_\kappa \in \left[-\frac{\pi+\rho}{2}, \frac{\pi+1-2\rho\kappa}{2}\right]$$

It is easy to verify that the set

$$\left\{\frac{\sin(\rho\kappa+\varphi)}{\sin(\varphi)} : \varphi \in \left[-\frac{\pi+\rho}{2}, \frac{\pi+\rho-2\rho\kappa}{2}\right]\right\}$$

is strictly decreasing in κ and empty for $\kappa > \pi/\rho + 1$. Therefore A has a nondecreasing solution only within a bounded set of positive integers that is connected to 1. There is one equilibrium with κ intervals for all κ in this set, and there are no equilibria of infinite size.

8. Related literature

8.1. *Fixed points versus recurrence relations*

Full descriptions of the set of equilibria for cheap talk games of this type are usually obtained through the study of an arbitrage recurrence relation.[9] In this section, we argue that using fixed points is a radically different method. We will show that fixed-points can handle a broader set of situations. In this sense, they are a stronger tool.

The recursive method works as follows. An equilibrium of size κ can be represented by a nondecreasing sequence of boundary types $0 = x_0 \leq \ldots \leq x_\kappa = 1$. Such a sequence represents an equilibrium partition, if and only if it satisfies a recurrence relation, derived from the equilibrium "arbitrage conditions." This relation links together any three consecutive terms x_{k-1}, x_k and x_{k+1} for all $k = 1, \ldots, \kappa - 1$, and expresses the requirement that the sender of type x_k should be indifferent between the actions induced by the intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$.

In the uniform-quadratic example in Section 7, the recurrence relation is linear. As a result, it is solvable, and we were able to compute all equilibria. In many applications that rely on the uniform-quadratic specification, the recurrence relation is linear as well. This is not true, however, outside of the uniform-quadratic world. In the general case, there is no closed-form solution. Nevertheless, as we explain next, when the bias is either strictly upward or strictly downward, and only then, the recurrence relation is well-behaved enough to permit a description of the set of equilibria.

A *forward solution* is a finite sequence $0 = x_0 \leq x_1 \ldots \leq x_k \leq 1$, that satisfies the recurrence relation at each step. For any index $k \geq 1$, we can view x_k as a function of the variable $x := x_1 \in T$. This function may be well defined only within a

[9] The work by Dimitrakas and Sarafidis (2006) and this paper are the only exceptions in the literature.

subset of $U_k \subseteq T$. Clearly for each $k \geq 1$, we have $U_{k+1} \subseteq U_k$. Under some regularity assumptions, the function $x_k(x)$ is continuous. Next, let $k \geq 1$ be fixed. For each $x \in U_k$, let $z^k(x)$ be the infimum of the set of types above x for which x_k is not well defined, unless no such type exists, in which case, let $z^k(x) := 1$. In other words, let $z^k(x) := \inf[([x, 1] \setminus U_k) \cup \{1\}]$.

If the bias is strictly upward, then the following property holds. Suppose that $x \in U_k$. Then $z^k(x) \in U_k$, and in addition, $x_k(z^k(x)) = 1$. As a consequence, if $x_0, \ldots, x_k$ is a forward solution, then there exists a second forward solution $y_0, \ldots, y_k$ such that $x_1 \leq y_1$ and $y_k = 1$. This second forward solution is then an equilibrium of size k. In particular, we have shown that, if U_k is nonempty, then there is at least one equilibrium of size k. It then follows that the set of equilibrium sizes is connected to 1. Indeed, suppose that $(x_0, \ldots, x_{k+1})$ is an equilibrium of size $k + 1$. Then $x_1 \in U_k$, thus there is at least one equilibrium of size k. This is the core of the argument in Crawford and Sobel's (1982) Theorem 1, where these authors prove that the set of equilibrium sizes is connected to 1. A similar argument can be made, if the bias is strictly downward.[10]

Unfortunately, when the bias is neither upward nor downward, we may have $x_k(z^k(x)) < 1$, so that the construction of the previous paragraph does not produce an equilibrium.[11] In particular, the recurrence relation method has no bite in the outward bias case, for which we provide precise results in Theorem 4. In contrast, the fixed-point method is effective whether the bias is upward, downward or neither upward nor downward. Thus, the fixed points method can handle a broader set of situations than the recurrence method, and in this sense, it is stronger.

Even for cases that can be studied using recurrence relations, such as the strict upward bias case, we believe that the fixed point method is a better tool. This is because it exploits underlying structures of the problem that are not solicited by the recurrence relation approach, in particular the lattice and semi-lattice structures.

Theorem 1, and all results in Sections 6 and 7 apply, in particular, to the strict

[10]In the strictly downward bias case, one need to use downward solutions instead and define $z^k(x) := \sup[([0, x] \setminus U_k) \cup \{0\}]$.

[11]When this inequality holds, then $x_k(z^k(x)) = x_{k-1}(z^k(x))$. Thus the constraint that binds at $z^k(x)$ is not $x_k \leq 1$, as in the strict upward bias case, but $x_{k-1} \leq x_k$.

upward bias case. Many of these results yield a more precise description of the set of equilibria in that case than that provided by Crawford and Sobel (1982). We will now compare these author's description with ours, for the strictly upward bias case.

First, the construction in our Theorem 1 shows that if $(x_0, \ldots, x_{k+1})$ is an equilibrium of size $k + 1$, then there is an equilibrium of size k, within the interval $[(x_0, \ldots, x_k), (x_1, \ldots, x_{k+1})]$. In contrast, the recurrence relation construction only tells us that we can construct an equilibrium y of size k, such that $x_1 \leq y_1$.

Second, the algorithm we provide in Theorem 6 is a completely novel result. It applies, in particular, to the strictly upward bias case.

A third example is the comparative statics result in Corollary 1. Crawford and Sobel (1982) provided a similar result, but they had to rely on Condition M, a joint assumption on R and $\succeq$. It implies, in particular, that the equilibrium of each size κ is unique, but the condition itself is hard to interpret in economic terms. We obtain a more general result, that holds even when Condition M does not hold, and even if the equilibrium of each size is not unique.

A fourth example are the comparative statics results in Section 7.2. Crawford and Sobel (1982) provided similar results, but again, under Condition M. We replace this assumption by a weaker Condition N, which solely restricts the preferences of the receiver, and does not imply that the equilibrium of each size is unique. Thus, our results in Section 7 are more general than theirs.

8.2. *Equilibria as fixed points*

The techniques used in this paper are related to an old tradition in general equilibrium theory and game theory of studying equilibria as fixed points of an appropriately defined correspondence. In particular, we borrow tools from the theory of supermodular games. These games too have a monotone best response. Thus, their equilibria are studied as fixed points of a monotone correspondence. In fact, both the cheap talk games studied in this paper and the supermodular games belong to a larger class of games with a monotone best response. Our work is also closely related to a literature on monotone pure strategies equilibria in Bayesian games. Here, we use means that are very similar to those used in that literature, but to quite different ends. Athey

(2001) studies Bayesian games with finite action sets and a unidimensional continuous type set for each player.[12] Athey's objective is to prove that, under certain monotonicity and regularity conditions, any such game has an equilibrium in pure monotone-in-type strategies. This is a key difference with our work. In our setting, the existence of such an equilibrium is trivial, since babbling equilibria always exist. Rather, what we are after is a description of the set of equilibria. Athey represents strategies by the means of a vector of jump points. She defines a mapping from this set to itself and applies a fixed point theorem. The mapping we use is non-decreasing, allowing us to use Tarski's fixed-point theorem. Athey's mapping is not monotone, which leads her to invoke instead Kakutani's fixed point theorem. Finally, Athey uses the fixed-point theorem once and obtains the existence of at least one monotone equilibrium. In contrast, we use the fixed-point argument either as an induction step to prove existence of equilibria of inferior sizes (Theorem 1) or directly for all possible equilibrium sizes, i.e. an infinite number of times, to obtain an (at least countable) infinity of equilibria (Theorem 4).

9. Further possible applications of the method

As a conclusion, we discuss the possibility of using the fixed point method to address questions that lay beyond the scope of this paper.

One important problem with cheap talk models is that there typically are multiple equilibria, and all are robust to standard criteria commonly used in signalling games. To date, there is no well-established theory on how to select among cheap talk equilibria. This problem occurs even in the classic Crawford and Sobel's strictly upward bias case.[13] In a companion paper (2007b), we propose to select the equilibria that

[12]See also McAdams (2003) and Van Zandt and Vives (2006).

[13]In this context, Kartik (2005), Chen (2006) and Chen, Kartik and Sobel (2007) propose a criterion that selects any equilibrium such that the lowest type of sender would not want to reveal itself to the receiver if could. The criterion performs well in the uniform-quadratic case, and in the general case under condition M. In such cases, it selects the unique equilibrium with the largest size. When condition M is not satisfied, however, the set of equilibria that satisfy the criterion is not known. These authors show that the equilibria satisfying the criterion are the limits of a certain class of equilibria in perturbed games. It is not clear what the proper definition of the criterion should be for biases other than upward or downward.

are asymptotically stable fixed-points of the equilibrium correspondence that we use in this paper. In the classic strictly upward bias Crawford and Sobel (1982) case, we prove that a unique equilibrium satisfies this criterion. It is the maximal element of the set of equilibria of maximal size, sometimes referred to as the most informative.

How useful is the fixed point method in more sophisticated models, beyond the unidimensional framework? In most, if not all, cheap talk models, it is possible to describe the equilibria of the game as the fixed points of an equilibrium correspondence such as the one we introduced here. But for the method to work, one needs at least one of the two following conditions to hold. Either the mapping should be nondecreasing, or the mapping's domain should contain the mapping's image. In this paper, the second condition only holds when the sender's bias is outward. All of our results that do not take this as an assumption crucially rely on the monotonicity of the equilibrium mapping.

Unfortunately, in more sophisticated models, the equilibrium mapping is not likely to be monotone, ruling out the use of Tarski's Theorem or any of its variants. Even so, in some cases, it is still possible to use the fixed-point method provided that the image of the equilibrium correspondence is contained in its domain. In such cases Kakutani's theorem can sometimes be used.

Levy and Razin (2007) and Chakraborty and Harbaugh (2007a, 2007b) introduced a multidimensional version of Crawford and Sobel's (1982) game. Unfortunately, this model raises serious technical difficulties. These authors provide partial results on the equilibrium set, but not a detailed description of the equilibrium set. In a companion paper (2007a), we apply the fixed-point method to this multidimensional model. We define the equilibrium mapping in this context, which maps pavements of the multidimensional type space to pavements of the same space. As one expects, this mapping is not monotone. However, in the special case where the sender has an outward bias, the domain of the equilibrium mapping contains its image. In this context, the assumption says that the support of the sender's preferred action contains the support of the receiver's preferred action. For this case, we obtain a result similar to this paper's Theorem 4. We prove that this game has infinitely many equilibria, at least one of each finite size, and at least one of infinite size. Whether the method can be somehow adapted to the case where the sender's bias is not outward, to perhaps obtain a result analogous to this paper's Theorem 1, is an open question.

A simpler model where the equilibrium mapping is not monotone is the model of unidimensional cheap talk with an "uncertain bias," studied by Morgan and Stocken (2003), Li and Madarasz (2007) and Dimitrakas and Sarafidis (2006). In this model, the sender's privately known type has two dimensions and actions have one dimension. One of the type dimensions is relevant to both the receiver's and the sender's preferred decision, and the other type dimension is only relevant to the sender's preferred decision, and is therefore interpreted as an uncertain sender's bias. These authors all restrict attention to the case where both the sender and the receiver have quadratic preferences. Despite the equilibrium mapping not being monotone, Dimitrakas and Sarafidis (2006) successfully apply the fixed-point method. To this end, they restrict attention to the case where the support of the marginal distribution of the sender's bias is of the form $[0, b]$, with $b > 0$. This assumption ensures that the equilibrium mapping's domain contains its image. As a result, a result similar to this paper's Theorem 4 holds, and they prove it via Brouwer's theorem. This yields a detailed description of the equilibrium set. The authors prove that there are infinitely many equilibria, at least one of each finite size, and at least one of infinite size. Also here, whether the method can be adapted to other marginal distributions of the bias,[14], to perhaps obtain results analogous to this paper's Theorem 1 or Theorem 4, is an open question.

As an unfortunate consequence of the non monotonicity of the equilibrium mapping, both in Gordon (2007a) and Dimitrakas and Sarafidis (2006), the rich structure studied in sections 5 and 6 in this paper for the unidimensional model is not inherited by these more complex models.

Finally, in the model studied in this paper, as in most cheap talk models, messages are interchangeable, in the sense that the equilibrium allocation of messages to pools is irrelevant. In our exposition of the method in this paper, we strongly rely on this property of the model. There are, however, cheap talk models where the messages encoding is not irrelevant. One recent example is a model of noisy cheap talk signalling

[14]For example, Morgan and Stocken (2003) consider marginal distributions of the bias with a support of the form $\{0, b\}$, where $b > 0$. Li and Madarasz (2007) consider marginal distributions of the bias with a support of the form $\{-b, +b\}$, where $b > 0$.

by Board, Blume and Kawamura (2006). In their framework, the presence of noise gives some messages an endogenous meaning in equilibrium. Therefore, as these authors point out,[15] in its current form, the method can only be used to describe a certain class of Pareto-suboptimal equilibria of the noisy talk model. Whether it could be adapted to settings such as theirs, or even to costly signalling games, is yet another question we leave open for future research.

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[15] Board, Blume and Kawamura (2006), p. 38.

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