# Optimal Auctions with Participation Costs

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**Abstract**

We study the optimal auction problem with participation costs in the symmetric independent private values setting, where bidders know their valuations when they make independent participation decisions. After characterizing the optimal auction in terms of participation cutoffs, we provide an example where it is asymmetric. We then investigate when the optimal auction will be symmetric/asymmetric and the nature of possible asymmetries. We also show that, under some conditions, the seller obtains her maximal profit in an (asymmetric) equilibrium of an anonymous second price auction. In general, the seller can also use non-anonymous auctions that resemble the ones that are actually observed in practice.

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# 1 Introduction

In many auctions, bidders incur participation costs even if they know their valuations for the object being sold or how much they will bid. Bidders are sometimes required to purchase bid documents, to pre-qualify or register for the auction, or to be present at the auction site, all of which may be costly. In procurement and sales of public assets, a "bid" is often more than a dollar amount; it must also include a detailed plan with the requisite documentation. Procurement auctions usually require the posting of bid bonds by all bidders before the auction and a performance bond by the winner immediately after. There may be fixed costs associated with securing bid bonds and making arrangements in advance for performance bonds, or for financing in general in other environments.

In this paper, we study the optimal (profit-maximizing) auction problem with participation costs in the standard symmetric independent private values setting.[1] We assume that (potential) bidders know their valuations when they make *independent* participation decisions.[2,3] Those who choose to submit a bid incur a fixed real resource cost.

We first show that the search for optimal auction need not involve considering stochastic bidder participation decisions. In particular, each bidder will participate in the optimal auction iff her valuation is greater than a cutoff point. If we treat these participation cutoffs as fixed, the seller's problem, and hence its solution, will be familiar: The bidder with the highest valuation *among participants* will receive the object. A revenue equivalence result immediately follows: The seller will obtain the same expected profit in any equilibrium of any auction satisfying this optimal allocation rule as long as bidders' cutoffs are identical across auctions.[4]

We next turn our attention to optimal cutoffs. We provide an example

[1] Asymmetries in participation costs or valuation distributions do not present any conceptual difficulties in what follows. We assume that bidders are ex-ante symmetric to keep the notation simple, since we will later focus on properties of optimal auctions in a symmetric environment.

[2] As we will elaborate later, our results are also relevant for the efficient auction problem.

[3] We therefore use the term "auction" in a more restrictive sense than Myerson (1981): We only allow mechanisms where each bidder's participation decision depends solely on her own valuation. Unlike in the standard setup, this constraint may be binding because of the participation cost.

[4] As usual, expected payoffs of bidders with valuations given by their respective cutoffs need to be identical across auctions as well.

where the optimal auction in our (symmetric) setup is asymmetric, i.e., bidders have different cutoffs.[5] We then give a sufficient condition for this to happen in general. As an immediate corollary, this result identifies valuation distribution functions for which the optimal auction is asymmetric *independent* of the magnitude of the participation cost $c$ and the number of bidders $n$. Note that in asymmetric auctions the object is not necessarily assigned to the highest valuation bidder (who may be a nonparticipant). The optimal auction does not have this type of allocative inefficiency when there are no participation costs.[6] We then characterize distribution functions for which the optimal auction is symmetric *independent* of $c$ and $n$. We also have some results about the nature of possible asymmetries that simplify the task of finding the optimal cutoffs.

We analyze the case of uniformly distributed valuations in detail, where it is possible to give a complete characterization of optimal auctions by using our results. In particular, depending on the support of the distribution, the optimal auction will be either symmetric or it will have only two distinct cutoffs where the smaller one is used by one bidder only. Given these, we can easily calculate the cutoffs as well. An interesting result is that whenever the optimal auction is asymmetric the seller will exclusively deal with a single bidder, i.e., "sole-source," when the participation cost is high enough or when there are many bidders.

The implementation of asymmetric optimal auctions is another issue we address. We show that, under some conditions, the seller will obtain her maximal profit in an (asymmetric) equilibrium of a second price auction that is *anonymous*, i.e., its rules treat all bidders identically. In general, the seller can also use non-anonymous (first or second price) auctions that resemble the ones that are actually observed in practice, where bidders face different participation costs (by design) or some of them are given explicit

[5] Since the environment is symmetric, *ex-ante* randomization by the seller among all auctions with the same set of cutoffs, with bidders' identities permuted, will restore symmetry (pre-randomization) in a trivial sense. Our use of the term "optimal auction" refers to the auction that ends up being used, which the seller might have chosen through such a randomization.

[6] We are referring to the "regular" symmetric bidders case. However, there is a difference also with the asymmetric bidders case: In our setup, the optimal auction does not necessarily assign the object to the bidder with the highest *virtual valuation*. We provide some intuition on why the seller may benefit from creating asymmetries among (symmetric) bidders in Section 2.3.

bidding preferences.[7]

In our model, the cost incurred by participating bidders is independent of the auction chosen by the seller. Yet, in many cases, this cost is endogenous; it is the seller who requires pre-qualification, a detailed plan with documents, or bid and performance bonds. However, there are good reasons for these types of requirements that are outside of our standard models, like making sure that the winner can and will do as she promises, and securing, or at least improving, the integrity of the process.[8] The participation cost in our setup can be thought as the smallest amount necessary for running *any* auction as in our textbook models, where doing so is preferable to the alternatives.[9]

We assume that bidders make their own participation decisions independently after the seller announces the auction rules, and thus study a constrained problem. The class of mechanisms allowed by this assumption, which includes standard auctions and their variations, is large enough and has received considerable interest both in academia and in practice. However, it leaves out sequential mechanisms that will, generally, be better for the seller if the cost of her contacting, or searching for, a buyer were identical to the bidder participation cost of our setup.[10] Note that even in this case auctions may be favored because of their transparency, as we mentioned above.[11]

There are a few papers that use our setup where bidders know their valuations when they make their participation decisions.[12] Samuelson (1985)

[7] Examples include government-run auctions where domestic/in-state/small businesses are preferentially treated, see Section 3. We are not arguing that the goal of these and other examples of bidder discrimination is to maximize the seller's profit. Instead, the point is that they *may* not hurt it as much as one may have thought *even in a symmetric environment*. McAfee and McMillan (1989) and Ayres and Cramton (1996) make the same point in asymmetric environments.

[8] The last one may be critically important when an agent must run the auction for the principal, which is the case for government procurement or sales of public assets. This issue is also relevant when comparing auctions to private negotiations.

[9] The seller would like the participation cost to be as small as possible in our setup.

[10] Sequential (costly) search mechanisms are considered by, among others, McAfee and McMillan (1988), Ehrman and Peters (1994), and Cremer, Spiegel and Zheng (2006).

[11] For example, the general rule for government procurement in the US, as well as in many other countries, is "full and open competition," see the Federal Acquisition Regulation.

[12] There is an important strand of literature where costly entry, or information acquisition, decisions are made *ex ante*. See, among others, Matthews (1984), McAfee and McMillan (1987), Harstad (1990), Tan (1992), Engelbrecht-Wiggans (1993), Levin and Smith (1994), Persico (2000), and Bergeman and Valimaki (2002).

shows that both ex-ante total surplus and the seller's revenue may decline with $n$ in symmetric equilibria of first price auctions with reserve prices, which are chosen optimally (given the respective criterion) for fixed $n$.[13]

Stegeman (1996) studies (ex-ante) efficient auctions (maximizing social surplus). He shows that the efficient auction is also characterized by participation cutoffs and provides an example where it is asymmetric. He further shows that the second price auction always has an efficient equilibrium, whereas the first price auction has one iff the symmetric equilibrium of the second price auction is efficient. One obvious way our paper differs from Stegeman's (1996) is that we consider optimal auctions, which necessitates using somewhat different techniques: Transfers from bidders to the seller are central to our problem even though they do not affect the social surplus. More importantly, we investigate the conditions under which the optimal auction will be symmetric, the nature of possible asymmetries, and the implementation question. In addressing these issues, we benefited significantly from the methods used by Tan and Yilankaya (2006) who study equilibria of second price auctions and identify sufficient conditions for uniqueness (respectively, multiplicity) in undominated strategies.

This might be a good place to point out that our results about the properties of optimal cutoffs are applicable also to the efficient auction problem. In particular, corresponding results in this problem can be obtained via a simple substitution in ours, which we will identify after the formal analysis.

Finally, in a recent work independent of ours, Lu (2003) studies optimal symmetric auctions. He observes that the seller's profit may be decreasing in $n$, and thus concludes that the (unrestricted) optimal auction may be asymmetric for given $n$. Another way to interpret this result is to note that the optimal symmetric auction can be implemented using a first price auction with a reserve price, and so Samuelson's (1985) observations apply. We remark again that symmetry here is a restriction on outcomes, not just mechanisms: As we show in this paper, the seller can actually obtain a higher profit in asymmetric equilibria of anonymous second price auctions.

The rest of the paper is organized as follows: We study optimal auctions in Section 2 and how to implement them in Section 3. All the proofs, except that of Proposition 1, are in the Appendix.

[13] His finding also applies to any symmetric and increasing equilibrium of any anonymous auction where the highest bidder receives the object and others obtain nothing. Note that Samuelson (1985) considered procurement and we have adjusted the terminology to facilitate comparison with other results.

# 2 Optimal Auctions

## 2.1 The Environment

We consider a symmetric independent private values environment. There is a risk-neutral seller who wants to sell an indivisible object that she owns and values at zero. There are $n \geq 2$ risk-neutral potential buyers, or "bidders". Let $v_i$ denote the valuation of bidder $i \in N = \{1, ..., n\}$, the set of bidders. Each bidder's valuation is independently distributed according to the cumulative distribution function $F(.)$ with full support and continuously differentiable density $f(.)$ on $[v_l, v_h]$, where $0 \leq v_l < v_h$. We assume throughout that the virtual valuation function, i.e., $J(v) = v - \frac{1-F(v)}{f(v)}$, with domain $[v_l, v_h]$, is increasing.[14] Bidders know their own valuations.

We depart from this standard setup by assuming the existence of participation costs, which are real resource costs. In particular, each bidder who participates in an auction incurs a cost of $c \in (0, v_h)$. Each bidder knows her valuation when she makes her participation decision independently of other bidders' participation decisions. Bidders who do not participate in the auction do not receive the object.[15] All of this is common knowledge.[16]

## 2.2 Optimal Auction up to Participation Cutoffs

In this section, we will show that, when searching for optimal auctions, the seller can, without loss of generality, restrict attention to those with deterministic participation decisions.[17] In particular, each bidder will participate in the optimal auction iff her valuation is greater than her participation cutoff. Once we fix these bidder-specific cutoffs, the seller's problem becomes identical to that in the standard setup (where $c = 0$) except the requirement that nonparticipating types do not receive the object. Therefore, the solution

[14] Myerson (1981) shows how to dispense with this standard regularity assumption.

[15] Stegeman (1996) calls this the "no passive reassignment rule." Note that it may be seen as a consequence of the costly participation issue we are addressing: Voluntarily receiving the object (a premise we maintain throughout) negates the idea of nonparticipation.

[16] The setup we are considering can be represented as follows, without any loss of generality: The bidders simultaneously choose messages from $\{No\} \cup [v_l, v_h]$, where *No* (denoting nonparticipation) is costless and all others cost $c$ to send. The seller's mechanism consists of *assignment* and *transfer* rules that map message profiles. Bidders who send *No* receive the object with probability zero.

[17] Note that this is not necessarily true for arbitrary auctions; optimality is crucial.

is similar as well: The bidder with the highest valuation among participants will receive the object (Proposition 1). After this characterization of the optimal allocation rule given arbitrary participation cutoffs, we investigate the optimal cutoffs in Section 2.3.

Consider any equilibrium of any auction.[18] Since bidder $i$ is risk-neutral, she cares only about her probability of winning the object, $Q_i$, and her expected payment, $P_i$. Notice that $Q_i$ incorporates $i$'s probability of participating in the auction, $\rho_i$, and $P_i$ incorporates the expected participation cost that $i$ incurs. The equilibrium expected payoff of type-$v_i$ bidder $i$ ($v_i$ for short) can thus be written as

$$\pi_i(v_i) = Q_i(v_i)v_i - P_i(v_i). \tag{1}$$

It must be the case that $v_i$ does not want to imitate the equilibrium behavior (inclusive of the participation decision) of any $v_i'$. Using standard arguments, this implies

$$\pi_i(v_i) = \pi_i(v_l) + \int_{v_l}^{v_i} Q_i(y)dy. \tag{2}$$

However, in our setup, where bidders have full control of the participation decisions that they make, (2) does not capture all implications of incentive constraints. When considering $v_i$'s incentives to imitate the equilibrium behavior of $v_i'$, we also need to make sure that $v_i$ does not have an incentive to choose *any* participation probability, not only the participation probability actually chosen by $v_i'$. Instead of incorporating these additional constraints generated by bidders' participation decisions (which we call *participational incentive constraints*) into the seller's problem, we will ignore them, thus analyzing a "relaxed problem". We will later show that they are satisfied by the solution to this relaxed problem, i.e., they are nonbinding. Observe that, as usual, $Q_i(.)$ and $\pi_i(.)$ are weakly increasing, and $\pi_i(.)$ is increasing whenever $Q_i(.) > 0$.

The seller's expected profit (also revenue, since her valuation is zero) is

$$\pi_s = \sum_{i=1}^{n} \{ \int_{v_l}^{v_h} [J(v_i)Q_i(v_i) - \rho_i(v_i)c] f(v_i)dv_i - \pi_i(v_l) \}, \tag{3}$$

[18] In what follows, we are using standard (revelation principle) arguments. We benefited from the exposition in Matthews (1995), where the reader can also find missing details in some of the calculations.

where the term in braces is bidder $i$'s expected payment to the seller, calculated by using (1), (2), and the fact that the participation cost is incurred by bidders, but not received by the seller.

In the optimal auction, the lowest type of each bidder will obtain zero equilibrium expected payoff, i.e., $\pi_i(v_l) = 0 \ \forall i \in N$. Moreover, for each $i$, since $Q_i(.)$ is increasing, there exists a *cutoff point* $\widetilde{v}_i \in [v_l, v_h]$ such that $Q_i(v_i) = 0$ for $v_i < \widetilde{v}_i$ and $Q_i(v_i) > 0$ for $v_i > \widetilde{v}_i$. It follows from (2) that $\pi_i(v_i) = 0$ for $v_i \leq \widetilde{v}_i$ and $\pi_i(v_i) > 0$ for $v_i > \widetilde{v}_i$. Therefore, bidders' participation decisions in the optimal auction will be deterministic for almost all types. In particular, for each bidder $i$, it must be the case that $\rho_i(v_i) = 0$ for all but a measure zero set of $v_i < \widetilde{v}_i$. Notice that for these types the expected equilibrium probability of winning the object, $Q_i(v_i)$, and the expected equilibrium payoff, $\pi_i(v_i)$, are both zero. If a positive measure set of these types were participating in an auction, then the seller can simply save the participation costs that must be incurred to induce their participation without affecting anyone's incentives.[19] Furthermore, for each bidder $i$, $\rho_i(v_i) = 1$ for all $v_i > \widetilde{v}_i$. This follows from these types' optimal participation decisions: Since their overall payoff is strictly positive, their payoff from participation must be strictly positive as well (notice that payoff from nonparticipation is zero). Therefore, we conclude that each bidder will participate in the optimal auction with probability one (respectively, zero) if her valuation is greater (respectively, less) than her cutoff, $\widetilde{v}_i$. Incorporating these deterministic participation decisions into (3), we have

$$\pi_s = \sum_{i=1}^{n} \int_{v_l}^{v_h} J(v_i) Q_i(v_i) f(v_i) dv_i - c \sum_{i=1}^{n} [1 - F(\widetilde{v}_i)], \tag{4}$$

where $Q_i(v_i) = 0$ for $v_i < \widetilde{v}_i$. Let $q_i(v_1, ..., v_n)$ be $i$'s equilibrium probability of winning the object when the valuations are given by $(v_1, ..., v_n)$. We can rewrite the seller's expected profit as

$$\pi_s = \int_{v_l}^{v_h} ... \int_{v_l}^{v_h} [\sum_{i=1}^{n} J(v_i) q_i(v_1, ..., v_n)] \prod_{i=1}^{n} f(v_i) dv_i - c \sum_{i=1}^{n} [1 - F(\widetilde{v}_i)]. \tag{5}$$

It is useful to think the seller's problem in two steps. First, given bidders' cutoff points, we find equilibrium winning probabilities that maximize the

[19] In what follows we will let $\rho_i(v_i) = 0$ for all $v_i < \widetilde{v}_i$. Clearly, this is without loss of generality in terms of expected payoffs of the bidders and the seller.

seller's expected profit. We then turn our attention to the issue of optimal cutoffs in Section 2.3.

For the first step, consider arbitrary cutoff points at which virtual valuations are nonnegative.[20] The following notation will be useful throughout the paper. Let $v_0 \in [v_l, v_h]$ be the smallest valuation for which the virtual valuation is nonnegative. In other words, if $J(v_l) < 0$, then $v_0 \in (v_l, v_h)$ is given by $J(v_0) = 0$; if $J(v_l) \geq 0$, then $v_0 = v_l$. (Note that $J(.)$ is increasing and $J(v_h) = v_h > 0$.)

The seller's problem is to maximize (5) with respect to $q_i(.)$'s subject to the constraints that these are probabilities and nonparticipating bidders neither obtain the object nor affect any participating bidder's probability of obtaining the object.[21] In other words, for each $i$ and $(v_1, ..., v_n)$, $q_i(v_1, ..., v_n)$ must satisfy the following constraints:

- $q_i(v_1, ..., v_n) \geq 0$ and $\sum_{i=1}^{n} q_i(v_1, ..., v_n) \leq 1$.
- $q_i(v_1, ..., v_n) = 0$ if $v_i < \widetilde{v}_i$ and $q_i(v_1, ..., v_j, ...v_n) = q_i(v_1, ..., v_j', ...v_n)$ for all $j$ and $v_j, v_j' < \widetilde{v}_j$.

Since the cutoffs are fixed, total participation cost incurred (i.e., $c \sum_{i=1}^{n} [1 - F(\widetilde{v}_i)]$ in (5)) is fixed as well, and thus it can be ignored for the time being. The seller's problem is now identical to that of the standard optimal auction setup, except that participation cutoffs of the bidders must be respected. Maximizing (5) pointwise results in the object being assigned with positive probability only to bidders who have the highest virtual valuations, and hence valuations, *among participants*.[22]

The constraints we ignored are satisfied by this optimal allocation rule. For any given bidder, higher types have weakly higher probabilities of winning the object, i.e., $Q_i(.)$ is weakly increasing for every $i$. The participational incentive constraints that we discussed above are also satisfied. Every type of every bidder makes a deterministic participation decision; in particular, for every $i$, $\rho_i(v_i) = 0$ (and $Q_i(v_i) = 0$) for $v_i < \widetilde{v}_i$ and $\rho_i(v_i) = 1$ for $v_i > \widetilde{v}_i$. So, if it is not profitable for $v_i$ to imitate any $v_i'$ (inclusive of $\rho_i(v_i') \in \{0, 1\}$), then it will not be profitable for $v_i$ to use a nondegenerate participation

[20] Notice that this indeed has to be the case for optimal cutoffs: The seller is better off not selling to negative virtual types.

[21] We also have to check that the resulting $Q_i(.)$ is weakly increasing.

[22] If bidders are ex-ante asymmetric, the object will still be assigned to the bidder with the highest virtual valuation (who may not have the highest valuation anymore).

probability (and then imitate the action of $v_i'$ in the auction), since this will yield an expected payoff which is just a convex combination of what $v_i$ would receive if she were to imitate $v_i'$ and the nonparticipation payoff, zero.

We have characterized the optimal auction up to the level of participation cutoffs, which we summarize next.

**Proposition 1** *In the optimal auction there exists a cutoff point for each bidder such that she participates in the auction if and only if her valuation is greater than her cutoff, i.e.,* $\forall i \; \exists \widetilde{v}_i \geq v_0$ *such that* $\rho_i(v_i) = 0$ *(hence* $Q_i(v_i) = \pi_i(v_i) = 0$*) for* $v_i < \widetilde{v}_i$ *and* $\rho_i(v_i) = 1$ *for* $v_i > \widetilde{v}_i$*. For each* $(v_1, ..., v_n)$ *the equilibrium winning probabilities satisfy:*

*i) If* $v_j < \widetilde{v}_j \; \forall j \in N$*, then* $q_i(v_1, ..., v_n) = 0 \;\; \forall i \in N$*. If* $\exists j$ *s.t.* $v_j > \widetilde{v}_j$*, then* $\sum_{i=1}^{n} q_i(v_1, ..., v_n) = 1$*.*

*ii)* $q_i(v_1, ..., v_n) > 0 \Rightarrow v_i \geq v_j \; \forall j \in N$ *s.t.* $v_j \geq \widetilde{v}_j$*.*

**Remark 1** *(Revenue Equivalence) Consider two auctions, say* $A$ *and* $B$*, that, in equilibrium, assign the object to the highest-valuation participant and have the same participation cutoff for each bidder, i.e.,* $\widetilde{v}_i^A = \widetilde{v}_i^B \; \forall i \in N$ *(with the associated cutoff rule in participation we discussed above), where expected payoffs of the marginal types are equal as well, i.e.,* $\pi_i(\widetilde{v}_i^A) = \pi_i(\widetilde{v}_i^B)$ $\forall i \in N$*. The expected payoff of every type of every bidder, and hence that of the seller, is the same in both auctions.*

## 2.3 Optimal Participation Cutoffs

We now turn our attention to optimal cutoffs. For this purpose, we first express the seller's expected profit in terms of solely bidders' participation cutoffs, utilizing what we know about optimal auctions (Proposition 1). We show with an example that the optimal auction may be asymmetric, i.e., not all bidders have identical cutoffs, even though the environment is symmetric.[23] We then identify a sufficient condition for the optimal auction to be asymmetric given the number of bidders $n$, the participation cost $c$, and the distribution function of the valuations $F(.)$ (Proposition 2). As a corollary, this result gives a condition on $F(.)$ under which the optimal auction will be

[23] We say that the optimal auction is symmetric if bidders with identical valuations have identical equilibrium probabilities of winning (and hence expected payoffs). Proposition 1 implies that the optimal auction is symmetric iff all bidders have identical participation cutoffs.

asymmetric for all $c$ and $n$. We next provide a characterization result for the symmetry of the optimal auction for all $c$ and $n$ (Proposition 3). Finally, we have some results about the nature of possible asymmetries that considerably simplify the task of finding optimal cutoffs in certain cases (Proposition 4). Together these results enable us to completely characterize optimal auctions when bidders' valuations are uniformly distributed.

We start with indexing the set of bidders with respect to their participation cutoffs so that

$$v_l \leq \widetilde{v}_1 \leq \widetilde{v}_2 \leq ... \leq \widetilde{v}_n \leq v_h \tag{6}$$

We adopt the convention that $\widetilde{v}_{n+1} = v_h$. Recall that in the optimal auction the object is assigned to the bidder who has the highest valuation among participants (we can ignore ties). Consider an arbitrary bidder $i$ with valuation $v$ who is a participant, i.e., with $v > \widetilde{v}_i$. For her to receive the object in the optimal auction, all *participating* bidders must have valuations less than $v$. This means that bidders whose cutoffs are lower than $v$ need to have valuations lower than $v$. Bidders with cutoffs higher than $v$ on the other hand, need to have valuations lower than their respective cutoffs, not $v$. Therefore, bidder $i$'s probability of receiving the object in the optimal auction is given by

$$Q_i(v) = F(v)^{j-1} \prod_{k=j+1}^{n+1} F(\widetilde{v}_k) \text{ if } \widetilde{v}_j \leq v \leq \widetilde{v}_{j+1} \tag{7}$$

for $v > \widetilde{v}_i$, with $Q_i(v) = 0$ for $v < \widetilde{v}_i$. Notice that, for any pair of bidders, the probability of winning functions differ at only those valuations for which only one of them is a participant: For any $i$ and $j$ with $\widetilde{v}_i > \widetilde{v}_j$, $Q_i(v) = Q_j(v)$ for $v > \widetilde{v}_i$ or $v < \widetilde{v}_j$, and $Q_j(v) > Q_i(v) = 0$ for $v \in (\widetilde{v}_j, \widetilde{v}_i)$.

Using these probability of winning functions and (4), the expected profit of the seller can be expressed solely as a function of the cutoffs (suppressing the dependence on exogenous variables):

$$\pi_s(\widetilde{v}_1, ..., \widetilde{v}_n) = \sum_{i=1}^{n} i \int_{\widetilde{v}_i}^{\widetilde{v}_{i+1}} J(v)[F(v)^{i-1} \prod_{k=i+1}^{n+1} F(\widetilde{v}_k)] f(v)\, dv - c \sum_{i=1}^{n} (1 - F(\widetilde{v}_i)). \tag{8}$$

The seller's problem is thus reduced to choosing a cutoff for each bidder to maximize $\pi_s(\widetilde{v}_1, ...\widetilde{v}_n)$, which is continuous, subject to the ranking constraint of the cutoffs, i.e., (6), defining a nonempty and compact constraint set. Therefore, a solution exists.

Let $\widetilde{v}_i^*$ denote the optimal $\widetilde{v}_i$. Notice that we have the well-known problem, and its solution, if there are no participation costs.[24] The optimal auction will be symmetric and the object will be assigned to the bidder with the highest valuation as long as her virtual valuation is positive, i.e., $\widetilde{v}_i^* = v_0$ $\forall i \in N$ (Myerson, 1981).

In our setup where participation is costly the seller's profit maximization problem *always* admits a symmetric critical point, i.e., the first order necessary conditions for this problem are satisfied at $\widetilde{v}_i = v^s$ $\forall i \in N$, where

$$J(v^s)F(v^s)^{n-1} = c. \tag{9}$$

This condition has a straightforward interpretation. Suppose all the bidders have cutoff $v^s$. Increasing the cutoff of one of the bidders slightly will decrease the gross profit of the seller by $J(v^s)F(v^s)^{n-1}$ (losing $J(v^s)$, the virtual valuation, when all the others' valuations are less than $v^s$, i.e., with probability $F(v^s)^{n-1}$), while saving her $c$, the marginal cost of inducing participation.[25]

Notice that this symmetric cutoff is unique with $v_0 < v^s < v_h$. The existence or the uniqueness of this symmetric critical point does not depend on the data of the problem, namely $F(.)$, $c$, and $n$, but, naturally, its magnitude does.

If the seller is restricted to use a symmetric auction, it is easy to show that $\widetilde{v}_i = v^s$ $\forall i \in N$, is indeed the solution to her profit maximization problem.[26] For this reason, we call $v^s$ the optimal symmetric cutoff.

We want to remark at this point the connection between the optimal and efficient (maximizing ex-ante social surplus) auction problems. Stegeman (1996) shows that the efficient auction in this setup is characterized by participation cutoffs (with the associated allocation rule) as well. Given this, the efficient auction problem also reduces to the problem above once we replace $J(v)$ (virtual valuations, or "marginal revenue") by $v$ (valuations, or "marginal social surplus") in (8), and hence in (9). Therefore, with only this substitution, the results below about optimal auctions are directly applicable to efficient auctions, as inspection of their proofs will confirm.[27]

[24] This is perhaps clearer from the formulation in (5).

[25] These are normalized (by dividing by the density) marginal gross profit and the marginal cost. The marginal profit is given by $-J(v^s)F(v^s)^{n-1}f(v^s) + cf(v^s)$.

[26] This does not mean that the seller cannot do better in an asymmetric equilibrium of an anonymous auction. See the discussion in Section 3.

[27] Naturally, $v_0$ becomes irrelevant in this case, and so should be replaced by $v_l$ in the statements of the results.

Returning to our problem, we first show that the optimal auction may be asymmetric:

**Example 1** *There are two bidders whose valuations are distributed according to* $F(v) = v^4$ *on* $[0, 1]$, *and the participation cost is* $\frac{2}{5}$.

It turns out that, for this example, the optimal auction is asymmetric. The optimal cutoffs are $\widetilde{v}_1^* \approx .816$ and $\widetilde{v}_2^* \approx .92$, yielding a profit of .2525 for the seller. If we impose symmetry, however, the seller's profit decreases to .25155 (with the optimal symmetric cutoff $v^s \approx .868$). Notice the allocative inefficiency of the optimal auction that we mentioned before. When the valuations of both bidders are between $\widetilde{v}_1^*$ and $\widetilde{v}_2^*$, the first bidder will obtain the object even when her valuation is less than that of the second bidder.

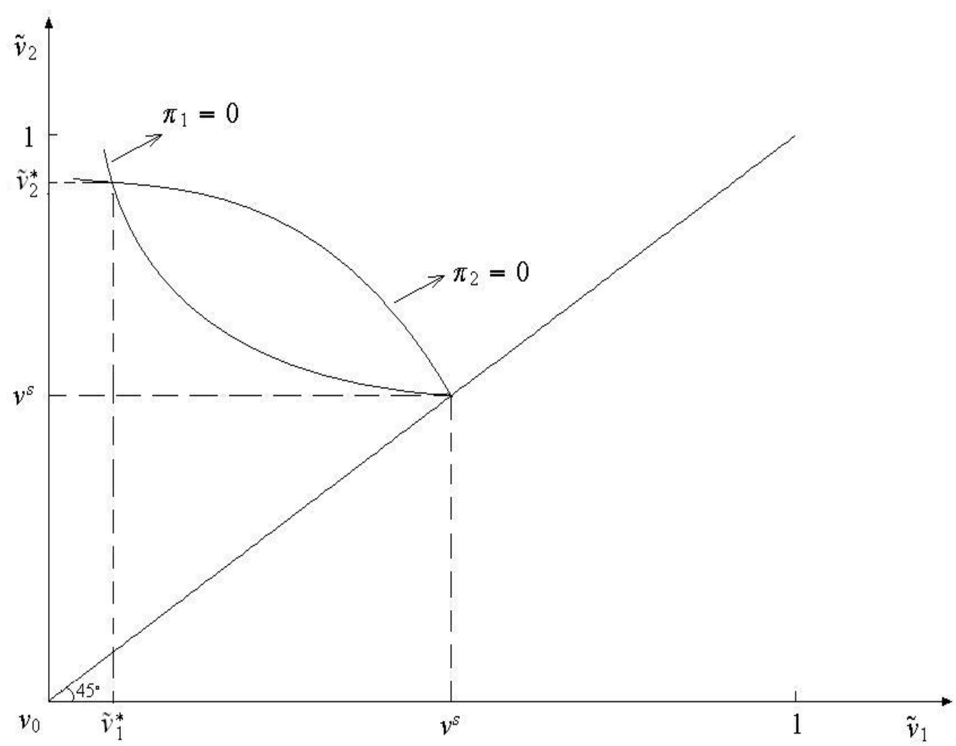


Figure 1

We use Figure 1 not only to explain why the optimal auction is asymmetric for this example, but also to provide some (pictorial) intuition for Proposition 2 below and its proof. Let $\pi_1$ (respectively, $\pi_2$) denote the marginal profit of the seller with respect to the first (respectively, second) bidder's cutoff, i.e., $\pi_1 = \frac{\partial \pi_s(\widetilde{v}_1, \widetilde{v}_2)}{\partial \widetilde{v}_1}$ and $\pi_2 = \frac{\partial \pi_s(\widetilde{v}_1, \widetilde{v}_2)}{\partial \widetilde{v}_2}$. First order necessary conditions for optimality are satisfied, i.e., $\pi_1 = \pi_2 = 0$, at two points: $(v^s, v^s)$ and $(\widetilde{v}_1^*, \widetilde{v}_2^*)$. However, $(v^s, v^s)$ does not give us even a local maximum. At any point to the right (respectively, left) of the $\pi_1 = 0$ curve, the seller can increase her profit by decreasing (respectively, increasing) the first bidder's cutoff while keeping the second bidder's cutoff constant. Similar arguments apply for the second bidder's cutoff above and below the $\pi_2 = 0$ curve.[28] Therefore, starting from the optimal symmetric cutoffs $(v^s, v^s)$, decreasing $\widetilde{v}_1$ while simultaneously increasing $\widetilde{v}_2$ by an appropriate amount, i.e., moving inside the lens-shaped area, will increase the seller's profit.[29]

From this discussion, it is clear that the existence of such a lens-shaped area emanating from $(v^s, v^s)$ in the admissible side of the constraint boundary (where $\widetilde{v}_2 \geq \widetilde{v}_1$) is a sufficient condition for the suboptimality of symmetric cutoffs, which we will utilize for our next result.

**Proposition 2** *If $\frac{J(v)}{F(v)}$ is decreasing at the optimal symmetric cutoff $v^s$, then the optimal auction is asymmetric. Moreover, for every $k$ such that $1 \leq k < n$, there is an auction where $k$ bidders use one cutoff ($\widetilde{v}_i = a < v^s$ for $i = 1, ..., k$) and the remaining bidders use another one ($\widetilde{v}_i = b > v^s$ for $i = k+1, ..., n$) that gives the seller higher profit than the optimal symmetric auction ($\widetilde{v}_i = v^s \ \forall i \in N$).*

We prove Proposition 2 (in the Appendix) by showing that, starting from the optimal symmetric cutoffs, as long as $\frac{J(v)}{F(v)}$ is decreasing, the seller can increase her profits by decreasing an arbitrary group of bidders' cutoffs and increasing the cutoffs of the complementary set of bidders. In other words, if $\frac{J(v)}{F(v)}$ is decreasing at $v^s$, then a lens-shaped improvement area, like that of Figure 1, will exist for *any* partition of bidders into two groups.

In order to gain some understanding of the sufficient condition for the asymmetry of the optimal auction, consider the two-bidders case, and start with optimal symmetric cutoffs, $(v^s, v^s)$. As we observed before, the first

[28]Note that we have $\pi_{11}, \pi_{22} < 0$, using the standard notation for second derivatives.

[29]The optimal cutoffs are indeed given by $(\widetilde{v}_1^*, \widetilde{v}_2^*)$, where the second order sufficient conditions are satisfied, as can also be seen in Figure 1.

order conditions are satisfied at $(v^s, v^s)$, so any explanation we provide will be about second order effects. Remembering that the highest-valuation participant obtains the object, consider the impact on *marginal* profit of using cutoffs $(v^s - \epsilon, v^s + \epsilon)$, where $\epsilon > 0$ is arbitrarily small. There are two opposite effects. Bidder 1 with a type $v$ in $(v^s - \epsilon, v^s + \epsilon)$ now obtains the object with a higher probability ($F(v^s + \epsilon)$ instead of $F(v)$), so the marginal profit increases by $2J(v^s)\ f(v^s)$ as $\epsilon$ approaches zero. Also, there is a decrease in the marginal profit due to selling to low virtual valuation bidder 1 types instead of high valuation bidder 2 types. As $\epsilon \to 0$, the net effect (the rest is offset by changes in the participation costs incurred) of selling to lower virtual valuation bidder 1 (with probability $F(v^s)$) is $-2J'(v^s)F(v^s)$. Therefore, the seller benefits from implementing $(v^s - \epsilon, v^s + \epsilon)$ instead of the optimal symmetric cutoffs $(v^s, v^s)$, if

$$J(v^s)f(v^s) - J'(v^s)F(v^s) > 0, \tag{10}$$

or, equivalently, if $\frac{J(v)}{F(v)}$ is decreasing at $v^s$.

An asymmetric optimal auction does not always assign the object to the bidder with the highest valuation, causing allocative inefficiency. If there are no participation costs, the optimal auction will have this type of inefficiency only when bidders are heterogenous. However, even in that case, the object is assigned to the bidder with the highest virtual valuation.[30] In contrast, in our setup it is not necessarily the bidder with the highest virtual valuation who gets the object. The seller can profit from this, since there is also the indirect effect of implementing asymmetric cutoffs: The bidders with lower cutoffs will receive the object with higher probabilities, thereby increasing what the seller can extract from these types. When our sufficient condition is satisfied, this indirect effect dominates the direct effect.

The sufficient condition for the asymmetry of the optimal auction, i.e., (10), and our discussions of it, seem to be independent of the magnitude of the participation cost, $c$. How can we reconcile this with the fact that the optimal auction is symmetric when $c = 0$? First note that the sufficient condition is not independent of $c$; the optimal symmetric cutoff $v^s$ depends on both $c$ and $n$, the number of bidders; see (9). More importantly, when $c = 0$ we have $v^s = v_0$, so that $J(v^s) = 0$ (unless $v_0 = v_l$ with $J(v_l) > 0$, in which case it is impossible to even create the type of asymmetry we are considering). Therefore, the sufficient condition, (10), is never satisfied. The

[30]We are considering "regular" cases in which virtual valuations are increasing.

reason is that when $v^s = v_0 > v_l$ the positive effect of creating an asymmetry does not exist at all. It is still true that the low valuation bidder is going to obtain the object with a higher probability, but the impact of this on the marginal profit is nil, i.e., $2J(v^s)\ f(v^s) = 0$.

When there are more than two bidders, Proposition 2 goes further than identifying a sufficient condition for the suboptimality of symmetric cutoffs. It shows that, whenever this condition is satisfied, even an arbitrary classification of the bidders into only two groups and implementation of a different cutoff for each group would improve over the optimal symmetric outcome. We find this observation relevant for analyzing the performance of auctions where one group of bidders receives preferential treatment from the seller. For example, domestic firms are sometimes given a price preference in government procurement (see McAfee and McMillan (1989)), and minority and women owned businesses received bidding credits and guaranteed financing in some FCC auctions (see Ayres and Cramton (1996)). We will come back to the preferential treatment issue when we discuss implementing asymmetric auctions in Section 3.

As we observed above, our sufficient condition for the asymmetry of the optimal auction depends on both the magnitude of the participation cost and the number of bidders through the optimal symmetric cutoff, $v^s$. For certain distribution functions (for example, uniformly distributed valuations, with $v_h < 2v_l$) this sufficient condition will always be satisfied, i.e., the optimal auction will be asymmetric regardless of the participation cost level and the number of bidders.[31]

**Corollary 1** *If $\frac{J(v)}{F(v)}$ is decreasing on $(v_l, v_h)$, then the optimal auction is asymmetric (independent of $c$ and $n$).*

We know that the optimal auction is symmetric when $c = 0$, where all the bidders have the cutoff $v_0$. In some cases, even an infinitesimally small $c$ causes the optimal auction to be asymmetric. However, for very small $c$, naturally, the asymmetry will be very small as well. As $c$ approaches to 0, bidders' optimal cutoffs all approach to $v_0$. In other words, even though there

[31] Since $v_0 < v^s < v_h$, we need $\frac{J(v)}{F(v)}$ to be decreasing only on $(v_0, v_h)$ for this result. However, when $v_0 > v_l$, $\frac{J(v)}{F(v)}$ cannot be decreasing on $(v_0, v_h)$ (since $\frac{J(v_0)}{F(v_0)} = 0$ and $\frac{J(v_h)}{F(v_h)} = v_h$), so this case is irrelevant.

is no "continuity" in the symmetry property of the optimal auction at $c = 0$, there is continuity in terms of outcomes, and hence the seller's profit.

We next turn our attention to conditions under which the optimal auction is symmetric.

**Proposition 3** *The optimal auction is symmetric for all $c$ (and $n$), i.e., $\widetilde{v}_i^* = v^s \ \forall i \in N$, if and only if $\frac{J(v)}{F(v)}$ is weakly increasing on $(v_0, v_h)$.*

The necessity part of the result is a consequence of Proposition 2. If $\frac{J(v)}{F(v)}$ is *not* weakly increasing at some $v'$ in $(v_0, v_h)$, then, for any given number of bidders, we can find a participation cost level for which the optimal symmetric cutoff $v^s$ equals to $v'$, so that the sufficient condition of Proposition 2 is satisfied, i.e., the optimal auction is asymmetric.[32]

The main interest in Proposition 3 stems from the sufficiency part. If the distribution of valuations is such that $\frac{J(v)}{F(v)}$ is weakly increasing on the relevant range, then the optimal auction is symmetric and hence completely characterized: Each bidder has the same participation cutoff $v^s$, as defined in (9). For this result, obviously, it is not enough to consider only local improvements around $v^s$, since we want to show that all bidders using $v^s$ yields a global maximum. In order to gain some understanding for the result and the condition, consider the two bidders case with asymmetric cutoffs, i.e., $\widetilde{v}_2 > \widetilde{v}_1$. Suppose the seller increases $\widetilde{v}_1$ and decreases $\widetilde{v}_2$ slightly in such a way that total participation cost incurred stays the same. As a result of these changes in the cutoffs, the seller's profit from bidder 1 (net of the participation cost) decreases by $J(\widetilde{v}_1)F(\widetilde{v}_2) + \int_{\widetilde{v}_1}^{\widetilde{v}_2} J(v)f(v)dv$, where the first term arises from increasing $\widetilde{v}_1$ slightly and the second term is the result of types in $(\widetilde{v}_1, \widetilde{v}_2)$ receiving the object with a lower probability due to a decrease in $\widetilde{v}_2$. This loss is bounded above by $J(\widetilde{v}_1)F(\widetilde{v}_2) + J(\widetilde{v}_2)[F(\widetilde{v}_2) - F(\widetilde{v}_1)]$. On the other hand, the profit from bidder 2 (again, net of the participation cost) increases by $J(\widetilde{v}_2)F(\widetilde{v}_2)$ due to the decrease in $\widetilde{v}_2$. Therefore, the seller's profit will increase if $J(\widetilde{v}_2)F(\widetilde{v}_1) - J(\widetilde{v}_1)F(\widetilde{v}_2) \geq 0$, or $\frac{J(\widetilde{v}_2)}{F(\widetilde{v}_2)} \geq \frac{J(\widetilde{v}_1)}{F(\widetilde{v}_1)}$.

**Remark 2** *For distribution functions that satisfy the monotone hazard rate condition ($\frac{1-F(v)}{f(v)}$ is decreasing), if $\frac{v}{F(v)}$ is increasing, then so is $\frac{J(v)}{F(v)}$. Therefore, if $v_l = 0$ and $F(v)$ is concave (and satisfies the monotone hazard rate condition), then the optimal auction will be symmetric.*

[32] We can see from the definition of $v^s$ in (9) that $v^s$ is a continuous and increasing function of $c$ (for any given $n$), where $v^s \to v_0$ as $c \to 0$ and $v^s \to v_h$ as $c \to v_h$.

We next present two results about the nature of (possible) asymmetries in the optimal auction. First, we identify a class of distribution functions for which the optimal auction is either symmetric or uses only two cutoffs. Second, when the sufficient condition for the asymmetry of the optimal auction in Corollary 1 is satisfied, only one bidder will have the lowest cutoff. Notice that both of these results are independent of the number of bidders and the magnitude of the participation cost, and they simplify the task of finding the optimal auction considerably whenever they apply.

**Proposition 4** *i) If $J'(v)\frac{F(v)}{f(v)}$ is weakly increasing on $(v_0, v_h)$, then the optimal auction has at most two distinct cutoffs.*

*ii) If $\frac{J(v)}{F(v)}$ is decreasing on $(v_l, v_h)$, then in the optimal auction only one bidder has the lowest cutoff, i.e., $\widetilde{v}_i^* > \widetilde{v}_1^*$ for all $i > 1$.*

## 2.4 Uniform Distributions

In this section, using our previous results, we completely characterize optimal auctions when bidders' valuations are uniformly distributed and provide some comparative statics.

We have $n \geq 2$ bidders whose valuations are uniformly distributed on $[v_l, v_h]$, where $0 \leq v_l < v_h$, i.e., $F(v) = \frac{v-v_l}{v_h-v_l}$. The participation cost is $c \in (0, v_h)$. The virtual valuation function is given by $J(v) = 2v - v_h$, which is increasing. If $2v_l - v_h \geq 0$, then $v_0 = v_l$; otherwise $v_0 = \frac{v_h}{2}$. When $c = 0$, in the optimal auction, the object is assigned to the highest valuation bidder as long as her valuation is higher than $v_0$. When $c > 0$ it is still true that a bidder with a negative virtual valuation will never get the object. In other words, all of the optimal cutoffs will be greater than $v_0$.

We first observe that $J'(v)\frac{F(v)}{f(v)} = 2(v - v_l)$ is increasing. Therefore, at most two distinct cutoffs will be used in the optimal auction (Proposition 4$i$). We next note that $\frac{J(v)}{F(v)} = \frac{(2v-v_h)(v_h-v_l)}{v-v_l}$ is either weakly increasing (if $v_h \geq 2v_l$) or decreasing (if $v_h < 2v_l$) on the entire support $[v_l, v_h]$. So, if $v_h \geq 2v_l$, then it follows from Proposition 3 that the optimal auction is symmetric. The optimal cutoffs are given by $\widetilde{v}_1^* = ... = \widetilde{v}_n^* = v^s$, where

$$J(v^s)F(v^s)^{n-1} = (2v^s - v_h)(\frac{v^s - v_l}{v_h - v_l})^{n-1} = c.$$

If $v_h < 2v_l$, then the optimal auction is asymmetric (Corollary 1) with exactly two cutoffs. Moreover, only one bidder will have the lower cutoff (Proposition $4ii$). Using these, solving the seller's problem becomes a straightforward exercise. We provide the solution here for completeness. Let $\widetilde{v}_1^* = a$ and $\widetilde{v}_2^* = ... = \widetilde{v}_n^* = b > a$.

- If $c \leq \min\{v_h - v_l, \frac{(2v_l - v_h)^n}{(v_h - v_l)^{n-1}}\}$, then $a = v_l$ and $b = v_l + c^{\frac{1}{n}}(v_h - v_l)^{\frac{n-1}{n}}$.
- If $v_h - v_l < c < 2v_l - v_h$, then $a = v_l$ and $b = v_h$.
- If $\frac{(2v_l - v_h)^n}{(v_h - v_l)^{n-1}} < c < 3v_h - 4v_l$, then $a$ satisfies $(2a - v_h)(\frac{a + v_l - v_h}{v_h - v_l})^{n-1} = c$ and $b = a + 2v_l - v_h$.
- If $c \geq \max\{2v_l - v_h, 3v_h - 4v_l\}$, then $a = \frac{v_h + c}{2}$ and $b = v_h$.[33]

Note that the optimal cutoffs are weakly increasing in $n$. If $v_h \geq 2v_l$, then the optimal auction is symmetric, and as $n$ increases the seller chooses to restrict participation symmetrically, i.e., $v^s$ is increasing in $n$ with $v^s \to v_h$ as $n \to \infty$. If $v_h < 2v_l$, both $a$ and $b$ are weakly increasing in $n$, and $b \to v_h$ as $n \to \infty$.

The optimal cutoffs are also weakly increasing in $c$. All cutoffs approach $v_0$ as $c \to 0$ and approach $v_h$ as $c \to v_h$.

Whenever the optimal auction is asymmetric, the seller deals with one of the bidders exclusively when the participation cost is high enough or when there are many bidders. In particular, when $v_h < 2v_l$, $b = v_h$ if $c$ is high enough for any fixed $n$, and $b \to v_h$ as $n \to \infty$ for any fixed $c$. Dealing exclusively with one bidder, or "sole-sourcing" is a commonly observed phenomenon in government procurement. In our setting, sole source contracting emerges as an optimal response to high participation costs in certain cases.

# 3 Implementing the Optimal Auction

We showed earlier that to maximize her profit the seller need to only consider auctions where bidders use cutoff rules in participation and the object is assigned to the highest-valuation participant. Given these participation and assignment rules, the seller's problem is reduced to choosing (bidder-specific) cutoffs optimally. As we remarked before, the seller's revenue will be identical

[33] Note that when $v_h < 2v_l$ we have, $v_h - v_l < \frac{(2v_l - v_h)^n}{(v_h - v_l)^{n-1}} \Leftrightarrow v_h - v_l < 2v_l - v_h \Leftrightarrow \frac{(2v_l - v_h)^n}{(v_h - v_l)^{n-1}} > 3v_h - 4v_l \Leftrightarrow 2v_l - v_h > 3v_h - 4v_l$.

in auctions that induce the same cutoffs and assign the object to the highest-valuation participant in equilibrium, an instance of the revenue equivalence theorem.[34]

Our objective in this section is to show that using common auction formats augmented with appropriately chosen "familiar" instruments or variations could indeed be optimal for the seller.[35] This task is trivial if the optimal auction is symmetric, i.e., each bidder has the same cutoff $v^s$, defined in (9). The standard auctions, e.g., first and second price auctions (FPA and SPA, respectively), with appropriately chosen reserve price and/or entry fee (or subsidy) will be optimal.[36] To see this, let $r$ denote the reserve price and $c^e$ *effective participation cost,* i.e., $c^e$ is the sum of the participation cost $c$ and the entry fee (which could be negative, implying an entry subsidy). Suppose $r$ and $c^e$ satisfy the following equation (obviously, there are many such $r$ and $c^e$):

$$(v^s - r)F(v^s)^{n-1} = c^e. \tag{11}$$

FPA and SPA, with $r$ and $c^e$ satisfying (11) are both optimal, since each has a symmetric equilibrium where bidders use the cutoff $v^s$ (at which their expected payoffs are zero) and their bids are increasing in their valuations, implying that the highest-valuation participant receives the object.

We only consider asymmetric optimal auctions from this point on. The seller can accomplish her goal in a very simple way even in this case. Consider the SPA where each bidder has an individualized reserve price given by her optimal cutoff (only bids exceeding her reserve price are allowable), and an entry subsidy of $c$ is provided to any bidder who submits an allowable bid, i.e., the effective participation cost is zero. There is an equilibrium *in dominant strategies* where bidders participate (and bid their valuations) iff their valuations are greater than their respective reserve prices. This equilibrium gives the seller her maximal profit, since the object is assigned to

[34] Expected payoffs of marginally participating types have to be the same as well. Also, implicit in our usage of the term "cutoff" is that the bidder will use the associated cutoff rule in participation.

[35] We will not be concerned with "strong implementation" in what follows. So, we call an auction form optimal if the seller obtains her maximal profit in one (as opposed to all) of its (Bayesian-Nash) equilibria.

[36] Assume that in the ascending price (or English) auction bidders incur the participation cost prior to the start of bid calling out (assumed to be costless), which is natural for most sources of participation costs. When this is the case, our results below concerning second price auctions will be valid for ascending price auctions as well.

the highest-valuation participant and bidders use the optimal cutoffs where their expected payoffs are zero. However, it may still be useful to investigate whether there are other auction formats that are optimal. Note that this SPA is not anonymous, i.e., the bidders are not treated identically by its rules. Moreover, even when non-anonymous auctions were used in practice (we provide a few examples below), they have never had, as far as we know, bidder-specific reserve prices.

We will first show that under some conditions the seller can obtain her maximal profit by using an anonymous auction. Afterwards, we will discuss some non-anonymous auctions that resemble the ones that are actually observed in practice.

## 3.1 An Anonymous Second Price Auction

There may be multiple equilibria (in undominated strategies) in SPAs with costly participation even in the symmetric independent private values environment we are considering.[37] In any equilibrium in undominated strategies, bidders employ cutoff rules in participation and bid their valuations whenever they submit a bid. There is always a symmetric equilibrium where the cutoffs used are all identical, but there may be asymmetric equilibria as well. Therefore, it may be possible for the seller to achieve her optimal profit level in an asymmetric equilibrium of an anonymous SPA. To demonstrate this point, we shall use Example 1, where there are two bidders, $F(v) = v^4$, and $c = \frac{2}{5}$. The optimal auction is asymmetric, with $\widetilde{v}_1^* \approx .816$ and $\widetilde{v}_2^* \approx .92$. Now, consider a SPA with reserve price $r \approx .598$ and effective participation cost $c^e \approx .156$, so that there is an entry subsidy. There is an equilibrium where one of the bidders participate iff her valuation is greater than .816, the other use .92 as her cutoff, and both bid their valuations whenever they participate. In this equilibrium, the highest-valuation participant receives the object. In addition, the expected payoffs of bidders are zero at their respective cutoffs, since these are determined by indifference (to participation) conditions. Therefore, the seller obtains her optimal profit.

This example can be generalized as follows: Suppose the optimal auction has two cutoffs. If the monotone hazard rate condition is satisfied, then the

[37] See Tan and Yilankaya (2006) for conditions under which this would happen. For this, it is immaterial whether participation cost is a real resource cost incurred by bidders or is an entry fee charged by the seller.

SPA, with appropriately chosen reserve price and effective participation cost, has an equilibrium that is optimal for the seller.[38]

Two cutoff requirement is obviously a restriction. However, we know that under some conditions the optimal auction will indeed have at most two distinct cutoffs (Proposition $4i$ provides a sufficient condition). Moreover, whenever our sufficient condition for the asymmetry of the optimal auction is satisfied, the seller needs to implement only two distinct cutoffs to improve over the optimal symmetric cutoff $v^s$ (Proposition 2), which can again be accomplished by using an anonymous SPA.

## 3.2 Differential Effective Participation Costs

Not all bidders incur the same participation cost in all auctions, and sometimes this happens by the design of the seller. One obvious way of doing this is by charging bidders different entry fees. There are also indirect ways. The seller may provide guaranteed financing for some bidders, thus saving them the fixed costs associated with credit arrangements. This was done, for example, in the FCC spectrum auctions; see, e.g., Ayres and Cramton (1996). Also, the rules of the auction may be such that some bidders face higher participation costs. For example, participation costs of foreign firms are sometimes increased in government procurement by imposing residency requirements, giving a very tight deadline for submission of bids, etc., see, e.g., McAfee and McMillan (1989).

If the seller can induce differential effective participation costs, then a SPA or FPA will be optimal for the seller. We demonstrate these for the two-bidders case for expositional simplicity. Let $\widetilde{v}_1^*$ be the cutoff of bidder 1 and $\widetilde{v}_2^* > \widetilde{v}_1^*$ that of bidder 2 in the optimal auction. Consider the SPA with $r = \widetilde{v}_1^*$, $c_1^e = 0$, and $c_2^e = \int_{\widetilde{v}_1^*}^{\widetilde{v}_2^*} F(v)dv$, where $c_i^e$ is the effective participation cost of bidder $i$. It is a dominant strategy for the first bidder to participate and bid her valuation iff her valuation is greater than $\widetilde{v}_1^*$. Given this, the second bidder's expected payoff (for $v_2 > \widetilde{v}_1^*$) if she participates and bids her valuation is

$$(v_2 - \widetilde{v}_1^*)F(\widetilde{v}_1^*) + \int_{\widetilde{v}_1^*}^{v_2} (v_2 - v)dF(v) - c_2^e = \int_{\widetilde{v}_1^*}^{v_2} F(v)dv - c_2^e.$$

[38] In the Appendix, we prove both this claim and the one in the next paragraph in the text.

Note that $c_2^e$ is chosen in such a way that bidder 2 participates (and bids her valuation) iff her valuation is greater than $\widetilde{v}_2^*$. Therefore, the seller obtains her maximal profit.[39]

The seller can also achieve her goal by using the FPA with $r = \widetilde{v}_1^*$, $c_1^e = 0$, and $c_2^e = \int_{\widetilde{v}_1^*}^{\widetilde{v}_2^*} F(\widetilde{v}_2^*)dv$, since there is an equilibrium of this auction where $i$ uses $\widetilde{v}_i^*$ as her cutoff (at which her expected payoff is zero) and both bidders use the same strictly increasing bid function for types greater than $\widetilde{v}_2^*$, so that the highest-valuation participant receives the object.[40] To calculate the bid functions, and to see where these effective participation costs are coming from, suppose such an equilibrium exists.[41] Let $Q_i^*(.)$ be $i$'s probability of winning function in this equilibrium (and hence in the optimal auction). From the incentive compatibility conditions, we have, for $v \geq \widetilde{v}_i^*$,

$$\pi_i(v) = Q_i^*(v)v - P_i(v) = \int_{\widetilde{v}_i^*}^{v} Q_i^*(y)dy, \tag{12}$$

where

$$P_i(v) = c_i^e + b_i(v)Q_i^*(v) \tag{13}$$

is $i$'s equilibrium expected payment and $b_i(.)$ is $i$'s equilibrium bid. Combining (12) and (13),

$$b_i(v) = v - \frac{\int_{\widetilde{v}_i^*}^{v} Q_i^*(y)dy + c_i^e}{Q_i^*(v)}. \tag{14}$$

Notice that $b_i'(v) > 0$. Consider $v > \widetilde{v}_2^*$. We have $Q_1^*(v) = Q_2^*(v) = F(v)$, since both participate and the highest-valuation participant wins, and so $b_1(v) = b_2(v)$ if $c_1^e = 0$ and $c_2^e = \int_{\widetilde{v}_1^*}^{\widetilde{v}_2^*} Q_1^*(y)dy = \int_{\widetilde{v}_1^*}^{\widetilde{v}_2^*} F(\widetilde{v}_2^*)dv$.

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[39] For arbitrary $n$, the same method would yield the SPA with $r = \widetilde{v}_1^*$ and $c_i^e = \sum_{j=1}^{i-1} \prod_{\substack{k=j+1 \\ k\neq i}}^{n+1} F(\widetilde{v}_k^*) \int_{\widetilde{v}_j^*}^{\widetilde{v}_{j+1}^*} F(v)^j dv, \forall i \in N$.

[40] The equilibrium bid functions for arbitrary $n$ are given by (14) as well, so the FPA with $r = \widetilde{v}_1^*$ and $c_i^e = \int_{\widetilde{v}_1^*}^{\widetilde{v}_i^*} Q_1^*(v)dv = \sum_{j=1}^{i-1} \prod_{k=j+1}^{n+1} F(\widetilde{v}_k^*) \int_{\widetilde{v}_j^*}^{\widetilde{v}_{j+1}^*} F(v)^{j-1}dv, \forall i \in N$, will be optimal.

[41] The bid functions we find below indeed constitute an equilibrium. The proof is identical to that of the similar claim for standard FPAs.

### 3.3 Bidding Preferences

In some government auctions certain groups of bidders are given explicit bidding preferences. For example, the Buy American Act of the US (and comparable provisions in other countries) gives bidding preferences to domestic firms over foreign firms in government procurement. Similarly, small businesses or in-state bidders are favored in some government auctions.

We now show that, in our setup, a FPA with bidding preferences could be optimal for the seller. To see this, first note that in the optimal auction, bidder $i$'s expected payment is given by, see (12) for example,

$$P_i^*(v) = Q_i^*(v)v - \int_{\widetilde{v}_i^*}^{v} Q_i^*(y)dy,$$

where $Q_i^*(.)$ is $i$'s probability of winning function (given by (7) and the optimal cutoffs). Now consider the FPA with $r = \widetilde{v}_1^*$ and effective bid functions, for all $i \in N$,

$$\delta_i(b) = \begin{cases} \widetilde{v}_i^* - \frac{c}{Q_i^*(v_i^*)} & \widetilde{v}_1^* \leq b < \widetilde{v}_i^* \\ b - \frac{\int_{\widetilde{v}_i^*}^{b} Q_i^*(v)dv + c}{Q_i^*(b)} & \widetilde{v}_i^* \leq b \leq v_h \\ b - (\int_{\widetilde{v}_i^*}^{v_h} Q_i^*(v)dv + c) & v_h < b \end{cases},$$

so that bidder $i$ receives the object if her bid $b$ is the highest bid (as long as it is higher than the reserve price $\widetilde{v}_1^*$), but pays only her effective bid $\delta_i(b)$ rather than her actual bid $b$. There is an equilibrium of this auction where each bidder $i$ participates iff her valuation is higher than $\widetilde{v}_i^*$ and all participating bidders bid their valuations, giving the seller her maximal profit. To see that this is indeed an equilibrium, suppose that all bidders but $i$ are following their equilibrium strategies. Bidding $\widetilde{v}_i^*$ is better than bidding anything lower, since the winning probability is higher (strictly, unless $i = 1$) and the effective bid, i.e., the payment conditional on winning, is the same. Similarly, bidding $v_h$ is better than bidding anything higher, since the winning probability is constant and the effective bid is lower. Finally, note that $i$'s effective bidding function is constructed so that if she bids $v' \in [\widetilde{v}_i^*, v_h]$, then her expected probability of winning is $Q_i^*(v')$ and her expected payment is $P_i^*(v')$, i.e., we have $\delta_i(v')\, Q_i^*(v') + c = P_i^*(v')$. Since the optimal auction is incentive compatible and individually rational, it is a best-response for $i$ to participate (and bid her valuation) iff her valuation is higher than $\widetilde{v}_i^*$.

# 4 Appendix

**Proof of Proposition 2.** Fix $k$ such that $1 \leq k < n$. Suppose that the seller considers only two cutoff auctions, where the cutoff of the first $k$ bidders is $a$ and the others' is $b \geq a$. The expected profit of the seller in terms of $a$ and $b$ is

$$R(a,b) = \int_a^b J(v)F(b)^{n-k}dF(v)^k + \int_b^{v_h} J(v)dF(v)^n - kc(1-F(a)) - (n-k)c(1-F(b)).$$

$R_{aa}, R_{bb} < 0$ at $v^s$, using the standard notation for second derivatives. We will show that, if $\frac{J(v)}{F(v)}$ is strictly decreasing at $v^s$, then at $a = b = v^s$ we have

$$0 < \frac{R_{aa}}{R_{ab}} < \frac{R_{ab}}{R_{bb}}, \tag{15}$$

proving the proposition. Note that, at $v^s$ we are on the boundary of the feasible set ($b \geq a$ constraint), so showing that the Hessian is not negative definite at $v^s$ would not be sufficient; (15) (which implies that the Hessian is *not* negative definite, but *not* implied by it) ensures that there is an improvement by "moving towards the right side of the boundary", i.e., we can find $\epsilon_1, \epsilon_2 > 0$ such that $R(v^s - \epsilon_1, v^s + \epsilon_2) > R(v^s, v^s)$.

It is straightforward to show that at $a = b = v^s$ (using the fact that $R_a = R_b = 0$),

$$\frac{R_{aa}}{R_{ab}} = \frac{J'(v^s)F(v^s) + (k-1)J(v^s)f(v^s)}{(n-k)J(v^s)f(v^s)} > 0,$$

$$\frac{R_{ab}}{R_{bb}} = \frac{kJ(v^s)f(v^s)}{J'(v^s)F(v^s) + (n-k-1)J(v^s)f(v^s)} > 0.$$

Therefore, if $\frac{J(v)}{F(v)}$ is strictly decreasing at $v^s$, i.e., $J'(v^s)F(v^s) < J(v^s)f(v^s)$, then $\frac{R_{aa}}{R_{ab}} < \frac{R_{ab}}{R_{bb}}$ at $a = b = v^s$. ■

**Proof of Proposition 3.** The necessity part is immediate and was discussed in the text. For sufficiency, suppose to the contrary that $\frac{J(v)}{F(v)}$ is weakly increasing on $(v_0, v_h)$, but the optimal auction is asymmetric, so that at least two distinct cutoffs are chosen. Consider two smallest cutoffs: $a \geq v_0$ is used for bidders $1, ..., m$, and $b > a$ is used for bidders $m+1, ..., m'$, where $1 \leq m < m' \leq n$. From the first order condition for $a$,

$$c - J(a)F(a)^{m-1}F(b)^{m'-m} \prod_{k=m'+1}^{n+1} F(\widetilde{v}_k^*) \leq 0. \tag{16}$$

which is satisfied with equality whenever $a > v_l$.

From the first order condition with respect to $b$,

$$c - J(b)F(b)^{m'-1}\prod_{k=m'+1}^{n+1} F(\widetilde{v}_k^*) + F(b)^{m'-m-1}\prod_{k=m'+1}^{n+1} F(\widetilde{v}_k^*)\int_a^b J(v)dF(v)^m \geq 0,$$

which is satisfied with equality whenever $b < v_h$. Combining these, we have

$$\begin{aligned} J(a)F(a)^{m-1}F(b) &\geq J(b)F(b)^m - \int_a^b J(v)dF(v)^m \\ &> J(b)F(b)^m - J(b)(F(b)^m - F(a)^m) \\ &= J(b)F(a)^m, \end{aligned}$$

or,

$$\frac{J(a)}{F(a)} > \frac{J(b)}{F(b)},$$

which is a contradiction. ■

**Proof of Proposition 4.** i) The proof is by contradiction. Suppose to the contrary that at least three cutoffs are used in the optimal auction, and consider three smallest of these cutoffs, $v_l \leq a_1 < a_2 < a_3 \leq v_h$, where the number of bidders using these cutoffs are $n_1, n_2$, and $n_3$ respectively. From the first order condition with respect to the cutoffs of $n_1$ bidders who use $a_1 \geq v_0$ (using (8)), we have

$$c - J(a_1)F(a_1)^{n_1-1}F(a_2)^{n_2}F(a_3)^{n_3}\prod_{j=n_1+n_2+n_3+1}^{n+1} F(\widetilde{v}_j^*) \leq 0, \qquad (17)$$

with equality if $a_1 > v_l$.

From the first order condition with respect to the cutoffs of bidders using $a_2$,

$$c - [J(a_2)F(a_2)^{n_1} - n_1\int_{a_1}^{a_2} J(v)F(v)^{n_1-1}f(v)dv]F(a_2)^{n_2-1}F(a_3)^{n_3}\prod_{j=n_1+n_2+n_3+1}^{n+1} F(\widetilde{v}_j^*) = 0,$$

or, after integration by parts,

$$c = [J(a_1)F(a_1)^{n_1} + \int_{a_1}^{a_2} J'(v)F(v)^{n_1}dv]F(a_2)^{n_2-1}F(a_3)^{n_3}\prod_{j=n_1+n_2+n_3+1}^{n+1} F(\widetilde{v}_j^*) \qquad (18)$$

Finally, from the first order condition with respect to $a_3$ bidders,

$$c-[J(a_3)F(a_3)^{n_1+n_2}-n_1\int_{a_1}^{a_2}J(v)F(v)^{n_1-1}F(a_2)^{n_2}f(v)dv-$$
$$(n_1+n_2)\int_{a_2}^{a_3}J(v)F(v)^{n_1+n_2-1}f(v)dv]F(a_3)^{n_3-1}\prod_{j=n_1+n_2+n_3+1}^{n+1}F(\widetilde{v}_j^*)\geq 0$$

or, after integration by parts,

$$\begin{aligned} c \;\geq\; & [J(a_1)F(a_1)^{n_1}F(a_2)^{n_2}+\int_{a_1}^{a_2}J'(v)F(v)^{n_1}F(a_2)^{n_2}dv+ \\ & \int_{a_2}^{a_3}J'(v)F(v)^{n_1+n_2}dv]F(a_3)^{n_3-1}\prod_{j=n_1+n_2+n_3+1}^{n+1}F(\widetilde{v}_j^*) \end{aligned} \quad (19)$$

From (17) and (18),

$$J(a_1)F(a_1)^{n_1-1}[F(a_2)-F(a_1)]\geq\int_{a_1}^{a_2}J'(v)F(v)^{n_1}dv$$

with equality if $a_1 > v_l$. Multiply both sides with $F(a_1)$. Now, either $F(a_1)=0$ or the above inequality holds as an equality. In either case,

$$J(a_1)F(a_1)^{n_1}=\frac{F(a_1)}{F(a_2)-F(a_1)}\int_{a_1}^{a_2}J'(v)F(v)^{n_1}dv$$

Adding $\int_{a_1}^{a_2}J'(v)F(v)^{n_1}dv$ to both sides,

$$J(a_1)F(a_1)^{n_1}+\int_{a_1}^{a_2}J'(v)F(v)^{n_1}dv=\frac{F(a_2)}{F(a_2)-F(a_1)}\int_{a_1}^{a_2}J'(v)F(v)^{n_1}dv. \quad (20)$$

Similarly, from (18) and (19), we have

$$J(a_1)F(a_1)^{n_1}+\int_{a_1}^{a_2}J'(v)F(v)^{n_1}dv\geq\frac{\int_{a_2}^{a_3}J'(v)F(v)^{n_1+n_2}dv}{F(a_2)^{n_2-1}[F(a_3)-F(a_2)]}>\frac{F(a_2)\int_{a_2}^{a_3}J'(v)F(v)^{n_1}dv}{F(a_3)-F(a_2)},$$

where the strict inequality follows from the fact that $F(v)$ is larger than $F(a_2)$ on $[a_2,a_3]$. Together with equality (20), this last inequality yields

$$\frac{\int_{a_1}^{a_2}J'(v)F(v)^{n_1}dv}{F(a_2)-F(a_1)}>\frac{\int_{a_2}^{a_3}J'(v)F(v)^{n_1}dv}{F(a_3)-F(a_2)},$$

or,

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} > \frac{\varphi(x_3) - \varphi(x_2)}{x_3 - x_2}, \tag{21}$$

where $x_i = F(\widetilde{v}_i^*)$ and $\varphi(x) = \int_0^{F^{-1}(x)} J'(v)F(v)^{n_1}dv$. Now notice that,

$$\varphi'(x) = \frac{J'(F^{-1}(x))F(F^{-1}(x))^{n_1}}{f(F^{-1}(x))} > 0,$$

and $\varphi''(x) \geq 0$ (since $J'(v)\frac{F(v)}{f(v)}$ is weakly increasing), which contradicts (21).

ii) Suppose by contradiction that $\widetilde{v}_1^*$ is the cutoff of the first $m > 1$ bidders in the optimal auction. Let $k$ be an arbitrary positive integer smaller than $m$. Consider the class of auctions, where the first $k$ cutoffs are equal to $a$, the following $m-k$ cutoffs are equal to $b$, and cutoffs $m+1$ to $n$ are given as $\widetilde{v}_{m+1}^*$ to $\widetilde{v}_n^*$, such that $a < b < \widetilde{v}_{m+1}^*$. We can write the expected profit from such an auction as a function of $a$ and $b$:

$$\begin{aligned} R(a,b) &= k\int_a^b J(v)[F(v)^{k-1}F(b)^{m-k}\prod_{j=m+1}^{n+1} F(\widetilde{v}_j^*)]f(v)dv \\ &\quad +m\int_b^{\widetilde{v}_{m+1}^*} J(v)[F(v)^{m-1}\prod_{j=m+1}^{n+1} F(\widetilde{v}_j^*)]f(v)dv \\ &\quad -kc(1-F(a)) - (m-k)c(1-F(b)) \\ &\quad +\sum_{i=m+1}^{n} i\int_{\widetilde{v}_i^*}^{\widetilde{v}_{i+1}^*} J(v)[F(v)^{i-1}\prod_{j=i+1}^{n+1} F(\widetilde{v}_j^*)]f(v)dv - c\sum_{i=m+1}^{n}(1-F(\widetilde{v}_i^*)) \end{aligned}$$

The optimal auction must also be optimal within this class. Therefore, $R(a,b)$ is maximized at $a = b = \widetilde{v}_1^*$. First, note that, since $\frac{J(v)}{F(v)}$ is decreasing on $[v_l, v_h]$, the optimal auction is asymmetric (Corollary 1), i.e., $\widetilde{v}_1^* < v_h$. Note also that $\widetilde{v}_1^* > v_l$, since when $a = b = v_l$, the first order condition for $a$ is violated, i.e.,

$$R_a(v_l, v_l) = kf(v_l)[c - J(v_l)F(v_l)^{m-1}\prod_{j=m+1}^{n+1} F(\widetilde{v}_j^*)] > 0,$$

since $F(v_l) = 0$ and $f(v_l) > 0$. Hence, $a = b = \widetilde{v}_1^*$ could satisfy the first order necessary conditions only at an interior point. Following the proof of Proposition 2, note that, at $a = b = \widetilde{v}_1^*$, we have

$$\frac{R_{aa}}{R_{ab}} = \frac{J'(v_1^*)F(v_1^*) + (k-1)J(v_1^*)f(v_1^*)}{(m-k)J(v_1^*)f(v_1^*)} > 0,$$

$$\frac{R_{ab}}{R_{bb}} = \frac{kJ(v_1^*)f(v_1^*)}{J'(v_1^*)F(v_1^*) + (m-k-1)J(v_1^*)f(v_1^*)} > 0.$$

Therefore, if $J'(\widetilde{v}_1^*)F(\widetilde{v}_1^*) < J(\widetilde{v}_1^*)f(\widetilde{v}_1^*)$, i.e., $\frac{J(v)}{F(v)}$ is decreasing at $\widetilde{v}_1^*$, then $\frac{R_{aa}}{R_{ab}} < \frac{R_{ab}}{R_{bb}}$ at $a = b = \widetilde{v}_1^*$, implying that $a = b = \widetilde{v}_1^*$ cannot be optimal, a contradiction. ■

**Proof of an anonymous SPA implementing the optimal auction.** Suppose that in the optimal auction $k$ bidders have the cutoff $a$ and $n-k$ bidders have the cutoff $b$, where $v_l \leq a < b \leq v_h$ and $1 \leq k \leq n-1$. Given $a$ and $b$, we will find $r$ and $c^e$ such that there is an equilibrium of the second price auction with reserve price $r$ and participation cost $c^e$ in which $k$ (respectively, $n-k$) bidders participate iff their valuation is greater than $a$ (respectively, $b$), and all the participating bidders bid their valuations. For this, it is sufficient to check (the rest is standard, see, for example, Tan and Yilankaya (2006)) that the expected payoffs of $k$ bidders who have $a$ as their cutoffs are nonnegative (zero if $a > v_l$) when their valuations are $a$, and similarly, the expected payoffs of $n-k$ bidders who have $b$ as their cutoffs are nonpositive (zero if $b < v_h$) when their valuations are $b$.

$$(a-r)F(a)^{k-1}F(b)^{n-k} - c^e \geq 0. \tag{22}$$

$$F(b)^{n-k-1}((b-r)F(a)^k + \int_a^b (b-v)dF(v)^k) - c^e \leq 0,$$

or, after using integration by parts,

$$F(b)^{n-k-1}((a-r)F(a)^k + \int_a^b F(v)^k dv) - c^e \leq 0. \tag{23}$$

(22) and (23) has an admissible solution in $r$ and $c^e$, i.e., with $0 < c^e, r, c^e + r < v_h$, iff

$$aF(a)^{k-1}F(b)^{n-k} > F(b)^{n-k-1}(aF(a)^k + \int_a^b F(v)^k dv),$$

or,

$$F(a)^{k-1}(F(b) - F(a)) > \int_a^b \frac{1}{a}F(v)^k dv. \tag{24}$$

The optimality of $a$ and $b$ implies the following first order conditions[42]:

$$J(a)F(a)^{k-1}F(b)^{n-k} - c \geq 0, \tag{25}$$

[42] Notice that these conditions must be satisfied even in the constrained problem where two distinct cutoffs (with $k$ bidders using the smaller one) will be used, and the only choice variables are the magnitudes of these cutoffs.

$$J(b)F(b)^{n-1} - F(b)^{n-k-1} \int_a^b J(v) dF(v)^k - c \leq 0. \tag{26}$$

Combining these, and using integration by parts,

$$F(a)^{k-1}(F(b) - F(a)) \geq \int_a^b \frac{J'(v)}{J(a)} F(v)^k dv. \tag{27}$$

Since $a > J(a)$ and $J'(v) \geq 1$ (because $\frac{1-F(v)}{f(v)}$ is decreasing), (27) implies (24), proving the result. ■

# References


[1] Ayres, I. and P. Cramton (1996), "Deficit Reduction Through Diversity: How Affirmative Action at the FCC Increased Auction Competition," *Stanford Law Review* **48**, 761- 815.

[2] Bergeman, D. and J. Valimaki (2002), "Information Acquisition and Efficient Mechanism Design," *Econometrica* **70**, 1007-1033.

[3] Cremer, J., Y. Spiegel and C. Z. Zheng (2006), "Optimal Search Auctions," forthcoming in *Journal of Economic Theory.*

[4] Ehrman, C. and M. Peters (1994), "Sequential Selling Mechanisms," *Economic Theory* **4**, 237-253.

[5] Engelbrecht-Wiggans, R. (1993), "Optimal Auctions Revisited," *Games and Economic Behavior* **93**, 227-239.

[6] Harstad, R. M. (1990), "Alternative Common-Value Auction Procedures: Revenue Comparison with Free Entry," *Journal of Political Economy* **98**, 421-429.

[7] Levin, D. and J. L. Smith (1994), "Equilibrium in Auctions with Entry," *American Economic Review* **84**, 585-599.

[8] Lu, J. (2003), "Auction Design with Communication Cost," *mimeo.*

[9] Matthews, S. A. (1984), "Information Acquisition in Discriminatory Auctions," in *Bayesian Models in Economic Theory*, ed. by M. Boyer and R. E. Kihlstrom, Elsevier Science Publishers, New York.

[10] Matthews, S. A. (1995), “A Technical Primer on Auction Theory I: Independent Private Values, CMSEMS Discussion Paper No. 1096, Northwestern University.

[11] McAfee, R. P. and J. McMillan (1987), “Auctions with Entry,” *Economics Letters* **23**, 343-347.

[12] McAfee, R. P. and J. McMillan (1988), “Search Mechanisms,” *Journal of Economic Theory* **44**, 99-123.

[13] McAfee, R. P. and J. McMillan (1989), “Government Procurement and International Trade,” *Journal of International Economics* **26**, 291-308.

[14] Myerson R. B. (1981), “Optimal Auction Design,” *Mathematics of Operations Research* **6**, 58-73.

[15] Persico, P. (2000), “Information Acquisition in Auctions,” *Econometrica* **68**, 135-148.

[16] Samuelson, W. F. (1985), “Competitive Bidding with Entry Costs,” *Economics Letters* **17**, 53-57.

[17] Stegeman, M. (1996), “Participation Costs and Efficient Auctions,” *Journal of Economic Theory* **71**, 228-259.

[18] Tan, G. (1992), “Entry and R&D in Procurement Contracting,” *Journal of Economic Theory* **58**, 41-60.

[19] Tan, G. and O. Yilankaya (2006), “Equilibria in Second Price Auctions with Participation Costs,” *Journal of Economic Theory* **130**, 205-219.