Delay in Strategic Information Aggregation

Ettore Damiano, Li, Hao
University of Toronto

Wing Suen*
The University of Hong Kong

May 22, 2008

Abstract. We study a model of collective decision making in which agents vote on the decision repeatedly until they agree, with the agents receiving no exogenous new information between two voting rounds but incurring a delay cost. Although preference conflict between the agents makes information aggregation impossible in a single round of voting, in the equilibrium of the repeated voting game agents are increasingly more willing to vote their private information after each disagreement. Information is efficiently aggregated within a finite number of rounds. As delay becomes less costly, agents are less willing to vote their private information, and efficient information aggregation takes longer. Even as the delay cost converges to zero, agents are strictly better off in the repeated voting game than in any single round game for moderate degrees of initial conflict.

* Li and Suen thank the Guanghua School of Management of Peking University, and especially Hongbin Cai, for the research support during their visit; Li thanks the Marshall School of Business of the University of Southern California for hosting his sabbatical when part of this paper was written. We have benefited from comments by Jacques Cremer, Jeff Ely, Roger Gordon, Jim Peck, Joel Sobel, and Asher Wolinsky, as well as by other seminar participants at Georgetown University, Northwestern University, University of Florida, Ohio State University, University of California at Los Angeles and at San Diego, and Singapore Management University. Chris Chan and Eric Mak provided excellent research assistance.

1. Introduction

Individuals may disagree with one another when they have different preferences or when they have different private information. Often, it is difficult to distinguish between these two types of disagreement because divergent preferences provide incentives for individuals to distort their information. Even though they may share a common interest in some states had the individuals known each other's private information, the strategic distortion of information can still cause disagreement in these states. When disagreements lead to delay in making decisions, it may seem that any decision is better than no decision and costly delay. We argue in this paper, however, that institutionalized delay in the decision making process can serve a useful purpose. In the context of a stylized model of repeated voting, the prospect of costly delay induces the parties to be more forthcoming with their private information. This enhances information aggregation and potentially improves the welfare of the agents relative to the case when the decision has to be made immediately. Even when the delay cost is arbitrarily small, there can be a significant welfare gain.

The constructive role of delay in strategic information aggregation is illustrated in the simplest model that captures the distinction between preference-driven and information-driven disagreements. Information aggregation is mutually beneficial precisely in states of nature in which agents disagree based on their own private information but would agree if perfectly informed. This particular configuration of preferences and information structures can be illustrated with the following story. Imagine that two managers of a corporation, of marketing and R&D divisions, must jointly decide how to enter an emerging market. One strategy focuses on pushing existing products through a marketing campaign, and the alternative mainly involves developing a new product that targets the emerging market. For some types of the market, one strategy is definitely more effective than the other, in which case the two managers both prefer the more effective strategy, but there are also other types of the market for which neither strategy is clearly better, in which case each manager prefers the strategy of his own division. Suppose that the marketing manager can distinguish the states for which the R&D strategy is more effective from the other states, and similarly the R&D manager can tell the states for which the marketing strategy is more effective from the other states. With these information structures, the two managers can disagree on the strategy based on their information only in the states when there is no clear effective strategy. But these are precisely the states where the two would disagree even if perfectly informed, and thus information aggregation is not an issue. In contrast, consider a different information structure for the marketing manager which allows him to distinguish the states in which the marketing strategy is more effective from the other states, and a symmetric information structure for the R&D manager. Now, it can happen that the two managers disagree based on their private information but would agree if perfectly informed. Here, information aggregation is valuable, but it may be precluded by strategic considerations. This is the environment we are interested in, where delay can potentially enhance information aggregation and improve welfare.

In many situations collective decisions can only be adopted under mutual consent. If two parties fail to reach an agreement, the only recourse is to keep trying. We introduce in this paper a model of repeated voting in which a decision will not be taken until both sides agree. In each voting round, two individuals vote simultaneously on two alternatives. If the two votes agree, the agreed alternative is implemented and the game ends; otherwise, each individual incurs an additive delay cost and voting proceeds to the next round, until an agreement is reached. There is one conflict state in which the two individuals prefer different alternatives, and two equally likely common interest states, one for each alternative, when their preferences coincide. Ex ante, each individual favors a different alternative, and the degree of conflict between the two individuals is captured by the prior probability of the conflict state. In each common interest state, the individual who ex ante favors the mutually preferred alternative is perfectly informed, while the other individual is uninformed and knows only that the state is not the other common interest state. In the conflict state, both individuals are uninformed and each knows only that his ex ante favorite alternative is not mutually preferred. The information structure and preferences are such that if the decision must be made without delay, there is no incentive compatible outcome that Pareto dominates a coin flip between the two alternatives when the degree of conflict is high, even though the state could be a common interest state.

In section 3 we construct a symmetric equilibrium of the repeated voting game. In equilibrium, the informed type votes his ex ante favorite alternative in every round, while the uninformed type may randomize between the two alternatives. After each "regular" disagreement in the previous voting round in which both individuals vote their ex ante favorite, uninformed types become more convinced that they are facing a common interest state and the other individual is informed, while after a "reverse" disagreement when they voted each other's favorite alternative, the state is revealed to be the conflict state. We show that in equilibrium uninformed types vote their ex ante favorite with a smaller probability after each regular disagreement. If we think of voting against one's favorite alternative as making concessions in a negotiation process, then the result is that uninformed types make increasingly large concessions to their opponents. Within a finite number of rounds, either the mutually preferred alternative is agreed upon, or the negotiation breaks down because the state is revealed to be the conflict state.

Section 4 considers welfare properties of the symmetric equilibrium. We show that the expected payoffs of the informed types and the uninformed types are both decreasing in the degree of conflict. In equilibrium information aggregation is perfectly achieved in the sense that the mutually preferred alternative is chosen whenever the state is a common interest state, but the price for this achievement is the delay before the decision is made. We show that the expected length of delay is an increasing function of the initial degree of conflict. A decrease in the delay cost causes the uninformed types to be less willing to make concessions, but increases their equilibrium payoff. Nonetheless, the expected payoff of the uninformed types is no greater than what they would obtain by an immediate coin flip for any delay cost. In contrast, the informed types can do better in the symmetric equilibrium than an immediate coin flip if the delay cost is not too large, and if the degree of conflict is not too high. When the delay cost converges to zero, the welfare gains for the informed types converge to strictly positive limits for moderate degrees of conflict. As a result, the ex ante equilibrium payoff of each individual in the repeated voting game is greater than what they would expect from an immediate coin flip when the degree of conflict is moderate. Even though the delay cost between two rounds of voting is arbitrarily small, the total expected payoff loss from delay is bounded away from zero in equilibrium. The constructive role played by delay in improving welfare and the quality of information aggregation is discussed further in section 5.

The problem of disagreement that we study in the repeated voting model resembles but is not identical to a pure bargaining problem. The two decision makers in our model have private information about which is the appropriate alternative to adopt. If they could perfectly aggregate their private information, there are some states of the world in which they still would disagree because of divergent preferences, but there are also some states of the world in which they would agree. Therefore the role of delay described in this paper is different from that in a pure bargaining model (Stahl 1972; Rubinstein 1982). In the Stahl-Rubinstein bargaining model, the trade off between getting a bigger share of the pie but at a later date helps pin down a unique solution to the bargaining problem which is plagued by multiple equilibria in a one-shot model, even though delay does not occur in equilibrium. There are numerous extensions to the Stahl-Rubinstein model that can generate delay as part of the equilibrium outcome. One strand of this literature relies on asymmetric information about the size of the pie that is being divided.[1] In a model of strikes, for example, a firm knows its own profitability but the firm's unionized workforce does not. Strike or delay is a signaling device in the sense that the willingness to endure a longer work stoppage can credibly signal the firm's low profitability and help it to arrive at a more favorable wage bargain. In this type of signaling models, each agent's gains from trade at a given price depend only on his own private information. In our model, disagreement over the alternatives is not a pure bargaining issue, because individuals in our model would sometimes agree on which is the best alternative had they known the true state. Put differently, voting outcomes in our setup determine the size as well as the division of the pie. We show that delay can play a constructive role in overcoming disagreements that arise from strategic considerations and improving the ex ante welfare of all individuals. Avery and Zemsky (1994) argue that if players are allowed to wait for new information before accepting or rejecting offers, then there is an option value to delay. In

[1] See, for example, Admanti and Perry (1987), Chatterjee and Samuelson (1987), Cho (1990), Cramton (1992), and Kennan and Wilson (1993). There are also bargaining models that generate equilibrium delay through commitment to not accepting offers poorer than past rejected ones (Freshtman and Seidmann 1993; Li 2007), simultaneous offers (Sakovic 1993), multi-lateral negotiations (Cai 2000), and excessive optimism (Yildiz 2004). More closely related to the present paper are recent models of bargaining with interdependent values where delay occurs in equilibrium. See Deneckere and Liang (2006), and Fuchs and Skrzypacz (2008).

our model, no new exogenous information arrives during the voting process. However, the way agents vote provides endogenous information that allow them to update their beliefs and reach better decisions.

Our paper is also related to the literature on debates (Austen-Smith 1990; Austen-Smith and Feddersen 2006; Ottaviani and Sorensen 2001) and voting (Li, Rosen and Suen 2001) in committees. Models of debate typically analyze repeated information transmission as cheap talk, while we emphasize the role of delay cost in multiple rounds of voting.[2] Our setup is the closest to Li, Rosen and Suen (2001). The focus there is on the impossibility of efficient information aggregation. Here, we choose to skirt issues such as quality of private signals and the trade-off between making the two different types of errors. We focus instead on how costly delay can help improve the quality of decisions and welfare.

2. The Model

Two players, called LEFT and RIGHT, have to make a joint choice between two alternatives, l and r. There are three possible states of the world: L, M, and R. The corresponding prior probabilities are denoted π_L, π_M, and π_R, with $\pi_L = \pi_R = \pi$ and $\pi_M = 1 - 2\pi$. The relevant payoffs for the two players are summarized in the following table:

	L	M	R
l	$(1, 1)$	$(1, 1-2\lambda)$	$(1-2\lambda, 1-2\lambda)$
r	$(1-2\lambda, 1-2\lambda)$	$(1-2\lambda, 1)$	$(1, 1)$

[2] Coughlan (2000) investigates conditions under which jurors vote their signals and their information is efficiently aggregated in a model where a mistrial leads to a retrial by a new independent jury. He does not consider the issues of delay that are the focus of the present paper. Farrell (1987) introduces a model in which repeated cheap talk helps players coordinate to arrive at a correlated equilibrium of a battle-of-the-sexes game. There is no issue of efficient information aggregation in that model. More recently, in a dynamic cheap talk model with multiple senders and a receiver who may choose to wait, Eso and Fong (2007) show that when the senders are perfectly informed there is an equilibrium with full revelation with no delay. When the senders are imperfectly informed, Eso and Fong establish conditions under which there exist equilibria converging to full revelation with no delay as the noises in the senders' signals disappear.

In each cell of this table, the first entry is the payoff to LEFT and the second is the payoff to RIGHT. We normalize the payoff from making the preferred decision to 1 and let the payoff from making the less preferred decision be $1 - 2\lambda$. The parameter $\lambda > 0$ is the loss from making the wrong decision. In state L both players prefer l to r, and in state R both prefer r to l. The two players' preferences are different when the state is M: LEFT prefers l while RIGHT prefers r. In this model there are elements of both common interest and conflict between these two players. Note that LEFT ex ante favors l, while RIGHT's ex ante favorite alternative is r.

The information structure is such that LEFT is able to distinguish whether the state is L or not, while RIGHT is able to distinguish whether the state is R or not. Such information is private and unverifiable. When LEFT knows that the state is L, or when RIGHT knows that the state is R, we say they are "informed;" otherwise, we say they are "uninformed."[3] Without information aggregation, the preference between l and r of an uninformed LEFT depends on the relative likelihood of state M versus state R. Let γ denote his belief that the state is M, given by

$$\gamma = \frac{\pi_M}{\pi_R + \pi_M} = \frac{1 - 2\pi}{1 - \pi}.$$

If LEFT could dictate the outcome, he strictly prefers l to r if and only if

$$\gamma + (1 - \gamma)(1 - 2\lambda) > \gamma(1 - 2\lambda) + (1 - \gamma),$$

or $\gamma > \frac{1}{2}$. Symmetrically, an uninformed RIGHT strictly prefers r to l if and only if $\gamma > \frac{1}{2}$. We note that γ can be interpreted as the ex ante degree of conflict. When γ is high, an uninformed player perceives that his opponent is likely to have different preferences regarding the correct decision to be chosen.

There is a potentially infinite number of rounds. In each round, LEFT and RIGHT vote simultaneously for either l or r. In any round if the votes agree, the agreed alternative

[3] It is not essential for our paper that the informed types are perfectly sure that the state is a common-interest state. The logic of our model remains the same as long as an informed and an uninformed type favor different alternatives on the basis on their private information only, but would recognize a mutually preferred alternative when information is shared. For example, suppose that each player observes a binary signal for or against his ex ante favorite alternative, and prefers his ex ante favorite if and only if there is at least one signal for it. Then, a player who receives a private signal for his ex ante favorite would be similar to an "informed" type in our setup, while a player who receives a signal against his ex ante favorite would be "uninformed."

is implemented immediately and the game ends. If the two votes disagree, each player incurs a delay cost $\delta > 0$ and moves to the next voting round. The cost of delay is modeled as as an additive fixed cost in this paper. Such cost may reflect the time and expenses of setting up a second round of meeting and negotiations. An alternative way to model delay cost is to apply a multiplicative discount factor to the payoffs if the decision is implemented in the second round. In this case, delaying a preferred decision is more costly than delaying an inferior decision. Consequently the analysis of the discounting case is slightly more cumbersome than the fixed cost case. We therefore adopt the more transparent assumption of fixed delay cost.[4] The basic insights of this paper do not depend on which of these two assumptions is used.

As a useful welfare benchmark for our repeated voting game, let us consider what happens if the decision must be made without delay. Imagine a game in which each player votes l or r simultaneously, with the agreed alternative implemented immediately and any disagreement leading to an immediate fair coin toss between l and r and a payoff of $1 - \lambda$ to each player. It is a dominant strategy for an informed player to vote for his ex ante favorite alternative. For uninformed LEFT or RIGHT, the optimal strategy depends on the degree of conflict, but not on the probability that the other player votes for his ex ante favorite alternative. Let $x \in [0, 1]$ be the probability that the uninformed LEFT votes for l, a measure of how "tough" he is playing. The expected payoff to the uninformed RIGHT from voting r is

$$\gamma(x(1-\lambda) + (1-x)) + (1-\gamma)(1-\lambda);$$

and his expected payoff from voting l is

$$\gamma(x(1-2\lambda) + (1-x)(1-\lambda)) + (1-\gamma).$$

It follows that if $\gamma < \frac{1}{2}$, then the dominant strategy for the uninformed players is to vote against their favorite decisions. In states L and R, such equilibrium voting strategies lead to the mutually preferred alternative being chosen, while in state M, the decision is

[4] The elapsed time between successive voting rounds can be quite short relative to the time for the actual implementation of a decision. In this context, modeling delay as a fixed cost may be more realistic than modeling it as a loss from impatience.

determined by flipping a coin, which is again Pareto efficient. In contrast, if $\gamma > \frac{1}{2}$, then it is a dominant strategy for each uninformed player to vote for his ex ante favorite. The equilibrium outcome is that the two players disagree in every state, and the decision is always determined by flipping a coin, with a payoff of $1 - \lambda$.

The result that information aggregation is impossible for $\gamma > \frac{1}{2}$ is a robust feature of the particular configuration of information structure and preferences. Indeed, the configuration is intentionally chosen to yield a stronger result that there is no incentive compatible outcome that Pareto dominates a coin toss when $\gamma > \frac{1}{2}$. To see this, we apply the revelation principle and consider any direct mechanism that satisfies the incentive compatibility constraints for truthful reporting of private information. Since in a truth-telling equilibrium the true state can be recovered from the reports submitted by the two players, we can write q_R, q_M, and q_L as the probabilities of implementing alternative r when the true states are R, M, and L, respectively. Finally, let $\tilde{q}$ be the probability of implementing r when the reports are inconsistent, that is, when both report that they are informed. The incentive constraints for, respectively, the informed Right, the informed Left, the uninformed Right and the uninformed Left, can be written as:

$$\begin{aligned}
&q_R + (1 - q_R)(1 - 2\lambda) \geq q_M + (1 - q_M)(1 - 2\lambda),\\
&q_L(1 - 2\lambda) + (1 - q_L) \geq q_M(1 - 2\lambda) + (1 - q_M),\\
&\gamma(q_M + (1 - q_M)(1 - 2\lambda)) + (1 - \gamma)(q_L(1 - 2\lambda) + (1 - q_L))\\
&\qquad \geq \gamma(q_R + (1 - q_R)(1 - 2\lambda)) + (1 - \gamma)(\tilde{q}(1 - 2\lambda) + (1 - \tilde{q})),\\
&\gamma(q_M(1 - 2\lambda) + (1 - q_M)) + (1 - \gamma)(q_R + (1 - q_R)(1 - 2\lambda))\\
&\qquad \geq \gamma(q_L(1 - 2\lambda) + (1 - q_L)) + (1 - \gamma)(\tilde{q} + (1 - \tilde{q})(1 - 2\lambda)).
\end{aligned}$$

The first two incentive constraints imply that $q_R \geq q_M$ and $q_M \geq q_L$; the last two imply that $(1 - \gamma)(\tilde{q} - q_L) \geq \gamma(q_R - q_M)$ and $(1 - \gamma)(q_R - \tilde{q}) \geq \gamma(q_M - q_L)$, and thus

$$(1 - \gamma)(q_R - q_L) \geq \gamma(q_R - q_L).$$

The above is inconsistent with $\gamma > \frac{1}{2}$ unless $q_R - q_L = 0$. It follows that $q_R = q_M = q_L$ when $\gamma > \frac{1}{2}$ in any incentive compatible outcome.[5] Since the two players are ex ante

[5] This result does not depend on the symmetry assumption that $\pi_L = \pi_R$. No information aggregation

symmetric, it is natural to focus on the outcome of $q_R = q_M = q_L = \frac{1}{2}$, which is equivalent to a fair coin toss. Thus, the no-delay payoff of $1 - \lambda$ is a natural welfare benchmark for comparison with the repeated voting game when $\gamma > \frac{1}{2}$.

The impossibility of information aggregation when $\gamma > \frac{1}{2}$ assumes that at least one of the two decisions must be taken and that there are no transfers. It is easy to see that efficient information aggregation can be achieved regardless of γ if a sufficiently large monetary penalty can be imposed on both players when their reports are inconsistent. In the equilibrium of the repeated voting game analyzed below, costly delay plays a similar role of incentive budget-breaking. Although the theoretical underpinning of the constructive role of delay is familiar, in many realistic environments of collective decision making delay is a more natural mechanism than transfers to improve the quality of information aggregation.[6] Furthermore, in our model incentive budget-breaking and welfare improvements occur even in the limit of the delay cost becoming arbitrarily small. In section 5 we offer further comments on the budget-breaking mechanism of costly delay in our model.

3. Equilibrium Construction and Characterization

A non-terminal history in the game of repeated voting consists of the first move by nature, which determine the permanent type of each player, followed by a sequence of disagreeing votes cast by the two players. An information set for a player of a given type is a collection of all histories that share the same sequence of disagreeing votes and begin with one of nature's move that yields that type. A strategy of a player is a sequence of randomizations

is possible for all π_L and π_R as long as both are less than π_M. In the present model of strategic information aggregation, the signal structure of each player is partitional and binary. This feature is responsible for the result that information aggregation is either ex post efficient, or impossible. In a more general model, ex post inefficiency does not necessarily take the form of impossibility of information aggregation. See Li, Rosen and Suen (2001).

[6] The importance of incentive budget breaking is well-known. See, for example, Holmstrom's (1982) model of moral hazard in teams, and Myerson and Satterthwaite's (1983) model of bilateral trading with asymmetric information. In the present model of strategic information aggregation, if the two players could commit to a mechanism that imposes an arbitrarily large cost of delay when their reports are inconsistent, then efficient information aggregation would be achieved, with no delay in the truth-telling equilibrium. It is also possible to achieve efficient information aggregation through transfers between the two players instead of costly delay: in the obvious quasi-linear extension of the present model, there is a truth-telling equilibrium with efficient information aggregation if each player is required to make a transfer equal to λ to the other player when his ex ante favorite alternative is chosen.

over the two votes for each of his information sets, and a belief system is a sequence of probability measures over the histories contained in each information set. Full equilibrium analysis is complicated, but note that the only unobserved component of a terminal history that affects the payoff to each player is the permanent type of his opponent, or equivalently, whether the state is M or not. We therefore focus on perfect Bayesian equilibria of the repeated voting game which have the property that the vote cast by each player at all information sets in a given round of voting depends only on his belief that the state is M. To simplify further, we restrict to equilibria with two additional properties: in each round of voting on and off the equilibrium path: (i) the informed types always vote for their ex ante favorite alternatives; (ii) for each pair of information sets of the uninformed types that share the same sequence of disagreeing votes, the two types have identical beliefs about the state being M and vote for their ex ante favorite alternative with the same probability.[7]

We will first construct an equilibrium and then argue that it is unique subject to the above restrictions and an additional continuity requirement. Since the game is symmetric and since the uninformed types vote for their ex ante favorites with the same probability on the equilibrium path, they have the same belief about the state being the conflict state after any observed sequence of disagreeing votes. For the equilibrium constructed below, it is sufficient to consider equilibrium play when the uninformed types hold the same beliefs. For each such common belief $\gamma \in [0,1]$ that the uninformed types hold regarding the conflict state M, we denote by $x(\gamma) \in [0,1]$ the equilibrium probability that the uninformed types vote for their ex ante favorite alternative. Let $U(\gamma)$ and $V(\gamma)$ be the equilibrium expected payoffs of the uninformed and informed types respectively.

In each round there are two kinds of disagreement. When Left votes l and Right votes r, we say that there is a "regular disagreement;" when the opposite occurs, we say there is a "reverse disagreement." The updating of beliefs of the uninformed types upon these two kinds of disagreement depends both on the equilibrium strategies and the kind of disagreement. Given the prior belief γ that the state is M, upon a regular disagreement,

[7] In general, there exist equilibria in which the uninformed types adopt different voting strategies, and equilibria in which the informed types vote against their ex ante favorite alternative with a positive probability. The analysis of these kinds of equilibria is outside the focus of the present paper.

the uninformed types revise their belief weakly downward to

$$\gamma' = \frac{\gamma x(\gamma)}{\gamma x(\gamma) + 1 - \gamma} \leq \gamma,$$

unless $\gamma = 1$ and $x(\gamma) = 0$. Upon a reverse disagreement, the uninformed types are sure that the state is M, unless $x(\gamma) = 1$.[8]

To construct an equilibrium, first we identify an equilibrium play when the uninformed players believe that the state is M with probability 1, in which they play mixed strategies with probability $x(1)$ of voting their favorite alternative. It follows from the indifference condition for the uninformed types between l and r that

$$U(1) = x(1)(-\delta + U(1)) + (1 - x(1)) = x(1)(1 - 2\lambda) + (1 - x(1))(-\delta + U(1)).$$

Solving these two equations gives a unique pair of equilibrium values

$$\begin{aligned} U(1) &= 1 - \lambda - \sqrt{\delta^2 + \lambda^2}, \\ x(1) &= \frac{-\delta + \lambda + \sqrt{\delta^2 + \lambda^2}}{2\lambda}. \end{aligned} \tag{1}$$

We note that $x(1) \in \left(\frac{1}{2}, 1\right)$ and $U(1) < 1 - 2\lambda$.

Next, we identify an equilibrium play when $\gamma = 0$. Since the uninformed Right believes that the state is L and his opponent (who is informed) votes l, voting l to obtain the preferred decision is strictly better than voting r. Thus, we have $x(0) = 0$ and $U(0) = 1$. Given this, we claim that it is an equilibrium when γ is positive but sufficiently small for the uninformed types to vote against their ex ante favorite alternative with probability 1. To see this, note that $x(\gamma) = 0$ implies that the updated belief upon a regular disagreement is $\gamma' = 0$. Therefore, the payoff to the uninformed Right from voting r is

$$\gamma + (1 - \gamma)(-\delta + U(0)),$$

and his payoff from voting l is

$$\gamma(-\delta + U(1)) + (1 - \gamma).$$

[8] Bayes' rule does not apply after a regular disagreement for $\gamma = 1$ if $x(1) = 0$, or after a reverse disagreement if $x(\gamma) = 1$ for any γ. In the equilibrium constructed below, neither scenario occurs so the issue of out-of-equilibrium belief specification does not arise.

Voting l is strictly preferred to voting r if and only if

$$\gamma < \frac{\delta}{(1+\delta-U(1))+\delta} \equiv G_1. \tag{2}$$

Therefore, when $\gamma < G_1$, it is an equilibrium for the uninformed types to "concede" by voting against their ex ante favorite alternative. The corresponding equilibrium payoff of the uninformed types takes the linear form of

$$U(\gamma) = 1-(1+\delta-U(1))\gamma. \tag{3}$$

We refer to the interval $[0, G_1]$ as the "compromise region."

For γ just above G_1, we conjecture that the equilibrium $x(\gamma)$ is such that the one-step updated belief γ' falls into the compromise region. We may then try to identify some one-step interval $[G_1, G_2]$, and so on. This conjecture turns out to be correct. That is, there exists an infinite sequence, $G_0 < G_1 < G_2 < \ldots$, with $G_0 = 0$ and $\lim_{k\to\infty} G_k = 1$, such that if $\gamma \in (G_k, G_{k+1}]$ for $k = 1, 2, \ldots$, then $x(\gamma) \in (0,1)$ is such that the updated belief after a regular disagreement satisfies

$$\gamma' = \frac{\gamma x(\gamma)}{\gamma x(\gamma)+1-\gamma} \in (G_{k-1}, G_k].$$

Furthermore, we conjecture that the payoff function for the uninformed types is piecewise linear of the form

$$U(\gamma) = a_k - b_k\gamma \tag{4}$$

for $\gamma \in (G_k, G_{k+1}]$, with $a_0 = 1$ and $b_0 = 1+\delta-U(1)$ from equation (3). Given the conjectures, we construct the sequences of $\{G_k\}$ and $\{(a_k, b_k)\}$ recursively, starting from G_1 and (a_0, b_0).

Fix any $\gamma \in (G_k, G_{k+1}]$ for $k \geq 1$. Assuming that the continuation payoff is given by equation (4), the expected payoff to the uninformed Right from voting r is

$$(\gamma x+1-\gamma)(-\delta+a_{k-1}-b_{k-1}\gamma')+\gamma(1-x) = (\gamma x+1-\gamma)(-\delta+a_{k-1})-\gamma x b_{k-1}+\gamma(1-x).$$

The payoff from voting l is

$$\gamma[x(1-2\lambda)+(1-x)(-\delta+U(1))]+(1-\gamma).$$

The uninformed RIGHT is indifferent between r and l when x is given by

$$x(\gamma) = \frac{\gamma b_0 - (1-\gamma)(1+\delta - a_{k-1})}{\gamma(b_0 + 1 + \delta - a_{k-1} + b_{k-1} - 2\lambda)}. \tag{5}$$

Using Bayes' rule

$$\frac{G_{k+1}x(G_{k+1})}{G_{k+1}x(G_{k+1}) + 1 - G_{k+1}} = G_k,$$

with $x(G_{k+1})$ given in equation (5), we can define G_{k+1} as follows:

$$G_{k+1} = \frac{1 + \delta - a_{k-1} + G_k(b_0 + b_{k-1} - 2\lambda)}{b_0 + 1 + \delta - a_{k-1} + G_k(b_{k-1} - 2\lambda)}. \tag{6}$$

Note that $x(\gamma)$ is increasing in γ from equation (5), implying that the updated belief γ' after a regular disagreement falls in the interval $(G_{k-1}, G_k]$. Finally, substituting equation (5) into the expression for the payoff from voting r, we can verify that $U(\gamma)$ is indeed piece-wise linear of the form given in equation (4), where

$$\begin{aligned} a_k &= 1 - \frac{(1+\delta - a_{k-1})(b_0 - 2\lambda)}{b_0 + 1 + \delta - a_{k-1} + b_{k-1} - 2\lambda}, \\ b_k &= 2\lambda + \frac{(b_0 - 2\lambda)(b_{k-1} - 2\lambda)}{b_0 + 1 + \delta - a_{k-1} + b_{k-1} - 2\lambda}. \end{aligned} \tag{7}$$

The above is a pair of difference equations for the sequence $\{(a_k, b_k)\}$. We have the following preliminary results regarding the sequences $\{G_k\}$ and $\{(a_k, b_k)\}$. The proof is in the appendix.

LEMMA 1. *(i)* $a_k \leq 1$ *and* $b_k > 2\lambda$ *for all* k*; (ii) both* a_k *and* b_k *are decreasing in* k*; (iii)* $\lim_{k\to\infty} a_k$ *exists and is given by* $a_\infty = 1 + \lambda - \sqrt{\delta^2 + \lambda^2}$*, and* $\lim_{k\to\infty} b_k$ *exists and is* $b_\infty = 2\lambda$*; (iv)* $0 < G_k < G_{k+1} < 1$ *for all* $k \geq 1$*; and (v)* $\lim_{k\to\infty} G_k = 1$.

The above piece-wise construction of $x(\gamma)$ and $U(\gamma)$ ensures that the strategy of the uninformed types is consistent with equilibrium. It remains to verify that the informed types have no incentive to deviate by voting against their ex ante favorite alternatives. This is established below by showing that given the equilibrium strategy of the uninformed types, the informed types have stronger incentives than the uninformed types to vote for their ex ante favorite alternative. We can now present the following equilibrium existence result.

Proposition 1. *There exists an equilibrium in which the strategy of the uninformed types is given by $x(\gamma)$ and their payoff is given by $U(\gamma)$.*

Proof. First, for $\gamma = 1$, since his opponent is choosing l with probability $x(1)$, the informed Right is indifferent between voting r and voting l, and his equilibrium payoff is $V(1) = U(1)$.

Next, for $\gamma \in [G_0, G_1]$, since his opponent is choosing $x(\gamma) = 0$, the payoff for the informed Right from voting r is 1, while his payoff from voting l is $-\delta + V(1) < 1$, implying $V(\gamma) = 1 \geq U(\gamma)$, with equality only if $\gamma = 0$.

Finally, for $\gamma \in (G_1, 1)$, we first establish by induction that $V(\gamma) > U(\gamma)$ for all $\gamma < 1$, as follows. Consider any $\gamma \in [G_k, G_{k+1}]$ and $k \geq 1$, with $\gamma' = \gamma x(\gamma)/(\gamma x(\gamma) + 1 - \gamma) \in (G_{k-1}, G_k]$. We obtain

$$\begin{aligned} V(\gamma) &> (\gamma x(\gamma) + 1 - \gamma)(-\delta + V(\gamma')) + \gamma(1 - x(\gamma)) \\ &> (\gamma x(\gamma) + 1 - \gamma)(-\delta + U(\gamma')) + \gamma(1 - x(\gamma)) \\ &= U(\gamma), \end{aligned}$$

where the first inequality follows from the fact that $x(\gamma) < \gamma x(\gamma) + 1 - \gamma$, the second inequality follows from the induction hypothesis, and the last equality follows because the uninformed Left is indifferent between l and r for $\gamma \in [G_k, G_{k+1}]$ for $k \geq 1$. Moreover, from the indifferent condition of the uninformed Left, we obtain

$$\gamma \left[x(\gamma)(-\delta + U(\gamma') - 1 + 2\lambda) + (1 - x(\gamma))(1 + \delta - U(1))\right] + (1 - \gamma)(-\delta + U(\gamma') - 1) = 0.$$

Note that the last term is strictly negative, and so the expression in the square bracket is strictly positive. Since $V(1) = U(1)$, and $V(\gamma') > U(\gamma')$, this implies that

$$x(\gamma)(-\delta + V(\gamma') - 1 + 2\lambda) + (1 - x(\gamma))(1 + \delta - V(1)) > 0,$$

or equivalently,

$$x(\gamma)(-\delta + V(\gamma')) + 1 - x(\gamma) > x(\gamma)(1 - 2\lambda) + (1 - x(\gamma))(-\delta + V(1)).$$

The left-hand-side of the above inequality is the equilibrium payoff for the informed Right from voting r. The right-hand-side is the deviation payoff from voting l, because after a reverse disagreement the uninformed LEFT is convinced that the state is M. Thus, the informed RIGHT strictly prefers r to l.

Q.E.D.

The equilibrium represented by equations (5) and (4) is continuous and monotone with respect to the degree of conflict γ. Note that the continuity of $x(\gamma)$ in γ is not required for the construction to be an equilibrium. Nor it is automatic from the construction, because the equilibrium strategy to the left and inclusive of $\gamma = G_k$ is constructed in the interval $(G_{k-1}, G_k]$ while $x(\gamma)$ just to the right of G_k is separately constructed in the next step of $[G_k, G_{k+1})$. The continuity and monotonicity of $x(\gamma)$ is indirectly established below by showing that the payoff function U is continuous.

PROPOSITION 2. *The equilibrium strategy $x(\gamma)$ is continuous and increasing for all $\gamma \in [0, 1]$.*

PROOF. We first establish the continuity of $U(\gamma)$ for all $\gamma < 1$. For each $k \geq 0$, the function $U(\gamma)$ is trivially continuous at any $\gamma \in (G_k, G_{k+1})$. We show by induction that $U(\gamma)$ is continuous at each G_{k+1}, that is,

$$a_{k+1} - b_{k+1}G_{k+1} = a_k - b_k G_{k+1}.$$

For $k = 0$, we have

$$a_1 - a_0 = -\frac{\delta(b_0 - 2\lambda)}{b_0 + \delta + b_0 - 2\lambda},$$

and

$$b_1 - b_0 = -\frac{(b_0 + \delta)(b_0 - 2\lambda)}{b_0 + \delta + b_0 - 2\lambda}.$$

Therefore,

$$\frac{a_1 - a_0}{b_1 - b_0} = \frac{\delta}{b_0 + \delta} = G_1.$$

Next, denote $w_k = 1 + \delta - a_k + b_k - 2\lambda$. We have

$$a_{k+1} - a_k = \frac{b_0 - 2\lambda}{w_k w_{k-1}}((a_k - a_{k-1})(b_0 + b_{k-1} - 2\lambda) + (1 + \delta - a_{k-1})(b_k - b_{k-1})),$$

and

$$b_{k+1} - b_k = \frac{b_0 - 2\lambda}{w_k w_{k-1}}((a_k - a_{k-1})(b_{k-1} - 2\lambda) + (b_0 + 1 + \delta - a_{k-1})(b_k - b_{k-1}).$$

Therefore,

$$\frac{a_{k+1} - a_k}{b_{k+1} - b_k} = \frac{1 + \delta - a_{k-1} + G_k(b_0 + b_{k-1} - 2\lambda)}{b_0 + 1 + \delta - a_{k-1} + G_k(b_{k-1} - 2\lambda)} = G_{k+1},$$

where the first equality follows from the induction hypothesis and the second equality follows from the law of motion of the sequence $\{G_k\}$ (equation 6). To show that $U(\gamma)$ is continuous at $\gamma = 1$, we note that $\lim_{k\to\infty} G_k = 1$ and $a_\infty - b_\infty = U(1)$. The continuity and monotonicity of $x(\gamma)$ follows immediately.

Q.E.D.

If we impose a continuity restriction on the equilibrium strategy, then we can establish that the equilibrium in Proposition 1 is unique. In other words, there is no equilibrium in which the strategy of uninformed types is represented by a continuous function that is different from $x(\gamma)$.

Proposition 3. *In any equilibrium if the strategy of the uninformed types is continuous in γ, then the strategy is given by $x(\gamma)$.*

Proof. Suppose that there is a continuous function $y(\gamma)$ defined on $\gamma \in [0, 1]$ such that it is an equilibrium that the uninformed types with belief γ vote for their ex ante favorite alternative with probability $y(\gamma)$.

First, in any equilibrium we must have $y(1) \in (0, 1)$, and thus $y(1) = x(1)$ as given by equation (1). This is because if the uninformed Left votes l with probability 1, then for the uninformed Right the outcome from voting r would be delay forever, which is strictly worse than conceding by voting l; and if the uninformed Left votes r with probability 1, then for the uninformed Right voting r would bring an immediate agreement and strictly dominate voting l. Thus, in any equilibrium the uninformed types must be indifferent between l and r, implying that $y(1) = x(1)$.

Next, we argue that in any equilibrium $y(\gamma) = 0$ for any $\gamma \in [0, G_1]$. This follows because regardless of the continuation plays, when γ is sufficiently small, the payoff to the uninformed RIGHT from voting r is strictly lower than the payoff from voting l regardless of the strategy of the uninformed LEFT. Further, for $\gamma < G_1$, if the uninformed LEFT votes for l with probability 1, then the uninformed RIGHT strictly prefers l to r, implying that in any equilibrium $y(\gamma)$ is bounded away from 1. Then, if $y(\gamma) > 0$ for some $\gamma \in [0, G_1]$, there would be some $\tilde{\gamma} \in (0, \gamma]$ such that $y(\tilde{\gamma}) \in (0, 1)$ and $y(\tilde{\gamma}') = 0$ where $\tilde{\gamma}'$ is the updated belief upon a regular disagreement, which makes it impossible to satisfy the indifference condition for the uninformed types between l and r at $\tilde{\gamma}$.

Finally, consider any $\gamma \in (G_k, G_{k+1}]$ for $k \geq 1$. Suppose that we have established the uniqueness of equilibrium for all beliefs lower than G_k. If $y(\gamma)$ is such that the updated belief γ' upon a regular disagreement is lower than G_k, then by assumption there is a unique continuation value $U(\gamma')$ as given by the proposition. Thus, we must have $y(\gamma) = x(\gamma)$, because for each $\gamma' \leq G_k$ and the corresponding $k' \leq k - 1$, only $x(\gamma)$ simultaneously satisfies the equilibrium indifference condition of the uninformed types and Bayes' rule. Suppose there is a subset of $(G_k, G_{k+1}]$ of a positive measure with the property that for each γ in this subset $y(\gamma)$ is such that the updated belief γ' is greater than G_k. Since y is continuous in γ, the infimum of this subset, $\underline{\gamma}$ also has the property that $y(\underline{\gamma})$ is such that the updated belief $\underline{\gamma}'$ is greater than or equal to G_k. Clearly, $y(\underline{\gamma}) < 1$; otherwise, we would have $U(\underline{\gamma}) = U(\underline{\gamma}) - \delta$, which is impossible. It then follows that $G_k \leq \underline{\gamma}' < \underline{\gamma}$, which contradicts the continuity of y at $\underline{\gamma}$ since we have already shown that for any $\gamma \in [G_k, \underline{\gamma})$ the equilibrium play is given by $x(\gamma)$ defined in Proposition 1. Thus, for all $\gamma \in (G_k, G_{k+1}]$, the updated belief γ' after a regular disagreement under the strategy $y(\gamma)$ falls below G_k, implying $y(\gamma) = x(\gamma)$ and completing the induction argument for the uniqueness of the equilibrium strategy $x(\gamma)$.

Q.E.D.

The monotonicity result of Proposition 2 provides an intuitive description of the equilibrium behavior. In each round of voting, there are four possible outcomes: an immediate agreement on r, an immediate agreement on l, a regular disagreement, or a reverse disagreement. We interpret a reverse disagreement as a breakdown of the negotiation process. Once a reverse disagreement occurs, it is revealed that what is a good decision for one player is necessarily an inferior decision for the other player. The continuation game is a version of war of attrition game, where each uninformed player chooses the stationary strategy represented by $x(1)$ until they reach a decision.[9] Upon a regular disagreement, on the other hand, the uninformed player becomes more convinced that he is playing against an informed type. The informed type continues to vote for his favorite alternative, but the uninformed player will "soften" his position as $x(\gamma') < x(\gamma)$. In a sense, the negotiation between the two players is making progress, because the probability of choosing the mutually preferred alternative rises if the state is L or R. Moreover, for any γ bounded away from 1, it only takes a finite number of rounds of regular disagreement before the uninformed player yields to his opponent completely by switching to voting against his ex ante favorite (i.e., $x(\gamma) = 0$), provided there is no breakdown of negotiation before that. Once the game reaches this compromise region, there is either an agreement on the mutually preferred alternative, or the negotiation breaks down and the two uninformed players engage in a war of attrition by adopting the strategy of voting for his ex ante favorite alternative with probability $x(1)$.

Although the equilibrium play is monotone in the sense that the uninformed types make gradually increasing concessions after each regular disagreement, it does not follow that on the equilibrium path the negotiation process on average speeds up after each disagreement. That is, the average "hazard rate," defined as the probability that the negotiation process will end in the next round conditional on it having not ended after T rounds, is not necessarily increasing in T. Starting from any initial degree of conflict, after T rounds of disagreement there are in general three possible scenarios, each with a different conditional hazard rate: first, the negotiation process may have already broken down and the state is revealed to be M, in which case the conditional hazard rate is a constant given by $2x(1)(1-x(1))$; second, the state is again M, but if T is smaller than the number of rounds of regular disagreements before the uninformed types concede (i.e., if the initial degree of conflict is above G_T), then it is possible that all previous disagreements

[9] In our version of the war of attrition game, "stopping" corresponds to voting against one's ex ante favorite alternative. Unlike the standard war of attrition game, when both players vote against their favorite, we have a reverse disagreement and the game continues.

are regular and the conditional hazard rate is given by $2x(\gamma)(1-x(\gamma))$, where γ is the belief of the uninformed types after T rounds of regular disagreements; third, the state is actually L or R, with a conditional hazard rate $1-x(\gamma)$. Since $x(\gamma)$ decreases to 0 in a finite number of rounds as γ continues to decrease after each regular disagreement, after observing a sufficiently long negotiation process, one must rule out the second and the third scenario and thus expect the average hazard rate to stay the same afterward. However, observe that $x(1) > \frac{1}{2}$ from equation (1), and the conditional hazard rate in the second scenario may either increase or decrease as γ decreases, depending on whether or not $x(\gamma)$ is greater than $\frac{1}{2}$. Moreover, as the negotiation continues, the probabilities of these three possible scenarios also vary, changing the relative weights attached to the three conditional hazard rates. Thus the average hazard rate can have complicated dynamics before it becomes a constant, even though the conditional hazard rate in the third scenario is monotonically increasing.

4. Equilibrium Welfare and Comparative Statics

To analyze the welfare properties of the equilibrium constructed in section 3, we first derive the payoff function of the informed types. Recall that for any belief γ of the uninformed, $V(\gamma)$ is the equilibrium expected payoff of the informed types. Given the equilibrium strategy $x(\gamma)$ of the uninformed, $V(\gamma)$ satisfies the following recursive formula:

$$V(\gamma) = x(\gamma)(V(\gamma') - \delta) + 1 - x(\gamma), \tag{8}$$

where $\gamma' = \gamma x(\gamma)/(\gamma x(\gamma) + 1 - \gamma)$ is the updated belief of the uninformed after a regular disagreement. Using the characterization of $x(\gamma)$ in Proposition 1, we have the following result about V.

LEMMA 2. *There exists a sequence $\{(c_k, d_k)\}$, with $c_k \leq 1$ decreasing and $d_k \geq 0$ increasing for all k, and $\lim_{k\to\infty} c_k = U(1)$, such that*

$$V(\gamma) = c_k + d_k \frac{1-\gamma}{\gamma} \tag{9}$$

for any $\gamma \in (G_k, G_{k+1}]$, $k \geq 1$.

The proof of the lemma is in the appendix, where we establish the following explicit system of difference equations for $\{(c_k, d_k)\}$:

$$c_k = 1 - \frac{b_0(1+\delta-c_{k-1})}{b_0+1+\delta-a_{k-1}+b_{k-1}-2\lambda},$$
$$d_k = d_{k-1} + \frac{(1+\delta-a_{k-1})(1+\delta-c_{k-1})}{b_0+1+\delta-a_{k-1}+b_{k-1}-2\lambda}.$$

By the proof of Proposition 2, the payoff function $V(\gamma)$ is continuous for all $\gamma \in [0,1]$, as is $U(\gamma)$. However, while $U(\gamma)$ is decreasing and piece-wise linear in γ, and is convex because b_k decreases with k, the payoff function $V(\gamma)$ is piece-wise convex but since d_k is increasing in k, at each kink G_k, the left derivative is smaller than the right derivative.[10] Further, from the proof of Proposition 1 we know that the two payoff functions satisfy $V(\gamma) \geq U(\gamma)$ for all $\gamma \in [0,1]$, with equality only at $\gamma = 0$ and $\gamma = 1$.[11]

Given any initial degree of conflict γ, the ex ante equilibrium payoff of each player, $W(\gamma)$, is given by

$$W(\gamma) = \frac{1}{2-\gamma}U(\gamma) + \frac{1-\gamma}{2-\gamma}V(\gamma), \tag{10}$$

where the equilibrium payoff functions for the uninformed and informed types are weighted by the prior probabilities of the types. By Proposition 1, $U(\gamma)$ is decreasing in γ; by Lemma 2, $V(\gamma)$ is also decreasing in γ. Further, from the proof of Proposition 1, $V(\gamma) \geq U(\gamma)$ for all γ. Since the weight on the informed types' expected payoff in (10) decreases in γ, we have the following result.

PROPOSITION 4. *The equilibrium ex ante expected payoff $W(\gamma)$ is a decreasing function of γ.*

In either common interest state, L or R, the mutually preferred alternative is always chosen in equilibrium, while in the conflict state M there is no mutually preferred alternative and in equilibrium l and r are chosen with equal probability. Thus, the payoffs of uninformed and informed types can be rewritten as the difference between the "first best"

[10] From the proof of Proposition 8 below, we can show that $V(\gamma)$ is concave in the limit as δ goes to 0.

[11] While the limit of d_k as k goes to infinity does not exist, the product $d_k(1-\gamma)/\gamma$ converges to 0 because γ goes to 1 as k grows arbitrarily large, which is why $V(1) = U(1)$.

expected payoff and the expected loss from delay. More precisely, let $I(\gamma)$ be the expected payoff loss from delay for the informed, and correspondingly $J(\gamma)$ for the uninformed types. We have:[12]

$$\begin{aligned} I(\gamma) &= 1 - V(\gamma); \\ J(\gamma) &= \gamma(1-\lambda) + (1-\gamma) - U(\gamma). \end{aligned} \tag{11}$$

Let $K(\gamma)$ be the equilibrium ex ante payoff loss from delay, given by

$$K(\gamma) = \frac{1}{2-\gamma} J(\gamma) + \frac{1-\gamma}{2-\gamma} I(\gamma). \tag{12}$$

Since $V(\gamma)$ is decreasing in γ by Proposition 3, $I(\gamma)$ is increasing in γ. Further, since $b_k > 2\lambda$ from Lemma 1, $U(\gamma)$ decreases in γ at a faster rate than the first best expected payoff for the uninformed, implying that $J(\gamma)$ is also increasing in γ. However, the weights on the informed and uninformed types change with the degree of conflict γ. Evaluating the overall effect of γ on $K(\gamma)$, we have the following result.

Proposition 5. *The equilibrium ex ante expected payoff loss from delay $K(\gamma)$ is an increasing function of γ.*

Proof. Fix any $\gamma \in (G_k, G_{k-1})$. Using (11) and (12), and taking derivatives of $K(\gamma)$ with respect to γ, we have

$$\frac{\mathrm{d}K(\gamma)}{\mathrm{d}\gamma} = \frac{1}{(2-\gamma)^2}(V(\gamma) - U(\gamma) - \lambda\gamma) + \frac{1}{2-\gamma}\left(-\lambda - \frac{\mathrm{d}U(\gamma)}{\mathrm{d}\gamma}\right) - \frac{1-\gamma}{2-\gamma}\frac{\mathrm{d}V(\gamma)}{\mathrm{d}\gamma}.$$

Using equations (4) and (9), we observe that the sign of $\mathrm{d}K(\gamma)/\mathrm{d}\gamma$ is the same as

$$2(b_k - \lambda) + \frac{2(1-\gamma)}{\gamma^2} d_k + c_k - a_k.$$

Since $V(\gamma) > U(\gamma)$ for all $\gamma \in (0,1)$ from the proof of Proposition 1, equations (4) and (9) imply

$$c_k - a_k > -b_k\gamma - d_k\frac{1-\gamma}{\gamma}.$$

[12] The loss functions I and J have explicit expressions using the equilibrium strategy $x(\gamma)$. For example, for any $\gamma \in (G_k, G_{k+1}]$, $k = 0, 1, \ldots$, the expected loss $I(\gamma)$ for the informed types can be written as $\sum_{k'=0}^{k} \delta(1 - x(\gamma_{k-k'}))\Pi_{m=0}^{k'-1} x(\gamma_{k-m})$, where $\gamma_k = \gamma$, and for each $k' = k-1, \ldots, 1$, $\gamma_{k'-1}$ is the updated belief for the uninformed types of $\gamma_{k'}$ after each regular disagreement. However, it is easier to characterize the loss functions indirectly through the equations below.

Since $b_k > 2\lambda$ by Lemma 1 and $d_k > 0$ by Lemma 2, it is immediate from the above inequality that the sign of $\mathrm{d}K(\gamma)/\mathrm{d}\gamma$ is positive.

Q.E.D.

The equilibrium welfare of the informed and uninformed types also critically depend on the delay cost δ. Before we present the main comparative statics results, we need the following lemma regarding the effects of changes in δ on the coefficients in $U(\gamma)$ and $x(\gamma)$; the proof is in the appendix.

LEMMA 3. *As δ decreases, for any k: (i) $(1+\delta-a_k+b_k-2\lambda)/b_0$ decreases; (ii) a_k increases; (iii) b_k decreases; (iv) $(1+\delta-a_k)/b_0$ decreases; and (v) $(1+\delta-a_k)/(b_0+1+\delta-a_k+b_k-2\lambda)$ decreases.*

In the following proposition, we establish that as δ decreases, the compromise region becomes smaller; further, the equilibrium voting by the uninformed types becomes tougher for any degree of conflict. Correspondingly, for any initial degree of conflict, as δ decreases, it takes a greater number of regular disagreements to reach the compromise region. However, in spite of the tougher positions taken by the uninformed types, their equilibrium expected payoffs increase unambiguously because the direct impact of a lower cost of delay per-round dominates.

PROPOSITION 6. *As δ decreases, G_k strictly decreases for each $k \geq 1$, $U(\gamma)$ strictly increases for all $\gamma \in (0,1]$, and $x(\gamma)$ strictly increases for all $\gamma \in (G_1, 1]$.*

PROOF. Fix any $k \geq 1$. Let $w_k = 1+\delta-a_k+b_k-2\lambda$. For the effects on G_k, rewrite the difference equation for G_k as:

$$\frac{G_{k+1}}{1-G_{k+1}} = \frac{1+\delta-a_{k-1}}{b_0} + \frac{b_0+w_{k-1}}{b_0}\frac{G_k}{1-G_k}.$$

From part (i) and part (iv) of Lemma 3, both w_k/b_0 and $(1+\delta-a_k)/b_0$ are increasing in δ. It is also clear that G_{k+1} is increasing in G_k. Finally, note that $G_1 = \delta/(b_0+\delta)$ is increasing in δ. An induction argument then establishes that G_k is strictly increasing in δ for each $k \geq 1$.

Next, for the effects on $U(\gamma)$, let $\tilde{d} > d$. Denote the sequence of threshold values of γ corresponding to $\tilde{d}$ as $\{\tilde{G}_k\}$, and denote the corresponding sequence of coefficients of the payoff function U as $\{(\tilde{a}_k, \tilde{b}_k)\}$. Suppose that $\gamma \in (G_k, G_{k+1}]$ while $\gamma \in (\tilde{G}_{\tilde{k}}, \tilde{G}_{\tilde{k}+1}]$. Then

$$\tilde{a}_{\tilde{k}} - \tilde{b}_{\tilde{k}}\gamma < a_{\tilde{k}} - b_{\tilde{k}}\gamma \leq a_k - b_k\gamma,$$

where the first inequality follows from part (ii) and part (iii) of Lemma 3, and the second inequality follows from the convexity of $U(\gamma)$. Thus, $U(\gamma)$ is decreasing in δ.

Finally, for the effects on $x(\gamma)$, fix any γ and let

$$x_k(\gamma) = \frac{b_0}{b_0 + w_{k-1}} - \frac{1-\gamma}{\gamma}\frac{1+\delta-a_{k-1}}{b_0+w_{k-1}}.$$

Since $x_k(G_{k+1}) = x_{k+1}(G_{k+1})$, and since

$$\frac{\partial x_k(\gamma)}{\partial \gamma} = \frac{1}{\gamma^2}\frac{1+\delta-a_{k-1}}{b_0+w_{k-1}} < \frac{1}{\gamma^2}\frac{1+\delta-a_k}{b_0+w_k} = \frac{\partial x_{k+1}(\gamma)}{\partial \gamma}$$

by part (v) of Lemma 3, we obtain $x_k(\gamma) \geq x_{k+1}(\gamma)$ for all $\gamma \leq G_{k+1}$. Iterating the argument establishes that $x_k(\gamma) \geq x_{\tilde{k}}(\gamma)$ for all $\gamma \leq G_{k+1}$ and all $\tilde{k} \geq k$. The same argument also proves that $x_k(\gamma) \geq x_{\tilde{k}}(\gamma)$ for all $\gamma \geq G_k$ and all $\tilde{k} \leq k$. Combining these two results, we have $x_k(\gamma) \geq x_{\tilde{k}}(\gamma)$ for all $\tilde{k}$ if $\gamma \in (G_k, G_{k+1}]$. Now, for any $\tilde{\delta} > \delta$, denote the corresponding equilibrium strategy as $\tilde{x}(\gamma)$, and define $\tilde{x}_k(\gamma)$ analogously. Then, for any $\gamma \in (G_k, G_{k+1}]$,

$$x(\gamma) = x_k(\gamma) \geq x_{\tilde{k}}(\gamma) > \tilde{x}_{\tilde{k}}(\gamma) = \tilde{x}(\gamma),$$

where the first inequality follows because $\gamma \in (G_k, G_{k+1}]$, and the second inequality comes from part (i) and part (v) of Lemma 3. Thus, $x(\gamma)$ is decreasing in δ for all γ.

Q.E.D.

For the informed types, the effect of a decrease in the delay cost δ turns out to be generally ambiguous. The uninformed types toughen their positions, which means longer delays before the mutually preferred alternative is chosen, but each round of disagreement is less costly.[13]

[13] Numerical examples of the non-monotonicity of $V(\gamma)$ in δ can be constructed; details are available upon request.

We now make the welfare comparison between the equilibrium payoffs of the uninformed and informed types with their corresponding benchmarks when there is no possibility of delay. As discussed in section 2, when the initial degree of conflict γ is below $\frac{1}{2}$, the first best outcome is achieved in the equilibrium of a voting game in which any disagreement leads to an immediate coin toss. The ex ante payoff from such an outcome is $1 - \lambda\gamma$. In contrast, if $\gamma > \frac{1}{2}$, no incentive compatible outcome is welfare superior to the coin toss and the ex ante payoff of $1 - \lambda$. For the uninformed, it is immediate that there is no possibility of welfare gain relative to the no-delay benchmark regardless of the degree of conflict γ or the delay cost δ. This is because $U(\gamma)$ is a decreasing function, with a slope b_k that is strictly larger than 2λ by Lemma 1. On the other hand, welfare gains are possible for the informed types (they are explicitly characterized below for the case of δ going to 0), but not if the degree of conflict is too large or if the delay cost is too great. To see why γ cannot be too large, note that $V(1) = U(1) < 1 - \lambda$, so the continuity of V implies that $V(\gamma)$ is smaller than the benchmark expected payoff of the informed types for γ close to 1. To see why δ cannot be too great, from equations (2) and (6) we can verify that for δ sufficiently great, $G_1 < \frac{1}{2} < G_2$, and then from (5) we can verify that $x\left(\frac{1}{2}\right)$ is bounded away from 0 for sufficiently great δ, implying that $V\left(\frac{1}{2}\right)$ falls below the benchmark payoff of $1 - \lambda$ if the delay cost δ is sufficiently great.

It is generally difficult to characterize the welfare gains of the informed types or the ex ante payoff function $W(\gamma)$. For the rest of this section we focus on the case of arbitrarily small delay cost δ, which turns out to be nicely behaved and yields clear and insightful results. From equation (1), we obtain $\lim_{\delta\to 0} U(1) = 1 - 2\lambda$, implying that $\lim_{\delta\to 0} b_0 = 2\lambda$. It immediately follows from the difference equations (7) for $\{(a_k, b_k)\}$ that $\lim_{\delta\to 0} a_k = 1$ and $\lim_{\delta\to 0} b_k = 2\lambda$ for any k. It is then straightforward to show from (2) that $\lim_{\delta\to 0} G_1 = 0$, and from (6) by induction that $\lim_{\delta\to 0}(G_{k+1} - G_k) = 0$ for any $k \geq 1$. Since $\lim_{k\to\infty} G_k = 1$ for any $\delta > 0$, the number of G_k's in any neighborhood of a fixed $\gamma \in (0, 1)$ grows arbitrarily large as δ becomes small. Although for any $\delta > 0$, the payoff functions U and V have a kink at each G_k, we are able to establish that the limits of U and V are differentiable for all $\gamma \in (0, 1)$, which allows us to characterize the limiting equilibrium behavior. To do so, we need the following preliminary result; the proof is in the appendix.

LEMMA 4. *For any* $\gamma \in (0,1)$, $\lim_{\delta\to 0} \delta x(\gamma)/(1-x(\gamma)) = 2\lambda\gamma$.

From (5) we have $\lim_{\delta\to 0} x(\gamma) = 1$ for any $\gamma \in (0,1)$.[14] Thus, as δ becomes arbitrarily small, the limit of $\delta x(\gamma)/(1-x(\gamma))$ has the interpretation of the total expected payoff loss from delay when the probability of each additional round of delay is $x(\gamma)$. The above result shows that this payoff loss is a linear function of the degree of conflict.

Now we are ready to characterize the limits of the payoff functions U and V as δ goes to 0. Let $U^0(\gamma) = \lim_{\delta\to 0} U(\gamma)$, and $V^0(\gamma) = \lim_{\delta\to 0} V(\gamma)$. These limits are well-defined because the equilibrium given in Proposition 1 is continuous in δ.

PROPOSITION 7. *For each* $\gamma \in (0,1)$, *the limits of the equilibrium payoff functions as* δ *goes to 0 are given by*

$$U^0(\gamma) = 1 - 2\lambda\gamma;$$
$$V^0(\gamma) = 1 - 2\lambda\left(1 + \frac{1-\gamma}{\gamma}\ln(1-\gamma)\right).$$

PROOF. First, we show that $U^0(\gamma)$ is differentiable. Fix any $\gamma \in (0,1)$. For any $\delta > 0$, let $x(\gamma)$ be the equilibrium strategy of the uninformed types, and let γ' be their updated belief after a reverse disagreement. We have:

$$U(\gamma) = \gamma[x(\gamma)(-\delta + U(\gamma')) + 1 - x(\gamma)] + (1-\gamma)(-\delta + U(\gamma')).$$

Using Bayes' rule, from the above we obtain:

$$\frac{U(\gamma) - U(\gamma')}{\gamma - \gamma'} = -\frac{\delta(\gamma x(\gamma) + 1 - \gamma)}{\gamma(1-\gamma')(1-x(\gamma))} + \frac{1-U(\gamma')}{1-\gamma'}.$$

As δ goes to 0, γ' converges to γ because $x(\gamma)$ goes to 1. Since $U^0(\gamma)$ is the limit of $U(\gamma)$ as δ goes to 0, the left-hand-side in the above equation converges to the derivative of $U^0(\gamma)$. By Lemma 3, the right-hand-side has a limit as δ goes to 0. We therefore have the following differential equation:

$$\frac{\mathrm{d}U^0(\gamma)}{\mathrm{d}\gamma} = -\frac{2\lambda}{1-\gamma} + \frac{1-U^0(\gamma)}{1-\gamma}.$$

[14] Also, from (1) we have $\lim_{\delta\to 0} x(1) = 1$. Since $x(0) = 0$ for all δ by construction, $x(\gamma)$ is discontinuous at $\gamma = 0$ in the limit of δ going to 0.

With the initial condition $U^0(0) = 1$, the solution is $U^0(\gamma) = 1 - 2\lambda\gamma$.

Following a similar argument as above, we can use equation (8) to get the following differential equation in $V^0(\gamma)$:

$$\frac{\mathrm{d}V^0(\gamma)}{\mathrm{d}\gamma} = -\frac{2\lambda}{1-\gamma} + \frac{1 - V^0(\gamma)}{\gamma(1-\gamma)}.$$

With the initial condition $\lim_{\gamma \to 0} V^0(\gamma) = 1$, we can verify that the solution is as given in the proposition.

Q.E.D.

With the explicit characterization of the limits of U and V, we can compare them to the benchmark payoffs when there is no delay. Note that $U^0\left(\frac{1}{2}\right) = 1 - \lambda$, implying that the uninformed types get the same benchmark expected payoff in the equilibrium when $\gamma = \frac{1}{2}$. In contrast, $V^0\left(\frac{1}{2}\right) = 1 - 2\lambda(1 - \ln 2)$, which is strictly larger than $1 - \lambda$. Thus, the ex ante equilibrium payoff to each player is strictly greater than the benchmark no-delay payoff for an interval of degrees of conflict above $\frac{1}{2}$. By continuity in the delay cost and in the degree of conflict, welfare gains over the benchmark case of no delay persist for small delay costs and for moderate conflict levels.

We can also analyze the limit behavior of payoff loss from delay. Let $I^0(\gamma)$ and $J^0(\gamma)$ be the limit of the expected payoff loss $I(\gamma)$ and $J(\gamma)$ for the informed types and uninformed types respectively, as δ goes to 0. For any delay cost δ, since $U(1) = V(1)$, from equation (11) we obtain $I(1) > J(1)$, and therefore by continuity the expected payoff loss is greater for the informed types than for the uninformed types when the degree of conflict is sufficiently great. Intuitively, an informed type knows that even though the state is a common interest state, his opponent believes the state is actually a conflict state with very high probability and thus it will take many rounds of regular disagreement for the latter to concede. For small degrees of conflict, however, the opposite comparison between $I(\gamma)$ and $J(\gamma)$ holds. In particular, so long as δ is strictly positive, we have $I(\gamma) = 0 < J(\gamma)$ for any γ in the compromise region of $(0, G_1]$. In this case, an informed type expects his opponent to concede immediately, while an uninformed type believes that the state is the conflict state with a positive probability, in which case there will be a reverse disagreement and delay in the future. Perhaps surprisingly, the following proposition establishes that in the limit of δ going to 0, the informed types always expect a greater loss from delay than the uninformed types.

PROPOSITION 8. *For all γ, the equilibrium expected loss functions satisfy $I^0(\gamma) \geq J^0(\gamma)$ with equality only at $\gamma = 0$.*

PROOF. From equations (11) and the expression for $U^0(\gamma)$, we immediately have $J^0(\gamma) = \lambda\gamma$. Similarly, the expected delay loss for the informed type is $I^0(\gamma) = 2\lambda(1 + ((1 - \gamma)/\gamma)\ln(1-\gamma))$. Using l'Hopital's rule, we get $I^0(1) = 2\lambda$, which is greater than $J^0(1) = \lambda$. For $\gamma < 1$, use the expansion $\ln(1 - \gamma) = -\sum_{k=1}^{\infty} \gamma^k/k$. We can write the difference in delay loss for the two types as

$$I^0(\gamma) - J^0(\gamma) = 2\lambda \sum_{k=2}^{\infty} \left(\frac{\gamma^k}{k} - \frac{\gamma^k}{k+1} \right) \geq 0,$$

with equality only at $\gamma = 0$.

Q.E.D.

As the delay cost δ converges to 0, the compromise region of $(0, G_1]$ disappears. Moreover, for any positive γ, it now takes arbitrarily long for the uninformed types to concede. Thus, even though for any positive δ, the informed types get their first best payoffs without delay in the compromise region and with a short delay when γ is near the compromise region, in the limit of δ going to 0, the expected payoff loss from delay is greater for the informed types for γ arbitrarily close to 0. Further, it can be easily verified from the expression of $I^0(\gamma)$ in the above proof that it is a convex function. When the delay cost δ is arbitrarily small, the payoff loss from delay for the informed types increases at an ever faster rate with the degree of conflict, while the loss increases linearly for the uninformed types. As a result, the ex ante payoff loss from delay also increases at an increasing rate with the degree of conflict. Since the benchmark no-delay payoff is a constant equal to $1 - \lambda$ for all $\gamma > \frac{1}{2}$, welfare gains from repeated voting with costly delay relative to the benchmark exist only for moderate degrees of conflict in the limit of the delay cost going to 0.

5. Discussion

Our repeated voting game with costly delay is cast in an environment where information aggregation is impossible in any incentive compatible outcome without delay. In particular, in the single-round voting game considered in section 2, where the two players vote between l and r with any agreement carried out immediately and disagreement resolved by a fair coin toss without delay, the equilibrium outcome is always a coin toss when the degree of conflict γ is greater than $\frac{1}{2}$. Imagine that this voting game is repeated, as in the present model, but that there is a hard deadline T, such that if disagreement persists after T rounds of voting a coin toss is used to decide between the two alternatives without further delay. In this game, if $\gamma > \frac{1}{2}$, then as the delay cost δ between two rounds of voting converges to 0, the only equilibrium outcome converges to $T - 1$ rounds of regular disagreements followed by a coin toss in the last round.[15] Therefore, there are no welfare gains relative to the single-round voting game in the limit of the delay cost going to 0 so long as there is a finite deadline. One interpretation of this result is that costless straw polls or other forms of cheap talk cannot bring about any improvement in information aggregation or welfare, which is simply another illustration that in the environment of the present model information aggregation is impossible in any incentive compatible outcome without costly delay.

Our result of welfare gains for moderate degrees of conflict in the repeated voting game with arbitrarily small delay cost hinges on the the assumption that there is no deadline. Even though in equilibrium the expected duration of disagreement is finite, and in fact it takes a finite number of rounds of regular disagreement for the uninformed types to concede completely, the assumption of no deadline in equilibrium creates a strictly positive payoff loss from delay as the delay cost goes to 0. This payoff loss reflects the role of incentive budget-breaking played by costly delay, even as the delay cost goes to 0. It explains the apparent discontinuity of the possibility of welfare gains relative to the no-delay benchmark in the length of the deadline at infinity.

[15] In an earlier version of the paper, we show that for any arbitrarily small but positive δ, there is a T such that for γ sufficiently close to $\frac{1}{2}$ in the repeated voting game, welfare gains relative to the no-delay benchmark increase with the length of the deadline up to T. This construction requires T to be arbitrarily large as δ converges to 0.

The budget-breaking role of costly delay is robust to the game form in the repeated voting game. Imagine a repeated voting game with costly delay which differs from our game only in that after a reverse disagreement the game ends with an immediate coin toss. Analysis for this game follows in a parallel fashion as what we have done in our paper. Somewhat surprisingly, in the limit of the delay cost converging to 0, this new game has the same equilibrium outcome as our game. The same is true for any game defined by replacing the equilibrium continuation payoff after a reverse disagreement with any feasible continuation payoff.[16] In a sense, the critical part of incentive budget-breaking has to do with the costly delay that arises after a regular disagreement in which the uninformed types vote their ex ante favorite alternative in hope of persuading each other to switch, rather than the costly delay that happens after a reverse disagreement resulting from each tentatively agreeing with the other side.

Appendix

Proof of Lemma 1

(i) For $k = 0$, we have $a_0 = 1$ and $b_0 = \delta + \lambda + \sqrt{\delta^2 + \lambda^2} > 2\lambda$. Next, if $a_{k-1} \leq 1$ and $b_{k-1} > 2\lambda$, the two fractions that appear in the difference equation (7) are both positive. Hence $a_k \leq 1$ and $b_k > 2\lambda$ by induction.

(ii) For the monotonicity of b_k, we can subtract b_{k-1} from both sides of the second equation in (7) to get:

$$b_k - b_{k-1} = -\frac{1 + \delta - a_{k-1} + b_{k-1}}{b_0 + 1 + \delta - a_k + b_k - 2\lambda}(b_{k-1} - 2\lambda) < 0.$$

To establish the monotonicity of a_k, we use induction. First, it is easy to see that

[16] In the equilibrium constructed in Proposition 1, the continuation payoff after a reverse disagreement is $-\delta + U(1)$. A continuation payoff after a reverse disagreement is feasible if it is smaller than or equal to $1 - \lambda$, which is the expected payoff from a coin flip without delay. The general analysis for all these games follows the same steps as in the present paper. The equilibrium payoff functions U and V are no longer continuous at $\gamma = 1$, but Lemma 4 still holds.

$a_1 < a_0 = 1$. Next, assume that $a_{k-1} < a_{k-2}$. We can write:

$$\begin{aligned}
a_k - a_{k-1} &= \frac{(1+\delta-a_{k-2})(b_0-2\lambda)}{b_0+1+\delta-a_{k-2}+b_{k-2}-2\lambda} - \frac{(1+\delta-a_{k-1})(b_0-2\lambda)}{b_0+1+\delta-a_{k-1}+b_{k-1}-2\lambda} \\
&< \frac{(1+\delta-a_{k-2})(b_0-2\lambda)}{b_0+1+\delta-a_{k-2}+b_{k-2}-2\lambda} - \frac{(1+\delta-a_{k-1})(b_0-2\lambda)}{b_0+1+\delta-a_{k-1}+b_{k-2}-2\lambda} \\
&< \frac{(1+\delta-a_{k-2})(b_0-2\lambda)}{b_0+1+\delta-a_{k-2}+b_{k-1}-2\lambda} - \frac{(1+\delta-a_{k-2})(b_0-2\lambda)}{b_0+1+\delta-a_{k-2}+b_{k-2}-2\lambda} \\
&= 0,
\end{aligned}$$

where the first inequality follows from $b_{k-1} < b_{k-2}$, and the second inequality follows from the induction hypothesis and the fact that the second term is decreasing in a_{k-1}.

(iii) Solving for the steady state version of the difference equation (7), we obtain the steady state values $a_\infty = 1+\lambda-\sqrt{\delta^2+\lambda^2}$ and $b_\infty = 2\lambda$. By the monotonicity of a_k and b_k, these steady state values are also the limit values of the sequence $\{(a_k, b_k)\}$.

(iv) By definition, we have $G_1 \in (0,1)$. Since $a_{k-1} \le 1$ and $b_{k-1} > 2\lambda$, an induction argument establishes that $G_k \in (0,1)$ for all $k \ge 1$. Next, subtracting G_k from both sides of (6), we obtain

$$G_{k+1} - G_k = \frac{(1+\delta-a_{k-1})(1-G_k)+(b_{k-1}-2\lambda)G_k}{b_0+1+\delta-a_{k-1}+(b_{k-1}-2\lambda)G_k} > 0.$$

(v) Since G_k is an increasing and bounded sequence, it has a limit value. By part (iii) established above, the limit is 1.

Proof of Lemma 2

From the proof of Proposition 1, $V(\gamma) = 1$ for $\gamma \in [0, G_1]$. Let $c_0 = 1$ and $d_0 = 0$. We derive difference equations for c_k and d_k by induction. For any $\gamma \in (G_k, G_{k+1}]$, $k \ge 1$, we can write

$$V(\gamma) = x(\gamma)\left(-\delta + c_{k-1} + d_{k-1}\frac{1-\gamma}{\gamma}\frac{1}{x(\gamma)}\right) + 1 - x(\gamma).$$

Using the formula (5) for $x(\gamma)$, we can verify the functional form of V and obtain a pair of difference equations in (c_k, d_k):

$$\begin{aligned}
c_k &= 1 - \frac{b_0(1+\delta-c_{k-1})}{b_0+1+\delta-a_{k-1}+b_{k-1}-2\lambda}, \\
d_k &= d_{k-1} + \frac{(1+\delta-a_{k-1})(1+\delta-c_{k-1})}{b_0+1+\delta-a_{k-1}+b_{k-1}-2\lambda}.
\end{aligned}$$

It is straightforward to show by induction that $c_k \le 1$ and $d_k \ge 0$ for all k. This implies that $\{d_k\}$ is an increasing sequence. Let $w_k = 1+\delta - a_k + b_k - 2\lambda$; that $\{c_k\}$ is a decreasing sequence follows immediately by induction if we establish that w_k is decreasing in k. To prove the latter claim, combine equations (7) to obtain

$$w_k = \delta + \frac{(b_0 - 2\lambda)w_{k-1}}{b_0 + w_{k-1}}. \tag{A.1}$$

The derivative of the right-hand-side with respect to w_{k-1} is positive. So $w_{k-1} < w_{k-2}$ implies $w_k < w_{k-1}$. Now,

$$w_1 - w_0 = \delta - \frac{(1+\delta - a_0 + b_0)w_0}{b_0 + w_0} = -\frac{b_0(b_0 - 2\lambda)}{2b_0 + \delta - 2\lambda} < 0.$$

An induction argument then establishes the claim.

The limit value of c_k as k goes to infinity is easily verified by using the limit values of a_k and b_k given in Lemma 1.

Proof of Lemma 3

(i) Let $v_k = 1+\delta - a_k$, $w_k = v_k + b_k - 2\lambda$, and $u_k = b_0 + w_k$. Recall that $b_0 = 1+\delta - U(1)$, and therefore $\mathrm{d}b_0/\mathrm{d}\delta = 1 + \delta/\sqrt{\delta^2+\lambda^2}$. Also, from the proof of Lemma 2 we know that w_k is decreasing in k.

First, we show that w_k is increasing in δ for each k. Take derivative of equation (A.1) to get

$$\frac{\partial w_k}{\partial \delta} = 1 + \frac{w_{k-1}(w_{k-1}+2\lambda)}{(b_0 + w_{k-1})^2}\frac{\mathrm{d}b_0}{\mathrm{d}\delta} > 0,$$

and

$$\frac{\partial w_k}{\partial w_{k-1}} = \frac{b_0(b_0 - 2\lambda)}{(b_0 + w_{k-1})^2} > 0.$$

Now,

$$\frac{\mathrm{d}w_k}{\mathrm{d}\delta} = \frac{\partial w_k}{\partial \delta} + \frac{\partial w_k}{\partial w_{k-1}}\frac{\mathrm{d}w_{k-1}}{\mathrm{d}\delta}.$$

An induction argument establishes that $\mathrm{d}w_k/\mathrm{d}\delta > 0$ if we can show that $\mathrm{d}w_0/\mathrm{d}\delta > 0$, which is true because $w_0 = \delta + b_0 - 2\lambda$ is increasing in δ.

To establish part (i) of the lemma, we write $f_k = w_k/(b_0 + w_k)$. Use equation (A.1) for w_k to write:

$$f_k = \frac{\delta + f_{k-1}(b_0 - 2\lambda)}{b_0 + \delta + f_{k-1}(b_0 - 2\lambda)}.$$

The partial derivative $\partial f_k/\partial\delta$ has the same sign as

$$b_0 + (2f_{k-1}\lambda - \delta)\frac{\mathrm{d}b_0}{\mathrm{d}\delta}.$$

Since w_k is decreasing in k, we have that f_k is decreasing in k. Therefore, this expression is greater than

$$b_0 + (2f_\infty\lambda - \delta)\frac{\mathrm{d}b_0}{\mathrm{d}\delta},$$

which is positive, where $f_\infty = \frac{1}{2} - \frac{1}{2}\lambda/(1 - \lambda + \delta - U(1))$ is the limit value of f_k as k goes to infinity. It is also easy to see that f_k is increasing in f_{k-1}. The claim then follows if we show $\mathrm{d}f_0/\mathrm{d}\delta > 0$, which we can verify by using the definition of f_0 and taking derivatives with respect to δ.

(ii) We claim that $(b_0 - 2\lambda)/u_k$ is increasing in δ for each k. To prove it, let $t_k = w_k + 2\lambda$. Write the difference equation for w_k in the form:

$$\frac{t_k}{b_0 - 2\lambda + t_k} = \frac{(\delta + 2\lambda)u_{k-1} + (b_0 - 2\lambda)(t_{k-1} - 2\lambda)}{(b_0 + \delta)u_{k-1} + (b_0 - 2\lambda)(t_{k-1} - 2\lambda)}.$$

Let $g_k = (b_0 - 2\lambda)/u_k = 1 - t_k/u_k$. Then the above equation can be transformed into:

$$g_k = \frac{b_0 - 2\lambda}{\delta + 2b_0 - 2\lambda - b_0 g_{k-1}}.$$

It is clear that $\partial g_k/\partial g_{k-1} > 0$. Moreover, $\partial g_k/\partial\delta$ has the same sign as:

$$-(b_0 - 2\lambda) + (\delta + 2\lambda - 2\lambda g_{k-1})\frac{\mathrm{d}b_0}{\mathrm{d}\delta}.$$

Since g_k is increasing in k, the above expression is greater than

$$-(b_0 - 2\lambda) + (\delta + 2\lambda - 2\lambda g_\infty)\frac{\mathrm{d}b_0}{\mathrm{d}\delta} > 0,$$

where $g_\infty = \frac{1}{2} - \frac{1}{2}\lambda/(1 - \lambda + \delta - U(1))$ is the limit value of g_k as k goes to infinity. So an induction argument will establish the monotonicity of g_k with respect to δ if we establish that $\mathrm{d}g_0/\mathrm{d}\delta > 0$, which we can verify by using the definition of g_0 and taking derivatives with respect to δ.

To establish part (ii) of the lemma, we write the difference equation for a_k as:

$$a_k = 1 - g_{k-1}(1 + \delta - a_{k-1}).$$

Thus,

$$\frac{\mathrm{d}a_k}{\mathrm{d}\delta} = -g_{k-1} - (1 + \delta - a_{k-1})\frac{\mathrm{d}g_{k-1}}{\mathrm{d}\delta} + g_{k-1}\frac{\mathrm{d}a_{k-1}}{\mathrm{d}\delta}.$$

Since $\mathrm{d}a_0/\mathrm{d}\delta = 0$, an induction argument establishes that $\mathrm{d}a_k/\mathrm{d}\delta \leq 0$ for each k.

(iii) We write the difference equation for b_k as:

$$b_k = 2\lambda + g_{k-1}(b_{k-1} - 2\lambda),$$

implying that

$$\frac{\mathrm{d}b_k}{\mathrm{d}\delta} = (b_{k-1} - 2\lambda)\frac{\mathrm{d}g_{k-1}}{\mathrm{d}\delta} + g_{k-1}\frac{\mathrm{d}b_{k-1}}{\mathrm{d}\delta}.$$

We have already shown that $\mathrm{d}g_{k-1}/\mathrm{d}\delta > 0$. Moreover, $\mathrm{d}b_0/\mathrm{d}\delta > 0$. So an induction argument shows that $\mathrm{d}b_k/\mathrm{d}\delta > 0$ for each k.

(iv) From part (ii) we have v_k is increasing in δ for each k. Write the difference equation for a_k as:

$$\frac{v_k}{b_0} = \frac{\delta}{b_0} + g_{k-1}\frac{v_{k-1}}{b_0}.$$

Note that $v_0/b_0 = \delta/b_0$ is increasing in δ. Also, g_{k-1} is increasing in δ. So an induction argument establishes the claim.

(v) First, we claim that $v_k/(b_0 - \lambda)$ is increasing in δ for each k. To prove it, write the difference equation for a_k as:

$$\frac{v_k}{b_0 - \lambda} = \frac{\delta}{b_0 - \lambda} + g_{k-1}\frac{v_{k-1}}{b_0 - \lambda}.$$

Note that $v_0/(b_0 - \lambda) = \delta/(b_0 - \lambda)$ is increasing in δ. So an induction argument establishes the claim.

Next, we show that $(b_k - \lambda)/(b_0 - \lambda)$ is decreasing in δ for each k. We can write the difference equation for b_k as:

$$\frac{b_k - \lambda}{b_0 - \lambda} = \frac{\lambda}{b_0 - \lambda} + \frac{(b_0 - 2\lambda)((b_{k-1} - \lambda) - \lambda)}{(b_0 - \lambda) + (1 + \delta - a_{k-1}) + (b_{k-1} - \lambda)}.$$

Define $p_k = (b_k - \lambda)/(b_0 - \lambda)$ and $q_k = (1 + \delta - a_k)/(b_0 - \lambda)$. Then we can write

$$p_k = (1 - g_{k-1}) + \frac{b_0 - 2\lambda}{b_0 - \lambda} \frac{p_{k-1}}{1 + q_{k-1} + p_{k-1}}.$$

Note that

$$\begin{aligned}
\frac{\partial p_k}{\partial p_{k-1}} &= \frac{b_0 - 2\lambda}{b_0 - \lambda} \frac{1 + q_{k-1}}{(1 + q_{k-1} + p_{k-1})^2} > 0, \\
\frac{\partial p_k}{\partial q_{k-1}} &= -\frac{b_0 - 2\lambda}{b_0 - \lambda} \frac{p_{k-1}}{(1 + q_{k-1} + p_{k-1})^2} < 0, \\
\frac{\partial p_k}{\partial g_{k-1}} &= -\frac{\lambda}{b_0 - \lambda} < 0, \\
\frac{\partial p_k}{\partial b_0} &= -\frac{\lambda}{(b_0 - \lambda)^2} \frac{1 + \delta - a_{k-1} + \lambda}{u_{k-1}} < 0.
\end{aligned}$$

Now,

$$\frac{\mathrm{d}p_k}{\mathrm{d}\delta} = \frac{\partial p_k}{\partial b_0}\frac{\mathrm{d}b_0}{\mathrm{d}\delta} + \frac{\partial p_k}{\partial g_{k-1}}\frac{\mathrm{d}g_{k-1}}{\mathrm{d}\delta} + \frac{\partial p_k}{\partial q_{k-1}}\frac{\mathrm{d}q_{k-1}}{\mathrm{d}\delta} + \frac{\partial p_k}{\partial p_{k-1}}\frac{\mathrm{d}p_{k-1}}{\mathrm{d}\delta}.$$

Since $\mathrm{d}b_0/\mathrm{d}\delta > 0$, $\mathrm{d}g_{k-1}/\mathrm{d}\delta > 0$, $\mathrm{d}q_{k-1}/\mathrm{d}\delta > 0$ and $\mathrm{d}p_0/\mathrm{d}\delta = 0$, an induction argument establishes that $\mathrm{d}p_k/\mathrm{d}\delta < 0$ for each k.

To establish the last part of the lemma, we divide both the denominator and numerator of v_k/u_k by $b_0 - \lambda$ to get:

$$\frac{v_k}{u_k} = \frac{q_k}{1 + q_k + p_k}.$$

Since q_k is increasing in δ and p_k is decreasing in δ, the result follows.

Proof of Lemma 4

Fix any $\gamma \in (0, 1)$. For each $\delta > 0$, let $k(\delta)$ be such that $\gamma \in (G_{k(\delta)}, G_{k(\delta)+1}]$. Note that as δ goes to 0, $k(\delta)$ becomes arbitrarily large. From equation (5), we have

$$\frac{\delta x(\gamma)}{1 - x(\gamma)} = \delta \frac{\gamma b_0 - (1 - \gamma)(1 + \delta - a_{k(\delta)-1})}{1 + \delta - a_{k(\delta)-1} + \gamma(b_{k(\delta)-1} - 2\lambda)}.$$

Since $k(\delta)$ goes to infinity as the delay cost δ converges to 0, and since $a_\infty = 1$, we have $\lim_{\delta\to 0} a_{k(\delta)-1} = 1$. Together with $\lim_{\delta\to 0} b_0 = 2\lambda$, we immediately obtain the lemma after we establish that

$$\lim_{\delta\to 0} \frac{1 - a_{k(\delta)-1}}{\delta} = \lim_{\delta\to 0} \frac{b_{k(\delta)-1} - 2\lambda}{\delta} = 0.$$

To verify the first limit, note that by Lemma 1, a_k is decreasing in k, and so

$$\frac{1-a_0}{\delta} < \frac{1-a_{k(\delta)-1}}{\delta} < \frac{1-a_\infty}{\delta}.$$

It is straightforward to verify that $\lim_{\delta\to 0}(1-a_\infty)/\delta = 0$. Since $a_0 = 1$, we have the desired result. For the second limit, note that by Lemma 1, b_k is decreasing in k, and therefore

$$\frac{b_\infty - 2\lambda}{\delta} < \frac{b_{k(\delta)-1} - 2\lambda}{\delta} < \frac{b_1 - 2\lambda}{\delta}.$$

Using the difference equations (7), and $a_0 = 1$ and $b_0 = 1 + \delta - U(1)$, we can easily verify that $\lim_{\delta\to 0}(b_1 - 2\lambda)/\delta = 0$. Since $b_\infty = 2\lambda$, we immediately obtain the desired result for the second limit.

References

Admanti, Anat and Motty Perry, "Strategic Delay in Bargaining," *Review of Economic Studies* 54, 1987, pp. 342–364.

Austen-Smith, David, "Information Transmission in Debate," *American Journal of Political Science* 34(1), 1990, pp. 124–152.

Austen-Smith, David and Timothy J. Feddersen, "Deliberation, Preference Uncertainty, and Voting Rules," *American Political Science Review* 100(2), May 2006, pp. 209–217.

Avery, Christopher and Peter Zemsky, "Option Values and Bargaining Delays," *Games and Economic Behavior* 7, 1994, pp. 139–153.

Cai, Hong-bin, "Delay in Multilateral Bargaining under Complete Information," *Journal of Economic Theory* 93, pp. 260-276.

Chatterjee, Kalyan and Larry Samuelson, "Bargaining with Two-sided Incomplete Information: An Infinite Horizon Model with Alternating Offers," *Review of Economic Studies* 54(2), April 1987, pp. 175–192.

Cho, In-Koo, "Uncertainty and Delay in Bargaining," *Review of Economic Studies* 57(4), October 1990, pp. 575–595.

Cho, In-Koo and David Kreps, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics* 102, pp. 179–221.

Coughlan, Peter J., "In Defense of Unanimous Jury Verdicts: Mistrials, Communication, and Strategic Voting," *American Political Science Review* 94(2), June 2000, pp. 375–393.

Cramton, Peter C., “Strategic Delay in Bargaining with Two-Sided Uncertainty,” *Review of Economic Studies* 59(1), January 1992, pp. 205–225.

Deneckere, Raymond and Meng-Yu Liang, “Bargaining with Interdependent Values,” *Econometrica* 74(5), pp. 1309-1364.

Eso, Peter and Yuk-fai Fong, “Wait and See,” Northwestern University working paper, 2007.

Farrell, Joseph, “Cheap Talk, Coordination, and Entry,” *RAND Journal of Economics* 18(1), 1987, pp. 34–39.

Fershtman, Chaim and Daniel Seidmann, “Deadline Effects and Inefficient Delay in Bargaining with Endogenous Commitment,” *Journal of Economic Theory* 60, 1993, pp. 306–321.

Fuchs, William and Andrzej Skrzypacz, “Bargaining with Arrival of New Traders,” University of Chicago and Stanford University working paper, 2008.

Kennan, John and Robert Wilson, “Bargaining with Private Information,” *Journal of Economic Literature* 31(1), March 1993, pp. 45–104.

Holmstrom, Bengt, “Moral Hazard in Teams,” *Bell Journal of Economics* 13(2), Autumn 1982, pp. 324–340.

Li, Duozhe, “Bargaining with History Dependent Preferences,” *Journal of Economic Theory* 136(1), 2007, pp. 695–708.

Li, Hao, Sherwin Rosen and Wing Suen, “Conflicts and Common Interests in Committees,” *American Economic Review* 91(5), December 2001, pp. 1478–1497.

Myerson, Roger and Mark Satterthwaite, “Efficient Mechanisms for Bilateral Trading,” *Journal of Economic Theory* 29(2), April 1983, pp. 265–281

Ottaviani, Marco and Peter Sorensen, “Information Aggregation in Debate: Who Should Speak First?” *Journal of Public Economics* 81, 2001, pp. 393–421.

Rubinstein, Ariel, “Perfect Equilibrium in a Bargaining Model,” *Econometrica* 50, 1982, pp. 97–109.

Sakovics, Jozsef, ”Delay in Bargaining Games with Complete Information,” *Journal of Economic Theory* 59, 1993, pp. 78–95.

Stahl, Ingolf, *Bargaining Theory,* Stockholm: Economics Research Institute, Stockholm School of Economics, 1972.

Yildiz, Muhamet, “Waiting to Persuade,” *Quarterly Journal of Economics* 119, 2004, pp. 223–248.