Instrumental Variable Estimation with Heteroskedasticity and Many Instruments*

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Abstract

It is common practice in econometrics to correct for heteroskedasticity of unknown form. This paper does so for instrumental variable estimators with many instruments. We give heteroskedasticity and many instrument robust versions of the limited information maximum likelihood (LIML) and Fuller (1977, FULL) estimators. We also give heteroskedasticity and many instrument consistent standard errors for these estimators. The estimators are based on removing the own observation terms in the numerator of the LIML variance ratio. We derive their properties under standard, many instrument, or many weak instrument asymptotics. Based on a series of Monte Carlo experiments, we find that the estimators perform as well as LIML or FULL under homoskedasticity, and have much lower bias and dispersion under heteroskedasticity, in nearly all cases considered.

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1 Introduction

It is common practice in econometrics to correct for heteroskedasticity of unknown form. In this paper we do this for instrumental variable (IV) estimation with many instruments. We propose computationally simple estimators with high efficiency. We also give heteroskedasticity and many instrument robust standard errors. These methods should prove useful in practice where many instruments are frequently used for efficient estimation.

We base our methods on those that work well with homoskedasticity and many instruments. There the limited information maximum likelihood (LIML) estimator is known to have low bias (Anderson, Kunitomo, and Sawa, 1982) and Bekker (1994) standard errors to lead to a good many instrument asymptotic approximation (Hahn and Hausman, 2002, Hahn and Inoue, 2002, and Hansen, Hausman, and Newey, 2008). Also, the random effects estimator of Chamberlain and Imbens (2004) leads to accurate inference. The Fuller (1977, FULL) estimator is even better, having fewer outliers than LIML, as pointed out by Hahn, Hausman, and Kuersteiner (2004).

Unfortunately, LIML and FULL are inconsistent under heteroskedasticity and many instruments, as shown by Bekker and van der Ploeg (2005) and Chao and Swanson (2004) in special cases, and fully characterized here. We modify these estimators to obtain heteroskedasticity and many instrument robust versions, HLIM and HFUL respectively. The modification consists of deleting the own observation terms in the numerator of the variance ratio that is minimized by LIML. The jackknife IV (JIV) estimators of Phillips and Hale (1977), Blomquist and Dahlberg (1999), and Angrist, Imbens, and Krueger (1999), also delete own observation terms. This deletion makes JIV robust to heteroskedasticity and many instruments, as shown by Ackerberg and Deveraux (2003) and Chao and Swanson (2004). HLIM and HFUL share this robustness property of JIV. Indeed HLIM is a linear combination of forward and reverse JIV estimators, similarly to a result of Hahn and Hausman (2002), that LIML is a linear combination of forward and reverse Nagar (1959) estimators.

An advantage of HLIM and HFUL is that they are asymptotically more efficient than JIV estimators under homoskedasticity and the many weak instrument sequence of Chao and Swanson (2005), being as efficient as LIML and FULL in this case. Also, in simulations reported below, HFUL and HLIM are more precise than JIV in all cases, including heteroskedasticity, nearly as precise as FULL and LIML under homoskedasticity, and remarkably more precise than FULL and LIML with heteroskedasticity. In particular, HLIM and HFUL overcome the criticisms of JIV made by Davidson and MacKinnon (2006).

The standard errors given here include the White (1982) heteroskedasticity robust standard errors and correction terms for many instruments. We prove their consistency and find in the simulations that they lead to accurate inference.

Under many weak instruments and heteroskedasticity, HLIM will be inefficient relative to the continuously updated estimator (CUE) of Hansen, Heaton, and Yaron (1996) and other generalized empirical likelihood (Smith, 1997) estimators. However, these estimators are much more difficult to compute than HFUL or HLIM, and in Monte Carlo work we do not find much advantage to using the CUE relative to HFUL and HLIM.

The asymptotic theory we consider allows for many instruments as in Kunitomo (1980) and Bekker (1994) or many weak instruments as in Chao and Swanson (2004, 2005), Stock and Yogo (2005), and Han and Phillips (2006). Asymptotic normality is obtained via a central limit theorem that imposes very weak conditions on instruments, given by Chao, Swanson, Hausman, Newey, and Woutersen (2007).

In Section 2, the model is outlined and the proposed estimators presented. In Section 3, the problem with LIML under heteroskedasticity is detailed and solutions discussed. An optimal estimator that is a two-step jackknife version of the CUE is presented in Section 4. An extension of the results to a restricted CUE is outlined in Section 5. Asymptotic theory is gathered in Section 6, and Monte Carlo findings are presented in Section 7. All the proofs are gathered in the Appendix.

2 The Model and Estimators

The model we consider is given by

$$\begin{aligned} \underset{n\times 1}{y} &= \underset{n\times G}{X}\underset{G\times 1}{\delta_0} + \underset{n\times 1}{\varepsilon}, \\ X &= \Upsilon + U, \end{aligned}$$

where n is the number of observations, G is the number of right-hand side variables, Υ is a matrix of observations on the reduced form, and U is the matrix of reduced form disturbances. For our asymptotic approximations, the elements of Υ will be implicitly allowed to depend on n, although we suppress dependence of Υ on n for notational convenience. Estimation of δ_0 will be based on an $n \times K$ matrix, Z, of instrumental variable observations with $rank(Z) = K$. We will assume that Z is nonrandom and that observations (ε_i, U_i) are independent across i and have mean zero. Alternatively, we could allow Z to be random, but condition on it, as in Chao et. al. (2007).

In this model some columns of X may be exogenous, with the corresponding columns of U being zero. Also, this model allows for Υ to be a linear combination of Z, i.e. $\Upsilon = Z\pi$ for some $K \times G$ matrix π. The model also permits Z to approximate the reduced form. For example, let X_i', Υ_i', and Z_i' denote the i^{th} row (observation) of X, Υ, and Z respectively. We could let $\Upsilon_i = f_0(w_i)$ be a vector of unknown functions of a vector w_i of underlying instruments, and $Z_i = (p_{1K}(w_i), ..., p_{KK}(w_i))'$ be approximating functions $p_{kK}(w)$, such as power series or splines. In this case, linear combinations of Z_i may approximate the unknown reduced form (e.g. as in Newey, 1990).

To describe HLIM and HFUL, let

$$P = Z(Z'Z)^{-1}Z'$$

and let P_{ij} denote the ij^{th} element of P. The HLIM estimator is given by

$$\tilde{\delta} = \arg\min_{\delta} \hat{Q}(\delta), \hat{Q}(\delta) = \frac{(y - X'\delta)'P(y - X'\delta) - \sum_{i=1}^{n} P_{ii}(y_i - X_i'\delta)^2}{(y - X\delta)'(y - X\delta)}.$$

The objective function $\hat{Q}(\delta)$ for HLIM is the same as the LIML objective function except that the $i = j$ terms have been deleted in the numerator. This adjustment to the numerator is what makes HLIM consistent under heteroskedasticity and many instruments, as further explained in Section 3.

Computation of this estimator is straightforward. Let $\bar{X} = [y, X]$. The minimized objective function $\tilde{\alpha} = \hat{Q}(\tilde{\delta})$ is the smallest eigenvalue of $(\bar{X}'\bar{X})^{-1}(\bar{X}'P\bar{X} - \sum_{i=1}^{n} P_{ii}\bar{X}_i\bar{X}_i')$. Solving the first order conditions gives

$$\tilde{\delta} = \left(X'PX - \sum_{i=1}^{n} P_{ii}X_iX_i' - \tilde{\alpha}X'X\right)^{-1}\left(X'Py - \sum_{i=1}^{n} P_{ii}X_iy_i - \tilde{\alpha}X'y\right).$$

Thus, the estimator can be computed by finding the smallest eigenvalue of a matrix and then using it in the above formula. This computation is analogous to that for LIML, except that the own observation terms have been deleted from the double sums involving P.

HLIM is a member of a class of estimators of the form

$$\hat{\delta} = \left(X'PX - \sum_{i=1}^{n} P_{ii}X_iX_i' - \hat{\alpha}X'X\right)^{-1}\left(X'Py - \sum_{i=1}^{n} P_{ii}X_iy_i - \hat{\alpha}X'y\right). \tag{2.1}$$

for some $\hat{\alpha}$ not necessarily equal to $\tilde{\alpha}$. HFUL is obtained by replacing $\tilde{\alpha}$ with $\hat{\alpha} = [\tilde{\alpha} - (1-\tilde{\alpha})C/T]/[1-(1-\tilde{\alpha})C/T]$ for some $C > 0$. The theoretical small sample properties of this estimator are unknown, but in the simulations in Section 5 its performance relative to HLIM is similar to that of FULL relative to LIML. As pointed out by Hahn, Hausman, and Kuersteiner (2004), FULL has much smaller dispersion than LIML with weak instruments, so we expect the same for HFUL. Monte Carlo results given below confirm these properties.

To describe the asymptotic variance estimator, let $\hat{\varepsilon}_i = y_i - X_i'\hat{\delta}$, $\hat{\gamma} = X'\hat{\varepsilon}/\hat{\varepsilon}'\hat{\varepsilon}$, $\hat{X} = X - \hat{\varepsilon}\hat{\gamma}'$, $\bar{X} = P\hat{X}$, and $\tilde{Z} = Z(Z'Z)^{-1}$. Also let

$$\begin{aligned}
\hat{H} &= X'PX - \sum_{i=1}^{n} P_{ii}X_iX_i' - \hat{\alpha}X'X, \\
\hat{\Sigma} &= \sum_{i=1}^{n}(\bar{X}_i\bar{X}_i' - \hat{X}_iP_{ii}\bar{X}_i' - \bar{X}_iP_{ii}\hat{X}_i')\hat{\varepsilon}_i^2 + \sum_{k=1}^{K}\sum_{\ell=1}^{K}\left(\sum_{i=1}^{n}\tilde{Z}_{ik}\tilde{Z}_{i\ell}\hat{X}_i\hat{\varepsilon}_i\right)\left(\sum_{j=1}^{n}Z_{jk}Z_{j\ell}\hat{X}_j\hat{\varepsilon}_j\right)',
\end{aligned}$$

be vectorized formulas that can be easily computed even when n is very large. The asymptotic variance estimator is

$$\hat{V} = \hat{H}^{-1}\hat{\Sigma}\hat{H}^{-1}.$$

Treating $\hat{\delta}$ as if it were normally distributed with mean δ_0 and variance $\hat{V}$ will lead to correct large sample inference, under conditions given in Section 4. In particular, defining q_α as the $1-\alpha/2$ quantile of a $N(0,1)$ distribution, an asymptotic $1-\alpha$ confidence interval for δ_{0k} is given by $\hat{\delta}_k \pm q_\alpha\sqrt{\hat{V}_{kk}}$.

HLIM is invariant to normalization, similarly to LIML, although HFUL is not. The vector $\tilde{d} = (1, -\tilde{\delta}')'$ solves

$$\min_{d:d_1=1} \frac{d'\left(\bar{X}'P\bar{X} - \sum_{i=1}^n P_{ii}\bar{X}_i\bar{X}_i'\right)d}{d'\bar{X}'\bar{X}d}.$$

Because of the ratio form of the objective function, another normalization, such as imposing that another d is equal to 1, would produce the same estimator, up to the normalization.

The HLIM and HFUL estimators are related to JIV estimators. In particular, consider the JIVE2 estimator of Angrist, Imbens, and Krueger (1999), given by

$$\bar{\delta} = \left(X'PX - \sum_{i=1}^n P_{ii}X_iX_i'\right)^{-1}\left(X'Py - \sum_{i=1}^n P_{ii}X_iy_i\right). \tag{2.2}$$

This estimator is a special case of $\hat{\delta}$, where $\hat{\alpha} = 0$. It can also be shown that the first-order conditions for HFUL are a linear combination of those for HLIM and this JIV estimator. Furthermore, we can interpret HLIM as a linear combination of forward and reverse JIV estimators.

For simplicity, we give this interpretation in the scalar δ case. Let $\tilde{\varepsilon}_i = y_i - X_i'\tilde{\delta}$ and $\tilde{\gamma} = \sum_i X_i\tilde{\varepsilon}_i / \sum_i \tilde{\varepsilon}_i^2$. The forward JIV estimator $\bar{\delta}$ is given in equation (2.2). The reverse JIV is obtained as follows. Dividing the structural equation by δ_0 gives

$$X_i = y_i/\delta_0 - \varepsilon_i/\delta_0.$$

Applying JIV to this equation in order to estimate $1/\delta_0$, and then inverting, gives the reverse JIV estimator

$$\bar{\delta}^r = \left(\sum_{i\neq j} y_iP_{ij}X_j\right)^{-1}\sum_{i\neq j} y_iP_{ij}y_j.$$

To interpret the HLIM estimator, note that its first-order conditions are

$$0 = -\frac{\partial \hat{Q}(\tilde{\delta})}{\partial \delta} \sum_i \tilde{\varepsilon}_i^2/2 = \sum_{i \neq j} (X_i - \tilde{\gamma}\tilde{\varepsilon}_i) P_{ij} (y_j - X_j'\tilde{\delta}) = \sum_{i \neq j} [(1+\tilde{\gamma}\tilde{\delta})X_i - \tilde{\gamma} y_i] P_{ij}(y_j - X_j'\tilde{\delta}).$$

Then, collecting terms gives

$$\begin{aligned} 0 &= (1+\tilde{\gamma}\tilde{\delta}) \sum_{i \neq j} X_i P_{ij}(y_j - X_j'\tilde{\delta}) - \tilde{\gamma} \sum_{i \neq j} y_i P_{ij}(y_j - X_j'\tilde{\delta}) \\ &= (1+\tilde{\gamma}\tilde{\delta}) \sum_{i \neq j} X_i P_{ij} X_j (\bar{\delta} - \tilde{\delta}) - \tilde{\gamma} \sum_{i \neq j} y_i P_{ij} X_j (\bar{\delta}^r - \tilde{\delta}). \end{aligned}$$

Dividing through by $\sum_{i \neq j} X_i P_{ij} X_j$ we then obtain

$$0 = (1+\tilde{\gamma}\tilde{\delta})(\bar{\delta} - \tilde{\delta}) - \tilde{\gamma}\bar{\delta}(\bar{\delta}^r - \tilde{\delta}).$$

Finally, solving for $\tilde{\delta}$ gives

$$\tilde{\delta} = \frac{(1+\tilde{\gamma}\tilde{\delta})\bar{\delta} - \left(\tilde{\gamma}\bar{\delta}\right)\bar{\delta}^r}{1 + \tilde{\gamma}(\tilde{\delta} - \bar{\delta})}.$$

As usual, the asymptotic variance of a linear combination of coefficients is unaffected by how the coefficients are estimated, so an asymptotically equivalent version of this estimator can be obtained by replacing $\tilde{\delta}$ in the coefficients with $\bar{\delta}$, giving

$$\bar{\delta}^* = (1+\tilde{\gamma}\bar{\delta})\bar{\delta} - \left(\tilde{\gamma}\bar{\delta}\right)\bar{\delta}^r.$$

Thus we see that, analogous to Hahn and Hausman (2002) for LIML, HFUL is asymptotically equivalent to a linear combination of forward and reverse bias corrected estimators.

3 The LIML Bias

To characterize the LIML bias we describe LIML as

$$\tilde{\delta}^* = \arg\min_{\delta} \hat{Q}^*(\delta), \hat{Q}^*(\delta) = \frac{(y - X\delta)'P(y - X\delta)}{(y - X\delta)'(y - X\delta)}.$$

The FULL estimator is

$$\breve{\delta}^* = (X'PX - \breve{\alpha}^* X'X)^{-1}(X'Py - \breve{\alpha}^* X'y),$$

for $\breve{\alpha}^* = [\tilde{\alpha}^* - (1-\tilde{\alpha}^*)C/T]/[1-(1-\tilde{\alpha}^*)C/T]$, $\tilde{\alpha}^* = \hat{Q}^*(\tilde{\delta}^*)$, and $C > 0$. FULL has moments of all orders, is approximately mean unbiased for $C = 1$, and is second order admissible for $C \geq 4$, under homoskedasticity and standard large sample asymptotics. Both LIML and FULL are members of a class of estimators of the form

$$\hat{\delta}^* = (X'PX - \hat{\alpha}^* X'X)^{-1}(X'Py - \hat{\alpha}^* X'y).$$

For example, LIML has this form for $\hat{\alpha}^* = \tilde{\alpha}^*$, FULL for $\hat{\alpha}^* = \breve{\alpha}^*$, and 2SLS for $\hat{\alpha}^* = 0$.

We can use the objective functions that these estimators minimize in order to characterize the problem with heteroskedasticity and many instruments. These objective functions are made up of quadratic forms that, like sample averages, will be close to their expectation in large samples. Thus, if the objective function with expectations substituted for quadratic forms is not minimized at the true parameter asymptotically, then the estimator will not be consistent. For expository purposes, first consider 2SLS, which has the following objective function

$$\hat{Q}_{2SLS}(\delta) = (y - X\delta)'P(y - X\delta)/n = \sum_{i \neq j}(y_i - X_i'\delta)P_{ij}(y_j - X_j'\delta)/n + \sum_{i=1}^{n} P_{ii}(y_i - X_i'\delta)^2/n.$$

By independence of the observations

$$E\left[\hat{Q}_{2SLS}(\delta)\right] = (\delta - \delta_0)'\sum_{i \neq j}\Upsilon_i P_{ij}\Upsilon_j'(\delta - \delta_0)/n + \sum_{i=1}^{n} P_{ii}E[(y_i - X_i'\delta)^2]/n$$

The first term following the above equality will be asymptotically minimized at δ_0 because $\sum_{i \neq j}\Upsilon_i P_{ij}\Upsilon_j'$ will be positive semi-definte under regularity conditions given below. The second term is an expected squared residual that will not be minimized at δ_0 due to endogeneity. With many instruments P_{ii} does not shrink to zero, so that the second term does not vanish asymptotically. Hence, with many instruments, 2SLS is not consistent, even under homoskedasticity, as pointed out by Bekker (1994).

For LIML, we can (asymptotically) replace the objective function, $\hat{Q}^*(\delta)$, with a corresponding ratio of expectations giving

$$\frac{E[(y - X\delta)'\,P\,(y - X\delta)]}{E[(y - X\delta)'\,(y - X\delta)]} = \frac{(\delta - \delta_0)'\sum_{i \neq j}P_{ij}\Upsilon_i\Upsilon_j'(\delta - \delta_0)}{\sum_{i=1}^{n} E[(y_i - X_i'\delta)^2]} + \frac{\sum_{i=1}^{n} P_{ii}E[(y_i - X_i'\delta)^2]}{\sum_{i=1}^{n} E[(y_i - X_i'\delta)^2]}.$$

Here, we again see that the first term following the equality will be minimized at δ_0 as long as $\sum_{i\neq j} P_{ij}\Upsilon_i\Upsilon_j'$ is positive semi-definite. Under heteroskedasticity, the second term may not have a critical value at δ_0, and so the objective function will not be minimized at δ_0. To see this let $\sigma_i^2 = E[\varepsilon_i^2]$, $\gamma_i = E[X_i\varepsilon_i]/\sigma_i^2$, and $\bar{\gamma} = \sum_{i=1}^n E[X_i\varepsilon_i]/\sum_{i=1}^n \sigma_i^2 = \sum_i \gamma_i\sigma_i^2/\sum_i \sigma_i^2$. Then

$$\begin{aligned} \frac{\partial}{\partial\delta}\frac{\sum_{i=1}^n P_{ii}E[(y_i - X_i\delta)^2]}{\sum_{i=1}^n E[(y_i - X_i\delta)^2]}\Bigg|_{\delta=\delta_0} &= \frac{-2}{\sum_{i=1}^n \sigma_i^2}\left[\sum_{i=1}^n P_{ii}E[X_i\varepsilon_i] - \sum_{i=1}^n P_{ii}\sigma_i^2\bar{\gamma}\right] \\ &= \frac{-2\sum_{i=1}^n P_{ii}(\gamma_i - \bar{\gamma})\sigma_i^2}{\sum_{i=1}^n \sigma_i^2} = -2\widehat{Cov_{\sigma^2}(P_{ii}, \gamma_i)}, \end{aligned}$$

where $\widehat{Cov_{\sigma^2}(P_{ii}, \gamma_i)}$ is the covariance between P_{ii} and γ_i, for the distribution with probability weight $\sigma_i^2/\sum_{i=1}^n \sigma_i^2$ for the i^{th} observation. When

$$\lim_{n\longrightarrow\infty} \widehat{Cov_{\sigma^2}(P_{ii}, \gamma_i)} \neq 0,$$

the objective function will not have zero derivative at δ_0 asymptotically so that it is not minimized at δ_0. When this covariance does have a zero limit then it can be shown that the ratio of expectations will be minimized at δ_0 as long as for $\Omega_i = E[U_iU_i']$ the matrix

$$\left(1 - \frac{\sum_{i=1}^n \sigma_i^2 P_{ii}}{\sum_{i=1}^n \sigma_i^2}\right)\sum \Upsilon_i\Upsilon_i'/n + \sum_i P_{ii}\Omega_i/n - \frac{\sum_{i=1}^n \sigma_i^2 P_{ii}}{\sum_{i=1}^n \sigma_i^2}\sum_{i=1}^n \Omega_i/n$$

has a positive definite limit.

Note that $\widehat{Cov_{\sigma^2}(P_{ii}, \gamma_i)} = 0$ when either γ_i or P_{ii} does not depend on i. Thus, it is variation in $\gamma_i = E[X_i\varepsilon_i]/\sigma_i^2$, the coefficients from the projection of X_i on ε_i, that leads to inconsistency of LIML, and not just any heteroskedasticity. Also, the case where P_{ii} is constant occurs with dummy instruments and equal group sizes. It was pointed out by Bekker and van der Ploeg (2005) that LIML is consistent in this case, under heteroskedasticity. Indeed, when P_{ii} is constant,

$$\hat{Q}^*(\delta) = \hat{Q}(\delta) + \frac{\sum_i P_{ii}(y_i - X_i'\delta)^2}{(y - X\delta)'(y - X\delta)} = \hat{Q}(\delta) + P_{11},$$

so that the LIML objective function equals the HLIM objective function plus a constant, and hence HLIM equals LIML.

LIML is inconsistent when $P_{ii} = Z_i'(Z'Z)^{-1}Z_i$ (roughly speaking this is the size of the i^{th} instrument observation) is correlated with γ_i. This can easily happen when (say) there is more heteroskedasticity in σ_i^2 than $E[X_i\varepsilon_i]$. Bekker and van der Ploeg (2005) and Chao and Swanson (2004) pointed out that LIML can be inconsistent with heteroskedasticity; the contribution here is to give the exact condition $\widehat{Cov_{\sigma^2}(P_{ii}}, \gamma_i) = 0$ for consistency of LIML.

Clearly with independent observations the own observation terms are the source of bias, so the bias can be eliminated by deleting those terms. For 2SLS this gives JIV, i.e.

$$\bar{\delta} = \arg\min_{\delta} \sum_{i \neq j} (y_i - X_i'\delta) P_{ij} (y_j - X_j'\delta)/n = \left(\sum_{i \neq j} X_i P_{ij} X_j' \right)^{-1} \sum_{i \neq j} X_i P_{ij} y_j.$$

Deleting the own observation terms from the numerator of the LIML objective function gives the HLIM objective function. We call this a jackknife bias correction because the jackknife also deletes own observation terms in forming estimates.

4 Optimal Estimation with Heteroskedasticity

HLIM and HFUL are not asymptotically efficient under heteroskedasticity and many weak instruments. In generalized method of moments (GMM) terminology, they use a nonoptimal weighting matrix, one that is not heteroskedasticity consistent for the inverse of the variance of the moments. In addition, they do not use a heteroskedasticity consistent projection of the endogenous variables on the disturbance, which leads to inefficiency in the many instruments correction term. Efficiency can be obtained by modifying the estimator so that the weight matrix and the projection are heteroskedasticity consistent. Let $\hat{\delta}$ be a preliminary estimator such as HLIM or HFUL, $\hat{\varepsilon}_i = y_i - X_i'\hat{\beta}$, and let

$$\begin{aligned} \hat{\Omega} &= \sum_{i=1}^{n} Z_i Z_i' \hat{\varepsilon}_i^2, \hat{B}_k = \left(\sum_{i=1}^{n} Z_i Z_i' \hat{\varepsilon}_i X_{ik}/n \right) \hat{\Omega}^{-1}, \\ \hat{D}_{ik} &= Z_i X_{ik} - \hat{B}_k Z_i \hat{\varepsilon}_i, \hat{D}_i = \left[\hat{D}_{i1}, ..., \hat{D}_{iG} \right]. \end{aligned}$$

An estimator that will be efficient under many weak moments is

$$\bar{\delta}^h = \left(\sum_{i \neq j} \hat{D}_i' \hat{\Omega}^{-1} Z_j X_j' \right)^{-1} \sum_{i \neq j} \hat{D}_i' \hat{\Omega}^{-1} Z_j y_j.$$

This is like a JIV estimator with weighting matrix $\hat{W} = \hat{\Omega}^{-1}$, where $\hat{D}_i$ replaces $X_i Z_i'$. We refer to it as a jackknife CUE because of its relationshp to the CUE. The use of $\hat{D}_i$ makes the estimator as efficient as the CUE under many weak instruments.

The asymptotic variance of $\bar{\delta}^h$ can be estimated by

$$\hat{V}^h = \left(\hat{H}^h\right)\hat{\Sigma}^h\left(\hat{H}^h\right)^{-1}, \hat{H}^h = \sum_{i \neq j} X_i Z_i' \hat{\Omega}^{-1} Z_j X_j', \hat{\Sigma}^h = \sum_{i,j=1}^{n} \hat{D}_i' \hat{\Omega}^{-1} \hat{D}_j.$$

This estimator has a jackknife, sandwich form similar to that given in Newey and Windmeijer (2008) for the CUE.

To explain the relationship of $\bar{\delta}^h$ to the CUE, note that $\bar{\delta}^h$ is the solution to

$$\sum_{i \neq j} \hat{D}_i' \hat{\Omega}^{-1} Z_j (y_j - X_j' \bar{\delta}^h) = 0.$$

From Donald and Newey (2000), we see that this equation is identical to the first-order conditions for the CUE when $\bar{\delta}^h$ equals $\hat{\delta}$. Thus, if this equation were iterated, by repeatedly setting $\hat{\delta} = \hat{\delta}^h$ and then recalculating $\hat{\delta}^h$, and that iteration converged to a point where $\hat{\delta} = \hat{\delta}^h$, the estimator would satisfy the same first-order conditions as the CUE. Also, because of the jackknife form of $\hat{\delta}^h$, the asymptotic variance of $\hat{\delta}^h$ will be the same as if $\hat{\delta} = \delta_0$ under many weak moments, so $\hat{\delta}^h$ will be asymptotically equivalent to the CUE. As shown by Newey and Windmeijer (2007), the CUE is efficient relative to jackknife GMM under many weak moments.

Asymptotic theory for $\bar{\delta}^h$ will be provided elsewhere. Here we focus on theory that allows for either many instruments or many weak instruments. We do not yet know how to analyze $\bar{\delta}^h$ under many instruments because of the presence of $\hat{\Omega}^{-1}$. See Newey and Windmeijer (2008) for further discussion.

5 Jackknife Bias Correction of Continuous Updated GMM Estimators

The jackknife bias correction for LIML can be extended to the continuously updated estimator (CUE) in the generalized method of moments (GMM) framework. To explain, consider a general GMM setup where δ denotes a $G \times 1$ parameter vector and $g_i(\delta)$ is a $K \times 1$ vector of functions of the data and parameters satisfying $E[g_i(\delta_0)] = 0$. For example, in the linear IV environment, $g_i(\delta) = Z_i(y_i - X_i'\delta)$. Let $\tilde{\Omega}(\delta)$ denote an estimator of $\Omega(\delta) = \sum_{i=1}^n E[g_i(\delta)g_i(\delta)']/n$, where an n subscript on $\Omega(\delta)$ is suppressed for notational convenience. Here we define a CUE to satisfy

$$\hat{\delta} = \arg\min_{\delta} \hat{g}(\delta)'\tilde{\Omega}(\delta)^{-1}\hat{g}(\delta),$$

minimizing the quadratic form simultaneously over δ in $\hat{g}(\delta)$ and $\tilde{\Omega}(\delta)$. When $\tilde{\Omega}(\delta) = \sum_{i=1}^n g_i(\delta)g_i(\delta)'/n$ this is an unrestricted CUE that is given by Hansen, Heaton, and Yaron (1996). For other choices of $\tilde{\Omega}(\delta)$, this estimator is a generalization that allows for restrictions on $\tilde{\Omega}(\delta)$.

For example, in the IV setting where $g_i(\delta) = Z_i(y_i - X_i'\delta)$, we may specify $\tilde{\Omega}\,(\delta)$ to be only consistent under homoskedasticity,

$$\tilde{\Omega}(\delta) = (y - X\delta)'\,(y - X\delta)\,Z'Z/n^2.$$

In this case the CUE objective function is

$$\hat{g}(\delta)'\tilde{\Omega}(\delta)^{-1}\hat{g}(\delta) = \frac{(y - X\delta)'\,P\,(y - X\delta)}{(y - X\delta)'\,(y - X\delta)},$$

which is the LIML objective function. This example and the small bias properties of LIML were the orginal motivation for the CUE in Hansen, Heaton, and Yaron (1996).

Two motivations for a CUE with restricted $\tilde{\Omega}(\delta)$ are computational simplicity and finite sample efficiency. For example, LIML is easy to compute while the unrestricted CUE seems to have a nearly flat objective function over a wide range of δ values that include $\hat{\delta}$. Also, imposing restrictions on $\tilde{\Omega}(\delta)$ may improve the asymptotic approximation to the distribution of $\hat{\delta}$. On asymptotic efficiency grounds the unrestricted CUE is preferred when the restrictions on $\Omega(\delta)$ are not satisfied.

The unrestricted CUE is also preferred on bias grounds when the restrictions on $\Omega(\delta)$ are not satisfied. We can explain this using a calculation similar to that for the LIML bias above. Consider an objective function where $\tilde{\Omega}(\delta)$ is replaced by its expectation,

$\bar{\Omega}(\delta) = E[\tilde{\Omega}(\delta)]$, similarly to replacing the denominator of the LIML objective function by its expectation. The expectation of the objective function is then

$$E[\hat{g}(\delta)'\bar{\Omega}(\delta)^{-1}\hat{g}(\delta)] = (1 - n^{-1})\bar{g}(\delta)'\bar{\Omega}(\delta)^{-1}\bar{g}(\delta) + tr(\bar{\Omega}(\delta)^{-1}\Omega(\delta))/n,$$

where $\bar{g}(\delta) = E[g_i(\delta)]$ and $\Omega(\delta) = E[g_i(\delta)g_i(\delta)']$. The first term in the above expression is minimized at δ_0, where $\bar{g}(\delta_0) = 0$. When $\bar{\Omega}(\delta) = \Omega(\delta)$, then

$$tr(\bar{\Omega}(\delta)^{-1}\Omega(\delta))/n = K/n,$$

so that the second term does not depend on δ. In this case the expected value of the CUE objective function is minimized at δ_0. When $\bar{\Omega}(\delta) \neq \Omega(\delta)$, the second term may depend on δ, and so the expected value of the CUE objective function will not be minimized at δ_0. This effect will lead to bias in the CUE, because the expected objective function is not minimized at the truth. It is also interesting to note that this bias effect will tend to increase with K. This bias term was noted by Han and Phillips (2005) for two-stage GMM, who referred to it as a "noise" term, and to the other term as a "signal" term.

We can modify the restricted CUE to produce an estimator that has small bias even when the restrictions on $\Omega(\delta)$ are not satisfied by jackknifing, i.e. deleting the own observation terms. Note that

$$E[\sum_{i \neq j} g_i(\delta)'\bar{\Omega}(\delta)^{-1}g_j(\delta)/n^2] = (1 - n^{-1})\bar{g}(\delta)'\bar{\Omega}(\delta)^{-1}\bar{g}(\delta),$$

which is always minimized at δ_0, no matter what $\bar{\Omega}(\delta)$ is. A corresponding estimator is obtained by replacing $\bar{\Omega}(\delta)$ by $\tilde{\Omega}(\delta)$ and minimizing. Namely,

$$\hat{\delta} = \arg\min_{\delta} \sum_{i \neq j} g_i(\delta)'\tilde{\Omega}(\delta)^{-1}g_j(\delta)/n^2.$$

This is a bias corrected, restricted CUE, that should have small bias by virtue of the jackknife form of the objective function. The HLIM estimator is precisely of this form, for $\tilde{\Omega}(\delta) = (y - X\delta)'(y - X\delta)Z'Z/n^2$. The jackknife CUE estimator should also prove useful in other settings.

6 Asymptotic Theory

Theoretical justification for the estimators is provided by asymptotic theory where the number of instruments grows with the sample size. Some regularity conditions are important for this theory. Let $Z_i', \varepsilon_i, U_i'$, and Υ_i' denote the i^{th} row of Z, ε, U, and Υ respectively. Here, we will consider the case where Z is constant, which can be viewed as conditioning on Z (see e.g. Chao et. al. 2007).

Assumption 1: Z includes among its columns a vector of ones, $rank(Z) = K$, and there is a constant C such that $P_{ii} \leq C < 1$, $(i = 1, ..., n)$, $K \longrightarrow \infty$.

The restriction that $rank(Z) = K$ is a normalization that requires excluding redundant columns from Z. It can be verified in particular cases. For instance, when w_i is a continuously distributed scalar, $Z_i = p^K(w_i)$, and $p_{kK}(w) = w^{k-1}$, it can be shown that $Z'Z$ is nonsingular with probability one for $K < n$.[1] The condition $P_{ii} \leq C < 1$ implies that $K/n \leq C$, because $K/n = \sum_{i=1}^n P_{ii}/n \leq C$.

Assumption 2: There is a $G \times G$ matrix, $S_n = \tilde{S}_n \operatorname{diag}(\mu_{1n}, ..., \mu_{Gn})$, and z_i such that $\Upsilon_i = S_n z_i/\sqrt{n}$, $\tilde{S}_n$ is bounded and the smallest eigenvalue of $\tilde{S}_n \tilde{S}_n'$ is bounded away from zero, for each j either $\mu_{jn} = \sqrt{n}$ or $\mu_{jn}/\sqrt{n} \longrightarrow 0$, $\mu_n = \min_{1 \leq j \leq G} \mu_{jn} \longrightarrow \infty$, and $\sqrt{K}/\mu_n^2 \longrightarrow 0$. Also, $\sum_{i=1}^n \|z_i\|^4 / n^2 \longrightarrow 0$, and $\sum_{i=1}^n z_i z_i'/n$ is bounded and uniformly nonsingular.

Setting $\mu_{jn} = \sqrt{n}$ leads to asymptotic theory like that in Kunitomo (1980) and Bekker (1994), where the number of instruments K can grow as fast as the sample size. In that case, the condition $\sqrt{K}/\mu_n^2 \longrightarrow 0$ would be automatically satisfied. Allowing for K to grow, and for μ_n to grow more slowly than $\sqrt{n}$, allows for many instruments without strong identification. This condition then allows for some components of the reduced form to give only weak identification (corresponding to $\mu_{jn}/\sqrt{n} \longrightarrow 0$), and other components

[1] The observations $w_1, ..., w_n$ are distinct with probability one and therefore, by $K < n$, cannot all be roots of a K^{th} degree polynomial. It follows that for any nonzero a there must be some i with $a'Z_i = a'p^K(w_i) \neq 0$, implying that $a'Z'Za > 0$.

(corresponding to $\mu_{jn} = \sqrt{n}$) to give strong identification. In particular, this condition allows for fixed constant coefficients in the reduced form.

Assumption 3: $(\varepsilon_1, U_1), ..., (\varepsilon_n, U_n)$ are independent with $E[\varepsilon_i] = 0$, $E[U_i] = 0$, $E[\varepsilon_i^4]$ and $E[\|U_i\|^4]$ are bounded in i, $Var((\varepsilon_i, U_i')') = diag(\Omega_i^*, 0)$, and $\sum_{i=1}^n \Omega_i^*/n$ is uniformly nonsingular.

This condition includes moment existence assumptions. It also requires the average variance of the nonzero reduced form disturbances to be nonsingular, and is useful for the proof of consistency contained in the appendix.

Assumption 4: There is a π_{Kn} such that $\sum_{i=1}^n \|z_i - \pi_{Kn} Z_i\|^2 / n \longrightarrow 0$.

This condition allows for an unknown reduced form that is approximated by a linear combination of the instrumental variables. It is possible to replace this assumption with the condition that $\sum_{i \neq j} z_i P_{ij} z_j'/n$ is uniformly nonsingular.

We can easily interpret all of these conditions for the important example of a linear model with exogenous covariates and a possibly unknown reduced form. This example is given by

$$X_i = \begin{pmatrix} \pi_{11} Z_{1i} + \mu_n f_0(w_i)/\sqrt{n} \\ Z_{1i} \end{pmatrix} + \begin{pmatrix} v_i \\ 0 \end{pmatrix}, Z_i = \begin{pmatrix} Z_{1i} \\ p^K(w_i) \end{pmatrix},$$

where Z_{1i} is a $G_2 \times 1$ vector of included exogenous variables, $f_0(w)$ is a $G - G_2$ dimensional vector function of a fixed dimensional vector of exogenous variables, w, and $p^K(w) \overset{def}{=} (p_{1K}(w), ..., p_{K-G_2,K}(w))'$. The variables in X_i other than Z_{1i} are endogenous with reduced form $\pi_{11} Z_{1i} + \mu_n f_0(w_i)/\sqrt{n}$. The function $f_0(w)$ may be a linear combination of a subvector of $p^K(w)$, in which case $z_i = \pi_{Kn} Z_i$, for some π_{Kn} in Assumption 4; or it may be an unknown function that can be approximated by a linear combination of $p^K(w)$. For $\mu_n = \sqrt{n}$, this example is like the model in Newey (1990), where Z_i includes approximating functions for the optimal (asymptotic variance minimizing) instruments Υ_i, but the number of instruments can grow as fast as the sample size. When $\mu_n^2/n \longrightarrow 0$, it is a modified version where the model is more weakly identified.

To see precise conditions under which the assumptions are satisfied, let

$$z_i = \begin{pmatrix} f_0(w_i) \\ Z_{1i} \end{pmatrix}, S_n = \tilde{S}_n diag\left(\mu_n, ..., \mu_n, \sqrt{n}, ..., \sqrt{n}\right), \text{ and } \tilde{S}_n = \begin{pmatrix} I & \pi_{11} \\ 0 & I \end{pmatrix}.$$

By construction we have that $\Upsilon_i = S_n z_i/\sqrt{n}$. Assumption 2 imposes the requirements that

$$\sum_{i=1}^n \|z_i\|^4/n^2 \longrightarrow 0,$$

and that $\sum_{i=1}^n z_i z_i'/n$ is bounded and uniformly nonsingular. The other requirements of Assumption 2 are satisfied by construction. Turning to Assumption 3, we require that $\sum_{i=1}^n Var(\varepsilon_i, U_i')/n$ is uniformly nonsingular. For Assumption 4, let $\pi_{Kn} = [\tilde{\pi}_{Kn}', [I_{G_2}, 0]']'$. Then Assumption 4 will be satisfied if, for each n, there exists a $\tilde{\pi}_{Kn}$ with

$$\sum_{i=1}^n \|z_i - \pi_{Kn}' Z_i\|^2/n = \sum_{i=1}^n \|f_0(w_i) - \tilde{\pi}_{Kn}' Z_i\|^2/n \longrightarrow 0.$$

Theorem 1: *If Assumptions 1-4 are satisfied and* $\hat{\alpha} = o_p(\mu_n^2/n)$ *or* $\hat{\delta}$ *is HLIM or HFUL then* $\mu_n^{-1} S_n'(\hat{\delta} - \delta_0) \xrightarrow{p} 0$ *and* $\hat{\delta} \xrightarrow{p} \delta_0$.

This result gives convergence rates for linear combinations of $\hat{\delta}$. For instance, in the above example, it implies that $\hat{\delta}_1$ is consistent and that $\pi_{11}'\hat{\delta}_1 + \hat{\delta}_2 = o_p(\mu_n/\sqrt{n})$.

The asymptotic variance of the estimator will depend on the growth rate of K relative to μ_n^2. The following condition allows for two cases.

Assumption 5: Either I) K/μ_n^2 is bounded and $\sqrt{K} S_n^{-1} \longrightarrow S_0$ or; II) $K/\mu_n^2 \longrightarrow \infty$ and $\mu_n S_n^{-1} \longrightarrow \bar{S}_0$.

To state a limiting distribution result it is helpful to also assume that certain objects converge. Let $\sigma_i^2 = E[\varepsilon_i^2]$, $\gamma_n = \sum_{i=1}^n E[U_i \varepsilon_i]/\sum_{i=1}^n \sigma_i^2$, $\tilde{U} = U - \varepsilon \gamma_n'$, having i^{th} row $\tilde{U}_i'$; and let $\tilde{\Omega}_i = E[\tilde{U}_i \tilde{U}_i']$.

Assumption 6: $H_P = \lim_{n \longrightarrow \infty} \sum_{i=1}^n (1 - P_{ii}) z_i z_i'/n$, $\Sigma_P = \lim_{n \longrightarrow \infty} \sum_{i=1}^n (1 - P_{ii})^2 z_i z_i' \sigma_i^2/n$ and $\Psi = \lim_{n \longrightarrow \infty} \sum_{i \neq j} P_{ij}^2 \left(\sigma_i^2 E[\tilde{U}_j \tilde{U}_j'] + E[\tilde{U}_i \varepsilon_i] E[\varepsilon_j \tilde{U}_j']\right)/K$.

This convergence condition can be replaced by an assumption that certain matrices are uniformly positive definite without affecting the limiting distribution result for t-ratios given in Theorem 3 below (see Chao et. al. 2007).

We can now state the asymptotic normality results. In Case I we have that

$$S_n'(\hat{\delta} - \delta_0) \xrightarrow{d} N(0, \Lambda_I), \tag{6.3}$$

where

$$\Lambda_I = H_P^{-1} \Sigma_P H_P^{-1} + H_P^{-1} S_0 \Psi S_0' H_P^{-1}.$$

In Case II, we have that

$$(\mu_n/\sqrt{K}) S_n'(\hat{\delta} - \delta_0) \xrightarrow{d} N(0, \Lambda_{II}), \tag{6.4}$$

where

$$\Lambda_{II} = H_P^{-1} \bar{S}_0 \Psi \bar{S}_0' H_P^{-1}.$$

The asymptotic variance expressions allow for the many instrument sequence of Kunitomo (1980) and Bekker (1994) and the many weak instrument sequence of Chao and Swanson (2004, 2005). In Case I, the first term in the asymptotic variance, Λ_I, corresponds to the usual asymptotic variance, and the second is an adjustment for the presence of many instruments. In Case II, the asymptotic variance, Λ_{II}, only contains the adjustment for many instruments. This is because K is growing faster than μ_n^2. Also, Λ_{II} will be singular when included exogenous variables are present.

We can now state an asymptotic normality result.

Theorem 2: *If Assumptions 1-6 are satisfied, $\hat{\alpha} = \tilde{\alpha} + O_p(1/T)$ or $\hat{\delta}$ is HLIM or HFUL, then in Case I, equation (6.3) is satisfied, and in Case II, equation (6.4) is satisfied.*

It is interesting to compare the asymptotic variance of the HLIM estimator with that of LIML when the disturbances are homoskedastic. Under homoskedasticity the variance of $Var((\varepsilon_i, U_i'))$ will not depend on i (e.g. so that $\sigma_i^2 = \sigma^2$). Then, $\gamma_n = E[X_i\varepsilon_i]/\sigma^2 = \gamma$ and $E[\tilde{U}_i\varepsilon_i] = E[U_i\varepsilon_i] - \gamma\sigma^2 = 0$, so that

$$\Sigma_P = \sigma^2 \tilde{H}_P, \tilde{H}_P = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} (1 - P_{ii})^2 z_i z_i'/n, \Psi = \sigma^2 E[\tilde{U}_j \tilde{U}_j'](1 - \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} P_{ii}^2/K).$$

Focusing on Case I, letting $\Gamma = \sigma^2 S_0 E[\tilde{U}_i \tilde{U}_i'] S_0'$, the asymptotic variance of HLIM is then

$$V = \sigma^2 H_P^{-1} \tilde{H}_P H_P^{-1} + \lim_{n\longrightarrow\infty} (1 - \sum_{i=1}^{n} P_{ii}^2/K) H_p^{-1} \Gamma H_P^{-1}.$$

For the variance of LIML, assume that third and fourth moments obey the same restrictions that they do under normality. Then from Hansen, Hausman, and Newey (2008), for $H = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} z_i z_i'/n$ and $\tau = \lim_{n\longrightarrow\infty} K/n$, the asymptotic variance of LIML is

$$V^* = \sigma^2 H^{-1} + (1 - \tau)^{-1} H^{-1} \Gamma H^{-1}.$$

With many weak instruments, where $\tau = 0$ and $\max_{i\leq n} P_{ii} \longrightarrow 0$, we will have $H_P = \tilde{H}_P = H$ and $\lim_{n\longrightarrow\infty} \sum_i P_{ii}^2/K \longrightarrow 0$, so that the asymptotic variances of HLIM and LIML are the same and equal to $\sigma^2 H^{-1} + H^{-1}\Gamma H^{-1}$. This case is most important in practical applications, where K is usually very small relative to n. In such cases we would expect from the asymptotic approximation to find that the variance of LIML and HLIM are very similar. Also, the JIV estimators will be inefficient relative to LIML and HLIM. As shown in Chao and Swanson (2004), under many weak instruments the asymptotic variance of JIV is

$$V_{JIV} = \sigma^2 H^{-1} + H^{-1} S_0 (\sigma^2 E[U_i U_i'] + E[U_i\varepsilon_i] E[\varepsilon_i U_i']) S_0' H^{-1},$$

which is larger than the asymptotic variance of HLIM because $E[U_i U_i'] \geq E[\tilde{U}_i \tilde{U}_i']$.

In the many instruments case, where K and μ_n^2 grow as fast as n, it turns out that we cannot rank the asymptotic variances of LIML and HLIM. To show this, consider an example where $p = 1$, z_i alternates between $-\bar{z}$ and $\bar{z}$ for $\bar{z} \neq 0$, $S_n = \sqrt{n}$ (so that $\Upsilon_i = z_i$), and z_i is included among the elements of Z_i. Then, for $\tilde{\Omega} = E[\tilde{U}_i^2]$ and $\kappa = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} P_{ii}^2/K$ we find that

$$V - V^* = \frac{\sigma^2}{\bar{z}^2 (1 - \tau)^2} (\tau\kappa - \tau^2) \left(1 - \frac{\tilde{\Omega}}{\bar{z}^2}\right).$$

Since $\tau\kappa - \tau^2$ is the limit of the sample variance of P_{ii}, which we assume to be positive, $V \geq V^*$ if and only if $\bar{z}^2 \geq \tilde{\Omega}$. Here, $\bar{z}^2$ is the limit of the sample variance of z_i. Thus, the asymptotic variance ranking can go either way depending on whether the sample variance of z_i is bigger than the variance of $\tilde{U}_i$. In applications where the sample size is large relative to the number of instruments, these efficiency differences will tend to be quite small, because P_{ii} is small.

For homoskedastic, non-Gaussian disturbances, it is also interesting to note that the asymptotic variance of HLIM does not depend on third and fourth moments of the disturbances, while that of LIML does (see Bekker and van der Ploeg (2005) and van Hasselt (2000)). This makes estimation of the asymptotic variance simpler for HLIM than for LIML.

It remains to establish the consistency of the asymptotic variance estimator, and to show that confidence intervals can be formed for linear combinations of the coefficients in the usual way. The following theorem accomplishes this, under additional conditions on z_i.

THEOREM 3: *If Assumptions 1-6 are satisfied, and $\hat{\alpha} = \tilde{\alpha} + O_p(1/T)$ or $\hat{\delta}$ is HLIM or HFUL, there exists a C with $\|z_i\| \leq C$ for all i, and there exists a π_n, such that $\max_{i \leq n} \|z_i - \pi_n Z_i\| \longrightarrow 0$, then in Case I, $S_n' \hat{V} S_n \xrightarrow{p} \Lambda_I$ and in Case II, $\mu_n^2 S_n' \hat{V} S_n / K \xrightarrow{p} \Lambda_{II}$.. Also, if $c' S_0' \Lambda_I S_0 c \neq 0$ in Case I or $c' \bar{S}_0' \Lambda_{II} \bar{S}_0 c \neq 0$ in Case II, then*

$$\frac{c'(\hat{\delta} - \delta_0)}{\sqrt{c' \hat{V} c}} \xrightarrow{d} N(0,1).$$

This result allows us to form confidence intervals and test statistics for a single linear combination of parameters in the usual way.

7 Monte Carlo Results

In this Monte Carlo simulation, we provide evidence concerning the finite sample behavior of HLIM and HFUL. The model that we consider is

$$y_i = \delta_{10} + \delta_{20} x_{2i} + \varepsilon_i, x_{2i} = \pi z_{1i} + U_{2i}$$

where $z_{i1} \sim N(0,1)$ and $U_{2i} \sim N(0,1)$. The i^{th} instrument observation is

$$Z_i' = (1, z_{1i}, z_{1i}^2, z_{1i}^3, z_{1i}^4, z_{1i}D_{i1}, ..., z_{1i}D_{i,K-5}),$$

where $D_{ik} \in \{0,1\}$, $\Pr(D_{ik} = 1) = 1/2$, and $z_{i1} \sim N(0,1)$. Thus, the instruments consist of powers of a standard normal up to the fourth power plus interactions with dummy variables. Only z_1 affects the reduced form, so that adding the other instruments does not improve asymptotic efficiency of the LIML or FULL estimators, though the powers of z_{i1} do help with asymptotic efficiency of the CUE.

The structural disturbance, ε, is allowed to be heteroskedastic, being given by

$$\varepsilon = \rho U_2 + \sqrt{\frac{1-\rho^2}{\phi^2 + (0.86)^4}}(\phi v_1 + 0.86 v_2), v_1 \sim N(0, z_1^2), v_2 \sim N(0, (0.86)^2),$$

where v_1 and v_2 are independent of U_2. This is a design that will lead to LIML being inconsistent with many instruments. Here, $E[X_i\varepsilon_i]$ is constant and σ_i^2 is quadratic in z_{i1}, so that $\gamma_i = (C_1 + C_2 z_{i1} + C_3 z_{i1}^2)^{-1}A$, for a constant vector A and constants C_1, C_2, C_3. In this case, P_{ii} will be correlated with $\gamma_i = E[X_i\varepsilon_i]/\sigma_i^2$ so that LIML is not consistent.

We report properties of estimators and t-ratios for δ_2. We set $n = 800$ and $\rho = 0.3$ throughout and choose $K = 2, 10, 30$. We choose π so that the concentration parameter is $n\pi^2 = \mu^2 = 8, 16, 32$. We also choose ϕ so that the R-squared for the regression of ε^2 on the instruments is 0, 0.1, or 0.2.

Below, we report results on median bias and the range between the .05 and .95 quantiles for LIML, HLIM, the jackknife CUE, JIV, HFUL ($C = 1$), HFUL$1/k$ ($C = 1/K$), CUE, and FULL. Interquartile range results were similar. We find that under homoskedasticity, LIML and HFUL have quite similar properties, though LIML is slightly less biased. Under heteroskedasticity, HFUL is much less biased and also much less dispersed than LIML. Thus, we find that heteroskedasticity can bias LIML. We also find that the dispersion of LIML is substantially larger than HFUL. Thus we find a lower bias for HFUL under heteroskedasticity and many instruments, as predicted by the theory, as well as substantially lower dispersion, which though not predicted by the theory may turn out to be important in practice. In additional tables following the references, we also find that coverage probabilities using the heteroskedasticity and many instrument consistent standard errors are quite accurate.

Median Bias $\mathcal{R}^2_{\varepsilon^2|z_1^2} = 0.00$

μ^2	K	$LIML$	$HLIM$	$FULL1$	$HFUL$	$HFUL\frac{1}{k}$	$JIVE$	CUE	$JCUE$
8	0	0.005	0.005	0.042	0.043	0.025	−0.034	0.005	0.005
8	8	0.024	0.023	0.057	0.057	0.027	0.053	0.025	0.032
8	28	0.065	0.065	0.086	0.091	0.067	0.164	0.071	0.092
32	0	0.002	0.002	0.011	0.011	0.007	−0.018	0.002	0.002
32	8	0.002	0.001	0.011	0.011	0.002	−0.019	0.002	0.002
32	28	0.003	0.002	0.013	0.013	0.003	−0.014	0.006	0.006

***Results based on 20,000 simulations.

Nine Decile Range: .05 to .95 $\mathcal{R}^2_{\varepsilon^2|z_1^2} = 0.00$

μ^2	K	$LIML$	$HLIM$	$FULL1$	$HFUL$	$HFUL\frac{1}{k}$	$JIVE$	CUE	$JCUE$
8	0	1.470	1.466	1.072	1.073	1.202	3.114	1.470	1.487
8	8	2.852	2.934	1.657	1.644	2.579	5.098	3.101	3.511
8	28	5.036	5.179	2.421	2.364	4.793	6.787	6.336	6.240
32	0	0.616	0.616	0.590	0.589	0.602	0.679	0.616	0.616
32	8	0.715	0.716	0.679	0.680	0.713	0.816	0.770	0.767
32	28	0.961	0.985	0.901	0.913	0.983	1.200	1.156	1.133

***Results based on 20,000 simulations.

Median Bias $\mathcal{R}^2_{\varepsilon^2|z_1^2} = 0.20$

μ^2	K	$LIML$	$HLIM$	$FULL1$	$HFUL$	$HFUL\frac{1}{k}$	$JIVE$	CUE	$JCUE$
8	0	−0.001	0.050	0.041	0.078	0.065	−0.031	−0.001	0.012
8	8	−0.623	0.094	−0.349	0.113	0.096	0.039	0.003	−0.005
8	28	−1.871	0.134	−0.937	0.146	0.134	0.148	−0.034	0.076
32	0	−0.001	0.011	0.008	0.020	0.016	−0.021	−0.001	−0.003
32	8	−0.220	0.015	−0.192	0.024	0.016	−0.021	0.000	−0.019
32	28	−1.038	0.016	−0.846	0.027	0.017	−0.016	−0.017	−0.021

***Results based on 20,000 simulations.

Nine Decile Range: .05 to .95 $\mathcal{R}^2_{\varepsilon^2|z_1^2} = 0.20$

μ^2	K	$LIML$	$HLIM$	$FULL1$	$HFUL$	$HFUL\frac{1}{k}$	$JIVE$	CUE	$JCUE$
8	0	2.219	1.868	1.675	1.494	1.653	4.381	2.219	2.582
8	8	26.169	5.611	4.776	2.664	4.738	7.781	16.218	8.586
8	28	60.512	8.191	7.145	3.332	7.510	9.975	1.5E+012	12.281
32	0	0.941	0.901	0.903	0.868	0.884	1.029	0.941	0.946
32	8	3.365	1.226	2.429	1.134	1.217	1.206	1.011	1.086
32	28	18.357	1.815	5.424	1.571	1.808	1.678	3.563	1.873

***Results based on 20,000 simulations.

8 Appendix: Proofs of Consistency and Asymptotic Normality

Throughout, let C denote a generic positive constant that may be different in different uses and let M, CS, and T denote the conditional Markov inequality, the Cauchy-Schwartz inequality, and the Triangle inequality respectively. The first Lemma is proved in Hansen, Hausman, and Newey (2006).

Lemma A0: *If Assumption 2 is satisfied and* $\left\|S_n'(\hat{\delta}-\delta_0)/\mu_n\right\|^2 / \left(1+\left\|\hat{\delta}\right\|^2\right) \xrightarrow{p} 0$ *then* $\left\|S_n'(\hat{\delta}-\delta_0)/\mu_n\right\| \xrightarrow{p} 0.$

We next give a result from Chao et al. (2007) that is used in the proof of consistency.

Lemma A1 (Lemma A1 of Chao et al., 2007): *If* $(W_i, Y_i), (i = 1, ..., n)$ *are independent,* W_i *and* Y_i *are scalars, and* P *is symmetric, idempotent of rank* K *then for* $\bar{w} = E[(W_1, ..., W_n)']$, $\bar{y} = E[(Y_1, ..., Y_n)']$, $\bar{\sigma}_{Wn} = \max_{i\le n} Var(W_i)^{1/2}$, $\bar{\sigma}_{Yn} = \max_{i\le n} Var(Y_i)^{1/2}$,

$$\sum_{i\neq j} P_{ij}W_iY_j = \sum_{i\neq j} P_{ij}\bar{w}_i\bar{y}_j + O_p(K^{1/2}\bar{\sigma}_{Wn}\bar{\sigma}_{Yn} + \bar{\sigma}_{Wn}\sqrt{\bar{y}'\bar{y}} + \bar{\sigma}_{Yn}\sqrt{\bar{w}'\bar{w}}).$$

For the next result let $\bar{S}_n = diag(\mu_n, S_n)$, $\tilde{X} = [\varepsilon, X]\bar{S}_n^{-1\prime}$, and $H_n = \sum_{i=1}^n (1 - P_{ii})z_iz_i'/n$.

Lemma A2: *If Assumptions 1-4 are satisfied and* $\sqrt{K}/\mu_n^2 \longrightarrow 0$ *then*

$$\sum_{i\neq j} \tilde{X}_iP_{ij}\tilde{X}_j' = diag(0, H_n) + o_p(1).$$

Proof: Note that

$$\tilde{X}_i = \begin{pmatrix} \mu_n^{-1}\varepsilon_i \\ S_n^{-1}X_i \end{pmatrix} = \begin{pmatrix} 0 \\ z_i/\sqrt{n} \end{pmatrix} + \begin{pmatrix} \mu_n^{-1}\varepsilon_i \\ S_n^{-1}U_i \end{pmatrix}.$$

Since $\|S_n^{-1}\| \le C\mu_n^{-1}$ we have $Var(\tilde{X}_{ik}) \le C\mu_n^{-2}$ for any element $\tilde{X}_{ik}$ of $\tilde{X}_i$. Then applying Lemma A1 to each element of $\sum_{i\neq j}\tilde{X}_iP_{ij}\tilde{X}_j'$ gives

$$\begin{aligned}\sum_{i\neq j} \tilde{X}_iP_{ij}\tilde{X}_j' &= diag(0, \sum_{i\neq j} z_iP_{ij}z_j'/n) + O_p(K^{1/2}/\mu_n^2 + \mu_n^{-1}(\sum_i \|z_i\|^2/n)^{1/2}) \\ &= diag(0, \sum_{i\neq j} z_iP_{ij}z_j'/n) + o_p(1).\end{aligned}$$

Also, note that

$$\begin{aligned} H_n - \sum_{i\neq j} z_i P_{ij} z_j'/n &= \sum_i z_i z_i'/n - \sum_i P_{ii} z_i z_i'/n - \sum_{i\neq j} z_i P_{ij} z_j'/n = z'(I-P)z/n \\ &= (z - Z\pi_{Kn}')'(I-P)(z - Z\pi_{Kn}')/n \leq (z - Z\pi_{Kn}')'(z - Z\pi_{Kn}')/n \\ &\leq I_G \sum_i \|z_i - \pi_{Kn} Z_i\|^2/n \longrightarrow 0, \end{aligned}$$

where the third equality follows by $PZ = Z$, the first inequality by $I - P$ idempotent, and the last inequality by $A \leq tr(A)I$ for any positive semi-definite (p.s.d.) matrix A. Since this equation shows that $H_n - \sum_{i\neq j} z_i P_{ij} z_j'/n$ is p.s.d. and is less than or equal to another p.s.d. matrix that converges to zero it follows that $\sum_{i\neq j} z_i P_{ij} z_j'/n = H_n + o_p(1)$. The conclusion follows by T. Q.E.D.

In what follows it is useful to prove directly that the HLIM estimator $\tilde{\delta}$ satisfies $S_n'(\tilde{\delta} - \delta_0)/\mu_n \xrightarrow{p} 0$.

Lemma A3: *If Assumptions 1-4 are satisfied then* $S_n'(\tilde{\delta} - \delta_0)/\mu_n \xrightarrow{p} 0$.

Proof: Let $\bar{\Upsilon} = [0, \Upsilon]$, $\bar{U} = [\varepsilon, U]$, $\bar{X} = [y, X]$, so that $\bar{X} = (\bar{\Upsilon} + \bar{U})D$ for

$$D = \begin{bmatrix} 1 & 0 \\ \delta_0 & I \end{bmatrix}.$$

Let $\hat{B} = \bar{X}'\bar{X}/n$. Note that $\|S_n/\sqrt{n}\| \leq C$ and by standard calculations $z'U/n \xrightarrow{p} 0$. Then

$$\left\|\bar{\Upsilon}'\bar{U}/n\right\| = \left\|\left(S_n/\sqrt{n}\right) z'U/n\right\| \leq C\,\|z'U/n\| \xrightarrow{p} 0.$$

Let $\bar{\Omega}_n = \sum_{i=1}^n E[\bar{U}_i \bar{U}_i']/n = diag(\sum_{i=1}^n \Omega_i^*/n, 0) \geq C diag(I_{G-G_2+1}, 0)$ by Assumption 3. By M we have $\bar{U}'\bar{U}/n - \bar{\Omega}_n \xrightarrow{p} 0$, so it follows that w.p.a.1.

$$\hat{B} = (\bar{U}'\bar{U} + \bar{\Upsilon}'\bar{U} + \bar{U}'\bar{\Upsilon} + \bar{\Upsilon}'\bar{\Upsilon})/n = \bar{\Omega}_n + \bar{\Upsilon}'\bar{\Upsilon}/n + o_p(1) \geq C diag(I_{G-G_2+1}, 0).$$

Since $\bar{\Omega}_n + \bar{\Upsilon}'\bar{\Upsilon}/n$ is bounded, it follows that w.p.a.1,

$$C \leq (1, -\delta')\hat{B}(1, -\delta')' = (y - X\delta)'(y - X\delta)/n \leq C\,\|(1, -\delta')\|^2 = C(1 + \|\delta\|^2).$$

Next, as defined preceding Lemma A2 let $\bar{S}_n = diag(\mu_n, S_n)$ and $\tilde{X} = [\varepsilon, X]\bar{S}_n^{-1\prime}$. Note that by $P_{ii} \le C < 1$ and uniform nonsingularity of $\sum_{i=1}^n z_i z_i'/n$ we have $H_n \ge (1-C)\sum_{i=1}^n z_i z_i'/n \ge CI_G$. Then by Lemma A2, w.p.a.1.

$$\hat{A} \overset{def}{=} \sum_{i\neq j} P_{ij}\tilde{X}_i\tilde{X}_j' \ge C diag(0, I_G),$$

Note that $\bar{S}_n' D(1,-\delta')' = (\mu_n, (\delta_0-\delta)'S_n)'$ and $\bar{X}_i = D'\bar{S}_n\tilde{X}_i$. Then w.p.a.1 for all δ

$$\begin{aligned}\mu_n^{-2}\sum_{i\neq j} P_{ij}(y_i - X_i'\delta)(y_j - X_j'\delta) &= \mu_n^{-2}(1,-\delta')\left(\sum_{i\neq j} P_{ij}\bar{X}_i\bar{X}_j'\right)(1,-\delta')' \\ &= \mu_n^{-2}(1,-\delta')D'\bar{S}_n\hat{A}\bar{S}_n' D(1,-\delta')' \ge C\left\|S_n'(\delta-\delta_0)/\mu_n\right\|^2.\end{aligned}$$

Let $\hat{Q}(\delta) = (n/\mu_n^2)\sum_{i\neq j}(y_i - X_i'\delta)P_{ij}(y_j - X_j'\delta)/(y-X\delta)'(y-X\delta)$. Then by the upper left element of the conclusion of Lemma A2, $\mu_n^{-2}\sum_{i\neq j}\varepsilon_i P_{ij}\varepsilon_j \xrightarrow{p} 0$. Then w.p.a.1

$$\left|\hat{Q}(\delta_0)\right| = \left|\mu_n^{-2}\sum_{i\neq j}\varepsilon_i P_{ij}\varepsilon_j / \sum_{i=1}^n \varepsilon_i^2/n\right| \xrightarrow{p} 0.$$

Since $\hat{\delta} = \arg\min_\delta \hat{Q}(\delta)$, we have $\hat{Q}(\hat{\delta}) \le \hat{Q}(\delta_0)$.Therefore w.p.a.1, by $(y-X\delta)'(y-X\delta)/n \le C(1+\|\delta\|^2)$, it follows that

$$0 \le \frac{\left\|S_n'(\hat{\delta}-\delta_0)/\mu_n\right\|^2}{1+\left\|\hat{\delta}\right\|^2} \le C\hat{Q}(\hat{\delta}) \le C\hat{Q}(\delta_0) \xrightarrow{p} 0,$$

implying $\left\|S_n'(\hat{\delta}-\delta_0)/\mu_n\right\|^2 / \left(1+\left\|\hat{\delta}\right\|^2\right) \xrightarrow{p} 0$. Lemma A0 gives the conclusion. Q.E.D.

Lemma A4: *If Assumptions 1-4 are satisfied,* $\hat{\alpha} = o_p(\mu_n^2/n)$, *and* $S_n'(\hat{\delta}-\delta_0)/\mu_n \xrightarrow{p} 0$ *then for* $H_n = \sum_{i=1}^n (1-P_{ii})z_i z_i'/n$,

$$S_n^{-1}\left(\sum_{i\neq j} X_i P_{ij} X_j' - \hat{\alpha}X'X\right)S_n^{-1\prime} = H_n + o_p(1), S_n^{-1}(\sum_{i\neq j} X_i P_{ij}\hat{\varepsilon}_j - \hat{\alpha}X'\hat{\varepsilon})/\mu_n \xrightarrow{p} 0.$$

Proof: By M and standard arguments $X'X = O_p(n)$ and $X'\hat{\varepsilon} = O_p(n)$. Therefore, by $\|S_n^{-1}\| = O(\mu_n^{-1})$,

$$\hat{\alpha}S_n^{-1}X'XS_n^{-1\prime} = o_p(\mu_n^2/n)O_p(n/\mu_n^2) \xrightarrow{p} 0, \hat{\alpha}S_n^{-1}X'\hat{\varepsilon}/\mu_n = o_p(\mu_n^2/n)O_p(n/\mu_n^2) \xrightarrow{p} 0.$$

Lemma A2 (lower right hand block) and T then give the first conclusion. By Lemma A2 (off diagonal) we have $S_n^{-1}\sum_{i\neq j} X_i P_{ij}\varepsilon_j/\mu_n \xrightarrow{p} 0$, so that

$$S_n^{-1}\sum_{i\neq j} X_i P_{ij}\hat{\varepsilon}_j/\mu_n = o_p(1) - \left(S_n^{-1}\sum_{i\neq j} X_i P_{ij} X_j' S_n^{-1\prime}\right) S_n'(\hat{\delta}-\delta_0)/\mu_n \xrightarrow{p} 0.Q.E.D.$$

LEMMA A5: *If Assumptions 1 - 4 are satisfied and* $S_n'(\hat{\delta}-\delta_0)/\mu_n \xrightarrow{p} 0$ *then* $\sum_{i\neq j}\hat{\varepsilon}_i P_{ij}\hat{\varepsilon}_j/\hat{\varepsilon}'\hat{\varepsilon} = o_p(\mu_n^2/n)$.

Proof: Let $\hat{\beta} = S_n'(\hat{\delta}-\delta_0)/\mu_n$ and $\breve{\alpha} = \sum_{i\neq j}\varepsilon_i P_{ij}\varepsilon_j/\varepsilon'\varepsilon = o_p(\mu_n^2/n)$. Note that $\hat{\sigma}_\varepsilon^2 = \hat{\varepsilon}'\hat{\varepsilon}/n$ satisfies $1/\hat{\sigma}_\varepsilon^2 = O_p(1)$ by M. By Lemma A4 with $\hat{\alpha} = \breve{\alpha}$ we have $\tilde{H}_n = S_n^{-1}(\sum_{i\neq j} X_i P_{ij} X_j' - \breve{\alpha} X'X) S_n^{-1\prime} = O_p(1)$ and $W_n = S_n^{-1}(X'P\varepsilon - \breve{\alpha}X'\varepsilon)/\mu_n \xrightarrow{p} 0$, so

$$\begin{aligned}
\frac{\sum_{i\neq j}\hat{\varepsilon}_i P_{ij}\hat{\varepsilon}_j}{\hat{\varepsilon}'\hat{\varepsilon}} - \breve{\alpha} &= \frac{1}{\hat{\varepsilon}'\hat{\varepsilon}}\left(\sum_{i\neq j}\hat{\varepsilon}_i P_{ij}\hat{\varepsilon}_j - \sum_{i\neq j}\varepsilon_i P_{ij}\varepsilon_j - \breve{\alpha}\left(\hat{\varepsilon}'\hat{\varepsilon} - \varepsilon'\varepsilon\right)\right) \\
&= \frac{\mu_n^2}{n}\frac{1}{\hat{\sigma}_\varepsilon^2}\left(\hat{\beta}'\tilde{H}_n\hat{\beta} - 2\hat{\beta}'W_n\right) = o_p(\mu_n^2/n),
\end{aligned}$$

so the conclusion follows by T. Q.E.D.

Proof of Theorem 1: First, note that if $S_n'(\hat{\delta}-\delta_0)/\mu_n \xrightarrow{p} 0$ then by $\lambda_{\min}\left(S_n S_n'/\mu_n^2\right) \geq \lambda_{\min}\left(\tilde{S}_n\tilde{S}_n'\right) \geq C$ we have

$$\left\|S_n'(\hat{\delta}-\delta_0)/\mu_n\right\| \geq \lambda_{\min}(S_n S_n'/\mu_n^2)^{1/2}\left\|\hat{\delta}-\delta_0\right\| \geq C\left\|\hat{\delta}-\delta_0\right\|,$$

implying $\hat{\delta} \xrightarrow{p} \delta_0$. Therefore, it suffices to show that $S_n'(\hat{\delta}-\delta_0)/\mu_n \xrightarrow{p} 0$. For HLIM this follows from Lemma A3. For HFUL, note that $\tilde{\alpha} = \hat{Q}(\tilde{\delta}) = \sum_{i\neq j}\tilde{\varepsilon}_i P_{ij}\tilde{\varepsilon}_j/\tilde{\varepsilon}'\tilde{\varepsilon} = o_p(\mu_n^2/n)$ by Lemma A5, so by the formula for HFUL, $\hat{\alpha} = \tilde{\alpha} + O_p(1/n) = o_p(\mu_n^2/n)$. Thus, the result for HFUL will follow from the most general result for any $\hat{\alpha}$ with $\hat{\alpha} = o_p(\mu_n^2/n)$. For any such $\hat{\alpha}$, by Lemma A4 we have

$$\begin{aligned}
S_n'(\hat{\delta}-\delta_0)/\mu_n &= S_n'(\sum_{i\neq j} X_i P_{ij} X_j' - \hat{\alpha}X'X)^{-1}\sum_{i\neq j}\left(X_i P_{ij}\varepsilon_j - \hat{\alpha}X'\varepsilon\right)/\mu_n \\
&= [S_n^{-1}(\sum_{i\neq j} X_i P_{ij} X_j' - \hat{\alpha}X'X)S_n^{-1\prime}]^{-1} S_n^{-1}\sum_{i\neq j}\left(X_i P_{ij}\varepsilon_j - \hat{\alpha}X'\varepsilon\right)/\mu_n \\
&= (H_n + o_p(1))^{-1} o_p(1) \xrightarrow{p} 0.Q.E.D.
\end{aligned}$$

Now we move on to asymptotic normality results. The next result is a central limit theorem that is proven in Chao et. al. (2007).

Lemma A6 (Lemma A2 of Chao et al., 2007): *If i) P is a symmetric, idempotent matrix with $rank(P) = K$, $P_{ii} \leq C < 1$; ii) $(W_{1n}, U_1, \varepsilon_1)$, ..., $(W_{nn}, U_n, \varepsilon_n)$ are independent and $D_n = \sum_{i=1}^{n} E[W_{in}W_{in}']$* is bounded; *iii) $E[W_{in}'] = 0$, $E[U_i] = 0$, $E[\varepsilon_i] = 0$ and there exists a constant C such that $E[\|U_i\|^4] \leq C$, $E[\varepsilon_i^4] \leq C$; iv) $\sum_{i=1}^{n} E[\|W_{in}\|^4] \longrightarrow 0$; v) $K \longrightarrow \infty$; then for $\bar{\Sigma}_n \stackrel{\text{def}}{=} \sum_{i \neq j} P_{ij}^2 \left(E[U_iU_i']E[\varepsilon_j^2] + E[U_i\varepsilon_i]E[\varepsilon_jU_j'] \right)/K$ and for any sequence of bounded nonzero vectors c_{1n} and c_{2n} such that $\Xi_n = c_{1n}'D_nc_{1n} + c_{2n}'\bar{\Sigma}_nc_{2n} > C$, it follows that*

$$Y_n = \Xi_n^{-1/2}(\sum_{i=1}^{n} c_{1n}'W_{in} + c_{2n}'\sum_{i \neq j} U_iP_{ij}\varepsilon_j/\sqrt{K}) \xrightarrow{d} N(0, 1).$$

Let $\tilde{\alpha}(\delta) = \sum_{i \neq j} \varepsilon_i(\delta)P_{ij}\varepsilon_j(\delta)/\varepsilon(\delta)'\varepsilon(\delta)$ and

$$\hat{D}(\delta) = \partial[\sum_{i \neq j} \varepsilon_i(\delta)P_{ij}\varepsilon_j(\delta)/2\varepsilon(\delta)'\varepsilon(\delta)]/\partial\delta = \sum_{i \neq j} X_iP_{ij}\varepsilon_j(\delta) - \tilde{\alpha}(\delta)X'\varepsilon(\delta).$$

A couple of other intermediate results are also useful.

Lemma A7: *If Assumptions 1 - 4 are satisfied and $S_n'(\bar{\delta} - \delta_0)/\mu_n \xrightarrow{p} 0$ then*

$$-S_n^{-1}[\partial\hat{D}(\bar{\delta})/\partial\delta]S_n^{-1\prime} = H_n + o_p(1).$$

Proof: Let $\bar{\varepsilon} = \varepsilon(\bar{\delta}) = y - X\bar{\delta}$, $\bar{\gamma} = X'\bar{\varepsilon}/\bar{\varepsilon}'\bar{\varepsilon}$, and $\bar{\alpha} = \tilde{\alpha}(\bar{\delta})$. Then differentiating gives

$$\begin{aligned} -\frac{\partial\hat{D}}{\partial\delta}(\bar{\delta}) &= \sum_{i \neq j} X_iP_{ij}X_j' - \bar{\alpha}X'X - \bar{\gamma}\sum_{i \neq j} \bar{\varepsilon}_iP_{ij}X_j' - \sum_{i \neq j} X_iP_{ij}\bar{\varepsilon}_j\bar{\gamma}' + 2(\bar{\varepsilon}'\bar{\varepsilon})\bar{\alpha}\bar{\gamma}\bar{\gamma}' \\ &= \sum_{i \neq j} X_iP_{ij}X_j' - \bar{\alpha}X'X + \bar{\gamma}\hat{D}(\bar{\delta})' + \hat{D}(\bar{\delta})\bar{\gamma}', \end{aligned}$$

where the second equality follows by $\hat{D}(\bar{\delta}) = \sum_{i \neq j} X_iP_{ij}\bar{\varepsilon}_j - (\bar{\varepsilon}'\bar{\varepsilon})\bar{\alpha}\bar{\gamma}$. By Lemma A5 we have $\bar{\alpha} = o_p(\mu_n^2/n)$. By standard arguments, $\bar{\gamma} = O_p(1)$ so that $S_n^{-1}\bar{\gamma} = O_p(1/\mu_n)$. Then by Lemma A4 and $\hat{D}(\bar{\delta}) = \sum_{i \neq j} X_iP_{ij}\bar{\varepsilon}_j - \bar{\alpha}X'\bar{\varepsilon}$

$$S_n^{-1}\left(\sum_{i \neq j} X_iP_{ij}X_j' - \bar{\alpha}X'X\right)S_n^{-1\prime} = H_n + o_p(1), S_n^{-1}\hat{D}(\bar{\delta})\bar{\gamma}'S_n^{-1\prime} \xrightarrow{p} 0,$$

The conclusion then follows by T. Q.E.D.

LEMMA A8: *If Assumptions 1-4 are satisfied then for* $\gamma_n = \sum_i E[U_i\varepsilon_i]/\sum_i E[\varepsilon_i^2]$ *and* $\tilde{U}_i = U_i - \gamma_n \varepsilon_i$

$$S_n^{-1}\hat{D}(\delta_0) = \sum_{i=1}^{n}(1-P_{ii})z_i\varepsilon_i/\sqrt{n} + S_n^{-1}\sum_{i\neq j}\tilde{U}_i P_{ij}\varepsilon_j + o_p(1).$$

Proof: Note that for $W = z'(P-I)\varepsilon/\sqrt{n}$ by $I-P$ idempotent and $E[\varepsilon\varepsilon'] \leq CI_n$ we have

$$\begin{aligned} E[WW'] &\leq Cz'(I-P)z/n = C(z - Z\pi_{Kn}')'(I-P)(z-Z\pi_{Kn}')/n \\ &\leq CI_G \sum_{i=1}^{n} \|z_i - \pi_{Kn}Z_i\|^2 /n \longrightarrow 0, \end{aligned}$$

so $z'(P-I)\varepsilon/\sqrt{n} = o_p(1)$. Also, by M

$$X'\varepsilon/n = \sum_{i=1}^{n} E[X_i\varepsilon_i]/n + O_p(1/\sqrt{n}), \varepsilon'\varepsilon/n = \sum_{i=1}^{n}\sigma_i^2/n + O_p(1/\sqrt{n}).$$

Also, by Assumption 3 $\sum_{i=1}^{n}\sigma_i^2/n \geq C > 0$. The delta method then gives $\tilde{\gamma} = X'\varepsilon/\varepsilon'\varepsilon = \gamma_n + O_p(1/\sqrt{n})$. Therefore, it follows by Lemma A1 and $\hat{D}(\delta_0) = \sum_{i\neq j} X_i P_{ij}\varepsilon_j - \varepsilon'\varepsilon\tilde{\alpha}(\delta_0)\tilde{\gamma}$ that

$$\begin{aligned} S_n^{-1}\hat{D}(\delta_0) &= \sum_{i\neq j} z_i P_{ij}\varepsilon_j/\sqrt{n} + S_n^{-1}\sum_{i\neq j}\tilde{U}_i P_{ij}\varepsilon_i - S_n^{-1}(\tilde{\gamma}-\gamma_n)\varepsilon'\varepsilon\tilde{\alpha}(\delta_0) \\ &= z'P\varepsilon/\sqrt{n} - \sum_i P_{ii}z_i\varepsilon_i/\sqrt{n} + S_n^{-1}\sum_{i\neq j}\tilde{U}_i P_{ij}\varepsilon_j + O_p(1/\sqrt{n}\mu_n)o_p(\mu_n^2/n) \\ &= \sum_{i=1}^{n}(1-P_{ii})z_i\varepsilon_i/\sqrt{n} + S_n^{-1}\sum_{i\neq j}\tilde{U}_i P_{ij}\varepsilon_j + o_p(1). Q.E.D. \end{aligned}$$

Proof of Theorem 2: Consider first the case where $\hat{\delta}$ is HLIM. Then by Theorem 1, $\hat{\delta} \xrightarrow{p} \delta_0$. The first-order conditions for LIML are $\hat{D}(\hat{\delta}) = 0$. Expanding gives

$$0 = \hat{D}(\delta_0) + \frac{\partial \hat{D}}{\partial \delta}\left(\bar{\delta}\right)(\hat{\delta} - \delta_0),$$

where $\bar{\delta}$ lies on the line joining $\hat{\delta}$ and δ_0 and hence $\bar{\beta} = \mu_n^{-1} S_n'(\bar{\delta} - \delta_0) \xrightarrow{p} 0$. Then by Lemma A7, $\bar{H}_n = S_n^{-1}[\partial \hat{D}(\bar{\delta})/\partial \delta] S_n^{-1\prime} = H_P + o_p(1)$. Then $\partial \hat{D}(\bar{\delta})/\partial \delta$ is nonsingular w.p.a.1 and solving gives

$$S_n'(\hat{\delta} - \delta) = -S_n'[\partial \hat{D}(\bar{\delta})/\partial \delta]^{-1} \hat{D}(\delta_0) = -\bar{H}_n^{-1} S_n^{-1} \hat{D}(\delta_0).$$

Next, apply Lemma A6 with $U_i = U_i$ and

$$W_{in} = (1 - P_{ii}) z_i \varepsilon_i / \sqrt{n},$$

By ε_i having bounded fourth moment, and $P_{ii} \leq 1$,

$$\sum_{i=1}^{n} E\left[\|W_{in}\|^4\right] \leq C \sum_{i=1}^{n} \|z_i\|^4 / n^2 \longrightarrow 0.$$

By Assumption 6, we have $\sum_{i=1}^n E[W_{in} W_{in}'] \longrightarrow \Sigma_P$. Let $\Gamma = \text{diag}\,(\Sigma_P, \Psi)$ and

$$A_n = \begin{pmatrix} \sum_{i=1}^n W_{in} \\ \sum_{i \neq j} \tilde{U}_i P_{ij} \varepsilon_j / \sqrt{K} \end{pmatrix}.$$

Consider c such that $c'\Gamma c > 0$. Then by the conclusion of Lemma A6 we have $c'A_n \xrightarrow{d} N(0, c'\Gamma c)$. Also, if $c'\Gamma c = 0$ then it is straightforward to show that $c'A_n \xrightarrow{p} 0$. Then it follows by the Cramer-Wold device that

$$A_n = \begin{pmatrix} \sum_{i=1}^n W_{in} \\ \sum_{i \neq j} \tilde{U}_i P_{ij} \varepsilon_j / \sqrt{K} \end{pmatrix} \xrightarrow{d} N(0, \Gamma), \Gamma = \text{diag}\,(\Sigma_P, \Psi).$$

Next, we consider the two cases. Case I) has K/μ_n^2 bounded. In this case $\sqrt{K} S_n^{-1} \longrightarrow S_0$, so that

$$F_n \stackrel{def}{=} [I, \sqrt{K} S_n^{-1}] \longrightarrow F_0 = [I, S_0], F_0 \Gamma F_0' = \Sigma_P + S_0 \Psi S_0'.$$

Then by Lemma A8,

$$\begin{aligned} S_n^{-1} \hat{D}(\delta_0) &= F_n A_n + o_p(1) \xrightarrow{d} N(0, \Sigma_P + S_0 \Psi S_0'), \\ S_n'(\hat{\delta} - \delta_0) &= -\bar{H}_n^{-1} S_n^{-1} \hat{D}(\delta_0) \xrightarrow{d} N(0, \Lambda_I). \end{aligned}$$

In case II we have $K/\mu_n^2 \longrightarrow \infty$. Here

$$(\mu_n / \sqrt{K}) F_n \longrightarrow \bar{F}_0 = [0, \bar{S}_0], \bar{F}_0 \Gamma \bar{F}_0' = \bar{S}_0 \Psi \bar{S}_0'$$

and $(\mu_n/\sqrt{K})o_p(1) = o_p(1)$. Then by Lemma A8,

$$\begin{aligned}(\mu_n/\sqrt{K})S_n^{-1}\hat{D}(\delta_0) &= (\mu_n/\sqrt{K})F_nA_n + o_p(1) \xrightarrow{d} N(0, \bar{S}_0\Psi\bar{S}_0'),\\ (\mu_n/\sqrt{K})S_n'(\hat{\delta}-\delta_0) &= -\bar{H}_n^{-1}(\mu_n/\sqrt{K})S_n^{-1}\hat{D}(\delta_0) \xrightarrow{d} N(0, \Lambda_{II}).Q.E.D.\end{aligned}$$

The next two results are useful for the proof of consistency of the variance estimator are taken from Chao et. al. (2007). Let $\bar{\mu}_{Wn} = \max_{i\leq n}|E[W_i]|$ and $\bar{\mu}_{Yn} = \max_{i\leq n}|E[Y_i]|$.

Lemma A9 (Lemma A3 of Chao et al., 2007): *If $(W_i, Y_i), (i = 1, ..., n)$ are independent, W_i and Y_i are scalars then*

$$\sum_{i\neq j} P_{ij}^2 W_i Y_j = E[\sum_{i\neq j} P_{ij}^2 W_i Y_j] + O_p(\sqrt{K}(\bar{\sigma}_{Wn}\bar{\sigma}_{Yn} + \bar{\sigma}_{Wn}\bar{\mu}_{Yn} + \bar{\mu}_{Wn}\bar{\sigma}_{Yn})).$$

Lemma A10 (Lemma A4 of Chao et al., 2007): *If W_i, Y_i, η_i, are independent across i with $E[W_i] = a_i/\sqrt{n}$, $E[Y_i] = b_i/\sqrt{n}$, $|a_i| \leq C$, $|b_i| \leq C$, $E[\eta_i^2] \leq C$, $Var(W_i) \leq C\mu_n^{-2}$, $Var(Y_i) \leq C\mu_n^{-2}$, there exists π_n such that $\max_{i\leq n}|a_i - Z_i'\pi_n| \longrightarrow 0$, and $\sqrt{K}/\mu_n^2 \longrightarrow 0$ then*

$$A_n = E[\sum_{i\neq j\neq k} W_i P_{ik}\eta_k P_{kj} Y_j] = O(1), \sum_{i\neq j\neq k} W_i P_{ik}\eta_k P_{kj} Y_j - A_n \xrightarrow{p} 0.$$

Next, recall that $\hat{\varepsilon}_i = Y_i - X_i'\hat{\delta}, \hat{\gamma} = X'\hat{\varepsilon}/\hat{\varepsilon}'\hat{\varepsilon}, \gamma_n = \sum_i E[X_i\varepsilon_i]/\sum_i \sigma_i^2$ and let

$$\begin{aligned}\breve{X}_i &= S_n^{-1}(X_i - \hat{\gamma}\hat{\varepsilon}_i) = S_n^{-1}\hat{X}_i, \dot{X}_i = S_n^{-1}(X_i - \gamma_n\varepsilon_i),\\ \breve{\Sigma}_1 &= \sum_{i\neq j\neq k}\breve{X}_i P_{ik}\hat{\varepsilon}_k^2 P_{kj}\breve{X}_j', \breve{\Sigma}_2 = \sum_{i\neq j} P_{ij}^2\left(\breve{X}_i\breve{X}_i'\hat{\varepsilon}_j^2 + \breve{X}_i\hat{\varepsilon}_i\hat{\varepsilon}_j\breve{X}_j'\right),\\ \dot{\Sigma}_1 &= \sum_{i\neq j\neq k}\dot{X}_i P_{ik}\varepsilon_k^2 P_{kj}\dot{X}_j', \dot{\Sigma}_2 = \sum_{i\neq j} P_{ij}^2\left(\dot{X}_i\dot{X}_i'\varepsilon_j^2 + \dot{X}_i\varepsilon_i\varepsilon_j\dot{X}_j'\right).\end{aligned}$$

Note that for $\hat{\Delta} = S_n'(\hat{\delta} - \delta_0)$ we have

$$\begin{aligned}
\hat{\varepsilon}_i - \varepsilon_i &= -X_i'(\hat{\delta} - \delta_0) = -X_i' S_n^{-1\prime}\hat{\Delta}, \\
\hat{\varepsilon}_i^2 - \varepsilon_i^2 &= -2\varepsilon_i X_i'(\hat{\delta} - \delta_0) + \left[X_i'(\hat{\delta} - \delta_0)\right]^2, \\
\breve{X}_i - \dot{X}_i &= -S_n^{-1}\hat{\gamma}(\hat{\varepsilon}_i - \varepsilon_i) - S_n^{-1}(\hat{\gamma} - \gamma_n)\varepsilon_i, \\
&= S_n^{-1}\hat{\gamma} X_i' S_n^{-1\prime}\hat{\Delta} - S_n^{-1}\mu_n(\hat{\gamma} - \gamma_n)(\varepsilon_i/\mu_n), \\
\breve{X}_i\hat{\varepsilon}_i - \dot{X}_i\varepsilon_i &= X_i\hat{\varepsilon}_i - \hat{\gamma}\hat{\varepsilon}_i^2 - X_i\varepsilon_i + \gamma_n\varepsilon_i^2, \\
&= -X_i X_i'(\hat{\delta} - \delta_0) - \hat{\gamma}\left\{-2\varepsilon_i X_i'(\hat{\delta} - \delta_0) + \left[X_i'(\hat{\delta} - \delta_0)^2\right]\right\} \\
&\quad -(\hat{\gamma} - \gamma_n)\varepsilon_i^2. \\
\left\|\breve{X}_i\breve{X}_i' - \dot{X}_i\dot{X}_i'\right\| &\leq \left\|\breve{X}_i - \dot{X}_i\right\|^2 + 2\left\|\dot{X}_i\right\|\left\|\breve{X}_i - \dot{X}_i\right\|
\end{aligned}$$

LEMMA A11: *If the hypotheses of Theorem 3 are satisfied then* $\breve{\Sigma}_2 - \dot{\Sigma}_2 = o_p(K/\mu_n^2)$.

Proof: Note first that $S_n/\sqrt{n}$ is bounded so by the Cauchy-Schwartz inequality, $\|\Upsilon_i\| = \|S_n z_i/\sqrt{n}\| \leq C$. Let $d_i = C + |\varepsilon_i| + \|U_i\|$. Note that $\hat{\gamma} - \gamma_n \xrightarrow{p} 0$ by standard arguments. Then for $\hat{A} = (1+\|\hat{\gamma}\|)(1+\left\|\hat{\delta}\right\|) = O_p(1)$, and $\hat{B} = \|\hat{\gamma} - \gamma_n\| + \left\|\hat{\delta} - \delta_0\right\| \xrightarrow{p} 0$, we have

$$\begin{aligned}
\|X_i\| &\leq C + \|U_i\| \leq d_i, |\hat{\varepsilon}_i| \leq |X_i'(\delta_0 - \hat{\delta}) + \varepsilon_i| \leq C d_i\hat{A}, \\
\left\|\dot{X}_i\right\| &= \left\|S_n^{-1}(X_i - \gamma_n\varepsilon_i)\right\| \leq C\mu_n^{-1}d_i, \left\|\breve{X}_i\right\| = \left\|S_n^{-1}(X_i - \hat{\gamma}\hat{\varepsilon}_i)\right\| \leq C\mu_n^{-1}d_i\hat{A}, \\
\left\|\breve{X}_i\breve{X}_i' - \dot{X}_i\dot{X}_i'\right\| &\leq \left(\left\|\breve{X}_i\right\| + \left\|\dot{X}_i\right\|\right)\left\|\breve{X}_i - \dot{X}_i\right\| \leq C\mu_n^{-2}d_i\hat{A}\,\|\hat{\gamma}\|\,\|\hat{\varepsilon}_i - \varepsilon_i\| + \|\hat{\gamma} - \gamma_n\|\,|\varepsilon_i| \\
&\leq C\mu_n^{-2}d_i^2\hat{A}^2\hat{B}, \\
\left|\hat{\varepsilon}_i^2 - \varepsilon_i^2\right| &\leq (|\varepsilon_i| + |\hat{\varepsilon}_i|)\,|\hat{\varepsilon}_i - \varepsilon_i| \leq C d_i^2\hat{A}\hat{B}, \\
\left\|\breve{X}_i\hat{\varepsilon}_i - \dot{X}_i\varepsilon_i\right\| &= \left\|S_n^{-1}\left(X_i\hat{\varepsilon}_i - \hat{\gamma}\hat{\varepsilon}_i^2 - X_i\varepsilon_i + \gamma_n\varepsilon_i^2\right)\right\| \\
&\leq C\mu_n^{-1}\left(\|X_i\|\,|\hat{\varepsilon}_i - \varepsilon_i| + \|\hat{\gamma}\|\,|\hat{\varepsilon}_i^2 - \varepsilon_i^2| + \left|\varepsilon_i^2\right|\|\hat{\gamma} - \gamma_n\|\right) \\
&\leq C\mu_n^{-1}d_i^2(\hat{B} + \hat{A}^2\hat{B} + \hat{B}) \leq C d_i^2\hat{A}^2\hat{B}, \\
\left\|\breve{X}_i\hat{\varepsilon}_i\right\| &\leq C\mu_n^{-1}d_i^2\hat{A}^2, \left\|\dot{X}_i\varepsilon_i\right\| \leq C\mu_n^{-1}d_i^2.
\end{aligned}$$

Also note that

$$E\left[\sum_{i\neq j} P_{ij}^2 d_i^2 d_j^2 \mu_n^{-2}\right] \leq C\mu_n^{-2}\sum_{i,j} P_{ij}^2 = C\mu_n^{-2}\sum_i P_{ii} = C\mu_n^{-2}K.$$

so that $\sum_{i\neq j} P_{ij}^2 d_i^2 d_j^2 \mu_n^{-2} = O_p(K/\mu_n^2)$ by the Markov inequality. Then it follows that

$$\begin{aligned}
\left\|\sum_{i\neq j} P_{ij}^2 \left(\breve{X}_i \breve{X}_i' \hat{\varepsilon}_j^2 - \dot{X}_i \dot{X}_i' \varepsilon_j^2\right)\right\| &\leq \sum_{i\neq j} P_{ij}^2 \left(\left|\hat{\varepsilon}_j^2\right| \left\|\breve{X}_i \breve{X}_i' - \dot{X}_i \dot{X}_i'\right\| + \left\|\dot{X}_i\right\|^2 \left|\hat{\varepsilon}_j^2 - \varepsilon_j^2\right|\right) \\
&\leq C\mu_n^{-2} \sum_{i\neq j} P_{ij}^2 d_i^2 d_j^2 (\hat{A}^4 \hat{B} + \hat{A}\hat{B}) = o_p\left(K/\mu_n^2\right).
\end{aligned}$$

We also have

$$\begin{aligned}
\left\|\sum_{i\neq j} P_{ij}^2 \left(\breve{X}_i \hat{\varepsilon}_i \hat{\varepsilon}_j \breve{X}_j' - \dot{X}_i \varepsilon_i \varepsilon_j \dot{X}_j\right)\right\| &\leq \sum_{i\neq j} P_{ij}^2 \left(\left\|\breve{X}_i \hat{\varepsilon}_i\right\| \left\|\breve{X}_j \hat{\varepsilon}_j - \dot{X}_j \varepsilon_j\right\| + \left\|\dot{X}_j \varepsilon_j\right\| \left\|\breve{X}_i \hat{\varepsilon}_i - \dot{X}_i \varepsilon_i\right\|\right) \\
&\leq C\mu_n^{-2} \sum_{i\neq j} P_{ij}^2 d_i^2 d_j^2 (1+\hat{A}^2)\hat{A}^2 \hat{B} = o_p\left(\frac{K}{\mu_n^2}\right).
\end{aligned}$$

The conclusion then follows by the triangle inequality. Q.E.D.

Lemma A12: *If the hypotheses of Theorem 3 are satisfied then* $\breve{\Sigma}_1 - \dot{\Sigma}_1 = o_p(K/\mu_n^2)$.

Proof: Note first that

$$\hat{\varepsilon}_i - \varepsilon_i = -X_i'(\hat{\delta} - \delta_0) = -X_i' S_n^{-1\prime} S_n'(\hat{\delta} - \delta_0) = -\left(z_i/\sqrt{n} + S_n^{-1} U_i\right)' \hat{\Delta} = -D_i' \hat{\Delta},$$

where $D_i = z_i/\sqrt{n} + S_n^{-1} U_i$ and $\hat{\Delta} = S_n'(\hat{\delta} - \delta_0)$. Also

$$\begin{aligned}
\hat{\varepsilon}_i^2 - \varepsilon_i^2 &= -2\varepsilon_i X_i'(\hat{\delta} - \delta_0) + \left[X_i'(\hat{\delta} - \delta_0)\right]^2, \\
\breve{X}_i - \dot{X}_i &= -\hat{\gamma}\hat{\varepsilon}_i + \gamma_n \varepsilon_i = S_n^{-1} \hat{\gamma} D_i' \hat{\Delta} - S_n^{-1} \mu_n (\hat{\gamma} - \gamma_n) \varepsilon_i/\mu_n.
\end{aligned}$$

We now have $\breve{\Sigma}_1 - \dot{\Sigma}_1 = \sum_{r=1}^7 T_r$ where

$$\begin{aligned}
T_1 &= \sum_{i\neq j\neq k} \left(\breve{X}_i - \dot{X}_i\right) P_{ik} \left(\hat{\varepsilon}_k^2 - \varepsilon_k^2\right) P_{kj} \left(\breve{X}_j - \dot{X}_j\right)', T_2 = \sum_{i\neq j\neq k} \dot{X}_i P_{ik} \left(\hat{\varepsilon}_k^2 - \varepsilon_k^2\right) P_{kj} \left(\breve{X}_j - \dot{X}_j\right)' \\
T_3 &= \sum_{i\neq j\neq k} \left(\breve{X}_i - \dot{X}_i\right) P_{ik} \varepsilon_k^2 P_{kj} \left(\breve{X}_j - \dot{X}_j\right)', T_4 = T_2', T_5 = \sum_{i\neq j\neq k} \left(\breve{X}_i - \dot{X}_i\right) P_{ik} \varepsilon_k^2 P_{kj} \dot{X}_j', \\
T_6 &= \sum_{i\neq j\neq k} \dot{X}_i P_{ik} \left(\hat{\varepsilon}_k^2 - \varepsilon_k^2\right) P_{kj} \dot{X}_j', T_7 = T_5'.
\end{aligned}$$

From the above expression for $\hat{\varepsilon}_i^2 - \varepsilon_i^2$ we see that T_6 is a sum of terms of the form $\hat{B} \sum_{i\neq j\neq k} \dot{X}_i P_{ik} \eta_i P_{kj} \dot{X}_j'$ where $\hat{B} \xrightarrow{p} 0$ and η_i is either a component of $-2\varepsilon_i X_i$ or of $X_i X_i'$. By Lemma A10 we have $\sum_{i\neq j\neq k} \dot{X}_i P_{ik} \eta_i P_{kj} \dot{X}_j' = O_p(1)$, so by the triangle inequality $T_6 \xrightarrow{p} 0$. Also, note that

$$T_5 = S_n^{-1} \hat{\gamma} \hat{\Delta}' \sum_{i\neq j\neq k} D_i P_{ik} \varepsilon_k^2 P_{kj} \dot{X}_j' + S_n^{-1} \mu_n (\hat{\gamma} - \gamma_n) \sum_{i\neq j\neq k} (\varepsilon_i/\mu_n) P_{ik} \varepsilon_k^2 P_{kj} \dot{X}_j'.$$

Note that $S_n^{-1}\hat{\gamma}\hat{\Delta}' \xrightarrow{p} 0$, $E[D_i] = z_i/\sqrt{n}$, $Var(D_i) = O(\mu_n^{-2})$, $E[\dot{X}_i] = z_i/\sqrt{n}$, and $Var(\dot{X}) = O(\mu_n^{-2})$. Then by Lemma A10 it follows that $\sum_{i\neq j\neq k} D_i P_{ik}\varepsilon_k^2 P_{kj}\dot{X}_j' = O_p(1)$ so that the $S_n^{-1}\hat{\gamma}\hat{\Delta}' \sum_{i\neq j\neq k} D_i P_{ik}\varepsilon_k^2 P_{kj}\dot{X}_j' \xrightarrow{p} 0$. A similar argument applied to the second term and the triangle inequality then give $T_5 \xrightarrow{p} 0$. Also $T_7 = T_5' \xrightarrow{p} 0$.

Next, analogous arguments apply to T_2 and T_3, except that there are four terms in each of them rather than two, and also to T_1 except there are eight terms in T_1. For brevity we omit details. Q.E.D.

Lemma A13: *If the hypotheses of Theorem 3 are satisfied then*

$$\dot{\Sigma}_2 = \sum_{i\neq j} P_{ij}^2 z_i z_i' \sigma_j^2/n + S_n^{-1}\sum_{i\neq j} P_{ij}^2 \left(E[\tilde{U}_i\tilde{U}_i']\sigma_j^2 + E[\tilde{U}_i\varepsilon_i]E[\varepsilon_j\tilde{U}_j']\right) S_n^{-1\prime} + o_p(K/\mu_n^2).$$

Proof: Note that $Var(\varepsilon_i^2) \le C$ and $\mu_n^2 \le Cn$, so that for $u_{ki} = e_k' S_n^{-1} U_i$,

$$\begin{aligned} E[(\dot{X}_{ik}\dot{X}_{i\ell})^2] &\le CE[\dot{X}_{ik}^4 + \dot{X}_{i\ell}^4] \le C\left\{z_{ik}^4/n^2 + E[u_k^4] + z_{i\ell}^4/n^2 + E[u_\ell^4]\right\} \le C\mu_n^{-4}, \\ E[(\dot{X}_{ik}\varepsilon_i)^2] &\le CE[(z_{ik}^2\varepsilon_i^2/n + u_{ki}^2\varepsilon_i^2)] \le Cn^{-1} + C\mu_n^{-2} \le C\mu_n^{-2}. \end{aligned}$$

Also, we have, for $\tilde{\Omega}_i = E[\tilde{U}_i\tilde{U}_i']$,

$$E[\dot{X}_i\dot{X}_i'] = z_i z_i'/n + S_n^{-1}\tilde{\Omega}_i S_n^{-1\prime}, E[\dot{X}_i\varepsilon_i] = S_n^{-1}E[\tilde{U}_i\varepsilon_i].$$

Next let W_i be $e_j'\dot{X}_i\dot{X}_i' e_k$ for some j and k, so that

$$\begin{aligned} E[W_i] &= e_j' S_n^{-1} E[\tilde{U}_i\tilde{U}_i'] S_n^{-1\prime} e_k + z_{ij} z_{ik}/n, |E[W_i]| \le C\mu_n^{-2}. \\ Var(W_i) &= Var\left\{\left(e_j' S_n^{-1} U_i + z_{ij}/\sqrt{n}\right)\left(e_k' S_n^{-1} U_i + z_{ik}/\sqrt{n}\right)\right\} \\ &\le C/\mu_n^4 + C/n\mu_n^2 \le C/\mu_n^4. \end{aligned}$$

Also let $Y_i = \varepsilon_i^2$. Then $\sqrt{K}(\bar{\sigma}_{Wn}\bar{\sigma}_{Yn} + \bar{\sigma}_{Wn}\bar{\mu}_{Yn} + \bar{\mu}_{Wn}\bar{\sigma}_{Yn}) \le CK^{1/2}/\mu_n^2$, so applying Lemma A9 for this W_i and Y_i gives

$$\sum_{i\neq j} P_{ij}^2 \dot{X}_i\dot{X}_i'\varepsilon_j^2 = \sum_{i\neq j} P_{ij}^2 \left(z_i z_i'/n + S_n^{-1}\tilde{\Omega}_i S_n^{-1\prime}\right)\sigma_j^2 + O_p(\sqrt{K}/\mu_n^2).$$

It follows similarly from Lemma A9 with W_i and Y_i equal to elements of $\dot{X}_i\varepsilon_i$ that

$$\sum_{i\neq j} P_{ij}^2 \dot{X}_i\varepsilon_i\varepsilon_j\dot{X}_j' = S_n^{-1}\sum_{i\neq j} P_{ij}^2 E[\tilde{U}_i\varepsilon_i]E[\varepsilon_j\tilde{U}_j']S_n^{-1\prime} + O_p(\sqrt{K}/\mu_n^2).$$

Also, by $K \longrightarrow \infty$ we have $O_p(\sqrt{K}/\mu_n^2) = o_p(K/\mu_n^2)$. The conclusion then follows by T. Q.E.D.

Lemma A14: *If the hypotheses of Theorem 3 are satisfied then*

$$\dot{\Sigma}_1 = \sum_{i\neq j\neq k} z_i P_{ik}\sigma_k^2 P_{kj} z_j'/n + o_p(1).$$

Proof: Apply Lemma A10 with W_i equal to an element of $\dot{X}_i$, Y_j equal to an element of $\dot{X}_j$, and $\eta_k = \varepsilon_k^2$. Q.E.D.

Proof of Theorem 3: Note that $\bar{X}_i = \sum_{j=1}^n P_{ij}\hat{X}_j$,

$$\begin{aligned}
&\sum_{i=1}^n (\bar{X}_i\bar{X}_i' - \hat{X}_i P_{ii}\bar{X}_i' - \bar{X}_i P_{ii}\hat{X}_i')\hat{\varepsilon}_i^2 \\
=& \sum_{i,j,k=1}^n \hat{X}_i P_{ik}\hat{\varepsilon}_k^2 P_{kj}\hat{X}_j' - \sum_{i,j=1}^n \hat{X}_i P_{ii}\hat{\varepsilon}_i^2 P_{ij}\hat{X}_j' - \sum_{i,j=1}^n \hat{X}_i P_{ij}\hat{\varepsilon}_j^2 P_{jj}\hat{X}_j' \\
=& \sum_{i,j,k=1}^n \hat{X}_i P_{ik}\hat{\varepsilon}_k^2 P_{kj}\hat{X}_j' - \sum_{i\neq j} \hat{X}_i P_{ii}\hat{\varepsilon}_i^2 P_{ij}\hat{X}_j' - \sum_{i\neq j} \hat{X}_i P_{ij}\hat{\varepsilon}_j^2 P_{jj}\hat{X}_j' - 2\sum_{i=1}^n \hat{X}_i P_{ii}^2\hat{\varepsilon}_i^2\hat{X}_i' \\
=& \sum_{i,j,k\notin\{i,j\}}^n \hat{X}_i P_{ik}\hat{\varepsilon}_k^2 P_{kj}\hat{X}_j' - \sum_{i=1}^n \hat{X}_i P_{ii}^2\hat{\varepsilon}_i^2\hat{X}_i' \\
=& \sum_{i\neq j\neq k}^n \hat{X}_i P_{ik}\hat{\varepsilon}_k^2 P_{kj}\hat{X}_j' + \sum_{i\neq j}^n P_{ij}^2\hat{X}_i\hat{X}_i\hat{\varepsilon}_j^2 - \sum_{i=1}^n \hat{X}_i P_{ii}^2\hat{\varepsilon}_i^2\hat{X}_i'.
\end{aligned}$$

Also, for Z_i' and $\tilde{Z}_i'$ equal to the ith row of Z and $\tilde{Z} = Z(Z'Z)^{-1}$ we have

$$\begin{aligned}
&\sum_{k=1}^K\sum_{\ell=1}^K \left(\sum_{i=1}^n \tilde{Z}_{ik}\tilde{Z}_{i\ell}\hat{X}_i\hat{\varepsilon}_i\right)\left(\sum_{j=1}^n Z_{jk}Z_{j\ell}\hat{X}_j\hat{\varepsilon}_j\right)' \\
=& \sum_{i,j=1}^n \left(\sum_{k=1}^K\sum_{\ell=1}^K \tilde{Z}_{ik}Z_{jk}\tilde{Z}_{i\ell}Z_{j\ell}\right)\hat{X}_i\hat{\varepsilon}_i\hat{\varepsilon}_j\hat{X}_j' = \sum_{i,j=1}^n (\sum_{k=1}^K \tilde{Z}_{ik}Z_{jk})^2\hat{X}_i\hat{\varepsilon}_i\hat{\varepsilon}_j\hat{X}_j' \\
=& \sum_{i,j=1}^n (\tilde{Z}_i'Z_j)^2\hat{X}_i\hat{\varepsilon}_i\hat{\varepsilon}_j\hat{X}_j' = \sum_{i,j=1}^n P_{ij}^2\hat{X}_i\hat{\varepsilon}_i\hat{\varepsilon}_j\hat{X}_j'
\end{aligned}$$

Adding this equation to the previous one then gives

$$\begin{aligned}\hat{\Sigma} &= \sum_{i\neq j\neq k}\hat{X}_iP_{ik}\hat{\varepsilon}_k^2P_{kj}\hat{X}_j' + \sum_{i\neq j}P_{ij}^2\hat{X}_i\hat{X}_i'\hat{\varepsilon}_j^2 - \sum_{i=1}^n\hat{X}_iP_{ii}^2\hat{\varepsilon}_i^2\hat{X}_i' + \sum_{i,j=1}^n P_{ij}^2\hat{X}_i\hat{\varepsilon}_i\hat{\varepsilon}_j\hat{X}_j' \\ &= \sum_{i\neq j\neq k}\hat{X}_iP_{ik}\hat{\varepsilon}_k^2P_{kj}\hat{X}_j' + \sum_{i\neq j}P_{ij}^2(\hat{X}_i\hat{X}_i'\hat{\varepsilon}_j^2 + \hat{X}_i\hat{\varepsilon}_i\hat{\varepsilon}_j\hat{X}_j').\end{aligned}$$

It then follows that $S_n^{-1}\hat{\Sigma}S_n^{-1\prime} = \breve{\Sigma}_1 + \breve{\Sigma}_2$, so that

$$S_n'\hat{V}S_n = (S_n^{-1}\hat{H}S_n^{-1\prime})^{-1}S_n^{-1}\hat{\Sigma}S_n^{-1\prime}(S_n^{-1}\hat{H}S_n^{-1\prime})^{-1} = (S_n^{-1}\hat{H}S_n^{-1\prime})^{-1}(\breve{\Sigma}_1+\breve{\Sigma}_2)(S_n^{-1}\hat{H}S_n^{-1\prime})^{-1}.$$

By Lemma A4 we have $S_n^{-1}\hat{H}S_n^{-1\prime} \xrightarrow{p} H_P$. Also, note that for $\bar{z}_i = \sum_j P_{ij}z_i = e_i'Pz$,

$$\begin{aligned}\sum_{i\neq j\neq k} z_iP_{ik}\sigma_k^2P_{kj}z_j'/n &= \sum_i\sum_{j\neq i}\sum_{k\notin\{i,j\}} z_iP_{ik}\sigma_k^2P_{kj}z_j'/n \\ &= \sum_i\sum_{j\neq i}\left(\sum_k z_iP_{ik}\sigma_k^2P_{kj}z_j' - z_iP_{ii}\sigma_i^2P_{ij}z_j' - z_iP_{ij}\sigma_j^2P_{jj}z_j'\right)/n \\ &= (\sum_k \bar{z}_k\sigma_k^2\bar{z}_k' - \sum_{i,k}P_{ik}^2z_iz_i'\sigma_k^2 - \sum_i z_iP_{ii}\sigma_i^2\bar{z}_i' + \sum_i z_iP_{ii}\sigma_i^2P_{ii}z_i' \\ &\quad -\sum_j \bar{z}_j\sigma_j^2P_{jj}z_j' + \sum_i z_jP_{jj}\sigma_j^2P_{jj}z_j')/n \\ &= \sum_i \sigma_i^2\left(\bar{z}_i\bar{z}_i' - P_{ii}z_i\bar{z}_i' - P_{ii}\bar{z}_iz_i' + P_{ii}^2z_iz_i'\right)/n - \sum_{i\neq j}P_{ij}^2z_iz_i'\sigma_j^2/n.\end{aligned}$$

Also, it follows similarly to the proof of Lemma A8 that $\sum_i \|z_i - \bar{z}_i\|^2/n \leq z'(I - P)z/n \longrightarrow 0$. Then by σ_i^2 and P_{ii} bounded we have

$$\begin{aligned}\left\|\sum_i \sigma_i^2(\bar{z}_i\bar{z}_i' - z_iz_i')/n\right\| &\leq \sum_i \sigma_i^2(2\|z_i\|\|z_i-\bar{z}_i\| + \|z_i-\bar{z}_i\|^2)/n \\ &\leq C(\sum_i\|z_i\|^2/n)^{1/2}(\sum_i\|z_i-\bar{z}_i\|^2/n)^{1/2} + C\sum_i\|z_i-\bar{z}_i\|^2/n \longrightarrow 0, \\ \left\|\sum_i \sigma_i^2P_{ii}(z_i\bar{z}_i' - z_iz_i')/n\right\| &\leq (\sum_i\sigma_i^4P_{ii}^2\|z_i\|^2/n)^{1/2}(\sum_i\|z_i-\bar{z}_i\|^2/n)^{1/2} \longrightarrow 0.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{i\neq j\neq k} z_iP_{ik}\sigma_k^2P_{kj}z_j'/n &= \sum_i \sigma_i^2(1-P_{ii})^2z_iz_i'/n + o(1) - \sum_{i\neq j}P_{ij}^2z_iz_i'\sigma_j^2/n \\ &= \Sigma_P - \sum_{i\neq j}P_{ij}^2z_iz_i'\sigma_j^2/n + o(1).\end{aligned}$$

It then follows by Lemmas A10-A14 and the triangle inequality that

$$
\begin{aligned}
\breve{\Sigma}_1 + \breve{\Sigma}_2 &= \sum_{i \neq j \neq k} z_i P_{ik} \sigma_k^2 P_{kj} z_j'/n + \sum_{i \neq j} P_{ij}^2 z_i z_i' \sigma_j^2/n \\
&\quad + S_n^{-1} \sum_{i \neq j} P_{ij}^2 \left(E[\tilde{U}_i \tilde{U}_i'] \sigma_j^2 + E[\tilde{U}_i \varepsilon_i] E[\varepsilon_j \tilde{U}_j'] \right) S_n^{-1\prime} + o_p(1) + o_p(K/\mu_n^2) \\
&= \Sigma_P + K S_n^{-1} (\Psi + o(1)) S_n^{-1\prime} + o_p(1) + o_p(K/\mu_n^2) \\
&= \Sigma_P + K S_n^{-1} \Psi S_n^{-1\prime} + o_p(1) + o_p(K/\mu_n^2).
\end{aligned}
$$

Then in case I) we have $o_p(K/\mu_n^2) = o_p(1)$ so that

$$
S_n' \hat{V} S_n = H^{-1} \left(\Sigma_P + K S_n^{-1} \Psi S_n^{-1\prime} \right) H^{-1} + o_p(1) = \Lambda_I + o_p(1).
$$

In case II) we have $(\mu_n^2/K)\, o_p(1) \xrightarrow{p} 0$, so that

$$
\left(\mu_n^2/K \right) S_n' \hat{V} S_n = H^{-1} \left(\left(\mu_n^2/K \right) \Sigma_P + \mu_n^2 S_n^{-1} \Psi S_n^{-1\prime} \right) H^{-1} + o_p(1) = \Lambda_{II} + o_p(1).
$$

Next, consider case I) and note that $S_n'(\hat{\delta} - \delta_0) \xrightarrow{d} Y \sim N(0, \Lambda_I)$, $S_n' \hat{V} S_n \xrightarrow{p} \Lambda_I$, $c' \sqrt{K} S_n^{-1\prime} \rightarrow c' S_0'$, and $c' S_0' \Lambda_I S_0 c \neq 0$. Then by the continuous mapping and Slutzky theorems,

$$
\begin{aligned}
\frac{c'(\hat{\delta} - \delta_0)}{\sqrt{c' \hat{V} c}} &= \frac{c' S_n^{-1\prime} S_n' (\hat{\delta} - \delta_0)}{\sqrt{c' S_n^{-1\prime} S_n' \hat{V} S_n S_n^{-1} c}} = \frac{c' \sqrt{K} S_n^{-1\prime} S_n' (\hat{\delta} - \delta_0)}{\sqrt{c' \sqrt{K} S_n^{-1\prime} S_n' \hat{V} S_n S_n^{-1} \sqrt{K} c}} \\
\xrightarrow{d} \frac{c' S_0' Y}{\sqrt{c' S_0' \Lambda_I S_0 c}} &\sim N(0,1).
\end{aligned}
$$

For case II), $\left(\mu_n/\sqrt{K} \right) S_n'(\hat{\delta} - \delta_0) \xrightarrow{d} \bar{Y} \sim N(0, \Lambda_{II})$, $(\mu_n^2/K)\, S_n' \hat{V} S_n \xrightarrow{p} \Lambda_{II}$, $c' \mu_n S_n^{-1\prime} \longrightarrow c' \bar{S}_0'$, and $c' \bar{S}_0' \Lambda_{II} \bar{S}_0 c \neq 0$. Then

$$
\begin{aligned}
\frac{c'(\hat{\delta} - \delta_0)}{\sqrt{c' \hat{V} c}} &= \frac{c' S_n^{-1\prime} \left(\mu_n/\sqrt{K} \right) S_n' (\hat{\delta} - \delta_0)}{\sqrt{c' S_n^{-1\prime} (\mu_n^2/K)\, S_n' \hat{V} S_n S_n^{-1} c}} \\
&= \frac{c' \mu_n S_n^{-1\prime} \left(\mu_n/\sqrt{K} \right) S_n' (\hat{\delta} - \delta_0)}{\sqrt{c' \mu_n S_n^{-1\prime} (\mu_n^2/K)\, S_n' \hat{V} S_n S_n^{-1} \mu_n c}} \xrightarrow{d} \frac{c' \bar{S}_0' \bar{Y}}{\sqrt{c' \bar{S}_0' \Lambda_{II} \bar{S}_0 c}} \sim N(0,1). Q.E.D.
\end{aligned}
$$

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