ASYMPTOTIC EQUIVALENCE OF PROBABILISTIC SERIAL AND RANDOM PRIORITY MECHANISMS

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Abstract. The random priority (random serial dictatorship) mechanism is a common method for assigning objects. The mechanism is easy to implement and strategy-proof. However this mechanism is inefficient, for all agents may be made better off by another mechanism that increases their chances of obtaining more preferred objects. Such an inefficiency is eliminated by the mechanism called probabilistic serial, but this mechanism is not strategy-proof. Thus, which mechanism to employ in practical applications is an open question. This paper shows that these mechanisms become equivalent when the market becomes large. More specifically, given a set of object types, the random assignments in these mechanisms converge to each other as the number of copies of each object type approaches infinity. Thus, the inefficiency of the random priority mechanism becomes small in large markets. Our result gives some rationale for the common use of the random priority mechanism in practical problems such as student placement in public schools. *JEL Classification Numbers*: C70, D61, D63.

Keywords: random assignment, random priority, probabilistic serial, ordinal efficiency, asymptotic equivalence.

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1. Introduction

Consider a mechanism design problem of assigning indivisible objects to agents who can consume at most one object each. University housing allocation, public housing allocation, office assignment, parking space assignment, and student placement in public schools are real-life examples.[1] A typical goal of the mechanism designer is to assign the objects efficiently and fairly, while eliciting the true preferences of the agents. The mechanism often needs to satisfy other constraints as well. For example, monetary transfers may be impossible or undesirable to use, as in the case of low income housing or placement to public schools. In such a case, random assignments are employed to achieve fairness. Further, the assignment often depends on agents' reports of ordinal preferences over objects rather than full cardinal preferences, as in placement in public schools in many cities.[2] Two mechanisms are regarded as promising solutions: the random priority mechanism (Abdulkadiroğlu and Sönmez 1998) and the probabilistic serial mechanism (Bogomolnaia and Moulin 2001).

In random priority, agents are ordered with equal probability and, for each realization of the ordering, the first agent in the ordering receives her most preferred object, the next agent receives his most preferred object among the remaining ones, and so on. Random priority is strategy-proof, that is, reporting ordinal preferences truthfully is a weakly dominant strategy for every agent. Moreover, random priority is ex-post efficient; the assignment after the ordering lottery is resolved is Pareto efficient. The random priority mechanism can also be easily tailored to accommodate other features, such as students

[1] See Abdulkadiroğlu and Sönmez (1999) and Chen and Sönmez (2002) for application to house allocation, and Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003b) for student placement. For the latter application, Abdulkadiroğlu, Pathak, and Roth (2005) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discuss practical considerations in designing student placement mechanisms in New York City and Boston.

[2] Why only ordinal preferences are used in many assignment rules seems unclear, and explaining it is outside the scope of this paper. Following the literature, we take it as given instead. Still, one reason may be that elicitation of cardinal preferences may be difficult (the pseudo-market mechanism proposed by Hylland and Zeckhauser (1979) is one of the few existing mechanisms incorporating cardinal preferences over objects.) Another reason may be that efficiency based on ordinal preferences is well justified regardless of agents' preferences; many theories of preferences over random outcomes (not just expected utility theory) agree that people prefer one assignment over another if the former first-order stochastically dominates the latter.

applying as roommates in college housing,[3] or respecting priorities of existing tenants in house allocation (Abdulkadiroğlu and Sönmez 1999) and non-strict priorities by schools in student placement (Abdulkadiroğlu, Pathak, and Roth 2005, Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005).

Perhaps more importantly for practical purposes, the random priority mechanism is straightforward and transparent, with the lottery used for assignment specified explicitly. Transparency of a mechanism can be crucial for ensuring fairness in the eyes of participants, who may otherwise be concerned about possible "covert selection."[4] These advantages explain the wide use of the random priority mechanism in many settings, such as house allocation in universities, student placement in public schools, and parking space assignment.

Despite the many advantages of the random priority mechanism, it may entail unambiguous efficiency loss ex ante. Bogomolnaia and Moulin (2001) provide an example in which the random priority assignment is dominated by another random assignment that improves the chance of obtaining a more preferred object for each agent, in the sense of first-order stochastic dominance. Bogomolnaia and Moulin introduce the ordinal efficiency concept: a random assignment is ordinally efficient if it is not first-order stochastically dominated for all agents by any other random assignment. Ordinal efficiency is perhaps the most relevant efficiency concept in the context of assignment mechanisms based solely on ordinal preferences.

Bogomolnaia and Moulin propose the probabilistic serial mechanism as an alternative to the random priority mechanism. The basic idea is to regard each object as a continuum of "probability shares." Each agent "eats" her most preferred available object (in probability share) with speed one at every point in time between 0 and 1. The probabilistic serial

[3]Applications by would-be roommates can be easily incorporated into the random priority mechanism by requiring each group to receive the same random priority order. For instance, non-freshman undergraduate students at Columbia University can apply as a group, in which case they draw the same lottery number. The lottery number, along with their seniority points, determines their priority. If no suite is available to accommodate the group or they do not like the available suite options, they can split up and make choices as individuals. This sort of flexibility between group and individual assignments seems difficult to achieve in other mechanisms such as the probabilistic serial mechanism.

[4]The concern of covert selection was pronounced in UK schools, which led to adoption of a new Mandatory Admission Code in 2007. The code, among other things, "makes the admissions system more straightforward, transparent and easier to understand for parents" ("Schools admissions code to end covert selection," *Education Guardian*, January 9, 2007). There had been numerous appeals by parents on schools assignments in the UK; there were 78,670 appeals in 2005-2006, and 56,610 appeals in 2006-2007.

random assignment is defined as the profile of object shares eaten by agents by time 1. The probabilistic serial random assignment is ordinally efficient if all the agents report their ordinal preferences truthfully.

However, the probabilistic serial mechanism is not strategy-proof. In other words, an agent may receive a more desirable random assignment (with respect to her true expected utility function) by misreporting her ordinal preferences. The mechanism is also less straightforward and less transparent for the participants than is random priority, since the lottery used for implementing the random assignment can be complicated and is not explicitly specified. The tradeoffs between the two mechanisms — random priority and probabilistic serial — are not easy to evaluate, leaving the choice between the two an important outstanding question in practical applications. Indeed, Bogomolnaia and Moulin (2001) show that no mechanism satisfies ordinal efficiency, strategy-proofness, and symmetry (equal treatment of equals) in all finite economies. So one cannot hope to resolve the tradeoffs by finding a mechanism with these three desiderata. Naturally, the previous studies have focused only on the choice between random priority and probabilistic serial.

The contribution of this paper is to offer a new perspective on the tradeoffs between the random priority and probabilistic serial mechanisms. We do so by showing that the two mechanisms become virtually equivalent in large markets. Specifically, we demonstrate that, given a set of arbitrary object types, the random assignments in these mechanisms converge to each other, as the number of copies of each object type approaches infinity.

Our result has several implications for both mechanisms. First, the result implies that the inefficiency of the random priority mechanism becomes small and disappears in the limit, as the economy becomes large. On the probabilistic serial mechanism, its equivalence to the random priority mechanism in a large market means that its incentive problem disappears in large economies. Taken together, these implications mean that we do not have as strong a theoretical basis by which to distinguish the two mechanisms in large markets as in small markets; indeed, both will be good candidates in large markets since they have good incentive, efficiency, and fairness properties.[5] Given its practical

[5]As mentioned above, Bogomolnaia and Moulin (2001) present three desirable properties, namely ordinal efficiency, strategy-proofness, and equal treatment of equals, and show that no mechanism satisfies all these three desiderata in finite economies. Random priority satisfies all but ordinal efficiency while probabilistic serial satisfies all but strategy-proofness. Our equivalence result implies that both mechanisms satisfy all these desiderata in the limit economy, thus overcoming impossibility in general finite economies.

merit, though, our result lends some support for the common use of the random priority mechanism in practical applications, such as student placement in public schools.

In our model, the large market assumption means that there exist a large number of copies of each object type. This model includes several interesting cases. For instance, a special case is the replica economies model wherein the copies of object types and of agent types are replicated a large number of times. Considering such a large economy is useful for many practical applications. In student placement in public schools, there are typically a large number of identical seats at each school. In the context of university housing allocation, the set of rooms may be partitioned into a number of categories by building and size, and all rooms of the same type may be treated to be identical.[6] Our model may be applicable to these markets.

We investigate a number of further issues as well. First, we define the random priority and probabilistic serial mechanisms directly in economies with a continuum of agents and objects. We show that random priority and probabilistic serial in finite economies converge to those in the continuum economy. In that sense, we provide a foundation of modeling approaches that study economies with a continuum of objects and agents directly. Second, we show that our equivalence is tight in the sense that in any finite economy, random priority and probabilistic serial can be different. We also present several extensions, such as cases with existing priority and multi-unit demands.

The rest of the paper proceeds as follows. Section 2 introduces the model. Section 3 defines the random priority mechanism and the probabilistic serial mechanism. Section 4 presents the main result. Section 5 investigates further topics. Section 6 discusses related literature. Section 7 concludes. Proofs are found in the Appendix unless stated otherwise.

2. Model

For each $q \in \mathbb{N}$, consider a q**-economy**, $\Gamma^q = (N^q, (\pi_i)_{i\in N^q}, O)$, where N^q represents the set of agents and O represents the set of **proper object types** (we assume that O is identical for all q). There are $|O| = n$ object types, and each object type $a \in O$ has **quota** q, that is, q copies of a are available.[7] There exist an infinite number of copies of a **null object** $\o$, which is not included in O. Let $\tilde{O} := O \cup \{\o\}$. Each agent $i \in N$ has a **strict preference** represented by a permutation $\pi_i \in \Pi$ of $\tilde{O}$, where a given permutation $\pi_i : \{1, ..., n+1\} \to \tilde{O}$ lists for its j-th element $\pi_i(j)$ the agent's j-th most preferred

[6]For example, the assignment of graduate housing at Harvard University is based on the preferences of each student over eight types of rooms: two possible sizes (large and small) and four buildings.

[7]Given a set X, we denote the cardinality of X by $|X|$ or $\#X$.

object. (That is, agent i prefers a over b if and only if $\pi_i^{-1}(a) < \pi_i^{-1}(b)$.) For preference type π and for any $O' \subset \tilde{O}$,

$$Ch_\pi(O') := \{a \in O' | \pi^{-1}(a) \leq \pi^{-1}(b) \ \forall b \in O'\},$$

is the object that an agent of preference type π chooses if the set O' of objects is available to her.

The agents are partitioned into different preference types: $N^q = \{N^q_\pi\}_{\pi \in \Pi}$, where N^q_π is the set of the agents with preference $\pi \in \Pi$ in the q-economy. Let $m^q_\pi := \frac{|N^q_\pi|}{q}$ be the per-unit number of agents of type π in the q-economy. We assume, for each $\pi \in \Pi$, there exists $m^\infty_\pi \in \mathbb{R}_+$ such that $m^q_\pi \to m^\infty_\pi$ as $q \to \infty$. For $q \in \mathbb{N} \cup \{\infty\}$, let $m^q := \{m^q_\pi\}_{\pi \in \Pi}$. For any $q \in \mathbb{N} \cup \{\infty\}$, $O' \subset O$ and $a \in O'$, let

$$m^q_a(O') = \sum_{\pi \in \Pi : a \in Ch_\pi(O')} m^q_\pi,$$

be the per unit number of agents whose most preferred object in O' is a in the q-economy. Throughout, we do not impose any restriction on the way in which the q-economy, Γ^q, grows with q (except for the existence of the limit $m^\infty_\pi = \lim_{q\to\infty} m^q_\pi$ for each $\pi \in \Pi$).

A special case of interest is when the economy grows at a constant rate with q. We say that the family $\{\Gamma^q\}_{q\in\mathbb{N}}$ are **replica economies** if $m^q_\pi = m^\infty_\pi$ (or equivalently, $|N^q_\pi| = q|N^1_\pi|$) for all $q \in \mathbb{N}$ and all $\pi \in \Pi$, and call Γ^1 a **base economy** and Γ^q its **q-fold replica**.

Throughout, we focus on a symmetric random assignment in which all the agents with the same preference type π receive the same lottery over the objects. Formally, a **symmetric random assignment in the q-economy** is a mapping $\phi^q : \Pi \to \Delta\tilde{O}$, where $\Delta\tilde{O}$ is the set of probability distributions over $\tilde{O}$, that satisfies the feasibility constraint $\sum_{\pi \in \Pi} \phi^q_a(\pi) \cdot |N^q_\pi| \leq q$, for each $a \in O$, where $\phi^q_a(\pi)$ represents the probability that a type π-agent receives the object a.[8]

It is useful to describe the limit economy (∞-economy) separately. For this purpose, we assume that there exists a continuum of copies of objects in O and agents in N^∞. More precisely, there exists a unit mass of each object in O, and the set of agent types Π is then endowed with a measure $\mu : \Pi \to \mathbb{R}_+$ such that $\mu(\pi) = m^\infty_\pi$. **A symmetric random assignment in the limit economy** is then defined as $\phi^\infty : \Pi \to \Delta\tilde{O}$ such that $\sum_{\pi\in\Pi} \phi^\infty_a(\pi) \cdot m^\infty_\pi \leq 1$ for each $a \in O$.

[8]The symmetry assumption that all the agents with the same preference type π receive the same lottery is often called the "equal treatment of equals" axiom.

2.1. **Ordinal Efficiency.** Consider a q-economy (where $q \in \mathbb{N} \cup \{\infty\}$). A symmetric random assignment ϕ^q **ordinally dominates** another random assignment $\hat{\phi}^q$ **at** m^q if, for each preference type π with $m^q_\pi > 0$, the lottery $\phi^q(\pi)$ first-order stochastically dominates the lottery $\hat{\phi}^q(\pi)$,

$$\sum_{\pi^{-1}(b) \leq \pi^{-1}(a)} \phi^q_b(\pi) \geq \sum_{\pi^{-1}(b) \leq \pi^{-1}(a)} \hat{\phi}^q_b(\pi) \qquad \forall \pi, m^q_\pi > 0, \forall a \in \tilde{O}, \tag{2.1}$$

with strict inequality for some (π, a). The random assignment ϕ^q is **ordinally efficient at** m^q if it is not ordinally dominated at m^q by any other random assignment. If ϕ^q ordinally dominates $\hat{\phi}^q$ at m^q, then every agent of every preference type prefers her assignment under ϕ^q to the one under $\hat{\phi}^q$ according to any expected utility function with utility index consistent with their ordinal preferences.

We say that ϕ^q is **non-wasteful at** m^q if there exists no preference type $\pi \in \Pi$ with $m^q_\pi > 0$ and objects $a, b \in \tilde{O}$ such that $\pi^{-1}(a) < \pi^{-1}(b)$, $\phi^q_b(\pi) > 0$ and $\sum_{\pi' \in \Pi} \phi^q_a(\pi') m^q_{\pi'} < 1$. That is, non-wastefulness requires that there be no good which some agent prefers to what she consumes but is not consumed fully, with positive probability. If there were such a good, the allocation would be ordinally inefficient.

Consider the binary relation $\rhd(\phi^q, m^q)$ on $\tilde{O}$ defined by

$$a \rhd (\phi^q, m^q)\, b \iff \exists \pi \in \Pi, m^q_\pi > 0, \pi^{-1}(a) < \pi^{-1}(b) \text{ and } \phi^q_b(\pi) > 0. \tag{2.2}$$

That is, $a \rhd (\phi^q, m^q)\, b$ if there are some agents who prefer a to b but are assigned to b with positive probability. If a relation $\rhd(\phi^q, m^q)$ admits a cycle, then the relevant agents can trade off shares of less preferred goods along the cycle and all do better, so the allocation would be ordinally inefficient.

One can show that ordinal efficiency is equivalent to the acyclicity of the binary relation and non-wastefulness. This is shown by Bogomolnaia and Moulin in a setting in which each object has quota 1 and there exist an equal number of agents and objects.[9] Their characterization extends straightforwardly to our setting as follows (the proof is omitted).

Proposition 1. The random assignment ϕ^q is ordinally efficient at m^q if and only if the relation $\rhd(\phi^q, m^q)$ is acyclic and ϕ^q is non-wasteful at m^q.

[9]This restriction means that non-wastefulness is trivially satisfied by every feasible allocation.

3. Two Competing Mechanisms: Random Priority and Probabilistic Serial

3.1. **Random Priority Mechanism.** We introduce the **random priority** mechanism (Bogomolnaia and Moulin 2001), also called the **random serial dictatorship** (Abdulkadiroğlu and Sönmez 1998), which is widely used in practice. The agents are ordered randomly, and each agent selects, according to the order, the most preferred object among the remaining ones. For our purpose, it is useful to model the random ordering procedure as follows: First, each agent i randomly draws a number f_i independently from a uniform distribution on $[0,1]$. Second, the agent with the smallest draw receives her most preferred object, the agent with the second-smallest draw receives his most preferred object from the remaining ones, and so forth (it suffices to only consider cases in which $f_i \neq f_j$ for any $i \neq j$, since $f_i = f_j$ occurs with probability zero). This procedure induces a random assignment. We let RP^q be the random assignment under the random priority mechanism in Γ^q.

The random assignment RP^q is characterized as follows. Fix an agent i of arbitrary preference π, and fix the draws $f_{-i} = (f_j)_{j \in N \setminus \{i\}} \in [0,1]^{(|N^q|-1)}$ for all agents other than i. We then ask how low agent i's draw should be for her to obtain a given object $a \in \tilde{O}$. Specifically, we characterize the **cutoff** $\hat{T}^q_a \in [0,1]$ for each object $a \in O$, which represents *the largest value of draw that would allow agent i to claim a.* It is the critical value in $[0,1]$ such that agent i can obtain a if and only if she draws f_i less than that value. The cutoffs depend on the random draws f_{-i}, so they are random. It is useful to characterize the cutoffs under RP^q, since, as will be seen later, they pin down its random assignment.

To begin, let $\hat{m}^q_{\pi'}(t,t') := \frac{\#\{j \in N^q_{\pi'} \setminus \{i\} | f_j \in (t,t']\}}{q}$ denote the per-unit number of agents of type π' (except i if $\pi' = \pi$) who have draws in $(t,t']$. For any $O' \subset O$ and $a \in O'$, we let

$$\hat{m}^q_a(O';t,t') = \sum_{\pi' \in \Pi : a \in Ch_{\pi'}(O')} \hat{m}^q_{\pi'}(t,t'),$$

be the per-unit number of agents in $N^q \setminus \{i\}$ whose most preferred object in O' is a and who have draws in $(t,t']$.

We then characterize the cutoffs for i by the following sequence of steps. (Note the cutoffs depend on the preference type of i, but this dependence will be suppressed for notational ease.) Let $\hat{O}^q(0) = \tilde{O}$, $\hat{t}^q(0) = 0$, and $\hat{x}^q_a(0) = 0$ for every $a \in \tilde{O}$. Given $\hat{O}^q(0), \hat{t}^q(0)$, $\{\hat{x}^q_a(0)\}_{a \in \tilde{O}}, \dots,$ $\hat{O}^q(v-1), \hat{t}^q(v-1), \{\hat{x}^q_a(v-1)\}_{a \in \tilde{O}}$, we let $\hat{t}^q_{\emptyset} := 1$ and for each $a \in O$, define

$$\hat{t}^q_a(v) = \sup\left\{t \in [0,1] \,\middle|\, \hat{x}^q_a(v-1) + \hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), t) < 1\right\}, \tag{3.1}$$

$$\hat{t}^q(v) = \min_{a \in \hat{O}(v-1)} \hat{t}^q_a(v), \tag{3.2}$$

$$\hat{O}^q(v) = \hat{O}^q(v-1) \setminus \{a \in \hat{O}^q(v-1) | \hat{t}^q_a(v) = \hat{t}^q(v)\}, \tag{3.3}$$

$$\hat{x}^q_a(v) = \hat{x}^q_a(v-1) + \hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), \hat{t}^q(v)). \tag{3.4}$$

The last step $\hat{v}^q := \min\{v' | \hat{t}^q(v') = 1\}$ is well defined since O is finite. For each $a \in O$, its cutoff is given by $\hat{T}^q_a := \{\hat{t}^q(v) | \hat{t}^q(v) = \hat{t}^q_a(v)\}$ if the set is nonempty, or else $\hat{T}^q_a = 1$.

Each step above determines the cutoff of an object. Suppose steps 1 through $v-1$ have determined the $v-1$ cutoffs for $v-1$ objects. In particular, by the end of step $v-1$, agents with draws less than $\hat{t}^q(v-1)$ have consumed the entire q copies of these objects and a fraction $x^q_b(v-1)$ of each remaining object $b \in O^q(v-1)$.

Suppose the object $a \in \hat{O}^q(v-1)$ is next to be "eaten up." Clearly, it will be consumed by agents who drew lottery numbers less than its cutoff, $\hat{t}^q(v) = \hat{T}^q_a$. An agent with draw $f \in (\hat{t}^q(v-1), \hat{T}^q_a]$ will consume the object if and only if she prefers a to all other remaining objects. The total number of all such agents is $q \cdot \hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), \hat{T}^q_a)$, and they consume a fraction $\hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), \hat{T}^q_a)$ of that object. Hence, the total fraction of a consumed by all agents with draws less than $\hat{T}^q_a$ must be

$$\hat{x}^q_a(v-1) + \hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), \hat{T}^q_a).$$

For $\hat{T}^q_a$ to be the cutoff for a, this fraction must be no greater than one and must equal one if $\hat{T}^q_a < 1$. This condition requires $\hat{T}^q_a$ to equal $\hat{t}^q_a(v)$, defined in (3.1). That a is the first to be eaten up among the remaining objects is given by (3.2). The last two equations reset the remaining set of objects and the fractions consumed by step v, thus continuing on the recursive procedure.

For each a,

$$\hat{\tau}^q_a(\pi) := \min\{\hat{t}^q(v) | \hat{t}^q(v) \leq \hat{T}^q_a \text{ and } a \in Ch_\pi(\hat{O}^q(v-1))\}$$

is the minimum value of draw for an agent with preference π, to choose a (again if the minimum is well defined, or else let $\hat{\tau}^q_a(\pi) := \hat{T}^q_a$). In other words, an agent with π has a better choice than a available if she draws a number lower than $\hat{\tau}^q_a(\pi)$. Hence, the minimum value $\hat{\tau}^q_a(\pi)$ is the highest cutoff of all objects that agent prefers to a, if that cutoff, say $\hat{T}^q_b$, is less than $\hat{T}^q_a$. In that case, agent i will obtain a if and only if her draw f_i lies between two cutoffs $\hat{T}^q_b$ and $\hat{T}^q_a$, as depicted in Figure 1.

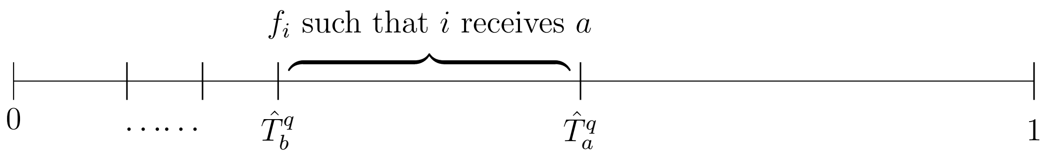

Figure 1: Cutoffs of objects under RP.

Therefore, the **random priority random assignment** is defined, for $i \in N_\pi^q$ and $a \in O$, as $RP_a^q(\pi) := \mathbb{E}[\hat{T}_a^q - \hat{\tau}_a^q(\pi)]$, where the expectation $\mathbb{E}$ is taken with respect to $f_{-i} = (f_j)_{j \neq i}$ which are distributed i.i.d uniformly on $[0,1]$.

The random priority mechanism is widely used in practice, as mentioned in the Introduction. Moreover, the mechanism is **strategy-proof**, that is, reporting true ordinal preferences is a dominant strategy for each agent. Furthermore, it is **ex post efficient**, that is, the assignment after random draws are realized is Pareto efficient. However, the mechanism may result in an ordinally inefficient allocation, as shown by the following example adapted from Bogomolnaia and Moulin (2001).

Example 1. Consider an economy Γ^1 with 2 types of proper objects, a and b, each with quota one. Let $N^1 = N_\pi^1 \cup N_{\pi'}^1$ be the set of agents, with $|N_\pi^1| = |N_{\pi'}^1| = 2$. Preferences of the agents are specified by

$$\begin{aligned} (\pi(1), \pi(2), \pi(3)) &= (a, b, \emptyset), \\ (\pi'(1), \pi'(2), \pi'(3)) &= (b, a, \emptyset). \end{aligned}$$

In this economy, the random assignments under RP^1 can be easily calculated to be

$$RP^1(\pi) = (RP_a^1(\pi), RP_b^1(\pi), RP_\emptyset^1(\pi)) = \left(\frac{5}{12}, \frac{1}{12}, \frac{1}{2}\right),$$

$$RP^1(\pi') = (RP_a^1(\pi'), RP_b^1(\pi'), RP_\emptyset^1(\pi')) = \left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}\right).$$

Each agent ends up with her less preferred object with positive probability, since two agents of any given preference type may get the two best draws, in which case the agent with the second best draw will take the less preferred object.[10] Obviously, any two agents of different preferences can benefit from trading off the probability share of the less preferred object with that of the most preferred. In other words, the RP assignment is

[10]For instance, let $N_\pi^1 = \{1,2\}$ and $N_{\pi'}^1 = \{3,4\}$. The draws can be $f_1 < f_2 < f_3 < f_4$, in which case 1 gets a, 2 gets b, and 3 and 4 get nothing.

ordinally dominated by

$$\phi^1(\pi) = \left(\frac{1}{2}, 0, \frac{1}{2}\right),$$
$$\phi^1(\pi') = \left(0, \frac{1}{2}, \frac{1}{2}\right).$$

Therefore the random priority assignment RP^1 is ordinally inefficient in this market.

Ordinal inefficiency of RP can be traced to the fact that the cutoffs of the objects are random and personalized. In Example 1, $\hat{T}^1_a < \hat{T}^1_b$ occurs to agents of type π with positive probability, and $\hat{T}^1_a > \hat{T}^1_b$ occurs to agents of type π' with positive probability. In the former case, an agent of type π may get b even though she prefers a to b. In the latter case, an agent of type π' may get a even though she prefers b to a. Hence both $a \rhd (RP^1, m^1) b$ and $b \rhd (RP^1, m^1) a$ occur, resulting in cyclicity of the relation $\rhd(RP^1, m^1)$ and hence ordinal inefficiency of RP^1. As will be seen, as $q \to \infty$, the cutoffs of the random priority mechanism converge in probability to deterministic limits that are common to all agents, and this feature ensures acyclicity of the binary relation $\rhd$ in the limit.

3.2. **Probabilistic Serial Mechanism.** Now we introduce the **probabilistic serial** mechanism, which is an adaptation of the mechanism proposed by Bogomolnaia and Moulin to our setting. The idea is to regard each object as a divisible object of "probability shares." Each agent "eats" a probability share of the best available object with speed one at every time $t \in [0, 1]$ (object a is available at time t if not all q shares of a have been eaten by time t). [11] The resulting profile of object shares eaten by agents by time 1 obviously induces a symmetric random assignment, which we call the **probabilistic serial random assignment**.

Formally, the probabilistic serial random assignment is defined as follows.

PS mechanism in the finite economy. For the q-economy Γ^q, the assignment under the probabilistic serial mechanism is defined by the following sequence of steps. Let $O^q(0) = \tilde{O}$, $t^q(0) = 0$, and $x^q_a(0) = 0$ for every $a \in \tilde{O}$. Given $O^q(0), t^q(0)$, $\{x^q_a(0)\}_{a \in \tilde{O}}, \ldots,$

[11] Bogomolnaia and Moulin (2001) consider a broader class of simultaneous eating algorithms, where eating speeds may vary across agents and time.

$O^q(v-1), t^q(v-1), \{x_a^q(v-1)\}_{a\in\tilde{O}}$, we let $t_{\o}^q := 1$ and for each $a \in O$, define

$$t_a^q(v) = \sup\left\{t \in [0,1] \,|x_a^q(v-1) + m_a^q(O^q(v-1))(t - t^q(v-1)) < 1\right\}, \tag{3.5}$$

$$t^q(v) = \min_{a\in O(v-1)} t_a^q(v), \tag{3.6}$$

$$O^q(v) = O^q(v-1) \setminus \{a \in O^q(v-1)|t_a^q(v) = t^q(v)\}, \tag{3.7}$$

$$x_a^q(v) = x_a^q(v-1) + m_a^q(O^q(v-1))(t^q(v) - t^q(v-1)). \tag{3.8}$$

The last step $\bar{v}^q := \min\{v'|t^q(v') = 1\}$ is again well defined since O is finite. For each $a \in \tilde{O}$, define its **expiration date**: $T_a^q := \{t^q(v)|t^q(v) = t_a^q(v)\}$. The expiration date for object a is the time at which the eating of a is complete. (When an agent starts eating a given object a depends on his preference.) Note that, unlike the cutoffs in the random priority mechanism, the expiration dates are deterministic and common to all agents. Aside from this important difference, though, expiration dates in PS play a similar role as cutoffs in RP. In particular, they completely pin down the random assignment for the agents. Agent of type $\pi \in \Pi$ starts eating a at time

$$\tau_a^q(\pi) := \min\{t^q(v)|t^q(v) \le T_a^q \text{ and } a \in Ch_\pi(O^q(v-1))\}$$

(if the minimum is well defined, or else let $\tau_a^q(\pi) := T_a^q$), and she consumes the good until it expires at time T_a^q. Hence, agent i's probability of getting assigned to $a \in \tilde{O}$ is simply its duration of consumption of its preference type; i.e., $PS_a^q(\pi) = T_a^q - \tau_a^q(\pi)$ if $i \in N_\pi^q$.

PS mechanism in the limit economy. Although our primary interest is in a large but finite economy, it is useful to define the PS mechanism in the limit economy, for it will act as a benchmark for subsequent analysis. We again do so recursively. Let $O^\infty(0) = \tilde{O}$, $t^\infty(0) = 0$, and $x_a^\infty(0) = 0$ for every $a \in \tilde{O}$. Given $O^\infty(0), t^\infty(0)$, $\{x_a^\infty(0)\}_{a\in\tilde{O}}, \dots,$ $O^\infty(v-1), t^\infty(v-1), \{x_a^\infty(v-1)\}_{a\in\tilde{O}}$, we let $t_{\o}^\infty := 1$ and for each $a \in O$, define

$$t_a^\infty(v) = \sup\left\{t \in [0,1] \,|x_a^\infty(v-1) + m_a^\infty(O^\infty(v-1))(t - t^\infty(v-1)) < 1\right\}, \tag{3.9}$$

$$t^\infty(v) = \min_{a\in O^\infty(v-1)} t_a^\infty(v), \tag{3.10}$$

$$O^\infty(v) = O^\infty(v-1) \setminus \{a \in O^\infty(v-1)|t_a^\infty(v) = t^\infty(v)\}, \tag{3.11}$$

$$x_a^\infty(v) = x_a^\infty(v-1) + m_a^\infty(O^\infty(v-1))(t^\infty(v) - t^\infty(v-1)). \tag{3.12}$$

Let $\bar{v}^\infty$ be such that $t^\infty(\bar{v}^\infty) = 1$. Consider the associated expiration dates: For each $a \in O$, $T_a^\infty := \{t^\infty(v)|t^\infty(v) = t_a^\infty(v)\}$ if the set is nonempty, or else $T_a^\infty := 1$. Likewise, the starting time for a for π is defined as

$$\tau_a^\infty(\pi) := \min\{t^\infty(v)|t^\infty(v) \le T_a^\infty \text{ and } a \in Ch_\pi(O^q(v-1))\}$$

if the minimum is well defined, or else let $\tau_a^\infty(\pi) := T_a^\infty$. The **PS random assignment in the limit** is then defined to be the duration of eating each object: for $a \in O$, $PS_a^\infty(\pi) := T_a^\infty - \tau_a^\infty(\pi)$.

Adapting the argument of Bogomolnaia and Moulin (2001), we can show the following result (whose proof is omitted).

Proposition 2. For any $q \in \mathbb{N} \cup \{\infty\}$, PS^q is ordinally efficient.

Example 2. Consider replica economies $\{\Gamma^q\}_{q\in\mathbb{N}}$ with 2 types of proper objects, a and b, each having quota q in the q-fold replica. Let $N^q = N^q_\pi \cup N^q_{\pi'}$ be the set of agents in the q-fold replica, with N^q_π and $N^q_{\pi'}$ containing $2q$ agents each. Assume that the preferences of the agents are specified by

$$\begin{aligned}(\pi(1), \pi(2), \pi(3)) &= (a, b, \emptyset),\\ (\pi'(1), \pi'(2), \pi'(3)) &= (b, a, \emptyset).\end{aligned}$$

Note that Γ^1 corresponds to the market in Example 1.

For any $q \in \mathbb{N}$, the random assignments under PS^q can be easily calculated to be

$$\begin{aligned}PS^q(\pi) = (PS^q_a(\pi), PS^q_b(\pi), PS^q_\emptyset(\pi)) = \left(\frac{1}{2}, 0, \frac{1}{2}\right),\\ PS^q(\pi') = (PS^q_a(\pi'), PS^q_b(\pi'), PS^q_\emptyset(\pi')) = \left(0, \frac{1}{2}, \frac{1}{2}\right),\end{aligned}$$

which is ordinally efficient.

Unlike the cutoffs in the random priority mechanism, the expiration dates in the probabilistic serial mechanism are deterministic and common to all agents. This explains, for instance, why there is no cycle on a binary relation $\triangleright(PS^q, m^q)$. To see this, suppose $a \triangleright (PS^q, m^q)\, b$. Then, there must be an agent who prefers a to b but ends up with b with positive probability. This is possible only if $T_a < T_b$; or else, by the time the agent finishes "eating" a (or something even better than a), b will have been completely eaten. Based on this logic, a cycle on $\triangleright(PS^q, m^q)$ will require the order on the expiration dates to be cyclic, which is impossible.

One main drawback of the probabilistic serial mechanism, as identified by Bogomolnaia and Moulin (2001), is that it is not strategy-proof. In other words, an agent may be better off by reporting a false ordinal preference.

Before proceeding to our main results, we show that PS^q converges to PS^∞ as $q \to \infty$. The convergence occurs in all standard metrics; for concreteness, we define the metric by $||\phi - \hat{\phi}|| := \sup_{\pi\in\Pi, a\in O} |\phi_a(\pi) - \hat{\phi}_a(\pi)|$ for any pair of symmetric random assignments ϕ

and $\hat{\phi}$. The convergence of PS^q to PS^∞ is immediate if $\{\Gamma^q\}_{q\in\mathbb{N}}$ are replica economies. In this case, $m_a^q(O') = m_a^\infty(O')$ for all q and a, so the recursive definitions, (3.5), (3.6), (3.7), and (3.8), of the PS procedure for each q-economy all coincide with those of the limiting economy, namely (3.9), (3.10), (3.11), and (3.12). The other cases are established.

Theorem 1. *$||PS^q - PS^\infty|| \to 0$ as $q \to \infty$. Further, $PS^q = PS^\infty$ for all $q \in \mathbb{N}$ if $\{\Gamma^q\}_{q\in\mathbb{N}}$ are replica economies.*

This theorem shows that PS in the limit economy captures the limiting behavior of PS in a large but finite economy. In this sense, Theorem 1 provides a foundation for a modeling approach that models PS directly in the continuum economy.

4. Main Result: Asymptotic Equivalence

The main purpose of this paper is to investigate the relationship between the random priority and probabilistic serial mechanisms in large markets. We now provide our main finding, beginning with an example.

Example 3. Consider replica economies $\{\Gamma^q\}_{q\in\mathbb{N}}$ of the base economy introduced in Example 1. That is, there are 2 proper object types, a and b, each having quota q in the q-fold replica. Let $N^q = N_\pi^q \cup N_{\pi'}^q$ be the set of agents in the q-fold replica, with N_π^q and $N_{\pi'}^q$ containing $2q$ agents each. Assume that the preferences of the agents are specified by

$$\begin{aligned}(\pi(1), \pi(2), \pi(3)) &= (a, b, \emptyset),\\ (\pi'(1), \pi'(2), \pi'(3)) &= (b, a, \emptyset).\end{aligned}$$

In Examples 1 and 2, we have seen that PS is ordinally efficient in all q-economies, while RP results in an ordinally inefficient random assignment in the base economy.

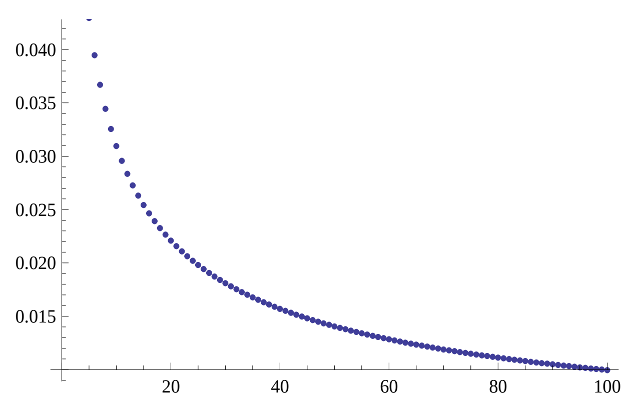

Figure 1. Horizontal axis: Market size q. Vertical axis: $RP_b^q(\pi) = ||RP^q - PS^q||$.

Figure 1 plots the misallocation probability $RP_b^q(\pi) = ||RP^q - PS^q||$ as a function of the size of the market q. Notice that the misallocation probability is positive for all q but declines and approaches zero as q becomes large. Correspondingly, one can see that the cutoff of each good converges to 1/2, the expiration date of both goods in probabilistic serial.

Figure 1 suggests the asymptotic equivalence of the two mechanisms in a specific example. The following theorem indeed shows that the equivalence holds generally for arbitrary preferences in the limit of any sequence of economies as $q \to \infty$ (beyond the simple cases of replica economies).

Theorem 2. *$||RP^q - PS^\infty|| \to 0$ as $q \to \infty$. Furthermore, $||RP^q - PS^q|| \to 0$ as $q \to \infty$.*

We shall give intuition of Theorem 2. The starting point is a recursive formulation of the random priority mechanism given by (3.1)-(3.4). The formulation suggests that the assignment under the random priority mechanism is similar to the one in the probabilistic serial mechanism, except that in the random priority mechanism the random cutoffs replace the expiration dates in the probabilistic serial mechanism. The basic idea of the proof is to show that the cutoff for each object type in RP converges to the expiration date of that object type in PS in probability as the size of the market approaches infinity. As noted earlier, since the cutoffs of RP and expiration dates of PS pin down their random assignments, the convergence of the former to the latter implies that the RP assignment is asymptotically equivalent to the PS assignment.

To gain intuition, it is useful to envision an (equivalent) implementation of the random priority mechanism in which an agent with lottery draw $f \in [0, 1]$ is instructed to make her selection at time f. Given this interpretation, the goods will be consumed over time interval $[0, 1]$ by the agents who arrive according to their lottery numbers. The convergence of the RP cutoffs to the PS expiration dates will then occur if, as the economy gets large, the rate at which each good is consumed under RP approaches its consumption rate under PS at each relevant time interval in probability. To see how this happens, fix any time $t \in [t^\infty(v), t^\infty(v + 1))$ for some v, and fix any good $a \in O$. Under PS^q with sufficiently large q, assuming that goods $O^\infty(v)$ are available at time t, the fraction of a consumed during time interval $[t, t + \delta]$ for small δ is approximately $\delta \cdot m_a^\infty(O^\infty(v))$, namely the measure of those who prefer a the most among $O^\infty(v)$ times the duration of their consumption of a.

In RP^q, assuming again that the same set $O^\infty(v)$ of goods is available at t, the measure $m_a^q(O^\infty(v); t, t + \delta)$ of agents (who prefer a most among $O^\infty(v)$) arrive during the (same) time interval $[t, t+\delta]$ and will consume a, so the fraction of a consumed during that interval is $m_a^q(O^\infty(v); t, t+\delta)$. As $q \to \infty$, this fraction converges to $\delta \cdot m_a^q(O^\infty(v))$, since by a law of large numbers, the arrival rate of these agents approaches $m_a^\infty(O^\infty(v))$.

The main challenge of the proof is to make this intuition precise when there are intertemporal linkages in the consumption of goods — namely, a change in consumption at one point of time alters the set of available goods, and thus the consumption rates of all goods, at later time. Our proof employs an inductive method to handle these linkages.

Example 3 also shows that the RP and PS assignments remain different for all finite values of q. This means that Theorem 2 is tight; we cannot generally expect that the RP assignment coincides with the PS assignment in a finite economy. In fact, we have a stronger characterization in this regard.

Proposition 3. Consider a family $\{\Gamma^q\}_{q\in\mathbb{N}}$ of replica economies. Then, RP^q is ordinally efficient for some $q \in \mathbb{N}$ if and only if $RP^{q'}$ is ordinally efficient for every $q' \in \mathbb{N}$. That is, for any given base economy, the random priority assignment is ordinally efficient for all replica economies or ordinally inefficient for all of them.

In particular, Proposition 3 implies that the ordinal inefficiency of RP does not disappear completely in any finitely replicated economy if the random priority assignment is ordinally inefficient in the base economy. More importantly, it may be misleading to simply examine whether a mechanism suffers ordinal inefficiencies; even if a mechanism is ordinally inefficient, the magnitude of the inefficiency may be very small, as is the case with RP.

5. Discussion

5.1. **Random Priority Mechanism in the Limit.** Our main result has been established without defining the RP in the limit economy. This omission entails no loss for our purpose, since we are primarily interested in the behavior of a large, but *finite*, economy. Further, defining the RP in the limit economy may require one to describe the aggregate behavior of independent lottery drawing for a continuum of population, which can be problematic.[12]

There is a way to define the RP in the limit economy, without appealing to a law of large numbers, as has been done by Abdulkadiroğlu, Che, and Yasuda (2008). To do so, we first augment the type of an agent to include his random draw, which is not observed until the random priority is drawn. Formally, a generic agent in the limit economy has

[12]See Judd (1985) for a classic reference for the associated conceptual problems, and Sun (2006) for a recent treatment.

type (π, f) representing his preference π and the draw f (which is possibly unobserved by the agent themselves until proper time). The set of agents in the limit economy is represented by the product space $\Pi \times [0,1]$ endowed with a product measure $\mu \times \nu$, such that $\mu(\pi) = m_\pi^\infty$ for all π and ν is uniform with $\nu([0,f]) = f$ for each $f \in [0,1]$. In words, the measure of agents with draws less than f is precisely f. This corresponds to the heuristics that the agents in the limit economy obtain random draws in $[0,1]$ and that a law of large numbers holds for the aggregate distribution (although it is never formally invoked). Again, we assume that a lower draw gives a higher priority for an agent.

As with q-economy with $q \in \mathbb{N}$, we characterize the RP^∞ via the cutoff values of the draws for each object. A cutoff $\hat{T}_a^\infty$ for object $a \in O$ is defined such that an agent can obtain a (when he/she wishes) if and only if $f < \hat{T}_a^\infty$. As before, we then define the cutoffs recursively by a sequence of steps. Let $\hat{O}^\infty(0) = \tilde{O}$, $\hat{t}^\infty(0) = 0$, and $\hat{x}_a^\infty(0) = 0$ for every $a \in \tilde{O}$. Given $\hat{O}^\infty(0), \hat{t}^\infty(0)$, $\{\hat{x}_a^\infty(0)\}_{a \in \tilde{O}}, \ldots,$ $\hat{O}^\infty(v-1), \hat{t}^\infty(v-1), \{\hat{x}_a^\infty(v-1)\}_{a \in \tilde{O}}$, we let $\hat{t}_\emptyset^\infty := 1$ and for each $a \in O$, define

$$\hat{t}_a^\infty(v) = \sup\left\{t \in [0,1] \,\middle|\, \hat{x}_a^\infty(v-1) + m_a^\infty(\hat{O}^\infty(v-1))(t - \hat{t}^\infty(v-1)) < 1\right\}, \tag{5.1}$$

$$\hat{t}^\infty(v) = \min_{a \in \hat{O}^\infty(v-1)} \hat{t}_a^\infty(v), \tag{5.2}$$

$$\hat{O}^\infty(v) = \hat{O}^\infty(v-1) \setminus \{a \in \hat{O}^\infty(v-1) | \hat{t}_a^\infty(v) = t^\infty(v)\}, \tag{5.3}$$

$$\hat{x}_a^\infty(v) = \hat{x}_a^\infty(v-1) + m_a^\infty(\hat{O}^\infty(v-1))(\hat{t}^\infty(v) - \hat{t}^\infty(v-1)). \tag{5.4}$$

Comparing (3.9)-(3.12) with (5.1)-(5.4) makes it plainly evident that $\hat{T}_a^\infty = T_a^\infty$, $\forall a \in O$, with the following conclusion:

Proposition 4. $RP^\infty = PS^\infty$.

This proposition and Theorem 2 imply

Corollary 1. $||RP^q - RP^\infty|| \to 0$ as $q \to \infty$.

Thus RP in the limit economy captures the limiting behavior of RP in a large but finite economy. In this sense, Proposition 4 gives a foundation for a modeling approach that models the random priority mechanism directly in the continuum economy, as has been done, for instance, by Abdulkadiroğlu, Che, and Yasuda (2008).

5.2. **Market Design in Large Economies.** A recurring theme in economics is that large economies can make things "right" in many settings, and our result shares the same theme. Nevertheless, no single economic insight appears to explain the benefit of large economies. And it is important to investigate what precisely the large economy buys.

First, it is often the case that the large economy limits individuals' abilities and incentives to manipulate the mechanism. This is clearly the case for the Walrasian mechanism in exchange economy, as has been shown by Roberts and Postlewaite (1976). It is also the case for the deferred acceptance algorithm in two-sided matching (Kojima and Pathak (2008)) and for the probabilistic serial mechanism in one-sided matching (Kojima and Manea (2008)). Even this property is not to be taken for granted, however. The so-called **Boston mechanism** (Abdulkadiroğlu and Sönmez 1999), which has been used to place students in public schools, provides an example. In that mechanism, a school first admits the students who rank it first, and if, and *only if*, there are seats left, admits those who rank it second, and so forth. It is well known that the students have incentives to misreport preferences in such a mechanism, and such manipulation incentives do not disappear as the economy becomes large.[13]

Second, one may expect that, with the diminished manipulation incentives, efficiency would be easier to obtain in a large economy. The asymptotic ordinal efficiency we find for the RP supports this impression. However, even some reasonable mechanisms fail to achieve asymptotic ordinal efficiency. Take the case of the **deferred acceptance algorithm with multiple tie-breaking (DA-MTB)**, an adaptation of the celebrated algorithm proposed by Gale and Shapley (1962) to the problem of assigning objects to agents, such as student assignment in public schools (see Abdulkadiroğlu, Pathak, and Roth (2005)). In DA-MTB, each object type randomly and independently orders agents and, given the ordering, the assignment is decided by conducting the agent-proposing deferred acceptance algorithm with respect to the submitted preferences and the randomly decided priority profile. It turns out DA-MTB fails even ex post efficiency, let alone ordinal efficiency. Moreover, these inefficiencies do not disappear in the limit economy, as shown by Abdulkadiroğlu, Che, and Yasuda (2008).

Third, one plausible conjecture may be that the asymptotic ordinal efficiency is a necessary consequence of a mechanism that produces an ex post efficient assignment in every finite economy. This conjecture turns out to be false. Consider a family $\{\Gamma^q\}_{q\in\mathbb{N}}$ of replica economies and what we call a **replication-invariant random priority** mechanism, $RIRP^q$, defined as follows. First, in the given q-economy, define a correspondence $\gamma : N^1 \twoheadrightarrow N^q$ such that $|\gamma(i)| = q$ for each $i \in N^1$, $\gamma(i) \cap \gamma(j) = \emptyset$ if $i \neq j$, and all agents in $\gamma(i)$ have the same preference as i. Call $\gamma(i)$ i's clones in the q-fold replica. Let each set $\gamma(i)$ of clones of agent i randomly draw a number f_i independently from a uniform

[13]See Kojima and Pathak (2008) for a concrete example on this point.

distribution on $[0, 1]$. Second, all the clones with the smallest draw receive their most preferred object, the clones with the second-smallest draw receive their most preferred object from the remaining ones, and so forth. This procedure induces a random assignment. It is clear that $RIRP^q = RP^1$ for any q-fold replica Γ^q. Therefore $||RIRP^q - RP^1|| \to 0$ as $q \to \infty$. Since RP^1 may not be ordinally efficient, the limit random assignment of $RIRP^q$ as $q \to \infty$ is not ordinally efficient in general.

Most importantly, our analysis shows the equivalence of two different mechanisms beyond showing certain asymptotic properties of given mechanisms. Such an equivalence is not expected even for a large economy, and has few analogues in the literature.

5.3. **Group-specific Priorities.** In some applications, the social planner may need to give higher priorities to a subset of agents over others. For example, when allocating graduate dormitory rooms, the housing office at Harvard University assigns rooms to first year students first, and then assigns remaining rooms to existing students. Other schools prioritize housing assignments based on students' seniorities and/or their academic performances.[14]

To model such a situation, assume that each student belongs to one of the classes C and, for each $c \in C$, let g_c be a density function over $[0, 1]$. The asymmetric random priority mechanism associated with $g = (g_c)_{c \in C}$ lets each agent i in class c to draw f_i according to the density function g_c independently from others, and the agent with the smallest draw among all agents receives her most preferred object, the agent with the second-smallest draw receives his most preferred object from the remaining ones, and so forth. The random priority mechanism is a special case in which g_c is a uniform distribution on $[0, 1]$ for each $c \in C$.

The asymmetric probabilistic serial mechanism associated with g is defined by simply letting agents in class c eat with speed $g_c(t)$ at each time $t \in [0, 1]$. The probabilistic serial mechanism is a special case in which g_c is a uniform distribution on $[0, 1]$ for each $c \in C$.

It is not difficult to see that our results generalize to a general profile of distributions g. In particular, given any g, the asymmetric random priority mechanism associated with

[14]For instance, Columbia University gives advantage in lottery draw based on seniority in its undergraduate housing assignment. The Technion university gives assignment priorities to students based on both seniority and academic performance (Perach, Polak, and Rothblum (2007)). Claremont McKenna College and Pitzer College give students assignment priority based on the number of credits they have earned.

g and the asymmetric probabilistic serial mechanism associated with g converge to the same limit as $q \to \infty$.

5.4. **Unequal Number of Copies.** We focused on a setting in which there are q copies of each object type in the q-economy. It is straightforward to extend our results to settings in which there are an unequal number of copies, as long as quotas of object types grow proportionately. More specifically, if there exist positive integers $(q_a)_{a \in O}$ such that the quota of object type a is $q_a q$ in the q-economy, then our results extend with little modification of the proof.

On the other hand, we need *some* assumption about the growth rate of quotas. Suppose that, for instance, quotas of some objects are q but quotas of others stay at one. Then, one can easily create an example in which random priority assignment of objects with quota one does not converge to those under the probabilistic serial mechanism. However, such an example does not pose a serious problem, since in the large market, assignment of object types with small quotas has only limited influence on overall welfare in the economy.

5.5. **Multi-Unit Demands.** Consider a generalization of our basic setting, in which each agent can obtain multiple units of objects. More specifically, we assume that there is a fixed integer k such that each agent can receive k objects. When $k = 1$, the model reduces to the model of the current paper. Assignment of popular courses in schools is one example of such a multiple unit assignment problem. See, for example, Kojima (2008) for formal definition of the model.

We consider two generalizations of the random priority mechanism to the current setting. In the **once-and-for-all random priority** mechanism, each agent i randomly draws a number f_i independently from a uniform distribution on $[0, 1]$ and, given the ordering, the agent with the smallest draw receives her most preferred k objects, the agent with the second-smallest draw receives his most preferred k objects from the remaining ones, and so forth. In the **draft random priority** mechanism, each agent i randomly draws a number f_i independently from a uniform distribution on $[0, 1]$. Second, the agent with the smallest draw receives her most preferred object, the agent with the second-smallest draw receives his most preferred object from the remaining ones, and so forth. Then agents obtain a random draw again and repeat the procedure k times.

We introduce two generalizations of the probabilistic serial mechanism. In the **multiunit-eating probabilistic serial** mechanism, each agent "eats" her k most preferred available objects with speed one at every time $t \in [0,1]$. In the **one-at-a-time probabilistic serial** mechanism, each agent "eats" the best available object with speed one at every time $t \in [0,k]$.

Our analysis can be adapted to this situation to show that the once-and-for-all random priority mechanism converges to the same limit as the multiunit-eating probabilistic serial mechanism, whereas the draft random priority mechanism converges to the same limit as the one-at-a-time probabilistic serial mechanism.

It is easy to see that the multiunit-eating probabilistic serial mechanism may not be ordinally efficient, while the one-at-a-time probabilistic serial mechanism is ordinally efficient. This may shed light on some issues in multiple unit assignment. It is well known that the once-and-for-all random priority mechanism is ex post efficient, but the mechanism is rarely used in practice. Rather, the draft mechanism is often used in application, for instance in sports drafting and allocations of courses in business schools. One of the reasons may be that the once-and-for-all random priority mechanism is ordinally inefficient even in the limit economy, whereas the draft random priority mechanism converges to an ordinally efficient mechanism as the economy becomes large, as in course allocation in schools.

6. Related literature

Pathak (2006) compares random priority and probabilistic serial using data on the assignment of about 8,000 students in the public school system of New York City. He finds that many students obtain a better random assignment in the probabilistic serial mechanism, but he notes that the difference seems small. The current paper complements his study, by explaining why the two mechanisms are not expected to differ much in some school choice settings.

Kojima and Manea (2008) find that truthtelling becomes a dominant strategy under probabilistic serial when there are a large number of copies of each object type. Their paper left the asymptotic behavior of random priority unanswered. The current paper gives an answer to that question, providing further understanding of random mechanisms in large markets. Furthermore, our analysis provides intuition for the result of Kojima and Manea (2008). To see this point, first recall that truthtelling is a dominant strategy in the random priority mechanism. Since our result shows that the probabilistic serial mechanism is close to the random priority mechanism in a large economy, this observation implies that it is difficult to profitably manipulate the probabilistic serial mechanism.[15]

Manea (2006) shows that the probability that the random priority assignment is ordinally inefficient approaches one as the market becomes large under a number of assumptions. We note that his result does not contradict ours because of a number of differences. Most notably, Manea (2006) focuses on whether there is *any* ordinal inefficiency in the random priority assignment, while the current paper investigates *how much* difference there is between the random priority and the probabilistic serial mechanisms, and hence (indirectly) how much ordinal inefficiency the random priority mechanism entails. As suggested by Proposition 3, this distinction is important particularly for the welfare assessment of RP.

While the analysis of large markets is relatively new in matching and resource allocation problems, it has a long tradition in many areas of economics. For example, Roberts and Postlewaite (1976) show that, under some conditions, the Walrasian mechanism is difficult to manipulate in large exchange economies.[16] Similarly, incentive properties of a large class of double auction mechanisms are studied by, among others, Gresik and Satterthwaite (1989), Rustichini, Satterthwaite, and Williams (1994), and Cripps and Swinkels (2006). Two-sided matching is an area closely related to our model. In that context, Roth and Peranson (1999), Immorlica and Mahdian (2005), and Kojima and Pathak (2008) show that the deferred acceptance algorithm proposed by Gale and Shapley (1962) becomes increasingly hard to manipulate as the number of participants becomes large. Many of these papers show particular properties of given mechanisms, such as incentive compatibility and efficiency. One of the notable features of the current paper is that we show the equivalence of apparently dissimilar mechanisms, beyond specific properties of each mechanism.

Finally, our paper is part of a growing literature on random assignment mechanisms.[17] The probabilistic serial mechanism is generalized to allow for weak preferences, existing property rights, and multi-unit demand by Katta and Sethuraman (2006), Yilmaz (2006), and Kojima (2008), respectively. Kesten (2008) introduces two mechanisms, one of which is motivated by the random priority mechanism, and shows that these mechanisms are

[15]However, the result of Kojima and Manea (2008) cannot be derived from the current paper since they establish a dominant strategy result in a large but finite economies, while our equivalence result holds only in the limit as the market size approaches infinity.

[16]See also Jackson (1992) and Jackson and Manelli (1997).

[17]Characterizations of ordinal efficiency are given by Abdulkadiroğlu and Sönmez (2003a) and McLennan (2002).

equivalent to the probabilistic serial mechanism. In the scheduling problem (a special case of the current environment), Crès and Moulin (2001) show that the probabilistic serial mechanism is group strategy-proof and ordinally dominates the random priority mechanism but these two mechanisms converge to each other as the market size approaches infinity, and Bogomolnaia and Moulin (2002) give two characterizations of the probabilistic serial mechanism.

7. Conclusion

Although the random priority (random serial dictatorship) mechanism is widely used for assigning objects to individuals, there has been increasing interest in the probabilistic serial mechanism as a potentially superior alternative. The tradeoffs associated with these mechanisms are multifaceted and difficult to evaluate in a finite economy. Yet, we have shown that the tradeoffs disappear, as the two mechanisms become effectively identical, in the large economy. More specifically, given a set of object types, the random assignments in these mechanisms converge to each other as the number of copies of each object type approaches infinity. This equivalence implies that the well-known concerns about the two mechanisms — the inefficiency of random priority and the incentive issue of probabilistic serial — abate in large markets.

Our equivalence is asymptotic, and the random priority and the probabilistic serial mechanisms do not exactly coincide in large but finite economies. How these competing mechanisms perform in such a case remains an interesting open question.

Appendix

A. Proof of Theorem 1

It suffices to show that $\sup_{a\in O}|T_a^q - T_a^\infty| \to 0$ as $q \to \infty$. To this end, let

$$L > 2\max\left\{\max\{\tfrac{1}{m_a^\infty(O')}, m_a^\infty(O')\}\middle| O' \subset O, a \in O', m_a^\infty(O') > 0\right\}, \tag{A1}$$

and let $K := \min\{1 - x_a^\infty(v) \mid a \in O^\infty(v), v < \bar{v}^\infty\} > 0$. Note (A1) implies $L > 2$.

Fix any $\epsilon > 0$ such that

$$2L^{4\bar{v}^\infty}\epsilon < \min\left\{K, \min_{v\in\{1,\ldots,\bar{v}^\infty\}} |t^\infty(v) - t^\infty(v-1)|\right\}. \tag{A2}$$

By assumption there exists Q such that, for each $q > Q$,

$$|m_a^q(O') - m_a^\infty(O')| < \epsilon, \forall O' \subset \tilde{O}, \forall a \in O'. \tag{A3}$$

Fix any such q. For each $v \in \{1, ..., \bar{v}^{\infty}\}$, consider the set $A^{\infty}(v) := \{a \in O | T_a^{\infty} = t^{\infty}(v)\}$ of objects that expire at step v of PS^{∞}. We show that $T_a^q \in (t^{\infty}(v) - L^{4v}\epsilon, t^{\infty}(v) + L^{4v}\epsilon)$ if and only if $a \in A^{\infty}(v)$. Let

$$J_v := \{i | t^q(i) = t_a^q(i) \text{ for some } a \in A^{\infty}(v)\}$$

be the steps at which the objects in $A^{\infty}(v)$ expire in PS^q. Clearly, it suffices to show that $t^q(i) \in (t^{\infty}(v) - L^{4v}\epsilon, t^{\infty}(v) + L^{4v}\epsilon)$ if and only if $i \in J_v$ We prove this recursively.

Suppose for each $v' \leq v - 1$, $t^q(i') \in (t^{\infty}(v') - L^{4v'}\epsilon, t^{\infty}(v') + L^{4v'}\epsilon)$ if and only if $i' \in J_{v'}$, and further that, for each $a \in O^{\infty}(v-1)$, $x_a^q(k) \in (x_a^{\infty}(v-1) - L^{4(v-1)}\epsilon, x_a^{\infty}(v-1) + L^{4(v-1)}\epsilon)$, where k is the largest element of J_{v-1}. We shall then prove that $t^q(i) \in (t^{\infty}(v) - L^{4v}\epsilon, t^{\infty}(v) + L^{4v}\epsilon)$ if and only if $i \in J_v$, and that, for each $a \in O^{\infty}(v)$, $x_a^q(l) \in (x_a^{\infty}(v) - L^{4v}\epsilon, x_a^{\infty}(v) + L^{4v}\epsilon)$, where l is the largest element of J_v.

Observe first $O^q(k) = O^{\infty}(v-1)$, since k is the largest element of J_{v-1}.

Claim 1. For any $i > k$, $t^q(i) > t^{\infty}(v) - L^{4v-2}\epsilon$.

Proof. Suppose object $a \in O^{\infty}(v-1) = O^q(k)$ expires at step $k+1$ of PS^q. It suffices to show $t_a^q(k+1) > t^{\infty}(v) - L^{4v-2}\epsilon$. Suppose to the contrary that

$$t_a^q(k+1) \leq t^{\infty}(v) - L^{4v-2}\epsilon. \tag{A4}$$

Recall, by the inductive assumption, that

$$x_a^q(k) < x_a^{\infty}(v-1) + L^{4(v-1)}\epsilon. \tag{A5}$$

Thus,

$$\begin{aligned} x_a^q(k+1) &= x_a^q(k) + m_a^q(O^q(k))(t_a^q(k+1) - t^q(k)) \\ &\leq x_a^q(k) + m_a^q(O^q(k))(t^{\infty}(v) - L^{4v-2}\epsilon - t^{\infty}(v-1) + L^{4(v-1)}\epsilon) \\ &\leq x_a^q(k) + m_a^q(O^q(k))[t^{\infty}(v) - t^{\infty}(v-1) - L^{4v-3}\epsilon] \\ &< x_a^{\infty}(v-1) + L^{4(v-1)}\epsilon + m_a^{\infty}(O^{\infty}(v-1))[t^{\infty}(v) - t^{\infty}(v-1) - L^{4v-3}\epsilon] + \epsilon, \end{aligned} \tag{A6}$$

where the first equality follows from definition of PS^q (3.8) and the fact that $t_a^q(k+1) = t^q(k+1)$, the first inequality follows from the inductive assumption and (A4), the second inequality holds since $L^{4v-2}\epsilon - L^{4(v-1)}\epsilon = L^{4v-3}(L - \frac{1}{L})\epsilon > L^{4v-3}\epsilon$ since $L > 2$, which follows from (A1), and the third inequality follows from (A2), (A3) and (A5).[18]

[18]By (A2), $t^{\infty}(v) - t^{\infty}(v-1) - L^{4v-3}\epsilon \in (0,1)$, so

$$\begin{aligned} & m_a^{\infty}(O^{\infty}(v-1))[t^{\infty}(v) - t^{\infty}(v-1) - L^{4v-3}\epsilon] - m_a^q(O^q(k))[t^{\infty}(v) - t^{\infty}(v-1) - L^{4v-3}\epsilon] \\ = & (m_a^{\infty}(O^{\infty}(v-1)) - m_a^q(O^q(k)))[t^{\infty}(v) - t^{\infty}(v-1) - L^{4v-3}\epsilon] < m_a^{\infty}(O^{\infty}(v-1)) - m_a^q(O^q(k)) < \epsilon, \end{aligned}$$

There are two cases. Suppose first $m_a^\infty(O^\infty(v-1)) = 0$. Then, the last line of (A6) becomes

$$x_a^\infty(v-1) + L^{4(v-1)}\epsilon + \epsilon,$$

which is strictly less than 1, by $a \in O^\infty(v-1)$ and (A2). Suppose next $m_a^\infty(O^\infty(v-1)) > 0$. Then, the last line of (A6) equals

$$\begin{aligned}
& x_a^\infty(v-1) + L^{4(v-1)}\epsilon + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1) - L^{4v-3}\epsilon] + \epsilon \\
< \;& x_a^\infty(v-1) + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1)] \\
\leq \;& 1,
\end{aligned}$$

where the first inequality holds since, by (A1), $m_a^\infty(O^\infty(v-1))L^{4v-3}\epsilon > 2L^{4(v-1)}\epsilon \geq L^{4(v-1)}\epsilon + \epsilon$, and the second follows since $a \in O^\infty(v-1)$. In either case, we have a contradiction to the fact that a expires at step $k+1$. ‖

Claim 2. For any $i \in J_v$, then $t^q(i) \leq t^\infty(v) + L^{4v-2}\epsilon$.

Proof. Suppose a expires at step $l \equiv \max J_v$ of PS^q. It suffices to show $t^q(l) = t_a^q(l) \leq t^\infty(v) + L^{4v-2}\epsilon$. If $t^\infty(v) = 1$, then this is trivially true. Thus, let us assume $t_a^\infty(v) < 1$. This implies $m_a^\infty(O^\infty(v-1)) > 0$. For that case, suppose for contradiction that

$$t_a^q(l) > t^\infty(v) + L^{4v-2}\epsilon. \tag{A7}$$

Then,

$$\begin{aligned}
x_a^q(l) &= x_a^q(k) + \sum_{j=k+1}^{l} m_a^q(O^q(j-1))[t^q(j) - t^q(j-1)] \\
&\geq x_a^q(k) + \sum_{j=k+1}^{l} m_a^q(O^q(k))[t^q(j) - t^q(j-1)] \\
&= x_a^q(k) + m_a^q(O^\infty(v-1))[t^q(l) - t^q(k)] \\
&> x_a^\infty(v-1) - L^{4(v-1)}\epsilon + m_a^q(O^\infty(v-1))[t^\infty(v) + L^{4v-2}\epsilon - t^\infty(v-1) - L^{4(v-1)}\epsilon] \\
&\geq x_a^\infty(v-1) - L^{4(v-1)}\epsilon + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1) + L^{4v-3}\epsilon] \\
&> x_a^\infty(v-1) + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1)] \\
&= x_a^\infty(v) = 1,
\end{aligned}$$

where the first equality follows from (3.8), the first inequality follows since $m_a^q(O^q(j-1)) \geq m_a^q(O^q(k))$ for each $j \geq k+1$ by $O^q(j-1) \subseteq O^q(k)$, the second equality from $O^q(k) = O^\infty(v-1)$, the second inequality follows from the inductive assumption and

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where the last inequality follows from (A3).

(A7), the third inequality follows from the assumption (A1), and the fourth inequality follows from (A1) and $m_a^\infty(O^\infty(v-1)) > 0$. Thus $x_a^q(l) > 1$, which contradicts the definition of $x_a^q(l)$. ‖

Claim 3. If $i \in J_{v'}$ for some $v' > v$, then $t^q(i) > t^\infty(v) + L^{4v}\epsilon$.

Proof. Suppose otherwise. Let c be the object that expires the first among $O^\infty(v)$ in PS^q. Let j be the step at which it expires. We must have

$$t^q(j) \leq t^\infty(v) + L^{4v}\epsilon. \tag{A8}$$

In particular, $t_c^q(j) < 1$ and $x_c^q(j) = 1$. Since c is the first object to expire in $O^\infty(v)$, at each of steps $k+1, \ldots, j-1$, some object in $A^\infty(v)$ expires. (If $j = k+1$, then no other object expires in between step k and step j.) Also, by Claim 1,

$$t^q(k+1) > t^\infty(v) - L^{4v-2}\epsilon. \tag{A9}$$

Therefore,

$$\begin{aligned}
x_c^q(j) &= x_c^q(k) + \sum_{i=k+1}^{j} m_c^q(O^q(i-1))(t^q(i) - t^q(i-1)) \\
&\leq x_c^q(k) + m_c^q(O^q(k))(t^q(k+1) - t^q(k)) + m_c^q(O^q(j-1))(t^q(j) - t^q(k+1)) \\
&\leq x_c^\infty(v-1) + L^{4(v-1)}\epsilon + (m_c^\infty(O^q(k)) + \epsilon)\left((t^\infty(v) + L^{4v-2}\epsilon) - (t^\infty(v-1) - L^{4(v-1)}\epsilon)\right) \\
&\qquad + (m_a^\infty(O^q(j)) + \epsilon)\left(L^{4v}\epsilon - L^{4v-2}\epsilon\right) \\
&\leq x_c^\infty(v) + L^{4v+1}\epsilon \\
&\leq 1 - K + L^{4\bar{v}^\infty}\epsilon \\
&< 1,
\end{aligned}$$

where the first equality follows from (3.8), the first inequality follows since $m_c^q(O^q(i-1)) \leq m_c^q(O^q(j-1))$ for any $i \leq j$ by $O^q(i-1) \subset O^q(j-1)$, the second inequality follows from the inductive assumption, (A3), (A9), and (A8), the third inequality follows from (A1), and the last inequality follows from (A2) and the definition of K. This contradicts the assumption that c expires at step j. ‖

Claims 1-3 prove that $t^q(i) \in (t^\infty(v) - L^{4v-2}\epsilon, t^\infty(v) + L^{4v-2}\epsilon) \subset (t^\infty(v) - L^{4v}\epsilon, t^\infty(v) + L^{4v}\epsilon)$ if and only if $i \in J_v$, which in turn implies that $T_a^q \in (t^\infty(v) - L^{4v}\epsilon, t^\infty(v) + L^{4v}\epsilon)$ if and only if $a \in A^\infty(v)$. It now remains to prove the following:

Claim 4. For each $a \in O^\infty(v)$, $x_a^q(l) \in (x_a^\infty(v) - L^{4v}\epsilon, x_a^\infty(v) + L^{4v}\epsilon)$, where l is the largest element of J_v.

Proof. Fix any $a \in O^\infty(v)$. Then,

$$\begin{aligned}
x_a^q(l) &= x_a^q(k) + \sum_{j=k+1}^{l} m_a^q(O^q(j-1))(t^q(j) - t^q(j-1)) \\
&\le x_a^q(k) + m_a^q(O^q(k))(t^q(k+1) - t^q(k)) + m_a^q(O^q(l-1))(t^q(l) - t^q(k+1)) \\
&\le x_a^\infty(v-1) + L^{4(v-1)}\epsilon + (m_a^\infty(O^q(k)) + \epsilon)\left(t^\infty(v) - t^\infty(v-1) + 2L^{4v-2}\epsilon\right) \\
&\qquad + (m_a^\infty(O^q(l-1)) + \epsilon)\left(2L^{4v-2}\epsilon\right) \\
&< x_a^\infty(v-1) + m_a^\infty(O^\infty(v-1))(t^\infty(v) - t^\infty(v-1)) + L^{4v}\epsilon \\
&= x_a^\infty(v) + L^{4v}\epsilon,
\end{aligned}$$

where the first equality follows from (3.8), the first inequality follows since $m_c^q(O^q(i-1)) \le m_c^q(O^q(l-1))$ for any $i \le l$ by $O^q(i-1) \subset O^q(l-1)$, the second inequality follows from the inductive assumption, (A3), Claims 1 and 2, the third inequality follows from (A1), and the last equality follows from (3.12).

A symmetric argument yields $x_a^q(l) \ge x_a^\infty(v) - L^{4v}\epsilon$. $\|$

We have thus completed the recursive argument, which taken together proves that $T_a^q \in (t^\infty(v) - L^{4v}\epsilon, t^\infty(v) + L^{4v}\epsilon)$ if and only if $t_a^\infty(v) = t^\infty(v)$, for any $q > Q$ for some $Q \in \mathbb{N}$. Since $\epsilon > 0$ can be arbitrarily small, $T_a^q \to T_a^\infty$ as $q \to \infty$. Since there are only a finite number of objects and a finite number of preference types, $||PS^q - PS^\infty|| \to 0$ as $q \to \infty$.

B. Proof of Theorem 2

As with the proof of Theorem 1, let L be a real number satisfying condition (A1) and let $K := \min\{1 - x_a^\infty(v) \,|\, a \in O^\infty(v), v < \bar{v}^\infty\} > 0$.

Fix an agent i_0 of preference type $\pi_0 \in \Pi$ and consider the random assignment for agents of type π_0. Consider the following events:

$E_1^q(\pi) : \hat{m}_\pi^q(t^\infty(v-1) - L^{4(v-1)}\epsilon, t^\infty(v) - L^{4v-2}\epsilon) < m_\pi^\infty[t^\infty(v) - t^\infty(v-1) - L^{4v-3}\epsilon]$, for all v,

$E_2^q(\pi) : \hat{m}_\pi^q(t^\infty(v-1) + L^{4(v-1)}\epsilon, t^\infty(v) + L^{4v-2}\epsilon) \ge m_\pi^\infty[t^\infty(v) - t^\infty(v-1) + L^{4v-3}\epsilon]$, for all $v \ne \bar{v}^\infty$,

$E_3^q(\pi) : \hat{m}_\pi^q(t^\infty(v-1) - L^{4(v-1)}\epsilon, t^\infty(v) + L^{4v-2}\epsilon) < m_\pi^\infty[t^\infty(v) - t^\infty(v-1) + 2L^{4v-2}\epsilon]$, for all v,

$E_4^q(\pi) : \hat{m}_\pi^q(t^\infty(v) - L^{4v-2}\epsilon, t^\infty(v) + L^{4v}\epsilon) < m_\pi^\infty \times 2L^{4v}\epsilon$, for all v,

$E_5^q(\pi) : \hat{m}_\pi^q(t^\infty(v) - L^{4v-2}\epsilon, t^\infty(v) + L^{4v-2}\epsilon) < m_\pi^\infty \times 3L^{4v-2}\epsilon$, for all v,

$E_6^q(\pi) : \hat{m}_\pi^q(t^\infty(v-1) + L^{4(v-1)}\epsilon, t^\infty(v) - L^{4v-2}\epsilon) \ge m_\pi^\infty[t^\infty(v) - t^\infty(v-1) - 2L^{4v-2}\epsilon]$ for all v.

Before presenting a formal proof of Theorem 2, we describe its outline. First, Lemma 1 below shows that all the cutoffs of RP^q become arbitrarily close to the corresponding expiration dates of PS^∞ as $q \to \infty$ when event $E_i^q(\pi)$ holds for every π and $i \in \{1, \dots, 6\}$. Then, in the proof of Theorem 2, (1) we use Lemma 1 to show that the conditional probability of obtaining an object under RP^q is close to the probability of receiving that object under PS^∞, given all the events of the form $E_i^q(\pi)$; and (2) we show that the probability that all the events of the form $E_i^q(\pi)$ hold approaches one as q goes to infinity, so the overall, unconditional probability of obtaining each object in RP^q is close to the conditional probability of receiving that object, given all the events of the form $E_i^q(\pi)$. We finally complete the proof of the Theorem by combining items (1) and (2) above.

Lemma 1. For any $\epsilon > 0$ such that

$$2L^{4\bar{v}^\infty}\epsilon < \min\left\{\min_{v \in \{1,\dots,\bar{v}^\infty\}}\{t^\infty(v) - t^\infty(v-1)\}, K\right\}, \tag{B1}$$

there exists Q such that the following is true for any $q > Q$: if the realization of $f_{-i_0} \in [0,1]^{|N^q|-1}$ is such that events $E_1^q(\pi)$, $E_2^q(\pi)$, $E_3^q(\pi)$, $E_4^q(\pi)$, $E_5^q(\pi)$, and $E_6^q(\pi)$ hold for all $\pi \in \Pi$ with $m_\pi^\infty > 0$, then $\hat{T}_a^q \in (t^\infty(v) - L^{4v}\epsilon, t^\infty(v) + L^{4v}\epsilon)$ if and only if $t_a^\infty(v) = t^\infty(v)$.

Before presenting a complete proof of Lemma 1, we note that the proof closely follows the proof of Theorem 1. More specifically, the proof of Theorem 1 shows inductively that the expiration date of each object type in PS^q is close to that of PS^∞ when q is large enough, while the proof of Lemma 1 shows inductively that the cutoff of each object type in RP^q is close to that of PS^∞ when all the events of the form $E_i^q(\pi)$ hold. Indeed, Claims 1, 2, 3, and 4 in the proof of Theorem 1 correspond to Claims 5, 6, 7, and 8 in the proof of Lemma 1, respectively. Both arguments utilize the fact that the average rates of consumption of each object type in PS^q and RP^q are close to those under PS^∞ during relevant time intervals. The main difference between the proofs of Theorem 1 and Lemma 1 is the following: consumption rates of PS^q are close to PS^∞ because m_π^q is close to m_π^∞ for all a and π when q is large, whereas consumption rates of RP^q are *assumed* to be close by all the events of the form $E_i^q(\pi)$, and Lemma 1 shows that these events in fact make the cutoffs in RP^q close to expiration dates in PS^∞. As mentioned above, the proof of Theorem 2 then shows that assuming all the events of the form $E_i^q(\pi)$ is not problematic, since the probability of these events converges to one as q approaches infinity.

Proof of Lemma 1. There exists Q such that

$$\sum_{\pi \in \Pi : m_\pi^\infty = 0} m_\pi^q < \epsilon, \tag{B2}$$

for any $q > Q$. Fix any such q and suppose that the realization of f_{-i_0} is such that $E_1^q(\pi)$, $E_2^q(\pi)$, $E_3^q(\pi)$, $E_4^q(\pi)$, $E_5^q(\pi)$, and $E_6^q(\pi)$ hold for all π with $m_\pi^\infty > 0$ as described in the statement of the Lemma. We first define the steps

$$\hat{J}_v := \{i | \hat{t}_a^q(i) = \hat{t}^q(i) \text{ for some } a \in A^\infty(v)\}$$

at which the goods in $A^\infty(v)$ expire in RP^q. The lemma shall be proven by showing that $\hat{t}^q(i) \in (t^\infty(v) - L^{4v}\epsilon, t^\infty(v) + L^{4v}\epsilon)$ if and only if $i \in \hat{J}_v$. We show this inductively.

Suppose for any $v' \leq v-1$, $\hat{t}^q(i') \in (t^\infty(v') - L^{4v'}\epsilon, t^\infty(v') + L^{4v'}\epsilon)$ if and only if $i' \in \hat{J}_{v'}$, and further that, for each $a \in O^\infty(v-1)$, $\hat{x}_a^q(k) \in (x_a^\infty(v-1) - L^{4(v-1)}\epsilon, x_a^\infty(v-1) + L^{4(v-1)}\epsilon)$, where k is the largest element of $\hat{J}_{v-1}$. We shall then prove that $\hat{t}^q(i) \in (t^\infty(v) - L^{4v}\epsilon, t^\infty(v) + L^{4v}\epsilon)$ if and only if $i \in \hat{J}_v$, and that, for each $a \in O^\infty(v)$, $\hat{x}_a^q(l) \in (x_a^\infty(v) - L^{4v}\epsilon, x_a^\infty(v) + L^{4v}\epsilon)$, where l is the largest element of $\hat{J}_v$.

Let k be the largest element of $\hat{J}_{v-1}$. It then follows that $\hat{O}^q(k) = O^\infty(v-1)$.

Claim 5. For any $i > k$, $\hat{t}^q(i) > t^\infty(v) - L^{4v-2}\epsilon$.

Proof. Suppose object $a \in O^\infty(v-1) = O^q(k)$ expires at step $k+1$ of RP^q. It suffices to show $\hat{t}_a^q(k+1) > t^\infty(v) - L^{4v-2}\epsilon$. Suppose to the contrary that

$$\hat{t}_a^q(k+1) \leq t^\infty(v) - L^{4v-2}\epsilon. \tag{B3}$$

Recall, by inductive assumption, that

$$\hat{x}_a^q(k) < x_a^\infty(v-1) + L^{4(v-1)}\epsilon. \tag{B4}$$

Thus,

$$\begin{aligned}
\hat{x}_a^q(k+1) &= \hat{x}_a^q(k) + \hat{m}_a^q(\hat{O}^q(k); \hat{t}^q(k), \hat{t}_a^q(k+1)) \\
&\leq \hat{x}_a^q(k) + \hat{m}_a^q(\hat{O}^q(k); t^\infty(v-1) - L^{4(v-1)}\epsilon, t^\infty(v) - L^{4v-2}\epsilon) \\
&< x_a^\infty(v-1) + L^{4(v-1)}\epsilon + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1) - L^{4v-3}\epsilon] + \epsilon, \qquad \text{(B5)}
\end{aligned}$$

where the first equality follows from (3.4) in the definition of RP^q, the first inequality follows from the inductive assumption and (B3), and the second inequality follows from the assumption that $E_1^q(\pi)$ holds for all $\pi \in \Pi$ and conditions (B2) and (B4).

There are two cases. Suppose first $m_a^\infty(O^\infty(v-1)) = 0$. Then, the last line of (B5) becomes

$$x_a^\infty(v-1) + L^{4(v-1)}\epsilon + \epsilon,$$

which is strictly less than 1, since $a \in O^\infty(v-1)$ and since (B1) holds. Suppose next $m_a^\infty(O^\infty(v-1)) > 0$. Then, the last line of (B5) equals

$$\begin{aligned} & x_a^\infty(v-1) + L^{4(v-1)}\epsilon + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1) - L^{4v-3}\epsilon] + \epsilon \\ < \; & x_a^\infty(v-1) + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1)] \\ \leq \; & 1, \end{aligned}$$

where the first inequality follows from (A1), and the second follows since $a \in O^\infty(v-1)$. In either case, we have a contradiction to the fact that a expires at step $k+1$. ‖

Claim 6. For any $i \in \hat{J}_v$, then $\hat{t}^q(i) \leq t^\infty(v) + L^{4v-2}\epsilon$.

Proof. Suppose a expires at step $l \equiv \max \hat{J}_v$ of RP^q. It suffices to show $\hat{t}^q(l) = \hat{t}_a^q(l) \leq t^\infty(v) + L^{4v-2}\epsilon$. If $t^\infty(v) = 1$, then the claim is trivially true. Thus, let us assume $t^\infty(v) < 1$. This implies $m_a^\infty(O^\infty(v-1)) > 0$. For that case suppose, for contradiction, that

$$\hat{t}^q(l) > t^\infty(v) + L^{4v-2}\epsilon. \tag{B6}$$

Then,

$$\begin{aligned} \hat{x}_a^q(l) &= \hat{x}_a^q(k) + \sum_{j=k+1}^{l} \hat{m}_a^q(\hat{O}^q(j-1); \hat{t}^q(j-1), \hat{t}^q(j)) \\ &\geq \hat{x}_a^q(k) + \sum_{j=k+1}^{l} \hat{m}_a^q(\hat{O}^q(k); \hat{t}^q(j-1), \hat{t}^q(j)) \\ &= \hat{x}_a^q(k) + \hat{m}_a^q(O^\infty(v-1); \hat{t}^q(k), \hat{t}^q(l)) \\ &> x_a^\infty(v-1) - L^{4(v-1)}\epsilon + \hat{m}_a^q(O^\infty(v-1); t^\infty(v-1) + L^{4(v-1)}\epsilon, t^\infty(v) + L^{4v-2}\epsilon) \\ &\geq x_a^\infty(v-1) - L^{4(v-1)}\epsilon + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1) + L^{4v-3}\epsilon] \\ &> x_a^\infty(v-1) + m_a^\infty(O^\infty(v-1))[t^\infty(v) - t^\infty(v-1)] \\ &= x_a^\infty(v) = 1, \end{aligned}$$

where the first equality follows from (3.4), the first inequality follows since $\hat{m}_a^q(\hat{O}^q(j-1); t, t') \geq m_a^q(\hat{O}^q(k); t, t')$ for any $j \geq k+1$ and $t \leq t'$ by $\hat{O}^q(j-1) \subseteq \hat{O}^q(k)$, the second equality from $\hat{O}^q(k) = O^\infty(v-1)$ and the definition of $\hat{m}_a^q$, the second inequality follows from the inductive assumption and (B6), the third inequality follows from the assumption that $E_2^q(\pi)$ holds, and the fourth inequality follows from (A1) and the assumption $m_a^\infty(O^\infty(v-1)) > 0$. Thus $\hat{x}_a^q(l) > 1$, which contradicts the definition of $x_a^q(l)$. ‖

Claim 7. If $i \in \hat{J}_{v'}$ for some $v' > v$, then $\hat{t}^q(i) > t^\infty(v) + L^{4v}\epsilon$.

Proof. Suppose otherwise. Let c be the object that expires the first among $O^{\infty}(v)$ in RP^q. Let j be the step at which it expires. Then, we must have

$$\hat{t}^q_c(j) \leq t^{\infty}(v) + L^{4v}\epsilon, \tag{B7}$$

and $\hat{x}^q_c(j) = 1$. Since c is the first object to expire in $O^{\infty}(v)$, at each of steps $k+1, \dots, j-1$, some object in $A^{\infty}(v)$ expires. (If $j = k+1$, then no other object expires in between step k and step j.) By Claim 5, this implies $\hat{t}^q(k+1) > t^{\infty}(v) - L^{4v-2}\epsilon$. Therefore,

$$\begin{aligned}
\hat{x}^q_c(j) &= \hat{x}^q_c(k) + \sum_{i=k+1}^{j} \hat{m}^q_c(\hat{O}^q(i-1); \hat{t}^q(i-1), \hat{t}^q(i)) \\
&\leq \hat{x}^q_c(k) + \hat{m}^q_c(\hat{O}^q(k); \hat{t}^q(k), \hat{t}^q(k+1)) + \hat{m}^q_c(\hat{O}^q(j-1); \hat{t}^q(k+1), \hat{t}^q(j)) \\
&\leq \hat{x}^q_c(k) + \hat{m}^q_c(\hat{O}^q(k); t^{\infty}(v-1) - L^{4(v-1)}\epsilon, t^{\infty}(v) + L^{4v-2}\epsilon) \\
&\quad + \hat{m}^q_c(\hat{O}^q(j-1); t^{\infty}(v) - L^{4v-2}\epsilon, t^{\infty}(v) + L^{4v}\epsilon) \\
&\leq x^{\infty}_c(v-1) + L^{4(v-1)}\epsilon + m^{\infty}_c(O^{\infty}(v-1))[t^{\infty}(v) - t^{\infty}(v-1) + 2L^{4v-2}\epsilon] \\
&\quad + m^{\infty}_c(\hat{O}^q(j-1) \times 2L^{4v}\epsilon + \epsilon \\
&\leq x^{\infty}_c(v) + L^{4v+1}\epsilon \\
&\leq 1 - K + L^{4\bar{v}^{\infty}}\epsilon \\
&< 1,
\end{aligned}$$

where the first equality follows from (3.4), the first inequality follows since $\hat{m}^q_c(\hat{O}^q(j-1); t, t') \geq m^q_c(\hat{O}^q(i-1); t, t')$ for any $j \geq i$ by $\hat{O}^q(j-1) \subseteq \hat{O}^q(i-1)$, the second inequality follows from the inductive assumption, and Claims 5 and 6, the third inequality follows from the inductive assumption, $E^q_3(\pi)$, $E^q_4(\pi)$ and (B2), the fourth inequality follows from (3.12) and (A1), the fifth inequality follows from the definition of K, and the last inequality follows from the assumption that $2L^{4\bar{v}^{\infty}}\epsilon < K$. Thus we obtain $\hat{x}^q_c(j) < 1$, which contradicts the assumption that c expires at step j. ‖

Claims 5, 6, and 7 prove that $\hat{t}^q(i) \in (t^{\infty}(v) - L^{4v-2}\epsilon, t^{\infty}(v) + L^{4v-2}\epsilon) \subset (t^{\infty}(v) - L^{4v}\epsilon, t^{\infty}(v) + L^{4v}\epsilon)$ if and only if $i \in \hat{J}_v$. This implies that $\hat{T}^q_a \in (t^{\infty}(v) - L^{4v}\epsilon, t^{\infty}(v) + L^{4v}\epsilon)$ if and only if $a \in A^{\infty}(v)$. It now remains to show the following.

Claim 8. For each $a \in O^{\infty}(v)$, $x^q_a(l) \in (x^{\infty}_a(v) - L^{4v}\epsilon, x^{\infty}_a(v) + L^{4v}\epsilon)$, where l is the largest element of $\hat{J}_v$.

Proof. Fix any $a \in O^{\infty}(v)$. Then,

$$\begin{aligned}
\hat{x}_a^q(l) &= \hat{x}_a^q(k) + \sum_{j=k+1}^{l} \hat{m}_a^q(\hat{O}^q(j-1); \hat{t}^q(j-1), \hat{t}^q(j)) \\
&\leq \hat{x}_a^q(k) + \hat{m}_a^q(\hat{O}^q(k); \hat{t}^q(k), \hat{t}^q(k+1)) + \hat{m}_a^q(\hat{O}^q(l); \hat{t}^q(k+1), \hat{t}^q(l)) \\
&\leq \hat{x}_a^q(k) + \hat{m}_a^q(\hat{O}^q(k); t^{\infty}(v-1) - L^{4(v-1)}\epsilon, t^{\infty}(v) + L^{4v-2}\epsilon) \\
&\qquad + \hat{m}_a^q(\hat{O}^q(l); t^{\infty}(v) - L^{4v-2}\epsilon, t^{\infty}(v) + L^{4v-2}\epsilon) \\
&< x_a^{\infty}(v-1) + L^{4(v-1)}\epsilon + m_a^{\infty}(\hat{O}^q(k))(t^{\infty}(v) - t^{\infty}(v-1) + 2L^{4v-2}\epsilon) \\
&\qquad + m_a^{\infty}(\hat{O}^q(l)) \times 3L^{4v-2}\epsilon + 2\epsilon \\
&< x_a^{\infty}(v-1) + (m_a^{\infty}(O^{\infty}(v-1)))\,(t^{\infty}(v) - t^{\infty}(v-1)) + L^{4v}\epsilon \\
&= x_a^{\infty}(v) + L^{4v}\epsilon,
\end{aligned}$$

where the first equality follows from (3.4), the first inequality follows from $m_a^q(\hat{O}^q(l); t, t') \geq m_a^q(\hat{O}^q(j); t, t')$ for all $l \geq j$, the second inequality follows from the inductive assumption and Claims 5 and 6, the third inequality follows from the inductive assumption, (B2) and $E_3^q(\pi)$ and $E_5^q(\pi)$, the fourth inequality follows from $\hat{O}^q(k) = O^{\infty}(v-1)$ and (A1), and the last inequality follows from (3.12).

Next we obtain

$$\begin{aligned}
\hat{x}_a^q(l) &= \hat{x}_a^q(k) + \sum_{j=k+1}^{l} \hat{m}_a^q(\hat{O}^q(j-1); \hat{t}^q(j-1), \hat{t}^q(j)) \\
&\geq \hat{x}_a^q(k) + \hat{m}_a^q(\hat{O}^q(k); \hat{t}^q(k), \hat{t}^q(l)) \\
&\geq \hat{x}_a^q(k) + \hat{m}_a^q(\hat{O}^q(k); t^{\infty}(v-1) + L^{4(v-1)}\epsilon, t^{\infty}(v) - L^{4v-2}\epsilon) \\
&\geq x_a^{\infty}(v-1) - L^{4(v-1)}\epsilon + m_a^{\infty}(O^{\infty}(v-1))[t^{\infty}(v) - t^{\infty}(v-1) - 2L^{4v-2}\epsilon] \\
&> x_a^{\infty}(v) - L^{4v}\epsilon,
\end{aligned}$$

where the first inequality follows from $\hat{O}^q(j-1) \subseteq \hat{O}^q(k)$ for any $j \geq k+1$, the second inequality follows from the inductive assumption and Claim 5, the third inequality follows from the inductive assumption and $E_6^q(\pi)$, and the last inequality follows from (3.12) and (A1). These inequalities complete the proof. ‖

We have thus completed the recursive argument, which taken together proves that $\hat{T}_a^q \in (t^{\infty}(v) - L^{4v}\epsilon, t^{\infty}(v) + L^{4v}\epsilon)$ if and only if $a \in A^{\infty}(v)$, for any $q > Q$ for some $Q \in \mathbb{N}$. □

Proof of Theorem 2. We shall show that for any $\varepsilon > 0$ there exists Q such that, for any $q > Q$, for any $\pi_0 \in \Pi$ and $a \in O$,

$$|PS_a^\infty(\pi_0) - RP_a^q(\pi_0)| < (2L^{4(n+1)} + 6(n+1)!)\varepsilon. \tag{B8}$$

Since n is a finite constant, relation (B8) implies the Theorem.

To show this, first assume without loss of generality that ϵ satisfies (B1) and Q is so large that (B2) holds for any $q > Q$. We have

$$\begin{aligned}
RP_a^q(\pi_0) &= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\right] \\
&= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \times Pr\left[\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \\
&+ \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\overline{\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] \times Pr\left[\overline{\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] \\
&= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \times \left(1 - Pr\left[\bigcup_{i=1}^{6}\bigcup_{\pi\in\Pi:m_\pi^\infty>0} \overline{E_i^q(\pi)}\right]\right) \\
&+ \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\overline{\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] \times Pr\left[\bigcup_{i=1}^{6}\bigcup_{\pi\in\Pi:m_\pi^\infty>0} \overline{E_i^q(\pi)}\right] \\
&= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \\
&+ \left\{\mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\overline{\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] - \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right]\right\} \\
&\times Pr\left[\bigcup_{i=1}^{6}\bigcup_{\pi\in\Pi:m_\pi^\infty>0} \overline{E_i^q(\pi)}\right],
\end{aligned} \tag{B9}$$

where for any event E, $\mathbb{E}[\cdot|E]$ denotes the conditional expectation given E, and $\bar{E}$ is the complement event of E.

First, we bound the first term of expression (B9). Since $\bar{v}^\infty \le n+1$, Lemma 1 implies that

$$\mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\middle|\bigcap_{i=1}^{6}\bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \in [T_a^\infty - \tau_a^\infty(\pi_0) - 2L^{4(n+1)}\epsilon, T_a^\infty - \tau_a^\infty(\pi_0) + 2L^{4(n+1)}\epsilon].$$

Second, we bound the second term of expression (B9). By the weak law of large numbers, for any $\varepsilon > 0$, there exists Q such that $Pr\left[\overline{E_i^q(\pi)}\right] < \epsilon$ for any $i \in \{1,2,3,4,5,6\}$, $q > Q$ and $\pi \in \Pi$ with $m_\pi^\infty > 0$. Since there are at most $6(n+1)!$ such events and, in general, the sum of probabilities of a number of events is weakly larger than the probability of the union of the events (Boole's inequality), we obtain

$$\begin{aligned} Pr\left[\bigcup_{i=1}^{6} \bigcup_{\pi \in \Pi : m_\pi^\infty > 0} \overline{E_i^q(\pi)}\right] &\leq \sum_{i=1}^{6} \sum_{\pi \in \Pi : m_\pi^\infty > 0} Pr\left[\overline{E_i^q(\pi)}\right] \\ &\leq 6(n+1)!\epsilon. \end{aligned}$$

Since $\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \in [0,1]$ for any a, q and π_0, the second term of equation (B9) is in $[-6(n+1)!\epsilon, 6(n+1)!\epsilon]$.

From the above arguments and the definition $PS_a^\infty(\pi_0) = T_a^\infty - \tau_a^\infty(\pi_0)$ for every a and π_0, we have that

$$|PS_a^\infty(\pi_0) - RP_a^q(\pi_0)| < (2L^{4(n+1)} + 6(n+1)!)\epsilon,$$

completing the proof. □

C. Proof of Proposition 3

The proposition uses the following two lemmas. Let $\{\Gamma^q\}$ be a family of replica economies. Given any q, define a correspondence $\gamma : N^1 \twoheadrightarrow N^q$ such that $|\gamma(i)| = q$ for each $i \in N^1$, $\gamma(i) \cap \gamma(j) = \emptyset$ if $i \neq j$, and all agents in $\gamma(i)$ have the same preference as i. Call $\gamma(i)$ i's clones in the q-fold replica.

Lemma 2. For all $q \in \mathbb{N}$ and $a, b \in \tilde{O}$, $a \rhd (RP^1, m^1)\, b \iff a \rhd (RP^q, m^q)\, b$.

Proof. We proceed in two steps.

(i) $a \rhd (RP^1, m^1)\, b \Longrightarrow a \rhd (RP^q, m^q)\, b$: Suppose first $a \rhd (RP^1, m^1)\, b$. There exists an individual $i^* \in N^1$ and an ordering $(i^1_{(1)}, \ldots, i^1_{(|N^1|)})$ (implied by some draw $f^1 \in [0,1]^{|N^1|}$) such that the agents in front of i^* in that ordering consume all the objects that i^* prefers to b but not b, and i^* consumes b.

Now consider the q-fold replica. With positive probability, we have an ordering $(\bar{\gamma}(i^1_{(1)}), \ldots, \bar{\gamma}(i^1_{(|N^1|)}))$, where $\bar{\gamma}(i)$ is an arbitrary permutation of $\gamma(i)$. Under this ordering, each agent in $\gamma(i^1_{(j)})$ will consume a copy of the object agent $i^1_{(j)}$ will consume in the base economy, and hence all the agents in $\gamma(i^*)$ will consume b (despite preferring a to b). This proves that $a \rhd (RP^q, m^q)\, b$.

(ii) $a \rhd (RP^q, m^q)\, b \Longrightarrow a \rhd (RP^1, m^1)\, b$: Suppose $a \rhd (RP^q, m^q)\, b$. Then, with positive probability, a draw $f^q \in [0,1]^{|N^q|}$ entails an ordering in which the agents ahead of $i^* \in N^q$

consume all of the objects that i^* prefers to b, but not all of the copies of b have been consumed by them. List these objects in the order that their last copies are consumed, and let the set of these objects be $\hat{O} := \{o_1, ..., o_m\} \subset O$, where o_l is completely consumed before o_{l+1} for all $l = 1, \ldots, m-1$. (Note that $a \in \hat{O}$.) Let i^{**} be such that $i^* \in \gamma(i^{**})$.

We first construct a correspondence $\xi : \hat{O} \to N^1 \setminus \{i^{**}\}$ defined by

$$\xi(o) := \left\{ i \in N^1 \setminus \{i^{**}\} \mid \exists j \in \gamma(i) \text{ who consumes } o \text{ under } f^q \right\}.$$

Claim 9. Any agent in N^q who consumes o_l prefers o_l to all objects in $\tilde{O} \setminus \{o_1, ..., o_{l-1}\}$ under f^q. Hence, any agent in $\xi(o_l)$ prefers o_l to all objects in $\tilde{O} \setminus \{o_1, ..., o_{l-1}\}$.

Claim 10. For each $O' \subset \hat{O}$, $| \cup_{o \in O'} \xi(o)| \geq |O'|$.

Proof. Suppose otherwise. Then, there exists $O' \subset \hat{O}$ such that $k := |\cup_{o \in O'} \xi(o)| < |O'| =: l$. Reindex the sets so that $\cup_{o \in O'} \xi(o) = \{a^1, ..., a^k\}$ and $O' = \{o^1, ..., o^l\}$. Let x_{ij} denote the number of clones of agent $a^j \in \xi(o^i)$ who consume o^i in the q-fold replica under f^q.

Since $\sum_{i=1}^{l} x_{ij} \leq |\gamma(a^j)| = q$,

$$\sum_{j=1}^{k} \sum_{i=1}^{l} x_{ij} \leq kq.$$

At the same time, all q copies of each object in O' are consumed, and at most $q-1$ clones of i^{**} could be those contributing to that consumption. Therefore,

$$\sum_{i=1}^{l} \sum_{j=1}^{k} x_{ij} \geq lq - (q-1) = (l-1)q + 1 > kq,$$

We thus have a contradiction. ‖

By Hall's Theorem, Claim 10 implies that there exists a mapping $\mu : \hat{O} \to N^1 \setminus \{i^{**}\}$ such that $\mu(o) \in \xi(o)$ for each $o \in \hat{O}$ and $\mu(o) \neq \mu(o')$ for $o \neq o'$.

Now consider the base economy. With positive probability, f^1 has a priority ordering, $(\mu(o_1), ..., \mu(o_m), i^{**})$ followed by an arbitrary permutation of the remaining agents. Given such a priority ordering, the objects in $\hat{O}$ will be all consumed before i^{**} gets her turn but b will not be consumed before i^{**} gets her turn, so she will consume b. This proves that $a \rhd (RP^1, m^1)\, b$. □

Lemma 3. RP^1 is wasteful if and only if RP^q is wasteful for any $q \in \mathbb{N}$.

Proof. We proceed in two steps.

(i) The "only if" Part: Suppose that RP^1 is wasteful. Then, there are objects $a, b \in \tilde{O}$ and an agent $i^* \in N^1$ who prefers a to b such that she consumes b under some ordering $(\tilde{i}^1_{(1)}, \ldots, \tilde{i}^1_{(|N^1|)})$ (implied by some $\tilde{f}^1$) and that a is not consumed by any agent under $(\hat{i}^1_{(1)}, \ldots, \hat{i}^1_{(|N^1|)})$ (implied by some $\hat{f}^1$). (This is the necessary implication of the "wastefulness" under RP^1.)

Now consider its q-fold replica, RP^q. With positive probability, an ordering $(\bar{\gamma}(\tilde{i}^1_{(1)}), \ldots, \bar{\gamma}(\tilde{i}^1_{(|N^1|)}))$ arises, where $\bar{\gamma}(i)$ is an arbitrary permutation of $\gamma(i)$. Clearly, each agent in $\gamma(i^*)$ must consume b even though she prefers a over b (since all copies of all objects the agents in $\gamma(i^*)$ prefer to b are all consumed by the agents ahead of them). Likewise, with positive probability, an ordering $(\bar{\gamma}(\hat{i}^1_{(1)}), \ldots, \bar{\gamma}(\hat{i}^1_{(|N^1|)}))$ arises. Clearly, under this ordering, no copies of object a are consumed. It follows that RP^q is wasteful.

(ii) The "if" Part: Suppose next that RP^q is wasteful. Then, there are objects $a, b \in \tilde{O}$ and an agent $i^{**} \in N^q$ who prefers a over b such that she consumes b under some ordering $(\tilde{i}^q_{(1)}, \ldots, \tilde{i}^q_{(|N^q|)})$ (implied by some $\tilde{f}^q$) and that not all copies of object a are consumed under $(\hat{i}^q_{(1)}, \ldots, \hat{i}^q_{(|N^q|)})$ (implied by some $\hat{f}^q$).

Now consider the corresponding base economy and associated RP^1. The argument of Part (ii) of Lemma 2 implies that there exists an ordering $(\tilde{i}^1_{(1)}, \ldots, \tilde{i}^1_{(|N^1|)})$ under which agent $\tilde{i}^* = \gamma^{-1}(i^{**}) \in N^1$ consumes b even though she prefers a over b.

Next, we prove that RP^1 admits a positive-probability ordering under which object a is not consumed. Let $N'' := \{r \in N^1 | \exists j \in \gamma(r)$ who consumes the null object under $\hat{f}^q\}$. For each $r \in N''$, we let $\emptyset^r$ denote the null object some clone of $r \in N^1$ consumes. In other words, we use different notations for the null object consumed by the clones of different agents in N''. Given this convention, there can be at most q copies of each $\emptyset^r$.

Let $\bar{O} := O \cup (\cup_{r \in N''} \emptyset^r) \setminus \{a\}$, and define a correspondence $\psi : N^1 \to \bar{O}$ by

$$\psi(r) := \{b \in \bar{O} | \exists j \in \gamma(r) \text{ who consumes } b \text{ under } \hat{f}^q\}.$$

Claim 11. For each $N' \subset N^1$, $|\cup_{r \in N'} \psi(r)| \geq |N'|$.

Proof. Suppose not. Then, $k := |\cup_{r \in N'} \psi(r)| < |N'| =: l$. Reindex the sets so that $\cup_{r \in N'} \psi(r) =: \{o^1, ..., o^k\}$ and $N' = \{r^1, ..., r^l\}$. Let x_{ij} denote the number of copies of object $o^j \in \psi(r^i)$ consumed by the clones of r^i in the q-fold replica under $\hat{f}^q$.

Since there are at most q copies of each object, we must have

$$\sum_{j=1}^{k} \sum_{i=1}^{l} x_{ij} \leq kq.$$

At the same time, all q clones of each agent in N', excluding $q-1$ agents (who may be consuming a), are consuming some objects in O' under $\hat{f}^q$, so we must have

$$\sum_{i=1}^{l} \sum_{j=1}^{k} x_{ij} \geq lq + q - 1 = (l-1)q + 1 > kq,$$

We thus have a contradiction. $\|$

Claim 11 then implies, via Hall's theorem, that there exists a mapping $\nu : N^1 \to \bar{O}$ such that $\nu(r) \in \psi(r)$ for each $r \in N^1$ and $\nu(r) \neq \nu(r')$ if $r \neq r'$.

Let $O' \subset \bar{O}$ be the subset of all object types in $\bar{O}$ whose entire q copies are consumed under $\hat{f}^q$. Order O' in the order that the last copy of each object is consumed; i.e., label $O' = \{o^1, ..., o^m\}$ such that the last copy of object o^i is consumed prior to the last copy of o^j if $i < j$. Let $\hat{N}$ be any permutation of the agents in $\nu^{-1}(\bar{O} \setminus O')$. Now consider the ordering in RP^1: $(\hat{i}^1_{(1)}, \dots, \hat{i}^1_{(|N^1|)}) = (\nu^{-1}(o^1), \dots, \nu^{-1}(o^m), \hat{N})$, where the notational convention is as follows: for any $l \in \{1, \dots, m\}$, if $\nu^{-1}(o^l)$ is empty, then no agent is ordered.

Claim 12. Under the ordering $(\hat{i}^1_{(1)}, \dots, \hat{i}^1_{(|N^1|)}) = (\nu^{-1}(o^1), \dots, \nu^{-1}(o^m), \hat{N})$, a is not consumed.

Proof. For any $l = 0, \dots, m$, let O^l be the set of objects that are consumed by agents $\nu^{-1}(o^1), \dots, \nu^{-1}(o^l)$ under the current ordering (note that some of $\nu^{-1}(o^1), \dots, \nu^{-1}(o^l)$ may be nonexistent). We shall show $O^l \subseteq \{o^1, \dots, o^l\}$ by an inductive argument. First note that the claim is obvious for $l = 0$. Assume that the claim holds for $0, 1, \dots, l-1$. If $\nu^{-1}(o^l) = \emptyset$, then no agent exists to consume an object at this step and hence the claim is obvious. Suppose $\nu^{-1}(o^l) \neq \emptyset$. By definition of ν, agent $\nu^{-1}(o^l)$ weakly prefers o^l to any object in $\tilde{O} \setminus \{o^1, \dots, o^{l-1}\}$. Therefore $\nu^{-1}(o^l)$ consumes an object in $\{o^l\} \cup (\{o^1, \dots, o^{l-1}\} \setminus O^{l-1}) \subseteq \{o^1, \dots, o^l\}$. This and the inductive assumption imply $O^l \subseteq \{o^1, \dots, o^l\}$.

Next, consider agents that appears in the ordered set $\hat{N}$. By an argument similar to the previous paragraph, each agent i in $\hat{N}$ consumes an object in $\nu(i) \cup (\{o^1, \dots, o^m\} \setminus O^m)$. In particular, no agent in $\hat{N}$ consumes a. $\|$

Since the ordering $(\hat{i}^1_{(1)}, \dots, \hat{i}^1_{(|N^1|)}) = (\nu^{-1}(o^1), \dots, \nu^{-1}(o^m), \hat{N})$ realizes with positive probability under RP^1, Claim 12 completes the proof of Lemma 3. $\square$

Proof of Proposition 3. If RP^q is ordinally inefficient for some $q \in \mathbb{N}$, then either it is wasteful or there must be a cycle of binary relation $\triangleright(RP^q, m^q)$. Lemmas 2 and 3 then imply that RP^1 is wasteful or there exists a cycle of $\triangleright(RP^1, m^1)$, and that $RP^{q'}$ is wasteful or there exists a cycle of $\triangleright(RP^{q'}, m^{q'})$ for each $q' \in \mathbb{N}$. Hence, for each $q' \in \mathbb{N}$, $RP^{q'}$ is ordinally inefficient. $\square$

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