On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games*

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Abstract

We generalize Athey's (2001) and McAdams' (2003) results on the existence of monotone pure strategy equilibria in Bayesian games. We allow action spaces to be compact locally-complete metrizable semilattices, type spaces to be partially ordered complete separable metric spaces, and employ weaker conditions than the single-crossing condition used by Athey and McAdams and the quasisupermodularity condition used by McAdams. Our proof is based upon contractibility rather than convexity of best reply sets. Several examples illustrate the scope of the result, including new applications to multiunit auctions with risk-averse bidders.

1. Introduction

In an important paper, Athey (2001) demonstrates that a monotone pure strategy equilibrium exists whenever a Bayesian game satisfies a Spence-Mirlees single-crossing property. Athey's result is now a central tool for establishing the existence of monotone pure strategy equilibria in auction theory (see e.g., Athey (2001), Reny and Zamir (2004)). Recently, McAdams (2003) has shown that Athey's results, which exploit the assumed total ordering of the players' one-dimensional type and action spaces, can be extended to settings in which type and action spaces are multi-dimensional and only partially ordered. This permits new existence results in auctions with multi-dimensional types and multi-unit demands (see McAdams (2004)). The techniques employed by Athey and McAdams, while ingenious, have

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their limitations and do not appear to easily extend beyond the environments they consider. We therefore introduce a new approach.

The approach taken here exploits an important unrecognized property of a large class of Bayesian games. In these games, the players' pure-strategy best-reply sets, while possibly nonconvex, are always contractible.[1] This observation permits us to generalize the results of Athey and McAdams in several directions. First, we permit infinite-dimensional type spaces and infinite-dimensional action spaces. Both can occur, for example, in share-auctions where a bidder's type is a function expressing his marginal valuation at any quantity of the good, and where a bidder's action is a downward-sloping demand schedule. Second, even when type and action spaces are subsets of Euclidean space, we permit more general joint distributions over types, allowing one player to have private information about the support of another's private information, as well as permitting positive probability on lower dimensional subsets, which can be useful when modeling random demand in auctions. Third, our approach allows general partial orders on both type spaces and action spaces. This can be especially helpful because, while single-crossing may fail for one partial order, it might nonetheless hold for another, in which case our existence result can still be applied (see section 5 for two such applications). Finally, while single-crossing is helpful in establishing the hypotheses of our main theorem, it is not necessary; our hypotheses are satisfied even in instances where single-crossing fails.

The key to our approach is to employ a more powerful fixed point theorem than those employed in Athey (2001) and McAdams (2003). Both Athey and McAdams apply a fixed-point theorem to the product of the players' best-reply correspondences — Athey applies Kakutani's theorem, McAdams applies Glicksberg's theorem. In both cases, essentially all of the effort is geared toward proving that sets of monotone pure-strategy best replies are convex. Our central observation is that this impressive effort is unnecessary and, more importantly, that the additional structure imposed to achieve the desired convexity (i.e., *Euclidean* type spaces with the *coordinatewise* partial order, *Euclidean sublattice* action spaces, *absolutely continuous* type distributions), is unnecessary as well.

The fixed point theorem upon which our approach is based is due to Eilenberg and Montgomery (1946) and does not require the correspondence in question to be convex-valued. Rather, the correspondence need only be contractible-valued. Consequently, we need only demonstrate that monotone pure-strategy best-reply sets are contractible. While this task need not be straightforward in general, it turns out to be essentially trivial in the class of Bayesian games of interest here. To gain a sense of this, note first that a pure strategy — a function from types to actions — is a best reply for a player if and only if it

[1] A set is contractible if it can be continuously deformed, within itself, to a single point. Convex sets are contractible, but contractible sets need not be convex (e.g., the symbol "+" viewed as a subset of $\mathbb{R}^2$).

is a pointwise interim best reply for almost every type of that player. Consequently, any piecewise combination of two best replies — i.e., a strategy equal to one of the best replies on some subset of types and equal to the other best reply on the remainder of types — is also a best reply. Thus, by reducing the set of types on which the first best reply is employed and increasing the set of types on which the second is employed, it is possible to move from the first best reply to the second, all the while remaining within the set of best replies. With this simple observation, the set of best replies can be shown to be contractible.[2]

Because contractibility of best-reply sets follows almost immediately from the pointwise almost everywhere optimality of interim best replies, we are able to expand the domain of analysis well beyond Euclidean type and action spaces, and most of our additional effort is directed here. In particular, we require and prove two new results about the space of monotone functions from partially ordered complete separable metric spaces endowed with an appropriate probability measure into compact metric semilattices. The first of these results (Lemma A.10) is a generalization of Helley's selection theorem, stating that any sequence of monotone functions possesses a pointwise almost everywhere convergent subsequence. The second result (Lemma A.16) states that the space of monotone functions is an absolute retract, a property that, like convexity, renders a space amenable to fixed point analysis. In contrast, both of these results would be straightforward to establish with the additional structure imposed by Athey and McAdams.

Our main result, Theorem 4.1, is as follows. Suppose that action spaces are compact convex semilattices or compact locally-complete metric semilattices, that type spaces are partially ordered complete separable metric spaces, that payoffs are continuous in actions for each type vector, and that the joint distribution over types induces marginals for each player assigning probability zero to any set with no strictly ordered points.[3] If, whenever the others employ monotone pure strategies, each player's set of monotone pure-strategy best replies is nonempty and join-closed,[4] then a monotone pure strategy equilibrium exists.

We provide several applications yielding new existence results. First, we consider both uniform-price and discriminatory multi-unit auctions with independent private values. We depart from standard assumptions by permitting bidders to be risk averse. Under risk aversion, monotonicity of best replies is known to fail under the standard coordinatewise partial order over types. Nevertheless, by employing an alternative, yet natural, partial order over types, we are able to demonstrate the existence of a monotone pure strategy

[2] Because we are concerned with *monotone* pure strategy best replies, some care must be taken to ensure that one maintains monotonicity throughout the contraction. Further, continuity of the contraction requires appropriate assumptions on the distribution over players' types. In particular there can be no atoms.

[3] Two points are strictly ordered if every point in some neighborhood of one is greater than every point in some neighborhood of the other.

[4] That is, the pointwise supremum of any pair of best replies is also a best reply.

equilibrium with respect to this partial order. In the uniform-price auction, no additional assumptions are required, while in the discriminatory auction each bidder is assumed to have CARA preferences. Our next application considers a price-competition game between firms selling differentiated products. Firms have private information about their constant marginal cost as well as private information about market demand. While it is natural to assume that costs may be affiliated, in the context we consider it is less natural to assume that information about market demand is affiliated. Nonetheless, and again through a judicious choice of a partial order over types, we are able to establish the existence of a pure strategy equilibrium that is monotone in players' costs, but not necessarily monotone in their private information about demand. Our final application establishes the existence of monotone *mixed* strategy equilibria when type spaces have atoms.[5]

If the actions of distinct players are strategic complements – an assumption we do not impose – Van Zandt and Vives (2006) have shown that even stronger results can be obtained. They prove that monotone pure strategy equilibria exist under somewhat more general distributional, type-space and action-space assumptions than we employ here, and demonstrate that such an equilibrium can be obtained through iterative application of the best reply map.[6] In our view, Van Zandt and Vives (2006) obtain perhaps the strongest possible results for the existence of monotone pure strategy equilibria in Bayesian games when strategic complementarities are present. Of course, while many interesting economic games exhibit strategic complements, many do not. Indeed, many auction games satisfy the hypotheses required to apply our result here, but fail to satisfy the strategic complements condition.[7] The two approaches are therefore complementary.

The remainder of the paper is organized as follows. Section 2 presents the essential ideas as well as the corollary of Eilenberg and Montgomery's (1946) fixed point theorem that is central to our approach. Section 3 describes the formal environment, including semilattices and related issues. Section 4 contains our main result, section 6 contains its proof, and section 5 provides several applications.

[5] A player's mixed strategy is monotone if all actions in the totally ordered support of one of his types are weakly greater than all actions in the totally ordered support of any lower type.

[6] Related results can be found in Milgrom and Roberts (1990) and Vives (1990).

[7] In a first-price IPV auction, for example, a bidder might increase his bid if his opponent increases her bid slightly when her private value is high. However, for sufficiently high increases in her bid at high private values, the bidder might be better off reducing his bid (and chance of winning) to obtain a higher surplus when he does win. Such nonmonotonic responses to changes in the opponent's strategy are not possible under strategic complements.

2. The Main Idea[8]

As mentioned in the introduction, the proof of our main result is based upon a fixed point theorem that permits the correspondence for which a fixed point is sought — here, the product of the players' monotone pure best reply correspondences — to have contractible rather than convex values.

In this section, we introduce this fixed point theorem and also illustrate the ease with which contractibility can be established, focussing on the most basic case in which type spaces are $[0,1]$, action spaces are subsets of $[0,1]$, and the marginal distribution over each player's type space is atomless.

A subset X of a metric space is *contractible* if for some $x_0 \in X$ there is a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X$, $h(0,x) = x$ and $h(1,x) = x_0$. We then say that h is a *contraction* for X.

Note that every convex set is contractible since, choosing any point x_0 in the set, the function $h(\tau, x) = (1-\tau)x + \tau x_0$ is a contraction. On the other hand, there are contractible sets that are not convex (e.g., the symbol "+"). Hence, contractibility is a strictly more permissive condition than convexity.

A subset X of a metric space Y is said to be a *retract* of Y if there is a continuous function mapping Y onto X leaving every point of X fixed. A metric space (X, d) is an *absolute retract* if for every metric space (Y, δ) containing X as a closed subset and preserving its topology, X is a retract of Y. Examples of absolute retracts include closed convex subsets of Euclidean space or of any metric space, and many nonconvex sets as well (e.g., any contractible polyhedron).[9] The fixed point theorem we make use of is the following corollary of an even more general result due to Eilenberg and Montgomery (1946).[10]

Theorem 2.1. *Suppose that a compact metric space (X, d) is an absolute retract and that $F : X \twoheadrightarrow X$ is an upper hemicontinuous, nonempty-valued, contractible-valued correspondence. Then F has a fixed point.*

For our purposes, the correspondence F is the product of the players' monotone pure strategy best reply correspondences and X is the product of their sets of monotone pure

[8]Readers more interested in applying our main result than in its proof may wish to skip the present section.

[9]Indeed, a compact subset, X, of Euclidean space is an absolute retract if and only if it is contractible and locally contractible. The latter means that for every $x_0 \in X$ and every neighborhood U of x_0, there is a neighborhood V of x_0 and a continuous $h : [0,1] \times V \to U$ such that $h(0,x) = x$ and $h(1,x) = x_0$ for all $x \in V$.

[10]Theorem 2.1 follows directly from Eilenberg and Montgomery (1946) Theorem 1, because every absolute retract is a contractible absolute neighborhood retract (Borsuk (1966), V (2.3)) and every nonempty contractible set is acyclic (Borsuk (1966), II (4.11)).

strategies. While we must eventually establish all of the properties necessary to apply Theorem 2.1, our modest objective for the remainder of this section is to show, with remarkably little effort, that in the simple environment considered here, F is contractible-valued, i.e., that monotone pure best reply sets are contractible.

Suppose that player 1's type is drawn uniformly from the unit interval $[0,1]$. Fix monotone pure strategies for other players, and suppose that $\bar{s} : [0,1] \to A$ is a monotone best reply for player 1, where $A \subseteq [0,1]$ is player 1's compact action set. Indeed, suppose that $\bar{s}$ is player 1's *largest* monotone best reply in the sense that if s is any other monotone best reply, then $\bar{s}(t) \geq s(t)$ for every type t of player 1. We shall provide a contraction that shrinks player 1's entire set of monotone best replies, within itself, to the largest monotone best reply $\bar{s}$. The simple, but key, observation is that a pure strategy is a best reply for player 1 if and only if it is a pointwise best reply for almost every type $t \in [0,1]$ of player 1.

Consider the following candidate contraction. For $\tau \in [0,1]$ and any monotone best reply, s, for player 1, define $h(\tau, s) : [0,1] \to A$ as follows:

$$h(\tau, s)(t) = \begin{cases} s(t), & \text{if } t \leq 1 - \tau \text{ and } \tau < 1 \\ \bar{s}(t), & \text{otherwise.} \end{cases}$$

Note that $h(0,s) = s$, $h(1,s) = \bar{s}$, and $h(\tau,s)(t)$ is always either $\bar{s}(t)$ or $s(t)$ and so is a best reply for almost every t. Hence, by the key observation in the previous paragraph, $h(\tau,s)(\cdot)$ is a best reply. The pure strategy $h(\tau,s)(\cdot)$ is monotone because it is the smaller of two monotone functions for low values of t and the larger of them for high values of t. Moreover, because the marginal distribution over player 1's type is atomless, the monotone pure strategy $h(\tau,s)(\cdot)$ varies continuously in the arguments τ and s, when the distance between two strategies of player 1 is defined to be the integral with respect to his type distribution of their absolute pointwise difference (see section 6).[11] Consequently, h is a contraction under this metric, and so player 1's set of monotone best replies is contractible. It's that simple.

Figure 2.1 shows how the contraction works when player 1's set of actions A happens to be finite, so that his set of monotone best replies cannot be convex in the usual sense unless it is a singleton. Three monotone functions are shown in each panel, where 1's actions are on the vertical axis and 1's types are on the horizontal axis. The dotted line step function is s, the solid line step function is $\bar{s}$, and the thick solid line step function (red) is the step function determined by the contraction h.

In panel (a), $\tau = 0$ and h coincides with s. The position of the vertical line (blue) appearing in each panel represents the value of τ. The vertical line (blue) appearing in each

[11]This particular metric is important because it renders a player's payoff continuous in his strategy choice.

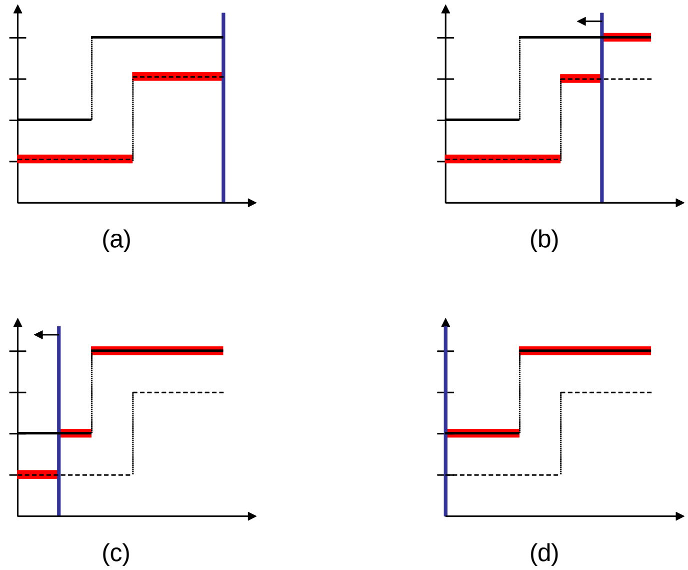

Figure 2.1: The Contraction

panel intersects the horizontal axis at the point $1-\tau$. When $\tau = 0$ the vertical line is at the far right-hand side, as shown in panel (a). As indicated by the arrow, the vertical line moves continuously toward the origin as τ moves from 0 to 1. The thick (red) step function determined by the contraction h is $s(t)$ for values of t to the left of the vertical line and is $\bar{s}(t)$ for values of t to the right; see panels (b) and (c). The step function h therefore changes continuously with τ because the areas between strategies change continuously. In panel (d), $\tau = 1$ and h coincides with $\bar{s}$. So altogether, as τ moves continuously from 0 to 1, the image of the contraction moves continuously from s to $\bar{s}$.

Two points are worth mentioning before moving on. First, single-crossing plays no role in establishing the contractibility of sets of monotone best replies. As we shall see, ensuring the existence of monotone pure strategy best replies is where single-crossing can be helpful. Thus, the present approach clarifies the role of single-crossing insofar as the existence of monotone pure strategy equilibrium is concerned.[12] Second, the action spaces employed in the above illustration are totally ordered, as in Athey (2001). Consequently, if two actions are optimal for some type of player 1, then the maximum of the two actions, being one or the other of them, is also optimal. The optimality of the maximum of two optimal actions is important for ensuring that a largest monotone best reply exists. When action spaces are only partially ordered (e.g., when actions are multi-dimensional with the coordinatewise partial order), the maximum of two optimal actions need not even be well-defined, let alone

[12] Both Athey (2001) and McAdams (2003) employ single-crossing to help establish the existence of monotone best replies *and* to establish the convexity of the set of monotone best replies. This accounts for why their single-crossing condition is more restrictive than necessary. See Subsection 4.1.

optimal. Therefore, to also cover partially ordered action spaces, we assume in the sequel (see section 3) that action spaces are semilattices — i.e., that for every pair of actions there is a least upper bound (l.u.b.) — and that the l.u.b. of two optimal actions is optimal. Stronger versions of both assumptions are employed in McAdams (2003).

3. The Environment

3.1. Partial Orders, Lattices and Semilattices

Let A be a nonempty set partially ordered by $\geq$.[13] For $a, b \in A$, if the set $\{a, b\}$ has a least upper bound (l.u.b.) in A, then it is unique and will be denoted by $a \vee b$, the *join* of a and b. In general, such a bound need not exist. However, if every pair of points in A has an l.u.b. in A, then we shall say that A is a *semilattice.* It is straightforward to show that, in a semilattice, every finite set, $\{a, b, ..., c\}$, has a least upper bound, which we denote by $\vee\{a, b, ..., c\}$ or $a \vee b \vee ... \vee c$.

If the set $\{a, b\}$ has a greatest lower bound (g.l.b.) in A, then it too is unique and it will be denoted by $a \wedge b$, the *meet* of a and b. Once again, in general, such a bound need not exist. If every pair of points in A has both an l.u.b.. in A and a g.l.b. in A, then we shall say that A is a *lattice.*[14]

Clearly, every lattice is a semilattice. However, the converse is not true. For example, employing the coordinatewise partial order on vectors in $\mathbb{R}^m$, the set of vectors whose sum is at least one is a semilattice, but not a lattice.

If A is a metric space, a partial order $\geq$ on A is called *measurable*, *closed*, or *convex* if the subset $\{(a, b) \in A \times A : b \geq a\}$ of $A \times A$ is, respectively, Borel measurable, closed, or convex.[15] Note that if the partial order $\geq$ is convex then A is convex because $a \geq a$ for every $a \in A$. Say that A is *upper-bound-convex* if it contains the convex combination of any two members whenever one of them, $\bar{a}$ say, is an upper bound for A – i.e., $\bar{a} \geq a$ for every $a \in A$. Because sets without upper bounds are trivially upper-bound-convex, every convex set is upper-bound-convex. Any two distinct points a, b in A are *strictly ordered* if there are neighborhoods U of a and V of b such that $u \geq v$ for every $u \in U$ and every $v \in V$.

A *metric semilattice* is a semilattice, A, endowed with a metric under which the join operator, $\vee$, is continuous as a function from $A \times A$ into A. In the special case in which A is a metric semilattice in $\mathbb{R}^m$ under the Euclidean metric, we say that A is a *Euclidean metric*

[13]Hence, $\geq$ is transitive ($a \geq b$ and $b \geq c$ imply $a \geq c$), reflexive ($a \geq a$), and antisymmetric ($a \geq b$ and $b \geq a$ imply $a = b$).

[14]Defining a semilattice in terms of the join operator, $\vee$, rather than the meet operator, $\wedge$, is entirely a matter of convention.

[15]Product spaces are endowed with the product topology throughout the paper.

semilattice. Note also that because in a semilattice $b \geq a$ if and only if $a \vee b = b$, a partial order in a metric semilattice is necessarily closed.[16]

A semilattice A is *complete* if every nonempty subset S of A has a least upper bound, $\vee S$, in A. A metric semilattice A is *locally complete* if for every $a \in A$ and every neighborhood U of a, there is a neighborhood W of a contained in U such that every nonempty subset S of W has a least upper bound, $\vee S$, contained in U. Lemma A.18 establishes that a compact metric semilattice A is locally complete if and only if for every $a \in A$ and every sequence $a_n \to a$, $\lim_m (\vee_{n \geq m} a_n) = a$.[17] A distinct sufficient condition for local completeness is given in Lemma A.20.

Some examples of compact locally-complete metric semilattices are,

- finite semilattices
- compact sublattices of $\mathbb{R}^m$– because the join of any two points is their coordinatewise maximum
- compact Euclidean metric semilattices (Lemma A.19)
- compact upper-bound-convex semilattices in $\mathbb{R}^m$ endowed with the coordinatewise partial order (Lemmas A.17 and A.19)
- The space of continuous functions $f : [0,1] \to [0,1]$ satisfying for some $\lambda > 0$ the Lipschitz condition $|f(x) - f(y)| \leq \lambda |x - y|$, endowed with the maximum norm $\|f\| = \max_x |f(x)|$, and partially ordered by $f \geq g$ if $f(x) \geq g(x)$ for all $x \in [0,1]$.

The last example is an infinite dimensional locally-complete metric semilattice. In general, and unlike compact Euclidean metric semilattices, infinite dimensional metric semilattices need not be locally complete even if compact and convex.[18]

3.2. A Class of Bayesian Games

There are N players, $i = 1, 2, ..., N$. Player i's type space is T_i and his action space is A_i, and both are nonempty and partially ordered. All partial orders, although possibly distinct, will be denoted by $\geq$. Player i's payoff function is $u_i : A \times T \to \mathbb{R}$, where $A = \times_{i=1}^N A_i$

[16] The converse can fail. For example, the set $A = \{(x, y) \in \mathbb{R}^2_+ : x + y = 1\} \cup \{(1, 1)\}$ is a semilattice with the coordinatewise partial order, and this order is closed under the Euclidean metric. But A is not a metric semilattice because whenever $a_n \neq b_n$ and $a_n, b_n \to a$, we have $(1, 1) = \lim(a_n \vee b_n) \neq (\lim a_n) \vee (\lim b_n) = a$.

[17] Hence, compactness and metrizability of a lattice under the order topology (see Birkohff (1967, p.244) is sufficient, but not necessary, for local completeness of the corresponding semilattice.

[18] No $\mathcal{L}_p$ space is locally complete when $p < +\infty$ and endowed with the pointwise partial order. See Hart and Weiss (2005) for a compact metric semilattice that is not locally complete. Their example can be modified so that the space is in addition convex and locally convex.

and $T = \times_{i=1}^N T_i$. The common prior over the players' types is a probability measure μ on the Borel subsets of T – see G.1 below for the topological structure on T. Let G denote this Bayesian game.

We shall make use of the following additional assumptions, where μ_i denotes the marginal of μ on T_i. For every player i,

G.1 T_i is a complete separable metric space endowed with a measurable partial order.

G.2 μ_i assigns probability zero to any Borel subset of T_i having no strictly ordered points.[19]

G.3 A_i is a compact metric space and a semilattice with a closed partial order.[20]

G.4 Either (i) A_i is a convex and locally convex topological space and the partial order on A_i is convex, or (ii) A_i is a locally-complete metric semilattice.[21]

G.5 $u_i(a, t)$ is bounded, jointly measurable, and continuous in $a \in A$ for every $t \in T$.

Assumptions G.1-G.5 strictly generalize the assumptions in Athey (2001) and McAdams (2003) who assume that each A_i is a compact sublattice of Euclidean space and hence a compact locally-complete metric semilattice, that each $T_i = [0, 1]^{m_i}$ is endowed with the coordinatewise partial order, and that μ is absolutely continuous with respect to Lebesgue measure.[22,23] This additional structure, which we do not require, is necessary for their Kakutani-Glicksberg-based approach.[24]

In addition to permitting infinite-dimensional type spaces, assumption G.1 permits the partial order on player i's type space to be distinct from the usual coordinatewise partial order when T_i is Euclidean. As we shall see, this flexibility is very helpful in providing new equilibrium existence results for multi-unit auctions with risk averse bidders.

[19] I thank Benjamin Weiss for suggesting this simplification of a closely related previous assumption.

[20] Note that G.3 does not require A_i to be a metric semilattice – its join operator need not be continuous.

[21] It is permissible for (i) to hold for some players and (ii) to hold for others. A topological space is convex if the operation of taking convex combinations of pairs of points yields a point in the space and is jointly continuous in the pair of points and in the weights on them. A topological space is locally convex if for every open set U, every point in U has a convex open neighborhood contained in U.

[22] McAdams (2003) assumes, further, that the joint density over types is everywhere strictly positive.

[23] If $T_i = [0, 1]^{m_i}$, then absolute continuity of μ implies G.2. Indeed, if no two members of some Borel subset B of i's type space are strictly ordered, then $B \cap [0, 1]t_i$ contains at most one point for every $t_i \in \text{int}T_i$. Fubini's theorem then implies that B has Lebesgue measure zero, and so $\mu_i(B) = 0$ by absolute continuity.

[24] Indeed, suppose a player's action set is the semilattice $A = \{(1, 0), (1/2, 1/2), (0, 1), (1, 1)\}$ in $\mathbb{R}^2$, with the coordinatewise partial order and note that A is not a sublattice of $\mathbb{R}^2$. It is not difficult to see that this player's set of monotone pure strategies from $[0, 1]$ into A, endowed with the metric $d(f, g) = \int_0^1 |f(x) - g(x)| \, dx$, is homeomorphic to three line segments joined at a common endpoint. Consequently, this strategy set is not homeomorphic to a convex set and so neither Kakutani's nor Glicksberg's theorems can be directly applied. On the other hand, this strategy set is an absolute retract (see Lemma A.16), which is sufficient for our approach.

Assumption G.2 implies that each μ_i is atomless because singleton sets have no strictly ordered points. In fact, when each player's type space is $[0,1]$ with its usual metric and total order, G.2 holds if and only if each μ_i is atomless. In general however, G.2 imposes additional restrictions as well. For example, if $T_i = [0,1]^2$ is endowed with the Euclidean metric and the coordinatewise partial order, then G.2 requires μ_i to assign probability zero to any negatively sloped line in T_i.[25] On the other hand, G.2 does not imply the Milgrom and Weber (1985) restriction that μ is absolutely continuous with respect to the product of its marginals $\mu_1 \times ... \times \mu_n$. In particular, G.2 holds when there are two players, each with unit interval type space, and the types are drawn according to Lebesgue measure conditional on any one of finitely many positively or negatively sloped lines in the unit square.

The role of assumption G.2 is twofold. First, it enters into the proof of contractibility of the player's sets of best replies by ensuring that each μ_i is atomless, which is needed for the continuity of our contraction in a topology in which payoffs are continuous. Second, and under this same topology, assumption G.2 – together with G.1 and G.3 – ensures the compactness of the players' sets of monotone pure strategies (Lemma A.10).[26] Indeed, without G.2, a player's type space could be the negative diagonal in $[0,1]^2$ endowed with the coordinatewise partial order. But then every measurable function from types to actions would be monotone because no two distinct types are ordered. Compactness in a useful topology is then effectively precluded.

Assumption G.4 is needed to help ensure that the set of monotone pure strategies is an absolute retract and therefore amenable to fixed point analysis.

Assumption G.5 ensures that best replies are well defined and that best-reply correspondences are upper hemicontinuous. Assumption G.5 is trivially satisfied when action spaces are finite. Thus, for example, it is possible to consider auctions here by supposing that players' bid spaces are discrete. We do so in section 5.

As functions from types to actions, best replies for any player i are determined only up

[25] Through a judicious adjustment of the metric, one can accommodate positive weight on vertical or horizontal lines. For example, suppose that player i's marginal distribution is uniform on $[0,1]^2$ with probability 1/2 and is uniform on $[0,1] \times \{0\}$ with probability 1/2. Thus, the horizontal line $[0,1] \times \{0\}$ receives positive probability, violating G.2. The Euclidean metric can be adjusted, leaving the coordinatewise partial order unchanged, so that G.2 is satisfied. Indeed, consider instead the metric on $[0,1]^2$ that, to any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$, assigns their Euclidean distance if both points are in $X = [0,1] \times \{0\}$, assigns the distance $\|x - y\| + \left|\frac{1}{x_2} - \frac{1}{y_2}\right|$ if both points are in $Y = [0,1] \times (0,1]$, and assigns the distance one if $x \in X$ and $y \in Y$. Under this metric, X and Y are "split apart," yet $[0,1]^2$ remains a complete separable metric space, the Borel sets are unchanged, and the coordinatewise partial order remains closed. Hence, G.1 is still satisfied. Moreover, the marginal distribution now satisfies G.2. In particular, distinct points in $[0,1] \times \{0\}$ are strictly ordered under the new metric because sets of the form $(a,b) \times \{0\}$ are now open. The horizontal line $[0,1] \times \{0\}$ is therefore permitted to have positive probability. This technique extends to many dimensions and beyond Euclidean type spaces.

[26] The Milgrom-Weber (1985) absolute-continuity assumption plays the same compactness role for distributional strategies.

to μ_i measure zero sets. This leads us to the following definitions. A *pure strategy* for player i is a function, $s_i : T_i \to A_i$, that is μ_i-a.e. (almost-everywhere) equal to a Borel measurable function, and is *monotone* if $t_i' \geq t_i$ implies $s_i(t_i') \geq s_i(t_i)$ for all $t_i, t_i' \in T_i$.[27] Let S_i denote player i's set of pure strategies and let $S = \times_{i=1}^N S_i$.

A vector of pure strategies, $(\hat{s}_1, ..., \hat{s}_N) \in S$ is an *equilibrium* if for every player i and every pure strategy s_i' for player i,

$$\int_T u_i(\hat{s}(t), t) d\mu(t) \geq \int_T u_i(s_i'(t_i), \hat{s}_{-i}(t_{-i}), t) d\mu(t),$$

where the left-hand side, henceforth denoted by $U_i(\hat{s})$, is player i's payoff given the joint strategy $\hat{s}$, and the right-hand side is his payoff when he employs s_i' and the others employ $\hat{s}_{-i}$.

It will sometimes be helpful to speak of the payoff to player i's type t_i from the action a_i given the strategies of the others, s_{-i}. This payoff, which we will refer to as i's *interim* payoff, is

$$V_i(a_i, t_i, s_{-i}) \equiv \int_T u_i(a_i, s_{-i}(t_{-i}), t) d\mu_i(t_{-i}|t_i),$$

where $\mu_i(\cdot|t_i)$ is a version of the conditional probability on T_{-i} given t_i. A single such version is fixed for each player i once and for all.

4. The Main Result

Call a subset of player i's pure strategies *join-closed* if for any pair of strategies, s_i, s_i', in the subset, the strategy taking the action $s_i(t_i) \vee s_i'(t_i)$ for each $t_i \in T_i$ is also in the subset.[28] We can now state our main result, whose proof is provided in section 6.

Theorem 4.1. *If G.1-G.5 hold, and each player's set of monotone pure best replies is non-empty and join-closed whenever the others employ monotone pure strategies, then G possesses a monotone pure strategy equilibrium.*

Remark 1. *Theorem 4.1 strictly generalizes the main results in Athey (2001) and McAdams (2003) – see Remark 3.*

A strengthening of Theorem 4.1 can be helpful when one wishes to demonstrate not merely the existence of a monotone pure strategy equilibrium but the existence of a monotone

[27] Our convention throughout is to say that property $P(t_i)$ holds μ_i-a.e. if the set of t_i for which $P(t_i)$ holds contains a Borel measurable subset having μ_i-measure one.

[28] Note that when the join operator is continuous, as it is in a metric semilattice, the resulting function is a.e.-measurable, being the composition of a.e.-measurable and continuous functions. But even when the join operator is not continuous, because the join of two monotone pure strategies is monotone, it is a.e.-measurable under the hypotheses of Lemma A.11.

pure strategy equilibrium within a particular subset of strategies. For example, in a uniform-price auction for m units, a strategy mapping a player's m-vector of marginal values into a vector of m bids is undominated only if his bid for a kth unit is no greater than his marginal value for a kth unit. As formulated, Theorem 4.1 does not directly permit one to demonstrate the existence of an undominated equilibrium.[29] The next result takes care of this. Its proof is a straightforward extension of the proof of Theorem 4.1, and is provided in Remark 6.

A subset of player i's pure strategies is called *pointwise-limit-closed* if whenever $s_i^1, s_i^2, ...$ are each in the set and $s_i^n(t_i) \to_n s_i(t_i)$ for μ_i almost-every $t_i \in T_i$, then s_i is also in the set. A subset of player i's pure strategies is called *piecewise-closed* if whenever s_i and s_i' are in the set, then so is any strategy s_i'' such that for every $t_i \in T_i$ either $s_i''(t_i) = s_i(t_i)$ or $s_i''(t_i) = s_i'(t_i)$.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, if for each player i, C_i is a join-closed, piecewise-closed and pointwise-limit-closed subset of pure strategies containing at least one monotone pure strategy, and the intersection of C_i with i's set of monotone pure best replies is nonempty whenever every other player j employs a monotone pure strategy in his C_j, then G possesses a monotone pure strategy equilibrium in which each player i's pure strategy is in C_i.*

Remark 2. *When player i's action space is a semilattice with a closed partial order (as implied by G.3) and C_i is defined by any collection of weak inequalities, i.e., if $\mathcal{F}_i$ and $\mathcal{G}_i$ are arbitrary collections of measurable functions from T_i into A_i and $C_i = \cap_{f\in\mathcal{F}_i, g\in\mathcal{G}_i}\{s_i \in S_i : g(t_i) \leq s_i(t_i) \leq f(t_i)$ for μ_i a.e. $t_i \in T_i\}$, then C_i is join-closed, piecewise-closed and pointwise-limit-closed.*

The next section provides conditions that are sufficient for the hypotheses of Theorem 4.1.

4.1. Sufficient Conditions

Both Athey (2001) and McAdams (2003), within the confines of a lattice, make use of quasisupermodularity and single-crossing conditions on interim payoffs. We now provide weaker versions of both of these conditions, as well as single condition that is weaker than their combination.

[29] Note that it is not possible to restrict the action space alone to ensure that the player chooses an undominated strategy since the bids that he must be permitted to choose will depend upon his private type, i.e., his vector of marginal values.

Suppose that player i's action space, A_i, is a lattice. We say that player i's interim payoff function V_i is *weakly quasisupermodular* if for all monotone pure strategies s_{-i} of the others, all $a_i, a_i' \in A_i$, and every $t_i \in T_i$,

$$V_i(a_i, t_i, s_{-i}) \geq V_i(a_i \wedge a_i', t_i, s_{-i}) \text{ implies } V_i(a_i \vee a_i', t_i, s_{-i}) \geq V_i(a_i', t_i, s_{-i}).$$

McAdams (2003) imposes the stronger assumption of quasisupermodularity – due to Milgrom and Shannon (1994) – which requires, in addition, that the second inequality must be strict if the first happens to be strict.[30] It is well-known that V_i is supermodular in actions – hence weakly quasisupermodular – when the coordinates of a player's *own* action vector are complementary, i.e., when $A_i = [0,1]^K$ is endowed with the coordinatewise partial order and the second cross-partial derivatives of $V_i(a_{i1}, ..., a_{iK}, t_i, s_{-i})$ with respect distinct action coordinates are nonnegative.[31]

We say that i's interim payoff function V_i satisfies *weak single-crossing* if for all monotone pure strategies s_{-i} of the others, for all player i action pairs $a_i' \geq a_i$, and for all player i type pairs $t_i' \geq t_i$,

$$V_i(a_i', t_i, s_{-i}) \geq V_i(a_i, t_i, s_{-i})$$

implies

$$V_i(a_i', t_i', s_{-i}) \geq V_i(a_i, t_i', s_{-i}).$$

Athey (2001) and McAdams (2003) assume that V_i satisfies the slightly more stringent single-crossing condition in which, in addition to the above, the second inequality is strict whenever the first one is.[32] We next present a condition that will be shown to be weaker than the combination of weak quasisupermodularity and weak single-crossing.

For any joint pure strategy for the others, player i's interim best reply correspondence is a mapping from his type into the set of optimal actions – or interim best replies – for that type. Say that player i's interim best reply correspondence is *monotone* if for every monotone joint pure strategy of the others, whenever action a_i is optimal for player i when his type is t_i, and a_i' is optimal when his type is $t_i' \geq_i t_i$, then $a_i \vee a_i'$ is optimal when his type is t_i'.[33]

[30] When actions are totally ordered, as in Athey (2001), interim payoffs are automatically supermodular, and hence both quasisupermodular and weakly quasisupermodular.

[31] Complementarities between the actions of distinct *players* is not implied. This is useful because, for example, many auction games satisfy only own-action complementarity.

[32] For conditions on the joint distribution of types, μ, and the players' payoff functions, $u_i(a,t)$, that imply the more stringent condition, see Athey (2001, pp.879-81), McAdams (2003, p.1197) and Van Zandt and Vives (2005).

[33] This is strictly weaker than requiring the interim best reply correspondence to be increasing in the strong set order, which in any case requires the additional structure of a lattice (see Milgrom and Shannon (1994)).

The following result relates the above conditions to the hypotheses of Theorem 4.1.

Proposition 4.3. *The hypotheses of Theorem 4.1 are satisfied if G.1-G.5 hold, and if for each player i and for each monotone joint pure strategy of the other players, at least one of the following three conditions is satisfied.*[34]

1. *Player i's action space is a lattice and i's interim payoff function is weakly quasisupermodular and satisfies weak single-crossing.*

2. *Player i's interim best reply correspondence is nonempty-valued and monotone.*

3. *Player i's set of monotone pure strategy best replies is nonempty and join-closed.*

Furthermore, the three conditions are in increasing order of generality, i.e., $1 \Longrightarrow 2 \Longrightarrow 3$.

Proof. Because, under G.1-G.5, the hypotheses of Theorem 4.1 hold if condition 3 holds for each player i, it suffices to show that $1 \Longrightarrow 2 \Longrightarrow 3$. So, fix some player i and some monotone pure strategy for every player but i for the remainder of the proof.

($1 \Longrightarrow 2$). Suppose i's action space is a lattice. By G.3 and G.5, for each of i's types, his interim payoff function is continuous on his compact action space. Player i therefore possesses an optimal action for each of his types and so his interim best reply correspondence is nonempty-valued. Suppose that action a_i is optimal for i when his type is t_i and a_i' is optimal when his type is $t_i' \geq t_i$. Then because $a_i \wedge a_i'$ is no better than a_i when i's type is t_i, weak quasisupermodularity implies that $a_i \vee a_i'$ is at least as good as a_i' when i's type is t_i. Weak single-crossing then implies that $a_i \vee a_i'$ is at least as good as a_i' when i's type is t_i'. Since a_i' is optimal when i's type is t_i' so too must be $a_i \vee a_i'$. Hence, i's interim best reply correspondence is monotone.

($2 \Longrightarrow 3$). Let $B_i : T_i \twoheadrightarrow A_i$ denote i's interim best reply correspondence. If a_i and a_i' are in $B_i(t_i)$, then $a_i \vee a_i'$ is also in $B_i(t_i)$ by the monotonicity of $B_i(\cdot)$ (set $t_i = t_i'$ in the definition of a monotone correspondence). Consequently, $B_i(t_i)$ is a subsemilattice of i's action space for each t_i and therefore i's set of monotone pure strategy best replies is join-closed (measurability of the pointwise join of two strategies follows as in footnote 28). It remains to show that i's set of monotone pure best replies is nonempty.

Let $\bar{a}_i(t_i) = \vee B_i(t_i)$, which is well-defined because G.3 and Lemma A.7 imply that A_i is a complete semilattice. Because i's interim payoff function is continuous in his action, $B_i(t_i)$ is compact. Hence $B_i(t_i)$ is a compact subsemilattice of A_i and so $B_i(t_i)$ is itself complete by Lemma A.7. Therefore, $\bar{a}_i(t_i)$ is a member of $B_i(t_i)$ implying that $\bar{a}_i(t_i)$ is optimal for

[34] Which of the three conditions is satisfied is permitted to depend both on the player, i, and on the joint pure strategy employed by the others.

every t_i. It remains only to show that $\bar{a}_i(t_i)$ is monotone (measurability follows from Lemma A.11).

So, suppose that $t_i' \geq_i t_i$. Because $\bar{a}_i(t_i) \in B_i(t_i)$ and $\bar{a}_i(t_i') \in B_i(t_i')$, the monotonicity of $B_i(\cdot)$ implies that $\bar{a}_i(t_i) \vee \bar{a}_i(t_i') \in B_i(t_i')$. Therefore, because $\bar{a}_i(t_i')$ is the largest member of $B_i(t_i')$ we have $\bar{a}_i(t_i') = \bar{a}_i(t_i) \vee \bar{a}_i(t_i') \geq \bar{a}(t_i)$, as desired. ■

Remark 3. *The environments considered in Athey (2001) and McAdams (2003) are strictly more restrictive than G.1-G.5 permit. Moreover, their conditions on interim payoffs are strictly more restrictive than condition 1 of Proposition 4.3. Theorem 4.1 is therefore a strict generalization of their main results.*

When G.1-G.5 hold, it is often possible to apply Theorem 4.1 by verifying condition 1 of Proposition 4.3. But there are important exceptions. For example, Reny and Zamir (2004) have shown in the context of asymmetric first-price auctions that, when bidders have distinct and finite bid sets, monotone best replies exist even though weak single-crossing fails. Further, since action sets (i.e., real-valued bids) there are totally ordered, best reply sets are necessarily join-closed and so the hypotheses of Theorem 4.1 are satisfied even though condition 1 of Proposition 4.3 is not. A similar situation arises in the context of multi-unit discriminatory auctions with risk averse bidders (see subsection 5 below). There, under CARA utility weak quasisupermodularity fails but sets of monotone best replies are nonetheless non-empty and join-closed because condition 2 of Proposition 4.3 is satisfied.

We now turn to several applications of our results.

5. Applications

5.1. Uniform-Price Multi-Unit Auctions with Risk Averse Bidders

Consider a uniform-price auction with n bidders and m homogeneous units of a single good for sale. Each bidder i simultaneously submits a bid, $b = (b_1, ..., b_m)$, where $b_{i1} \geq ... \geq b_{im}$ and each b_{ik} is taken from the finite set $B \subset [0,1]$. Call b_{ik} bidder i's kth unit-bid. The uniform price, p, is the $m+1$st highest of all nm unit-bids. Each unit-bid above p wins a unit at price p, and any remaining units are awarded to unit-bids equal to p according to a random-bidder-order tie-breaking rule.[35]

Bidder i's private type is his vector of nonincreasing marginal values, so that his type space is $T_i = \{t_i \in [0,1]^m : t_{i1} \geq ... \geq t_{im}\}$. Bidder i is risk averse with utility function for money $u_i : [-m, m] \to \mathbb{R}$, where $u_i' > 0$, $u_i'' \leq 0$. If bidder i's type is t_i and he wins k units

[35] The tie-breaking rule is as follows. Bidders are ordered randomly and uniformly. Then, one bidder at a time according to this order, each bidder's *total* remaining demand (i.e., his number of bids equal to p), or as much as possible, is filled at price p per unit until supply is exhausted.

at price p, his payoff is $u_i(t_{i1} + ... + t_{ik} - kp)$. Types are chosen independently across bidders and bidder i's type-vector is chosen according to the density f_i, which need not be positive on all of $[0,1]^m$.[36]

Multi-unit uniform-price auctions always have trivial equilibria in weakly dominated strategies in which some player always bids very high on all units and all others always bid zero. We wish to establish the existence of monotone pure strategy equilibria that are not trivial in this sense. But observe that, because the set of feasible bids is finite, bidding above one's marginal value on some unit need not be weakly dominated. Indeed, it might be a strict best reply for bidder i of type t_i to bid $b_k > t_{ik}$ for a kth unit so long as no feasible bid is in $[t_{ik}, b_k)$. Such a kth unit-bid might permit bidder i to win a kth unit and earn a surplus with high probability rather than risk losing the unit by bidding below t_{ik}. On the other hand, in this instance there is never any gain, and there might be a loss, from bidding above b_k on a kth unit.

Call a monotone pure strategy equilibrium *nontrivial* if for each bidder i, for f_i almost-every t_i, and for every k, bidder i's kth unit-bid does not exceed the smallest feasible bid greater than or equal to t_{ik}. As shown by McAdams (2006), under the coordinatewise partial order on type and action spaces, nontrivial monotone pure strategy equilibria need not exist when bidders are risk averse, as we permit here. Nonetheless, we will demonstrate that a nontrivial monotone pure strategy equilibrium does exist under an economically motivated partial order on type spaces that differs from the coordinatewise partial order; we maintain the coordinatewise partial order on action spaces.

Before introducing the new partial order, it is instructive to see what goes wrong with the coordinatewise partial order on types. The heart of the matter is that single-crossing fails. To see why, it is enough to consider the case of two units. Fix monotone pure strategies for the other bidders and consider two bids for bidder i, $\bar{b} = (\bar{b}_1, \bar{b}_2)$ and $\underline{b} = (\underline{b}_1, \underline{b}_2)$, where $\bar{b}_k > \underline{b}_k$ for $k = 1, 2$. Suppose that when bidder i employs the high bid, $\bar{b}$, he is certain to win both units and pay $\bar{p}$ for each, while he is certain to win only one unit when he employs the low bid, $\underline{b}$. Further, suppose that the low bid yields a price for the one unit he wins that is either $\underline{p}$ or $\underline{p}' > \underline{p}$, each being equally likely. Thus, the expected difference in his payoff from employing the high bid versus the low one can be written as,

$$\frac{1}{2}\left[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p}')\right] + \frac{1}{2}\left[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p})\right].$$

Single-crossing requires this difference, when nonnegative, to remain nonnegative when bid-

[36] By employing the technique described in footnote 25 it is possible to permit a bidder's total demand to be stochastic in the sense that, for each $k > 1$, his marginal value for a kth and higher unit may be zero with positive probability, as might occur if a bidder's endowment of the good were private information. We will not pursue this further here.

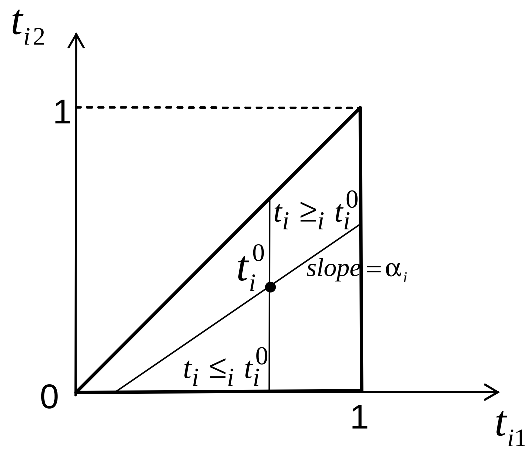

Figure 5.1: Types that are ordered with t_i^0 are bounded between two lines through t_i^0, one being vertical, the other having slope α_i.

der i's type increases according to the coordinatewise partial order, i.e., when t_{i1} and t_{i2} increase. But this can fail when risk aversion is strict because, whenever $t_{i1}+t_{i2}-2\bar{p} > t_{i1}-\underline{p}'$, the first utility difference above strictly falls when t_{i1} increases. Consequently, the expected difference can become negative if the second utility difference is negative to start with.

The economic intuition for the failure of single-crossing is straightforward. Under risk aversion, the marginal utility of winning a second unit falls when the dollar value of a first unit rises, giving the bidder an incentive to reduce his second unit bid so as to reduce the price paid on the first unit. We now turn to the new partial order, which ensures that a higher type is associated with a higher marginal utility of winning each additional unit.

For each bidder i, let $\alpha_i = \frac{u_i'(-m)}{u_i'(m)} - 1 \geq 0$, and consider the partial order, $\geq_i$, on T_i defined as follows: $t_i' \geq_i t_i$ if,

$$\begin{aligned} &1.\ t_{i1}' \geq t_{i1},\ \text{and} \\ &2.\ t_{ik}' - \alpha_i(t_{i1}' + ... + t_{ik-1}') \geq t_{ik} - \alpha_i(t_{i1} + ... + t_{ik-1}),\ \text{for all } k \in \{2, ..., m\}. \end{aligned} \tag{5.1}$$

Figure 5.1 shows which types are greater than and less than a typical type, t_i^0, when types are two-dimensional, i.e., when $m = 2$.

Under the Euclidean metric on the type space, the partial order $\geq_i$ defined by (5.1) is clearly closed so that G.1 is satisfied. To see that G.2 is satisfied, suppose that $\int_B f_i(t_i)dt_i > 0$ for some Borel subset B of $T_i = [0,1]^m$ Then B must have positive Lebesgue measure in $\mathbb{R}^m$. Consequently, by Fubini's theorem, there exists $z \in \mathbb{R}^m$ (indeed there is a positive Lebesgue measure of such z's) such that the line defined by $z + \mathbb{R}((1+\alpha_i), (1+\alpha_i)^2, ..., (1+\alpha_i)^m)$ intersects B in a set of positive one-dimensional Lebesgue measure on the line. Therefore we may choose two distinct points, t_i and t_i' in B that are on this line. Hence, $t_i' - t_i =$

$\beta((1+\alpha_i),(1+\alpha_i)^2,...,(1+\alpha_i)^m)$, where we may assume without loss that $\beta > 0$. But then, $t'_{i1} - t_{i1} = \beta(1+\alpha_i) > 0$ and for $k \in \{2,...,m\}$,

$$\begin{aligned} t'_{ik} - t_{ik} &= \beta(1+\alpha_i)^k \\ &= \beta\{1+\alpha_i[1+(1+\alpha_i)+(1+\alpha_i)^2+...+(1+\alpha_i)^{k-1}]\} \\ &= \beta(1+\alpha_i)+\alpha_i[\beta(1+\alpha_i)+\beta(1+\alpha_i)^2+...+\beta(1+\alpha_i)^{k-1}] \\ &= \beta(1+\alpha_i)+\alpha_i[(t'_{i1}-t_{i1})+(t'_{i2}-t_{i2})+...+(t'_{ik-1}-t_{ik-1})] \\ &> \alpha_i[(t'_{i1}-t_{i1})+(t'_{i2}-t_{i2})+...+(t'_{ik-1}-t_{ik-1})], \end{aligned}$$

from which we conclude that t'_i is strictly greater than t_i (since the strict inequality will hold for pairwise comparisons of points within sufficiently small balls around t'_i and t_i). This shows that any subset having positive f_i-measure contains at least two strictly ordered points according to the partial order $\geq_i$ defined by (5.1), and so G.2 is satisfied.

As noted in section 4.1, actions spaces, being finite sublattices, are locally complete compact metric semilattices. Hence, G.3 and G.4 (ii) hold. Also, G.5 holds because action spaces are finite. Thus, we have so far verified G.1-G.5.

McAdams (2004) shows that each bidder's interim payoff function is modular and hence quasisupermodular. By condition 1 of Proposition 4.3, the hypotheses of Theorem 4.1 will be satisfied if interim payoffs satisfy weak single crossing, which we now demonstrate. It is here where the new partial order $\geq_i$ in (5.1) is fruitfully employed.

To verify weak single crossing it suffices to show that ex-post payoffs satisfy increasing differences. So, fix the strategies of the other bidders, a realization of their types, and an ordering of the players for the purposes of tie-breaking. With these fixed, suppose that the bid, $\bar{b}$, chosen by bidder i of type t_i wins k units at the price $\bar{p}$ per unit, while the coordinatewise-lower bid, $\underline{b}$, wins $j \leq k$ units at the price $\underline{p} \leq \bar{p}$ per unit. The difference in i's ex-post utility from bidding $\bar{b}$ versus $\underline{b}$ is then,

$$u_i(t_{i1}+...+t_{ik}-k\bar{p}) - u_i(t_{i1}+...+t_{ij}-j\underline{p}). \tag{5.2}$$

Assuming that $t'_i \geq t_i$ in the sense of (5.1), it suffices to show that (5.2) is weakly greater at t'_i than at t_i. Noting that (5.1) implies that $t'_{il} \geq t_{il}$ for every l, it can be seen that, if $j = k$, then (5.2) is weakly greater at t'_i than at t_i by the concavity of u_i. It therefore remains only to consider the case in which $j < k$, where we have,

$$\begin{aligned} u_i(t'_{i1} + ... + t'_{ik} - k\bar{p}) - u_i(t_{i1} + ... + t_{ik} - k\bar{p}) &\geq u'_i(m)[(t'_{i1} - t_{i1}) + ... + (t'_{ik} - t_{ik})] \\ &\geq u'_i(m)[(t'_{i1} - t_{i1}) + ... + (t'_{ij+1} - t_{ij+1})] \\ &\geq u'_i(-m)[(t'_{i1} - t_{i1}) + ... + (t'_{ij} - t_{ij})] \\ &\geq u_i(t'_{i1} + ... + t'_{ij} - j\underline{p}) - u_i(t_{i1} + ... + t_{ij} - j\underline{p}), \end{aligned}$$

where the first and fourth inequalities follow from the concavity of u_i and because a bidder's surplus lies between m and $-m$, and the third inequality follows because $t'_i \geq t_i$ in the sense of (5.1). We conclude that weak single crossing holds and so the hypotheses of Theorem 4.1 are satisfied.

Finally, for each player i, let C_i denote the subset of his pure strategies such that for f_i almost-every t_i, and for every k, bidder i's kth unit-bid does not exceed $\phi(t_{ik})$, the smallest feasible unit-bid greater than or equal to t_{ik}. By Remark 2, each C_i is join-closed, piecewise-closed and pointwise-limit-closed. Further, because the hypotheses of Theorem 4.1 are satisfied, whenever the others employ monotone pure strategies player i has a monotone best reply, b'_i, say. Defining $b_i(t_i)$ to be the coordinatewise minimum of $b'_i(t_i)$ and $(\phi(t_{i1}), ..., \phi(t_{im}))$ for all $t_i \in T_i$ implies that b_i is a monotone best reply contained in C_i. This is because, ex-post, any units won by employing b'_i that are also won by employing b_i are won at a weakly lower price with b_i, and any units won by employing b'_i that are not won by employing b_i cannot be won at a positive surplus. Hence, the hypotheses of Theorem 4.2 are satisfied and we conclude that a nontrivial monotone pure strategy equilibrium exists. We may therefore state the following proposition.

Proposition 5.1. *Consider an independent private value uniform-price multi-unit auction with the random-bidder-order tie-breaking rule and in which bids are restricted to a finite grid. Suppose that each bidder i's vector of marginal values is decreasing and chosen according to the density f_i, and that each bidder is weakly risk averse.*

Then, there is a pure strategy equilibrium of the auction with the following properties. For each bidder i,

(i) the equilibrium is monotone under the type-space partial order $\geq_i$ defined by (5.1) and under the usual coordinatewise partial order on bids, and

(ii) the equilibrium is nontrivial in the sense that for f_i almost-all of his types, and for every k, bidder i's kth unit-bid does not exceed the smallest feasible unit-bid greater than or equal to his marginal value for a kth unit.

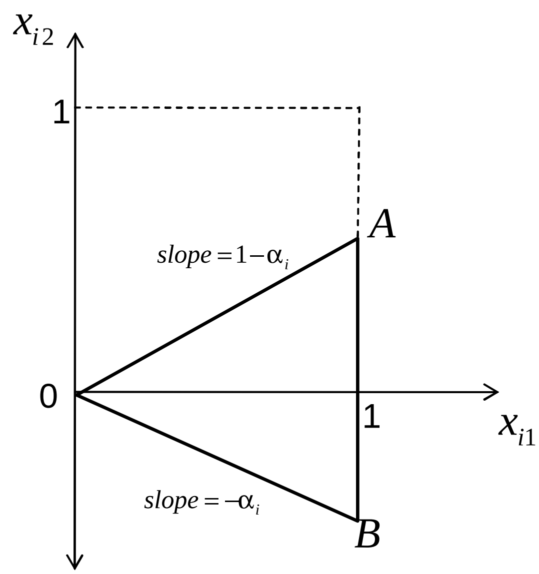

Figure 5.2: After performing the change of variable from t_i to x_i as described in Remark 5 bidder i's new type space is triangle OAB and it is endowed with the coordinatewise partial order. The figure is drawn for the case in which $\alpha_i \in (0, 1)$.

Remark 4. *The partial order defined by (5.1) reduces to the usual coordinatewise partial order under risk neutrality (i.e., when $\alpha_i = 0$), but is distinct from the coordinatewise partial order under strict risk aversion (i.e., when $\alpha_i > 0$), in which case McAdams (2003) does not apply since he employs the coordinatewise partial order.*

Remark 5. *The partial order defined by (5.1) can instead be thought of as a change of variable from t_i to say x_i, where $x_{i1} = t_{i1}$ and $x_{ik} = t_{ik} - \alpha_i(t_{i1} + ... + t_{ik-1})$ for $k > 1$, and where the coordinatewise partial order is applied to the new type space. Our results apply equally well using this change-of-variable technique. In contrast, McAdams (2003) still does not apply because the resulting type space is not the product of intervals, an assumption he maintains together with a strictly positive joint density.[37] See Figure 5.2 for the case in which $m = 2$.*

5.2. Discriminatory Multi-Unit Auctions with CARA Bidders

Consider the same setup as in Subsection 5.1 with two exceptions. First, change the payment rule so that each bidder pays his kth unit-bid for a kth unit won. Second, assume that each bidder's utility function, u_i, exhibits constant absolute risk aversion.

[37] Indeed, starting with the partial order defined by (5.1) there is no change of variable that, when combined with the coordinatewise partial order, is order-preserving and maps to a product of intervals. This is because, in contrast to a product of intervals with the coordinatewise partial order, under the new partial order there is never a smallest element of the type space and there is no largest element when $\alpha_i > 1$.

Despite these two changes, single-crossing still fails under the coordinatewise partial order on types for the same underlying reason as in a uniform-price auction with risk averse bidders. Nonetheless, just as in the previous section it can be shown here that assumptions G.1-G.5 hold and each bidder i's interim expected payoff function satisfies weak single-crossing under the partial order $\geq_i$, defined in (5.1).[38]

For the remainder of this section, we employ the type-space partial order $\geq_i$, defined in (5.1) and the coordinatewise partial order on the space of feasible bid vectors. Monotonicity of pure strategies is then defined in terms of these partial orders.

If it can be shown that interim expected payoffs are quasisupermodular, condition 1 of Proposition 4.3 would permit us to apply Theorem 4.1. However, quasisupermodularity does not hold in discriminatory auctions with strictly risk averse bidders – even CARA bidders.

The intuition for the failure of quasisupermodularity is as follows. Suppose there are two units, and let b_k denote a kth unit-bid. Fixing b_2, suppose that b_1 is chosen to maximize a bidder's interim payoff when his type is (t_1, t_2), namely,

$$P_1(b_1)[u(t_1 - b_1) - u(0)] + P_2(b_2)[u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)],$$

where $P_k(b_k)$ is the probability of winning at least k units.

There are two benefits from increasing b_1. First, the probability, $P_1(b_1)$, of winning at least one unit increases. Second, when risk aversion is strict, the marginal utility, $u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)$, of winning a second unit increases. The cost of increasing b_1 is that the marginal utility, $u(t_1 - b_1) - u(0)$, of winning a first unit decreases. Optimizing over the choice of b_1 balances this cost with the two benefits. For simplicity, suppose that the optimal choice of b_1 satisfies $b_1 > t_2$.

Now suppose that b_2 increases. Indeed, suppose that b_2 increases to t_2. Then the marginal utility of winning a second unit vanishes. Consequently, the second benefit from increasing b_1 is no longer present and the optimal choice of b_1 may fall — even with CARA utility.

This illustrates that the change in utility from increasing one's first unit-bid may be positive when one's second unit-bid is low, but negative when one's second unit-bid is high. Thus, the different coordinates of a bidder's bid are not necessarily complementary, and weak quasisupermodularity can fail. We therefore cannot appeal to condition 1 of Proposition 4.3.

Fortunately, we can instead appeal to condition 2 of Proposition 4.3 owing to the following lemma, whose proof is in the appendix.

Lemma 5.2. *Fix any monotone pure strategies for other bidders and suppose that the vector of bids b_i is optimal for bidder i when his type vector is t_i, and that b'_i is optimal*

[38] This statement remains true with any risk averse utility function. The CARA utility assumption is required for a different purpose.

when his type is $t_i' \geq_i t_i$, where $\geq_i$ is the partial order defined in (5.1). Then the vector of bids $b_i \vee b_i'$ is optimal when his type is t_i'.

Because Lemma 5.2 establishes condition 2 of Proposition 4.3, we may apply Theorem 4.1 to conclude that a monotone pure strategy equilibrium exists. Thus, despite the failure – even with CARA utilities – of both single-crossing with the coordinatewise partial order on types and of weak quasisupermodularity with the coordinatewise partial order on bids, we have established the following.

Proposition 5.3. *Consider an independent private value discriminatory multi-unit auction with the random-bidder-order tie-breaking rule and in which bids are restricted to a finite grid. Suppose that each bidder i's vector of marginal values is decreasing and chosen according to the density f_i, and that each bidder is weakly risk averse and exhibits constant absolute risk aversion.*

Then, there is a pure strategy equilibrium that is monotone under the type-space partial order $\geq_i$ defined by (5.1) and under the usual coordinatewise partial order on bids.

The two applications provided so far demonstrate that it is useful to have flexibility in defining the partial order on the type space since the mathematically natural partial order (in this case the coordinatewise partial order on the original type space) may not be the partial order that corresponds best to the economics of the problem. The next application shows that even when single crossing cannot be established for all coordinates of the type space jointly, it is enough for the existence of a pure strategy equilibrium if single-crossing holds strictly even for a single coordinate of the type space.

5.3. Price Competition with Non-Substitutes

Consider an n-firm differentiated-product price-competition setting. Firm i chooses price $p_i \in [0,1]$, and receives two pieces of private information – his constant marginal cost, $c_i \in [0,1]$, and information $x_i \in [0,1]$ about the state of demand in each of the n markets. The demand for firm i's product is $D_i(p,x)$ when the vector of prices chosen by all firms is $p \in [0,1]^n$ and when their joint vector of private information about market demand is $x \in [0,1]^n$. Demand functions are assumed to be twice continuously differentiable, and $D_i(p,x) > 0$ whenever $p_i < 1$.

Some products may be substitutes, but others need not be. More precisely, the n firms are partitioned into two subsets N_1 and N_2.[39] Products produced by firms within each subset are substitutes, so that $D_i(p,x)$ is nondecreasing in p_j whenever i and j are in the same N_k.

[39] The extension to any finite number of subsets is straightforward.

In addition, marginal costs are affiliated among firms within each N_k and are independent across the two subsets of firms. The joint density of costs is given by the continuously differentiable density $f(c)$ on $[0,1]^n$. Information about market demand may be correlated across firms, but is independent of all marginal costs and has continuously differentiable joint density $g(x)$ on $[0,1]^n$. We do not assume that market demands are nondecreasing in x because we wish to permit the possibility that information that increases demand for some products might decrease it for others.

We assume that demands are strictly downward sloping, i.e., that for all i, $\partial D_i/\partial p_i < 0$ and that $\partial D_i/\partial p_i$ is nondecreasing in p_j when i and j are in the same N_k.

Given pure strategies $p_j(c_j, x_j)$ for the others, firm i's interim expected profits are,

$$v_i(p_i, c_i, x_i) = \int (p_i - c_i) D_i(p_i, p_{-i}(c_{-i}, x_{-i}), x) g_i(x_{-i}|x_i) f_i(c_{-i}|c_i) dx_{-i} dc_{-i}, \tag{5.3}$$

so that,

$$\frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} = -E(\frac{\partial D_i}{\partial p_i}|c_i, x_i) + \frac{\partial}{\partial c_i} E(D_i|c_i, x_i) + (p_i - c_i)\frac{\partial}{\partial c_i} E(\frac{\partial D_i}{\partial p_i}|c_i, x_i).$$

Therefore, if $p_j(c_j, x_j)$ is nondecreasing in c_j for each firm $j \neq i$ and every x_j, then,

$$\frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} \geq -E(\frac{\partial D_i}{\partial p_i}|c_i, x_i) > 0 \tag{5.4}$$

for all $p_i, c_i, x_i \in [0,1]$ such that $p_i \geq c_i$, where the weak inequality follows because both partial derivatives with respect to c_i on the right-hand side of the first line are nonnegative. For example, consider the expectation in the first partial derivative. If $i \in N_1$, then

$$E(D_i|c_i, x_i) = E\left[E(D_i(p_i, p_{-i}(c_{-i}, x_{-i}), x)|c_i, x_i, (c_j, x_j)_{j \in N_2})|c_i, x_i\right].$$

The inner expectation is nondecreasing in c_i because the vector of marginal costs for firms in N_1 are affiliated, their prices are nondecreasing in their costs, and their goods are substitutes. That the entire expectation is nondecreasing in c_i now follows from the independence of (c_i, x_i) and $(c_j, x_j)_{j \in N_2}$.

Thus, according to (5.4), when $p_i \geq c_i$ single-crossing holds strictly for the marginal cost coordinate of the type space. On the other hand, single-crossing need not hold for the market-demand coordinate, x_i, since we have made no assumptions about how x_i affects demand.[40] Nonetheless, we shall now define a partial order on firm i's type space $T_i = [0,1]^2$

[40] We cannot simply restrict attention to strategies $p_i(c_i, x_i)$ that are monotone in c_i and jointly measurable in (c_i, x_i) because this set of pure strategies is not compact in a topology rendering ex-ante payoffs continuous.

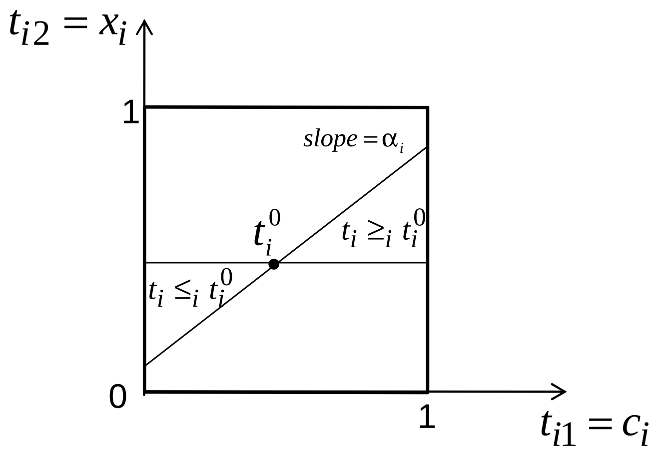

Figure 5.3: Types that are greater than and less than t_i^0 are bounded between two lines through t_i^0, one being horizontal, the other having slope α_i.

under which a monotone pure strategy equilibrium exists.

Note that, because $-\partial D_i/\partial p_i$ is positive and continuous on its compact domain, it is bounded strictly above zero with a bound that is independent of the pure strategies, $p_j(c_j, x_j)$ employed by other firms, so long as they are nondecreasing in c_j. Hence, because our continuity assumptions imply that $\partial^2 v_i(p_i, c_i, x_i)/\partial c_i \partial x_i$ is bounded, there exists $\alpha_i > 0$ such that for all $\beta \in [0, \alpha_i]$ and all pure strategies $p_j(c_j, x_j)$ nondecreasing in c_j,

$$\frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} + \beta \frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial x_i} > 0, \tag{5.5}$$

for all $p_i, c_i, x_i \in [0, 1]$ such that $p_i \geq c_i$.

For each player i, define the partial order $\geq_i$ on $T_i = [0, 1]^2$ as follows: $(c_i', x_i') \geq_i (c_i, x_i)$ if $\alpha_i c_i' - x_i' \geq \alpha_i c_i - x_i$ and $x_i' \geq x_i$. Figure 5.3 shows those types greater than and less than a typical type $t_i^0 = (c_i^0, x_i^0)$.

The partial order $\geq_i$ can be shown to satisfy type-space assumptions G.1 and G.2 as in Example 5.1. The action-space assumption G.3 clearly holds while G.4 (ii) holds by Lemma A.19 given the usual partial order over the reals. Assumption G.5 holds by our continuity assumption on demand. Also, because the action space $[0, 1]$ is totally ordered, the set of monotone best replies is join-closed because the join of two best replies is, for every t_i, equal at t_i to one of them or to the other. Finally, as is shown in the Appendix (see Lemma A.21), under the type-space partial order, $\geq_i$, firm i possesses a monotone best reply when the others employ monotone pure strategies.

Therefore, by Theorem 4.1, there exists a pure strategy equilibrium in which each firm's price is monotone in (c_i, x_i) according to $\geq_i$. In particular, there is a pure strategy equilibrium in which each firm's price is nondecreasing in his marginal cost, the coordinate in which strict single-crossing holds.

5.4. Type Spaces with Atoms

When type spaces contain atoms, assumption G.2 fails. In such cases, there may not exist any pure strategy equilibria, let alone a monotone pure strategy equilibrium. Thus, one must permit mixing and we show here how our results can be used to ensure the existence of a monotone mixed strategy equilibrium.

Let $\Delta(A_i)$ denote the Borel probability measures over player i's action space A_i. Call a mixed strategy $m_i : T_i \to \Delta(A_i)$ *monotone* if $m_i(t_i)$ is a totally ordered subset of A_i for every $t_i \in T_i$, and $\inf m_i(t_i) \geq \sup m_i(t_i')$ whenever $t_i \geq t_i'$. Consider the following weakening of assumption G.2.

G.2′. For each player i, there is a finite subset of types, T_i^0, such that G.2 holds for every Borel subset B of $T_i \backslash T_i^0$.

Theorem 5.4. *If G.1, G.2′, G.3-G.5 hold, and each player's set of monotone pure best replies is nonempty and join-closed whenever the others employ monotone mixed strategies, then G possesses a monotone mixed strategy equilibrium.*

Proof. Consider the following surrogate Bayesian game. Player i's type space is $Q_i = [(T_i \backslash T_i^0) \times \{0\}] \cup (T_i^0 \times [0,1])$. The joint distribution on types, ν, is determined as follows. Nature first chooses $t \in T$ according to the original type distribution μ. Then, for each i, Nature independently and uniformly chooses $x_i \in [0,1]$ if $t_i \in T_i^0$, and chooses $x_i = 0$ if $t_i \in T_i \backslash T_i^0$. Player i is informed of $q_i = (t_i, x_i)$. Action spaces are unchanged. The x_i are payoff irrelevant and so payoff functions are as before. This completes the description of the surrogate game.

The metric employed on Q_i is applied coordinatewise, being the sum of the given metric on T_i with the usual absolute-value metric on $[0,1]$. The partial order employed on Q_i is the lexicographic partial order. That is, $q_i' = (t_i', x_i') \geq (t_i, x_i) = q_i$ if either $t_i' \geq t_i$ and $t_i' \neq t_i$, or $t_i' = t_i$ and $x_i' \geq x_i$. The metrics and partial orders on the players' action spaces are unchanged.

It is straightforward to show that under the hypotheses above, all the hypotheses of Theorem 4.1 but perhaps G.2 hold in the surrogate game.[41] We now show that G.2 in fact holds in the surrogate game. So, suppose for some player i that $\nu_i(B) > 0$ for some Borel

[41] Observe that a monotone pure strategy in the surrogate game induces a monotone mixed strategy in the original game, and that a monotone pure strategy in the original game defines a monotone pure strategy in the surrogate game by viewing it to be constant in x_i.

subset B of Q_i. Then either $\nu_i(B \cap [(T_i \backslash T_i^0) \times \{0\}]) > 0$ or $\nu_i(B \cap (\{t_i^0\} \times [0,1])) > 0$ for some $t_i^0 \in T_i^0$. In the former case, $\mu_i(\{t_i \in T_i \backslash T_i^0 : (t_i, 0) \in B\}) > 0$ and G.2′ implies the existence of t_i' and t_i'' in $\{t_i \in T_i \backslash T_i^0 : (t_i, 0) \in B\}$ such that t_i' and t_i'' are strictly ordered according to the partial order on T_i. But then $(t_i', 0)$ and $(t_i'', 0)$ are strictly ordered according to the lexicographic partial order on Q_i. In the latter case there exist $x_i > x_i' > 0$ such that the distinct points (t_i^0, x_i) and (t_i^0, x_i') are in B. But any two such points are strictly ordered according to the lexicographic order on Q_i. Thus, the surrogate game satisfies G.2 and we may conclude, by Theorem 4.1, that it possesses a monotone pure strategy equilibrium. But any such equilibrium is a monotone mixed strategy equilibrium of the original game. ■

6. Proof of Theorem 4.1

Let M_i denote the nonempty set of monotone functions from T_i into A_i, and let $M = \times_{i=1}^N M_i$. By Lemma A.11, every element of M_i is equal μ_i almost-everywhere to a Borel measurable monotone function, and so M_i coincides with player i's set of monotone pure strategies. Let $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ denote player i's best-reply correspondence when all players must employ monotone pure strategies. Because, by hypothesis, each player possesses a monotone best reply (among all strategies) when the others employ monotone pure strategies, any fixed point of $\times_{i=1}^n \mathbf{B}_i : M \twoheadrightarrow M$ is a monotone pure strategy equilibrium. The following steps demonstrate that such a fixed point exists.

STEP I. (M is a nonempty, compact, metric, absolute retract.) Without loss, we may assume for each player i that the metric d_i on A_i is bounded.[42] Given d_i, define a metric δ_i on M_i as follows:[43]

$$\delta_i(s_i, s_i') = \int_{T_i} d_i(s_i(t_i), s_i'(t_i)) d\mu_i(t_i).$$

By Lemmas A.13 and A.16, each (M_i, δ_i) is a compact absolute retract.[44] Consequently, under the product topology – metrized by the sum of the δ_i – M is a nonempty compact metric space and, by Borsuk (1966) IV (7.1), an absolute retract.

STEP II. ($\times_{i=1}^n \mathbf{B}_i$ is nonempty-valued and upper-hemicontinuous.) We first demonstrate that, given the metric spaces (M_j, δ_j), each player i's payoff function, $U_i : M \to \mathbb{R}$, is continuous under the product topology. To see this, suppose that s^n is a sequence of joint

[42] For any metric, $d(\cdot,\cdot)$, a topologically equivalent bounded metric is $\min(1, d(\cdot,\cdot))$.

[43] Formally, the resulting metric space (M_i, δ_i) is the space of equivalence classes of functions in M_i that are equal μ_i almost everywhere – i.e., two functions are in the same equivalence class if the set on which they coincide contains a measurable subset having μ_i-measure one. Nevertheless, analogous to the standard treatment of $\mathcal{L}_p$ spaces, in the interest of notational simplicity we focus on the elements of the original space M_i rather than on the equivalence classes themselves.

[44] One cannot improve upon Lemma A.16 by proving, for example, that M_i, metrized by δ_{M_i}, is homeomorphic to a convex set. It need not be (e.g., see footnote 24). Evidently, the present approach can handle action spaces that the Athey-McAdams approach cannot easily accommodate, if at all.

strategies in M, and that $s^n \to s \in M$. By Lemma A.12, for each player i, $s_i^n(t_i) \to s_i(t_i)$ for μ_i almost every $t_i \in T_i$. Consequently, $s^n(t) \to s(t)$ for μ almost every $t \in T$.[45] Hence, since u_i is bounded, Lebesgue's dominated convergence theorem yields

$$U_i(s^n) = \int_T u_i(s^n(t), t)d\mu(t) \to \int_T u_i(s(t), t)d\mu(t) = U_i(s),$$

establishing the continuity of U_i.

Because each M_i is compact, Berge's theorem of the maximum implies that $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is nonempty-valued and upper-hemicontinuous. Hence, $\times_{i=1}^n \mathbf{B}_i$ is nonempty-valued and upper-hemicontinuous as well.

STEP III. ($\times_{i=1}^n \mathbf{B}_i$ is contractible-valued.) According to Lemma A.4, for each player i, assumptions G.1 and G.2 imply the existence of a monotone and measurable function $\Phi_i : T_i \to [0,1]$ such that $\mu_i\{t_i \in T_i : \Phi_i(t_i) = c\} = 0$ for every $c \in [0,1]$. Fixing such a function Φ_i permits the construction of a contraction map.[46]

Fix some monotone pure strategy, s_{-i}, for players other than i, and consider player i's set of monotone pure best replies, $\mathbf{B}_i(s_{-i})$. Because $\mathbf{B}_i(\cdot)$ is u.h.c., it is closed-valued and therefore $\mathbf{B}_i(s_{-i})$ is compact, being a closed subset of the compact metric space M_i. By hypothesis, $\mathbf{B}_i(s_{-i})$ is join-closed, and so $\mathbf{B}_i(s_{-i})$ is a compact semilattice under the partial order defined by $s_i \geq s_i'$ if $s_i(t_i) \geq s_i'(t_i)$ for μ_i-a.e. $t_i \in T_i$. By Lemma A.12, this partial order is closed. Therefore, Lemma A.7 implies that $\mathbf{B}_i(s_{-i})$ is a complete semilattice so that $\tilde{s}_i = \vee\mathbf{B}_i(s_{-i})$ is a well-defined member of $\mathbf{B}_i(s_{-i})$. Consequently for every $s_i \in \mathbf{B}_i(s_{-i})$, $\tilde{s}_i(t_i) \geq s_i(t_i)$ for μ_i-a.e. $t_i \in T_i$. By Lemma A.14, there exists $\bar{s}_i \in M_i$ such that $\bar{s}_i(t_i) = \tilde{s}_i(t_i)$ for μ_i-a.e. t_i – and hence $\bar{s}_i \in \mathbf{B}_i(s_{-i})$ – and such that $\bar{s}_i(t_i) \geq s_i(t_i)$ for every $t_i \in T_i$ and every s_i that is μ_i-a.e. less or equal to $\tilde{s}_i$ and therefore for every $s_i \in \mathbf{B}_i(s_{-i})$.[47]

Define $h : [0,1] \times \mathbf{B}_i(s_{-i}) \to \mathbf{B}_i(s_{-i})$ as follows: For every $t_i \in T_i$,

$$h(\tau, s_i)(t_i) = \begin{cases} s_i(t_i), & \text{if } \Phi_i(t_i) \leq 1 - \tau \text{ and } \tau < 1 \\ \bar{s}_i(t_i), & \text{otherwise.} \end{cases} \tag{6.1}$$

Note that $h(0, s_i) = s_i$, $h(1, s_i) = \bar{s}_i$, and $h(\tau, s_i)(t_i)$ is always either $\bar{s}_i(t_i)$ or $s_i(t_i)$ and so is a best reply for μ_i almost every t_i. Moreover, $h(\tau, s_i)$ is monotone because Φ_i is monotone and $\bar{s}_i(t_i) \geq s_i(t_i)$ for all $t_i \in T_i$. Hence, $h(\tau, s_i) \in \mathbf{B}_i(s_{-i})$. Therefore, h will be

[45] This is because if $Q_1, ..., Q_n$ are such that $\mu(Q_i \times T_{-i}) = \mu_i(Q_i) = 1$ for all i, then $\mu(\times_i Q_i) = \mu(\cap_i(Q_i \times T_{-i})) = 1$.

[46] For example, if $T_i = [0,1]^2$ and μ_i is absolutely continuous with respect to Lebesgue measure, we may take $\Phi_i(t_i) = (t_{i1} + t_{i2})/2$.

[47] Defining, for each $t_i \in T_i$, $\bar{s}_i(t_i) = \vee s_i(t_i)$, where the join is taken over all $s_i \in \mathbf{B}_i(s_{-i})$ appears more direct. However, one must show using an argument such as that given here that $\bar{s}_i$ is in $\mathbf{B}_i(s_{-i})$, which is not obvious since each member of $\mathbf{B}_i(s_{-i})$ is an interim best reply only μ_i almost everywhere.

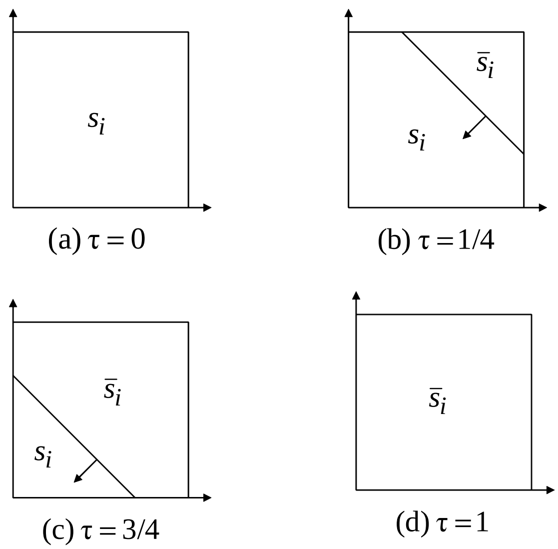

Figure 6.1: $h(\tau, s_i)$ as τ varies from 0 (panel (a)) to 1 (panel (d)) and the domain is the unit square.

a contraction for $\mathbf{B}_i(s_{-i})$ and $\mathbf{B}_i(s_{-i})$ will be contractible if $h(\tau, s_i)$ is continuous, which we establish next.[48]

Suppose $\tau_n \in [0,1]$ converges to τ and $s_i^n \in \mathbf{B}_i(s_{-i})$ converges to s_i, both as $n \to \infty$. By Lemma A.12, there is a Borel subset, D, of i's types such that $\mu_i(D) = 1$ and for all $t_i \in D$, $s_i^n(t_i) \to s_i(t_i)$. Consider any $t_i \in D$. There are three cases: (a) $\Phi_i(t_i) < 1 - \tau$, (b) $\Phi_i(t_i) > 1 - \tau$, and (c) $\Phi_i(t_i) = 1 - \tau$. In case (a), $\tau < 1$ and $\Phi_i(t_i) < 1 - \tau_n$ for n large enough and so $h(\tau_n, s_i^n)(t_i) = s_i^n(t_i) \to s_i(t_i) = h(\tau, s_i)$. In case (b), $\Phi_i(t_i) > 1 - \tau_n$ for n large enough and so for such large enough n, $h(\tau_n, s_i^n)(t_i) = \bar{s}_i(t_i) = h(\tau, s_i)(t_i)$. The remaining case (c) occurs only if t_i is in a set of types having μ_i-measure zero. Consequently, $h(\tau_n, s_i^n)(t_i) \to h(\tau, s_i)(t_i)$ for μ_i-a.e. t_i, which, by Lemma A.12 implies that $h(\tau_n, s_i^n) \to h(\tau, s_i)$, establishing the continuity of h.

Thus, for each player i, the correspondence $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is contractible-valued. Under the product topology, $\times_{i=1}^n \mathbf{B}_i$ is therefore contractible-valued as well.

Steps I-III establish that $\times_{i=1}^n \mathbf{B}_i$ satisfies the hypotheses of Theorem 2.1 and therefore possesses a fixed point. ∎

Remark 6. *The proof of Theorem 4.2 mimics that of Theorem 4.1, but where each M_i is replaced with $M_i \cap C_i$, and where each correspondence $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is replaced with*

[48] With Φ_i defined as in footnote 46, Figure 6.1 provides snapshots of the resulting $h(\tau, s_i)$ as τ moves from zero to one. The axes are the two dimensions of the type vector (t_{i1}, t_{i2}), and the arrow within the figures depicts the direction in which the negatively-sloped line, $(t_{i1} + t_{i2})/2 = 1 - \tau$, moves as τ increases. For example, panel (a) shows that when $\tau = 0$, $h(\tau, s_i)(t_i)$ is equal to $s_i(t_i)$ for all t_i in the unit square. On the other hand, panel (c) shows that when $\tau = 3/4$, $h(\tau, s_i)(t_i)$ is equal to $s_i(t_i)$ for t_i below the negatively-sloped line and equal to $\bar{s}_i(t_i)$ for t_i above it.

the correspondence $\mathbf{B}_i^* : M_{-i} \cap C_{-i} \twoheadrightarrow M_i \cap C_i$ defined by $\mathbf{B}_i^*(s_{-i}) = \mathbf{B}_i(s_{-i}) \cap C_i$. The proof goes through because the hypotheses of Theorem 4.2 imply that each $M_i \cap C_i$ is compact, nonempty, join-closed, piecewise-closed, and pointwise-limit-closed (and hence the proof that each $M_i \cap C_i$ is an absolute retract mimics the proof of Lemma A.16), and that each correspondence $\mathbf{B}_i^*$ is upper hemicontinuous, nonempty-valued and contractible-valued (the contraction is once again defined by 6.1). The result then follows from Theorem 2.1.

A. Appendix

To simplify the notation, we drop the subscript i from T_i, μ_i, and A_i throughout the appendix. Thus, in this appendix, T, μ, and A should be thought of as the type space, marginal distribution, and action space, respectively, of any one of the players, not as the joint type spaces, joint distribution, and joint action spaces of all the players. Of course, the theorems that follow are correct with either interpretation, but in the main text we apply the theorems below to the players individually rather than jointly and so the former interpretation is the more relevant. For convenience, we rewrite here without subscripts the assumptions from section 3.2 that will be used in this appendix.

G.1 T is a complete separable metric space endowed with a measurable partial order.

G.2 μ assigns probability zero to any Borel subset of T having no strictly ordered points.

G.3 A is a compact metric space and a semilattice with a closed partial order.

G.4 Either (i) A is a convex subset of a locally convex linear topological space, and the partial order on A is convex, or (ii) A is a locally-complete metric semilattice.

A.1. Partially Ordered Spaces

Preliminaries. If $\geq$ is a measurable partial order on a metric space T, Lemma 7.6.1 of Cohn (1980) implies that the sets $\geq(t) = \{t' \in T : t' \geq t)$ and $\leq(t) = \{t' \in T : t \geq t'\}$ are in $\mathcal{B}(T)$, the set of Borel subsets of T, for each $t \in T$. A totally ordered subset of a partially ordered set is called a *chain*. A *strict chain* is a chain in which every pair of distinct points are strictly ordered. Finally, if μ is a Borel measure on T, we say that $t \in T$ is in the *order-support* of μ if $\mu(U \cap \geq(t)) > 0$ and $\mu(U \cap \leq(t)) > 0$ for every neighborhood U of t.

Lemma A.1. *Under G.1 and G.2, there is a Borel measurable subset of the order-support of μ having μ-measure one.*

Proof. Let $\mathcal{A} = \{E \in \mathcal{B}(T \times T) : \mu(E_t)$ is a Borel measurable function of $t \in T\}$, where $E_t = \{t' \in T : (t,t') \in E\}$. Then $\mathcal{A}$ contains, in particular, all open sets of form $E = U \times V$, since the resulting function $\mu(E_t)$ is constant on U and on $T \backslash U$. Suppose that $E^1 \subseteq E^2 \subseteq ...$ is an increasing sequence of sets in $\mathcal{A}$. Then because $(E^2 \backslash E^1)_t = E_t^2 \backslash E_t^1$ and $(\cup_i E^i)_t = \cup_i E_t^i$, we have $\mu\left[(E^2 \backslash E^1)_t\right] = \mu(E_t^2) - \mu(E_t^1)$ and $\mu\left[(\cup_i E^i)_t\right] = \mu(\cup_i E_t^i) = \lim_i \mu(E_t^i)$. Consequently, $E^2 \backslash E^1$ and $\cup_i E^i$ are in $\mathcal{A}$. Hence, by Theorem 1.6.1 of Cohn (1980), $\mathcal{A}$ contains $\mathcal{B}(T) \times \mathcal{B}(T)$, the sigma algebra generated by all open sets of the form $U \times V$. But because T is a separable metric space, Proposition 8.1.5 of Cohn (1980) implies that $\mathcal{B}(T) \times \mathcal{B}(T) = \mathcal{B}(T \times T)$. Hence, $\mathcal{A} = \mathcal{B}(T \times T)$. In particular, because the measurability of $\geq$ implies that $E = (T \times U) \cap \{(t, t') \in T \times T : t' \geq t\}$ is a member of $\mathcal{B}(T \times T)$ for every open subset U of T, we may conclude that $\mu(E_t) = \mu(U \cap \geq(t))$ is a measurable function of $t \in T$ for each open subset U of T.

Let U be any open subset of T, and consider the measurable set $D = \{t \in U : \mu(U \cap \geq(t)) = 0\}$. We claim that $\mu(D) = 0$. Suppose, by way of contradiction, that $\mu(D) > 0$. Because T is a separable metric space, we may assume without loss that D is contained in the support of μ, so that every open set intersecting D has positive μ-measure. By G.2, D contains two strictly ordered points, $t_0 \leq t_1$. Hence, there are disjoint neighborhoods U_0 of t_0 and U_1 of t_1 such that $u_0 \leq u_1$ for every $u_0 \in U_0$ and every $u_1 \in U_1$. In particular, U_1 is contained in $\geq(t_0)$, so that $U \cap U_1 \subseteq U \cap \geq(t_0)$. The open set $U \cap U_1$ intersects D because both sets contain t_1, and so $\mu(U \cap U_1) > 0$. But then $\mu(U \cap \geq(t_0)) > 0$, contradicting $t_0 \in D$ and establishing the claim.

Let $\{U_1, U_2, ...\}$ be a countable base for the topology of T and consider the measurable set $S = \cap_i \left[\{t \in U_i : \mu(U_i \cap \geq (t)) > 0\} \cup U_i^c\right]$. The result established in the previous paragraph implies that $\mu(S) = 1$ since, for each i, the set in curly brackets has measure $\mu(U_i)$, and U_i^c has the complementary measure. Now consider any $t \in S$ and any neighborhood U of t. For some i, we have $t \in U_i \subseteq U$, and therefore $\mu(U \cap \geq(t)) \geq \mu(U_i \cap \geq(t)) > 0$, since $t \in S$.

Consequently, for every $t \in S$, $\mu(U \cap \geq(t)) > 0$ for every neighborhood U of t. A similar argument establishes the existence of a measurable set S' such that $\mu(S') = 1$ and every $t \in S'$ satisfies $\mu(U \cap \leq(t)) > 0$ for every neighborhood U of t. Therefore, $S \cap S'$ is a measurable subset of the order-support of μ having μ-measure one. ■

Lemma A.2. *Let C be a chain in a partially ordered separable metric space. Then c is an accumulation point of both $C \cap \geq(c)$ and $C \cap \leq(c)$ for all but perhaps countably many $c \in C$.*[49]

Proof. Since the given metric renders C separable, we may assume that the ambient space is C itself. Also without loss, we may assume that C is uncountable. Suppose first, and by way of contradiction, that there is no $c \in C$ that is an accumulation point of $\geq(c)$. Then, for every $c \in C$ there exists $\varepsilon_c > 0$ such that $B_{\varepsilon_c}(c)$, the open ball with radius ε_c around c, has only the point c in common with $\geq(c)$. Consequently, for some fixed $\varepsilon > 0$ there must be uncountably many $c \in C$ that are each the only common point of $B_\varepsilon(c)$ and $\geq(c)$. Let C' denote this uncountable subset of C, and consider the collection of open sets $\{B_{\varepsilon/2}(c)\}_{c \in C'}$. The separability of C implies that not all pairs of sets in this collection can be disjoint. Hence, there must be distinct $c, c' \in C'$ such that $B_{\varepsilon/2}(c) \cap B_{\varepsilon/2}(c')$ is nonempty. Then, by the triangle inequality, $d(c, c') < \varepsilon$, where d is the metric on C. However, because C' is a chain, we may assume without loss that $c' \geq c$ and so by the definition of C', $c' \notin B_\varepsilon(c)$, implying that $d(c, c') \geq \varepsilon$, a contradiction. We conclude that some $c \in C$ is an accumulation point of $\geq(c)$.

But then c is an accumulation of $\geq(c)$ for all but perhaps countably many $c \in C$ since, otherwise, we would be led to a contradiction by repeating the above argument on the uncountable number of exceptions. Similarly, c is an accumulation point of $\leq(c)$ for all but perhaps countably many $c \in C$. ■

[49] Recall that a point is an accumulation point of a set if every neighborhood of the point contains infinitely many points of the set.

Lemma A.3. *Assume G.1 and G.2. If $\mu(B) > 0$, then B contains a strict chain with uncountably many elements.*

Proof.[50] Assume that $\mu(B) > 0$. Because T is a complete separable metric space, B contains a compact subset having positive μ-measure. Hence, without loss, we may assume that B is compact. Replacing B if necessary with $B \cap V^c$, where V is the largest open set whose intersection with B has μ-measure zero, we may further assume without loss that $\mu(U \cap B) > 0$ for every open set U intersecting B.[51]

By G.2, B contains two strictly ordered points $t_0 \leq t_1$. Hence, there are disjoint neighborhoods U_0 of t_0 and U_1 of t_1 such that $u_0 \leq u_1$ for every $u_0 \in U_0$ and every $u_1 \in U_1$. Clearly, any two such u_0 and u_1 are strictly ordered. Therefore, by replacing the U_i if necessary with sufficiently small balls around t_0 and t_1, we may assume that u_0 and u_1 are strictly ordered for every $u_0 \in \bar{U}_0$ and every $u_1 \in \bar{U}_1$, where $\bar{U}_i$ denotes the closure of U_i. Because each $U_i \cap B$ is nonempty (t_i is a member), each has positive μ-measure. Hence, for $i = 0, 1$, we may repeat the construction on each $U_i \cap B$, giving rise to strictly ordered points t_{i0} and t_{i1} in $U_i \cap B$ and their strictly ordered neighborhood closures $\bar{U}_{i0}$ and $\bar{U}_{i1}$, both of which can be chosen to be subsets of $\bar{U}_i$. Continuing in this manner, we obtain a countably infinite collection of open sets U_0, U_1, U_{00}, U_{01}, U_{10}, $U_{11}, ..., U_s, ...$, where s runs over all finite sequences of zeros and ones. The open sets $\{U_s\}$ and T form a binary tree with T at its root, where succession is defined by set inclusion, because for each zero-one sequence s, U_s contains both U_{s0} and U_{s1}. Further, each set in $\{U_s\}$ intersects B and, without loss, we may choose the U_s so that their boundaries are mutually disjoint and so that the diameter of each U_s is no greater than the reciprocal of the length of the sequence s.

For each $\alpha \in [0, 1]$, consider its binary expansion (choose one expansion if there are two), $.i_1 i_2 i_3 ...$, and the infinite intersection $\bar{U}_{i_1} \cap \bar{U}_{i_1 i_2} \cap \bar{U}_{i_1 i_2 i_3} \cap ...$. The sets in the intersection form a decreasing sequence of closed sets whose diameters converge to zero. Hence, by the completeness of T, their intersection contains a single point, t_α. Moreover, $t_\alpha \in B$ because each set in the sequence intersects the compact set B. We claim that $\{t_\alpha : \alpha \in [0, 1]\}$ is an uncountable strict chain in B. To see this, suppose $\alpha, \beta \in [0, 1]$ are distinct. Their binary expansions must therefore differ for the first time at, say, the $n + 1$st digit. If their common first n digits are $i_1, ..., i_n$ and their $n + 1$st digits are j and k for α and β, respectively, then $t_\alpha \in \bar{U}_{i_1 ... i_n j}$ and $t_\beta \in \bar{U}_{i_1 ... i_n k}$. Hence, because the boundaries of the disjoint open sets $U_{i_1 ... i_n j}$ and $U_{i_1 ... i_n k}$ do not intersect, t_a and t_β are distinct elements of B. Moreover, by construction, every element of $\bar{U}_{i_1 ... i_n j}$ is strictly ordered with every element of $\bar{U}_{i_1 ... i_n k}$. Consequently, t_α and t_β are strictly ordered, proving the claim. ■

Lemma A.4. *Assume G.1 and G.2. There is a monotone and measurable function $\Phi : T \to [0, 1]$ such that $\mu(\Phi^{-1}(\alpha)) = 0$ for every $\alpha \in [0, 1]$.*

Proof. By separability, T admits a countable dense subset, $\{t_1, t_2, ...\}$. Define $\Phi : T \to [0, 1]$ as follows:

$$\Phi(t) = \sum_{i=1}^{\infty} 2^{-i} \mathbf{1}_{\geq (t_i)}(t). \tag{A.1}$$

[50] I am grateful to Benjamin Weiss for outlining the proof given here.

[51] To see that V is well-defined, let $\{U_i\}$ be a countable base for T. Then V is the union of all the U_i satisfying $\mu(U_i \cap B) = 0$.

Clearly, Φ is monotone and measurable, being the pointwise convergent sum of monotone and measurable functions. It remains to show that $\mu\{t \in T : \Phi(t) = \alpha\} = 0$ for every $\alpha \in [0,1]$.

By Lemma A.3, it suffices to show that for every $\alpha \in [0,1]$, every strict chain in $\{t \in T : \Phi(t) = \alpha\}$ is countable. In fact, we will show that every such strict chain contains no more than two elements. To see this, suppose by way of contradiction that for some $\alpha \in [0,1]$, $\{t \in T : \Phi(t) = \alpha\}$ contains a strict chain with three distinct elements, $t \geq t' \geq t''$. Hence, $\Phi(t) = \Phi(t') = \Phi(t'')$ and there are mutually disjoint neighborhoods U of t, U' of t' and U'' of t'', such that $u \geq u' \geq u''$ for every $u \in U$, $u' \in U'$ and $u'' \in U''$. The open set U' must contain a member, t_i say, of the dense set $\{t_1, t_2, ...\}$. Hence, $t \geq t_i \geq t''$ and $t'' \not\geq t_i$. But then $\Phi(t) \geq \Phi(t'') + 2^{-i} > \Phi(t'')$, a contradiction. ∎

A.2. Semilattices

The standard proofs of the next two lemmas are omitted.

Lemma A.5. *If G.3 holds, and a_n, b_n, c_n are sequences in A such that $a_n \leq b_n \leq c_n$ for every n and both a_n and c_n converge to a, then b_n converges to a.*

Lemma A.6. *If G.3 holds, then every nondecreasing sequence and every nonincreasing sequence in A converges.*

Lemma A.7. *If G.3 holds, then A is a complete semilattice.*

Proof. Let S be a nonempty subset of A. Because A is a compact metric space, S has a countable dense subset, $\{a_1, a_2, ...\}$. Let $a^* = \lim_n a_1 \vee ... \vee a_n$, where the limit exists by Lemma A.6. Suppose that $b \in A$ is an upper bound for S and let a be an arbitrary element of S. Then, some sequence, a_{n_k}, converges to a. Moreover, $a_{n_k} \leq a_1 \vee a_2 \vee ... \vee a_{n_k} \leq b$ for every k. Taking the limit as $k \to \infty$ yields $a \leq a^* \leq b$. Hence, $a^* = \vee S$. ∎

A.3. The Space of Monotone Functions from T into A

In this subsection we introduce a metric, δ, under which the space $\mathcal{M}$ of monotone functions from T into A will be shown to be a compact metric space. Further, it will be shown that under suitable conditions, the metric space $(\mathcal{M}, \delta)$ is an absolute retract. Some preliminary results are required.

We remind the reader of the following convention. We say that property $P(t)$ holds for μ-a.e. $t \in T$, if the set of $t \in T$ on which $P(t)$ holds contains a Borel measurable subset having μ-measure one.

Lemma A.8. *Assume G.1-G.3. If C is a strict chain in T and $f : C \to A$ is monotone, then f is continuous at all but perhaps countably many $t \in C$.*

Proof. If $a = f(t)$ and t is neither the smallest nor the largest element of the strict chain $f^{-1}(a)$, then there are $t', t'' \in f^{-1}(a)$ such that $t' \leq t \leq t''$, with all three points distinct. Because the three points are strictly ordered, there is a neighborhood U of t such that $t' \leq u \leq t''$ for every $u \in U$. Consequently, if t_k is a sequence in C converging to t, then $t' \leq t_k \leq t''$ and so also $a = f(t') \leq f(t_k) \leq f(t'') = a$ for all k large enough. Hence, $\lim_k f(t_k) = a = f(t)$, and we conclude that f is continuous at t and so at all but at most two points in $f^{-1}(a)$, the smallest and the largest if they exist. Consequently, if $D \subseteq C$ is the set of discontinuity points of f, then D will be countable if $f(D)$ is countable.

Suppose that $t \in D$. Then, focusing on one of two possibilities, we may assume that C contains a sequence $t_n \to t$ such that $t_n \geq t$ for all n and $f(t_n) \to a \geq f(t) \neq a$, where the weak inequality follows because $f(t_n) \geq f(t)$ by the monotonicity of f and because the partial order on A is closed – a fact that will be used repeatedly.[52] Because C is a strict chain, if $t' \in C$ is distinct from t and $t' \geq t$, there is a neighborhood U of t such that $t' \geq u$ for every $u \in U$. Hence, for all n sufficiently large, $t' \geq t_n$ and therefore also $f(t') \geq f(t_n)$. Taking the limit in n implies that $f(t') \geq a$. From this we may conclude that $f(t)$ is not an accumulation point of $f(C) \cap \geq(f(t))$. To see this, suppose otherwise that there is a sequence $t'_n \in C$ with $f(t) \neq f(t'_n) \geq f(t)$ and $f(t'_n) \to f(t)$. Because C is a strict chain and f is monotone, the first two relations imply $t \neq t'_n \geq t$ and so, as just shown, $f(t'_n) \geq a$ for every n. Taking limits yields $f(t) \geq a$. However, because $a \geq f(t)$, this would imply $a = f(t)$, a contradiction, establishing that $f(t)$ is not an accumulation point of $f(C) \cap \geq(f(t))$. But then, a fortiori, $f(t)$ is not an accumulation point of $f(D) \cap \geq(f(t))$. Because $f(t)$ is an arbitrary element of $f(D)$, we have shown that $f(D)$ is a chain such that no $a \in f(D)$ is an accumulation point of $f(D) \cap \geq(a)$. Because A, being a compact metric space, is complete and separable, Lemma A.2 implies that $f(D)$ is countable. ■

Lemma A.9. *Assume G.1-G.3. If $f : T \to A$ is measurable and monotone, then f is continuous μ almost everywhere.*

Proof. Let D denote the set of discontinuity points of f. Note that D is Borel measurable because its complement, the set of continuity points of f, is $\cap_{i=1}^{\infty} (\mathrm{int} f^{-1}(U_i) \cup [f^{-1}(U_i)]^c)$, where $\{U_i\}$ is a countable base for A.[53] It suffices to show that $\mu(D) = 0$. Letting C be a strict chain in D, it suffices by Lemma A.3 to show that C is countable. Let $f|_C$ be the restriction of f to C, and let C' be the set of $t \in C$ that are accumulation points of both $C \cap \geq(t)$ and $C \cap \leq(t)$ and also continuity points of $f|_C$. By Lemmas A.2 and A.8, C' contains all but countably many $t \in C$. Hence, it suffices to show that C' is empty. Suppose by way of contradiction that $t \in C'$. Then C contains sequences t'_n and t''_n converging to t such that $t'_n \leq t \leq t''_n$ and both t'_n and t''_n are distinct from t for all n. Let t_k be an arbitrary sequence in T converging to t such that $f(t_k)$ converges to some $a \in A$. Because C is a strict chain, for each n there is a neighborhood U_n of t such that $t'_n \leq u \leq t''_n$ for every $u \in U_n$. Hence, for each n, $t'_n \leq t_k \leq t''_n$ and therefore $f(t'_n) \leq f(t_k) \leq f(t''_n)$ for all large enough k. Taking the limit first as $k \to \infty$ and then as $n \to \infty$ implies that $f(t) \leq a \leq f(t)$, because t is a continuity point of $f|_C$ and the partial order on A is closed. But then $a = f(t)$ and we conclude, because A is compact, that $t \in C$ is a continuity point of f, contradicting the definition of C. ■

Lemma A.10. *(A Generalized Helley Selection Theorem). Assume G.1-G.3. If $f_n : T \to A$ is a sequence of monotone functions – not necessarily measurable – then there is a subsequence, f_{n_k}, and a measurable monotone function, $f : T \to A$, such that $f_{n_k}(t) \to_k f(t)$ for μ-a.e. $t \in T$.*

Proof. Let $\{t_1, t_2, ...\}$ be a countable dense subset of T. Choose a subsequence, f_{n_k}, of f_n such that, for every i, $\lim_k f_{n_k}(t_i)$ exists. Define $f(t_i) = \lim_k f_{n_k}(t_i)$ for every i, and extend f to all of T by defining $f(t) = \vee\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t\}$.[54] By Lemma A.7, this is

[52] The other possibility involves the reverse inequalities.

[53] Every compact metric space has a countable base.

[54] Note then that $f(t) = \vee A$ if no $t_i \geq t$.

well defined because $\{a \in A : a \leq f(t_i)$ for all $t_i \geq t\}$ is nonempty for each t since it contains any limit point of $f_{n_k}(t)$. Indeed, if $f_{n_{k_j}}(t) \to_j a$, then $a = \lim_j f_{n_{k_j}}(t) \leq \lim_j f_{n_{k_j}}(t_i) = f(t_i)$ for every $t_i \geq t$. Further, as required, the extension to T is monotone and leaves the values of f on $\{t_1, t_2, ...\}$ unchanged, where the latter follows because the monotonicity of f on $\{t_1, t_2, ...\}$ implies that $\{a \in A : a \leq f(t_i)$ for all $t_i \geq t_k\} = \{a \in A : a \leq f(t_k)\}$. To see that f is measurable, note first that $f(t) = \lim_m g_m(t)$, where $g_m(t) = \vee\{a \in A : a \leq f(t_i)$ for all $i = 1, ..., m$ such that $t_i \geq t\}$, and where the limit exists by Lemma A.6. Because $\geq$ is measurable, each g_m is a measurable simple function. Hence, f is measurable, being the pointwise limit of measurable functions.

Let t be a continuity point of f in the order-support of μ. By Lemmas A.1 and A.9, it suffices now to show that $f_{n_k}(t) \to f(t)$. So, suppose that $f_{n_{k_j}}(t) \to a \in A$ for some subsequence n_{k_j} of n_k. By the compactness of A, it suffices to show that $a = f(t)$. Because t is in the order support of μ, both $\mu(U\cap \geq(t))$ and $\mu(U\cap \leq(t))$ are positive for every neighborhood U of t. Hence, by G.2, $U\cap \geq(t)$ and $U\cap \leq(t)$ each contain a pair of strictly ordered points. In particular therefore, we may choose two distinct points $t' \geq t''$ in $U\cap \geq(t)$ and choose an open set U' contained in U and containing t' such that $u' \geq t'' \geq t$ for every $u' \in U'$. Because U' is open, it contains some t_i in the dense set $\{t_1, t_2, ...\}$ and so $t_i \geq t$. Similarly, by considering a pair of strictly ordered points in $U\cap \leq(t)$, we can find t_j in U such that $t_j \leq t$. Since U is an arbitrary open set containing t, this shows that there are sequences t_{i_m} and t_{j_m} each converging to t and contained in $\{t_1, t_2, ...\}$ and such that $t_{j_m} \leq t \leq t_{i_m}$ for every m. Hence, because the f_n are monotone, $f_{n_{k_j}}(t_{j_m}) \leq f_{n_{k_j}}(t) \leq f_{n_{k_j}}(t_{i_m})$ for every j and m. Taking the limit in j gives $f(t_{j_m}) \leq a \leq f(t_{i_m})$, and taking next the limit in m gives $f(t) \leq a \leq f(t)$, because t is a continuity point of f. Hence, $a = f(t)$ as desired. ∎

By setting $\{f_n\}$ in Lemma A.10 equal to a constant sequence, we obtain the following.

Lemma A.11. *Under G.1-G.3, every monotone function from T into A is μ almost everywhere equal to a Borel measurable monotone function.*

We can now introduce a metric on $\mathcal{M}$, the space of monotone functions from T into A. Denote the metric on A by d and assume without loss that $d(a, b) \leq 1$ for all $a, b \in A$. Define the metric, δ, on $\mathcal{M}$ by

$$\delta(f, g) = \int_T d(f(t), g(t))d\mu(t),$$

which is well-defined by Lemma A.11.

Formally, the resulting metric space $(\mathcal{M}, \delta)$ is the space of equivalence classes of monotone functions that are equal μ almost everywhere – i.e., two functions are in the same equivalence class if there is a measurable subset of T having μ-measure one on which they coincide. Nevertheless, and analogous to the standard treatment of $\mathcal{L}_p$ spaces, we focus on the elements of the original space $\mathcal{M}$ rather than on the equivalence classes themselves.

Lemma A.12. *Assume G.1-G.3. In $(\mathcal{M}, \delta)$, f_k converges to f if and only if in (A, d), $f_k(t)$ converges to $f(t)$ for μ-a.e. $t \in T$.*

Proof. (only if) Suppose that $\delta(f_k, f) \to 0$. By Lemmas A.1 and A.9, it suffices to show that $f_k(t) \to f(t)$ for all continuity points, t, of f in the order-support of μ.

Let t_0 be a continuity point of f in the order-support of μ. Because A is compact, it suffices to show that an arbitrary convergent subsequence, $f_{k_j}(t_0)$, of $f_k(t_0)$ converges to $f(t_0)$. So, suppose that $f_{k_j}(t_0)$ converges to $a \in A$. By Lemma A.10, there exists a further subsequence, $f_{k'_j}$ of f_{k_j} and a monotone measurable function, $g : T \to A$ such that $f_{k'_j}(t) \to g(t)$ for μ a.e. t in T. Because d is bounded, the dominated convergence theorem implies that $\delta(f_{k'_j}, g) \to 0$. But $\delta(f_{k'_j}, f) \to 0$ then implies that $\delta(f, g) = 0$ and so $f_{k'_j}(t) \to f(t)$ for μ a.e. t in T.

Because $f_{k'_j}(t) \to f(t)$ for μ a.e. t in T and because t_0 is in the order-support of μ, for every $\varepsilon > 0$ there exist $t_\varepsilon, t'_\varepsilon$ each within ε of t_0 such that $t_\varepsilon \leq t_0 \leq t'_\varepsilon$ and such that $f_{k'_j}(t_\varepsilon) \to_j f(t_\varepsilon)$ and $f_{k'_j}(t'_\varepsilon) \to_j f(t'_\varepsilon)$. Consequently, $f_{k'_j}(t_\varepsilon) \leq f_{k'_j}(t_0) \leq f_{k'_j}(t'_\varepsilon)$, and taking the limit as $j \to \infty$ yields $f(t_\varepsilon) \leq a \leq f(t'_\varepsilon)$, and taking next the limit as $\varepsilon \to 0$ yields $f(t_0) \leq a \leq f(t_0)$, so that $a = f(t_0)$, as desired.

(if) To complete the proof, suppose that $f_k(t)$ converges to $f(t)$ for μ-a.e. $t \in T$. Then, because d is bounded, the dominated convergence theorem implies that $\delta(f_k, f) \to 0$. ■

Combining Lemmas A.10 and A.12 we obtain the following.

Lemma A.13. *Under G.1-G.3, the metric space $(\mathcal{M}, \delta)$ is compact.*

Lemma A.14. *Suppose that G.1-G.3 hold and $f : T \to A$ is monotone. If for every $t \in T$, $\bar{f}(t) = \vee g(t)$, where the join is taken over all monotone $g : T \to A$ s.t. $g(t) \leq f(t)$ for μ-a.e. $t \in T$, then $\bar{f} : T \to A$ is monotone and $\bar{f}(t) = f(t)$ for μ-a.e. $t \in T$.*[55]

Proof. Note that $\bar{f}(t)$ is well-defined for each $t \in T$ by Lemma A.7, and $\bar{f}$ is monotone, being the pointwise join of monotone functions. It remains only to that $\bar{f}(t) = f(t)$ for μ-a.e. $t \in T$.

Suppose first that f is measurable. Let E denote the intersection of the order-support of μ and the set of continuity points of f, and let L_f denote the set of monotone $g : T \to A$ such that $g(t) \leq f(t)$ for μ-a.e. $t \in T$. We claim that $f(t) \geq g(t)$ for every $t \in E$ and every $g \in L_f$. To see this, fix $t \in E$ and $g \in L_f$. By Lemmas A.1 and A.9 E contains a measurable subset, D say, having μ-measure one such that $g(t) \leq f(t)$ for every $t \in D$. Consider any $t \in E$. Because t is in the order-support of μ, $\mu(U \cap \geq(t)) > 0$, and so also $\mu(D \cap U \cap \geq(t)) > 0$, for every open set U containing t. Consequently, there is a sequence of points $t_n \in D$ converging to t such that $t_n \geq t$, and therefore $f(t_n) \geq g(t_n) \geq g(t)$, for all n. The continuity of f at t implies that $f(t) = \lim_n f(t_n) \geq g(t)$, proving the claim. Consequently, because $f \in L_f$, $f(t) = \vee_{g \in L_f} g(t) = \bar{f}(t)$ for every $t \in E$ and hence for μ-a.e. $t \in T$.

If f is not measurable, then by Lemma A.11, we may repeat the argument replacing f with a measurable and monotone $\tilde{f} : T \to A$ that is almost everywhere equal to f, concluding that $\tilde{f}(t) = \vee_{g \in L_{\tilde{f}}} g(t)$ for μ-a.e. $t \in T$. But $L_f = L_{\tilde{f}}$ then implies that $f(t) = \tilde{f}(t) = \vee_{g \in L_{\tilde{f}}} g(t) = \vee_{g \in L_f} g(t) = \bar{f}(t)$ for μ-a.e. $t \in T$. ■

Lemma A.15. *Assume G.1-G.3. Suppose that the join operator on A is continuous and that $\Phi : T \to [0, 1]$ is a monotone and measurable function such that $\mu(\Phi^{-1}(c)) = 0$ for every $c \in [0, 1]$. Define $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by*

$$h(\tau, f, g)(t) = \begin{cases} f(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau < 1/2 \\ g(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau \geq 1/2 \\ f(t) \vee g(t), & \text{if } \Phi(t) > |1 - 2\tau| \end{cases} \tag{A.2}$$

Then $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is continuous.

[55] It can be further shown that, for all $t \in T$, $\bar{f}(t) = \vee\{a \in A : a \leq f(t')$ for all $t' \geq t$ s.t. $t' \in T$ is a continuity point of f in the order-support of $\mu\}$. But we will not need this result.

Proof. Suppose that $(\tau_k, f_k, g_k) \to (\tau, f, g) \in [0,1] \times \mathcal{M} \times \mathcal{M}$. By Lemma A.12, there is a full μ-measure subset, D, of T such that $f_k(t) \to f(t)$ and $g_k(t) \to g(t)$ for every $t \in D$. There are three cases: $\tau = 1/2$, $\tau > 1/2$ and $\tau < 1/2$.

Suppose that $\tau < 1/2$. For each $t \in D$ such that $\Phi(t) < |1 - 2\tau|$, we have $\Phi(t) < |1 - 2\tau_k|$ for all k large enough. Hence, $h(\tau_k, f_k, g_k)(t) = f_k(t)$ for all k large enough, and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \to f(t) = h(\tau, f, g)(t)$. Similarly, for each $t \in D$ such that $\Phi(t) > |1 - 2\tau|$, $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(\tau, f, g)(t)$, where the limit follows because $\vee$ is continuous. Because $\mu(\{t \in T : \Phi(t) = |1 - 2\tau|\}) = 0$, if $\tau < 1/2$, $h(\tau_k, f_k, g_k)(t) \to h(\tau, f, g)(t)$ for μ a.e. $t \in T$ and so, by Lemma A.12, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$.

Because the case $\tau > 1/2$ is similar to $\tau < 1/2$, we need only consider the remaining case in which $\tau = 1/2$. In this case, $|1 - 2\tau_k| \to 0$. Consequently, for any $t \in T$ such that $\Phi(t) > 0$, we have $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t)$ for k large enough and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(1/2, f, g)(t)$. Hence, because $\mu(\{t \in T : \Phi(t) = 0\}) = 0$, $h(\tau_k, f_k, g_k)(t) \to h(1/2, f, g)(t)$ for μ a.e. $t \in T$, and so again by Lemma A.12, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$. ■

Lemma A.16. *Under G.1-G.4, the metric space $(\mathcal{M}, \delta)$ is an absolute retract.*

Proof. Define $h : [0,1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by $h(\tau, s, s')(t) = \tau s(t) + (1 - \tau)s'(t)$ for all $t \in T$ if G.4(i) holds, and by (A.2) if G.4(ii) holds, where the monotone function $\Phi(\cdot)$ appearing in (A.2) is defined by (A.1). Note that h maps into $\mathcal{M}$ in case G.4(i) holds because the partial order on A is convex. We claim that, in either case, h is continuous. Indeed, if G.4(ii) holds, this follows from Lemmas A.4 and A.15. If G.4(i) holds and the sequence $(\tau_n, s_n, s'_n) \in [0,1] \times \mathcal{M} \times \mathcal{M}$ converges to (τ, s, s'), then by Lemma A.12, $s_n(t) \to s(t)$ and $s'_n(t) \to s'(t)$ for μ-a.e. $t \in T$. Hence, because A_i is a convex topological space, $\tau_n s_n(t) + (1 - \tau_n)s'_n(t) \to \tau s(t) + (1 - \tau)s'(t)$ for μ-a.e. $t \in T$. But then Lemma A.12 implies $\tau_n s_n + (1 - \tau_n)s'_n \to \tau s + (1 - \tau)s'$, as desired.

Also in either case, for any $g \in \mathcal{M}$, $h(\cdot, \cdot, g)$ is a contraction for $\mathcal{M}$ so that $(\mathcal{M}, \delta)$ is contractible. Hence, by Borsuk (1966, IV (9.1)) and Dugundji (1965), it suffices to show that in either case, for each $f' \in \mathcal{M}$ and each neighborhood U of f', there exists a neighborhood V of f' and contained in U such that the sets V^n, $n \geq 1$, defined inductively by $V^1 = h([0,1], V, V)$, $V^{n+1} = h([0,1], V, V^n)$, are all contained in U. We demonstrate this separately for each of the two cases, G.4(i) and G.4(ii) each with their respective definitions of h.

Case I. Suppose G.4(i) holds. For each n, $V^{n+1} \subset coV$, so it suffices to show that $coV \subset U$ for some neighborhood V of f'. Taking V to be $B_{1/k}(f')$, the $1/k$ ball around f', it suffices to show that $coB_{1/k}(f') \subset U$ for some $k = 1, 2, ...$. If no such k exists, then for each k, there exist $f_1^k, ..., f_{n_k}^k$ in $B_{1/k}(f')$ and nonnegative weights $\lambda_1^k, ..., \lambda_{n_k}^k$ summing to one such that $g_k = \sum_{j=1}^{n_k} \lambda_j^k f_j^k \notin U$. Hence, $g_k(t) = \sum_{j=1}^{n_k} \lambda_j^k f_j^k(t)$ for μ-a.e. $t \in T$ and so for all t in some Borel subset, E, having μ-measure one. Moreover, the sequence $f_1^1, ..., f_{n_1}^1, f_1^2, ..., f_{n_2}^2, ...$ converges to f'. Consequently, by Lemma A.12 the sequence $f_1^1(t), ..., f_{n_1}^1(t), f_1^2(t), ..., f_{n_2}^2(t), ...$ converges to $f'(t)$ for μ-a.e. $t \in T$ and so for all t in some Borel subset, D, having μ-measure one. But then for each $t \in D \cap E$ and every convex neighborhood W_t of $f'(t)$, each of $f_1^k(t), ..., f_{n_k}^k(t)$ is in W_t for all k large enough, and therefore $g_k(t) = \sum_{j=1}^{n_k} \lambda_j^k f_j^k(t)$ is in W_t for k large enough as well. But this implies, by the local convexity of A, that $g_k(t) \to f'(t)$ for every $t \in D \cap E$ and hence for μ-a.e. $t \in T$. Lemma A.12 then implies that $g_k \to f'$, a contradiction.

Case II. Suppose G.4(ii) holds. As a matter of notation, for $f, g \in \mathcal{M}$, write $f \leq g$ if $f(t) \leq g(t)$ for μ-a.e. $t \in T$. Also, for any sequence of monotone functions $f_1, f_2, ...,$ in $\mathcal{M}$, denote by $f_1 \vee f_2 \vee ...$ the monotone function taking the value $\lim_n[f_1(t) \vee f_2(t) \vee ... \vee f_n(t)]$ for each t in T. This is well-defined by Lemma A.6.

For each V, note that if $g \in V^1$, then $g = h(\tau, f_0, f_1)$ for some $\tau \in [0,1]$ and some $f_1, f_1 \in V$. Hence, by the definition of h, we have $g \leq f_0 \vee f_1$ and either $f_0 \leq g$ or $f_1 \leq g$. We may choose the indices so that $f_0 \leq g \leq f_0 \vee f_1$. Inductively, it can similarly be seen that if $g \in V^n$, then there exist $f_0, f_1, ..., f_n \in V$ such that

$$f_0 \leq g \leq f_0 \vee ... \vee f_n. \tag{A.3}$$

Suppose now, by way of contradiction, that there is no open set V containing $f' \in \mathcal{M}$ and contained in the neighborhood U of f' such that all the V^n as defined above are contained in U. Then, successively for each $k = 1, 2, ..$, taking V to be $B_{1/k}(f')$, the $1/k$ ball around f', there exists n_k such that some $g_k \in V^{n_k}$ is not in U. Moreover, by (A.3), there exist $f_0^k, ..., f_{n_k}^k \in V = B_{1/k}(f')$ such that

$$f_0^k \leq g_k \leq f_0^k \vee ... \vee f_{n_k}^k. \tag{A.4}$$

Consider the sequence $f_0^1, ..., f_{n_1}^1, f_0^2, ..., f_{n_2}^2, ...$. Because f_j^k is in $B_{1/k}(f')$, this sequence converges to f'. Let us reindex this sequence as $f_1, f_2, ...$. Hence, $f_j \to f'$.

Because for every n the set $\{f_n, f_{n+1}, ...\}$ contains the set $\{f_0^k, ..., f_{n_k}^k\}$ whenever k is large enough, we have

$$f_0^k \vee ... \vee f_{n_k}^k \leq \vee_{j \geq n} f_j,$$

for every n and all large enough k. Combined with (A.4), this implies that

$$f_0^k \leq g_k \leq \vee_{j \geq n} f_j \tag{A.5}$$

for every n and all large enough k.

Now, $f_0^k \to f'$ as $k \to \infty$. Hence, by Lemma A.12, $f_0^k(t) \to f'(t)$ for μ-a.e. $t \in T$. Consequently, if for μ-a.e. $t \in T$, $\vee_{j \geq n} f_j(t) \to f'(t)$ as $n \to \infty$, then (A.5) and Lemma A.5 would imply that $g_k(t) \to f'(t)$ for μ-a.e. $t \in T$. Then, Lemma A.12 would imply that $g_k \to f'$ contradicting the fact that no g_k is in U, and completing the proof that $(\mathcal{M}, \delta)$ is an absolute retract.

It therefore remains only to establish that for μ a.e. $t \in T$, $\vee_{j \geq n} f_j(t) \to f'(t)$ as $n \to \infty$. But, by Lemma A.18, because A is locally complete this will follow if $f_j(t) \to_j f'(t)$ for μ a.e. t, which follows from Lemma A.12 because $f_j \to f'$. ■

A.4. Locally Complete Metric Semilattices

Lemma A.17. *If A is a compact upper-bound-convex subset of Euclidean space and a semilattice under the coordinatewise partial order, then A is a metric semilattice, i.e., $\vee$ is continuous.*

Proof. Suppose that $a_n \to a$, $b_n \to b$, $a \vee b = c$, and $a_n \vee b_n \to d$, where all of these points are in A. By the compactness of A, it suffices to show that $c = d$. Because $a_n \leq a_n \vee b_n$, taking limits implies $a \leq d$. Similarly, $b \leq d$, so that $c = a \vee b \leq d$. Thus, it remains only to show that $c \geq d$.

Let $\bar{a} = \vee A$ denote the largest element of A, which is well defined by Lemma A.7. By the upper-bound-convexity of A, $\varepsilon\bar{a} + (1-\varepsilon)c \in A$ for every $\varepsilon \in [0,1]$. Because the coordinatewise partial order is closed, it suffices to show that $\varepsilon\bar{a} + (1-\varepsilon)c \geq d$ for every $\varepsilon > 0$ sufficiently small. So, fix $\varepsilon \in (0,1)$ and consider the kth coordinate, c_k, of c. If for some n, $a_{kn} > c_k$, then because $\bar{a}_k \geq a_{kn}$ we have $\bar{a}_k > c_k$ and therefore $\varepsilon\bar{a}_k + (1-\varepsilon)c_k > c_k$. Consequently, because $a_{kn} \to_n a_k \leq c_k$, we have $\varepsilon\bar{a}_k + (1-\varepsilon)c_k > a_{kn}$ for all n sufficiently large. On the other hand, suppose that $a_{kn} \leq c_k$ for all n. Then because $\bar{a}_k \geq c_k$ we have $\varepsilon\bar{a}_k + (1-\varepsilon)c_k \geq a_{kn}$ for all n. So, in either case $\varepsilon\bar{a}_k + (1-\varepsilon)c_k \geq a_{kn}$ for all n sufficiently large. Therefore, because k is arbitrary, $\varepsilon\bar{a} + (1-\varepsilon)c \geq a_n$ for all n sufficiently large. Similarly, $\varepsilon\bar{a} + (1-\varepsilon)c \geq b_n$ for all n sufficiently large. Therefore, because $\varepsilon\bar{a} + (1-\varepsilon)c \in A$, $\varepsilon\bar{a} + (1-\varepsilon)c \geq a_n \vee b_n$ for all n sufficiently large. Taking limits in n gives $\varepsilon\bar{a} + (1-\varepsilon)c \geq d$. ■

Lemma A.18. *If G.3 holds, then A is locally complete if and only if for every $a \in A$ and every sequence a_n converging to a, $\lim_n(\vee_{k\geq n}a_k) = a$.*

Proof. We first demonstrate the "only if" direction. Suppose that A is locally complete, that U is a neighborhood of $a \in A$, and that $a_n \to a$. By local completeness, there exists a neighborhood W of a contained in U such that every subset of W has a least upper bound in U. In particular, because for n large enough $\{a_n, a_{n+1}, ...\}$ is a subset of W, the least upper bound of $\{a_n, a_{n+1}, ...\}$, namely $\vee_{k\geq n}a_k$, is in U for n large enough. Since U was arbitrary, this implies $\lim_n(\vee_{k\geq n}a_k) = a$.

We now turn to the "if" direction. Fix any $a \in A$, and let $B_{1/n}(a)$ denote the open ball around a with diameter $1/n$. For each n, $\vee B_{1/n}(a)$ is well-defined by Lemma A.7. Moreover, because $\vee B_{1/n}(a)$ is nonincreasing in n, $\lim_n \vee B_{1/n}(a)$ exists by Lemma A.6. We first argue that $\lim_n \vee B_{1/n}(a) = a$. For each n, we may construct, as in the proof of Lemma A.7, a sequence $\{a_{n,m}\}$ of points in $B_{1/n}(a)$ such that $\lim_m(a_{n,1} \vee ... \vee a_{n,m}) = \vee B_{1/n}(a)$. We may therefore choose m_n sufficiently large so that the distance between $a_{n,1} \vee ... \vee a_{n,m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$. Consider now the sequence $\{a_{1,1}, ..., a_{1,m_1}, a_{2,1}, ..., a_{2,m_2}, a_{3,1}, ..., a_{3,m_3}, ...\}$. Because $a_{n,m}$ is in $B_{1/n}(a)$, this sequence converges to a. Consequently, by hypothesis,

$$\lim_n(a_{n,1} \vee ... \vee a_{n,m_n} \vee a_{(n+1),1} \vee ... \vee a_{(n+1),m_{(n+1)}} \vee ...) = a.$$

But because every $a_{k,j}$ in the join in parentheses on the left-hand side above (denote this join by b_n) is in $B_{1/n}(a)$, we have

$$a_{n,1} \vee ... \vee a_{n,m_n} \leq b_n \leq \vee B_{1/n}(a).$$

Therefore, because for every n the distance between $a_{n,1} \vee ... \vee a_{n,m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$, Lemma A.5 implies that $\lim_n \vee B_{1/n}(a) = \lim_n b_n$. But since $\lim_n b_n = a$, we have $\lim_n \vee B_{1/n}(a) = a$. Next, for each n, let S_n be an arbitrary nonempty subset of $B_{1/n}(a)$, and choose any $s_n \in S_n$. Then $s_n \leq \vee S_n \leq \vee B_{1/n}(a)$. Because $s_n \in B_{1/n}(a)$, Lemma A.5 implies that $\lim_n \vee S_n = a$. Consequently, for every neighborhood U of a, there exists n large enough such that $\vee S$ (well-defined by Lemma A.7) is in U for every subset S of $B_{1/n}(a)$. Since a was arbitrary, A is locally complete. ■

Lemma A.19. *Every compact Euclidean metric semilattice is locally complete.*

Proof. Suppose that $a_n \to a$ with every a_n and a in the semilattice. By Lemma A.18, it suffices to show that $\lim_n(\vee_{k\geq n}a_k) = a$. By Lemma A.6, $\lim_n(\vee_{k\geq n}a_k)$ exists and is equal to $\lim_n \lim_m(a_n \vee ... \vee a_m)$ since $a_n \vee ... \vee a_m$ is nondecreasing in m, and $\lim_m(a_n \vee ... \vee a_m)$

is nonincreasing in n. For each dimension $k = 1, ..., K$, let $a^k_{n,m}$ denote the first among $a_n, a_{n+1}, ..., a_m$ with the largest kth coordinate. Hence, $a_n \vee ... \vee a_m = a^1_{n,m} \vee ... \vee a^K_{n,m}$, where the right-hand side consists of K terms. Because $a_n \to a$, $\lim_m a^k_{n,m}$ exists for each k and n, and $\lim_n \lim_m a^k_{n,m} = a$ for each k. Consequently, $\lim_n \lim_m (a_n \vee ... \vee a_m) = \lim_n \lim_m (a^1_{n,m} \vee ... \vee a^K_{n,m}) = (\lim_n \lim_m a^1_{n,m}) \vee ... \vee (\lim_n \lim_m a^K_{n,m}) = a \vee ... \vee a = a$, as desired. ■

Lemma A.20. *If G.3 holds and for all $a \in A$, every neighborhood of a contains a' such that $b' \leq a'$ for all b' close enough to a, then A is locally complete.*

Proof. Suppose that $a_n \to a$. By Lemma A.18, it suffices to show that $\lim_n(\vee_{k \geq n} a_k) = a$. For every n and m, $a_m \leq a_m \vee a_{m+1} \vee ... \vee a_{m+n}$, and so taking the limit first as $n \to \infty$ and then as $m \to \infty$ gives $a \leq \lim_m \vee_{k \geq m} a_k$, where the limit in n exists by Lemma A.6 because the sequence is monotone. Hence, it suffices to show that $\lim_m \vee_{k \geq m} a_k \leq a$.

Let U be a neighborhood of a and let a' be chosen as in the statement of the lemma. Then, because $a_m \to a$, $a_m \leq a'$ for all m large enough. Consequently, for m large enough and for all n, $a_m \vee a_{m+1} \vee ... \vee a_{m+n} \leq a'$. Taking the limit first in n and then in m yields $\lim_m \vee_{k \geq m} a_k \leq a'$. Because for every neighborhood U of a this holds for some a' in U, $\lim_m \vee_{k \geq m} a_k \leq a$, as desired. ■

A.5. Proofs from Section 5

Lemma A.21. *In the price competition game from subsection 5.3, and given the partial orders on types, $\geq_i$, defined there, each firm possesses a monotone pure strategy best reply when the other firms employ monotone pure strategies.*

Proof. Suppose that all firms $j \neq i$ employ monotone pure strategies according to $\geq_j$ defined in subsection 5.3. Therefore, in particular, $p_j(c_j, x_j)$ is nondecreasing in c_j for each x_j, and (5.5) applies. For the remainder of this proof, we omit most subscripts i to keep the notation manageable.

Because firm i's interim payoff function is continuous in his price for each of his types and because his action space, $[0, 1]$, is totally ordered and compact, firm i possesses a largest best reply, $\hat{p}(c, x)$, for each of his types $(c, x) \in [0, 1]^2$. We will show that $\hat{p}(\cdot)$ is monotone according to $\geq_i$.

Let $\bar{t} = (\bar{c}, \bar{x})$, $\underline{t} = (\underline{c}, \underline{x})$ in $[0, 1]^2$ be two types of firm i, and suppose that $\bar{t} \geq_i \underline{t}$. Hence, $\bar{c} \geq \underline{c}$ and $\bar{x} - \underline{x} = \beta(\bar{c} - \underline{c})$ for some $\beta \in [0, \alpha_i]$. Let $\bar{p} = \hat{p}(\bar{c}, \bar{x})$, $\underline{p} = \hat{p}(\underline{c}, \underline{x})$, and $t^\lambda = (1 - \lambda)\underline{t} + \lambda\bar{t}$ for $\lambda \in [0, 1]$. We wish to show that $\bar{p} \geq \underline{p}$.

By the fundamental theorem of calculus,

$$v_i(\underline{p}, t^\lambda) - v_i(p', t^\lambda) = \int_{p'}^{\underline{p}} \frac{\partial v_i(p, t^\lambda)}{\partial p} dp,$$

so that

$$
\begin{aligned}
\frac{\partial\left[v_i(\underline{p},t^\lambda)-v_i(p',t^\lambda)\right]}{\partial\lambda} &= \int_{p'}^{\underline{p}} \frac{\partial^2 v_i(p,t^\lambda)}{\partial\lambda\partial p}dp \\
&= \int_{p'}^{\underline{p}} \left[\frac{\partial^2 v_i(p,t^\lambda)}{\partial c\partial p}(\bar{c}-\underline{c})+\frac{\partial^2 v_i(p,t^\lambda)}{\partial x\partial p}(\bar{x}-\underline{x})\right]dp \\
&= (\bar{c}-\underline{c})\int_{p'}^{\underline{p}} \left[\frac{\partial^2 v_i(p,t^\lambda)}{\partial c\partial p}+\beta\frac{\partial^2 v_i(p,t^\lambda)}{\partial x\partial p}\right]dp \\
&\geq 0,
\end{aligned}
$$

where the inequality follows by (5.5) if $\underline{p} \geq p' \geq \bar{c}$. Therefore, $v_i(\underline{p},\bar{t}) - v_i(p',\bar{t}) \geq v_i(\underline{p},\underline{t}) - v_i(p',\underline{t}) \geq 0$, where the first inequality follows because $t^0 = \underline{t}$, $t^1 = \bar{t}$, and the second because $\underline{p}$ is a best reply at $\underline{t}$. Therefore, we have shown the following: If $\underline{p} \geq \bar{c}$, then

$$v_i(\underline{p},\bar{t}) - v_i(p',\bar{t}) \geq 0, \text{ for all } p' \in [\bar{c},\underline{p}].$$

Hence, if $\underline{p} \geq \bar{c}$, then $\hat{p}(\bar{t}) = \bar{p} \geq \underline{p} = \hat{p}(\underline{t})$ because $\hat{p}(\bar{t})$ is the largest best reply at $\bar{t}$ and because no best reply at $\bar{t} = (\bar{c},\bar{x})$ is below $\bar{c}$. On the other hand, if $\underline{p} < \bar{c}$, then $\bar{p} = \hat{p}(\bar{t}) \geq \bar{c} > \underline{p} = \hat{p}(\underline{t})$, where the first inequality again follows because no best reply at $\bar{t}$ is below $\bar{c}$. We conclude that $\bar{p} \geq \underline{p}$, as desired. ∎

Proof of Lemma 5.2. (see subsection 5.2) Fix any monotone pure strategies of all players but i. For the remainder of this proof, we omit most subscripts i to keep the notation manageable. Let $v(b,t)$ denote bidder i's expected payoff from employing the bid vector $b = (b_1,...,b_m)$ when his type vector is $t = (t_1,...,t_m)$. Then, letting $P_k(b_k)$ denote the probability that bidder i wins at least k units – which depends only on his kth unit-bid b_k – we have, where e_k is an m-vector of k ones followed by $m-k$ zeros,

$$
\begin{aligned}
v(b,t) &= u(0) + \sum_{k=1}^{m} P_k(b_k)\left(u((t-b)\cdot e_k) - u((t-b)\cdot e_{k-1})\right) \\
&= \frac{1}{r}\sum_{k=1}^{m} e^{r(b_1+...+b_{k-1})}P_k(b_k)\left(1-e^{-r(t_k-b_k)}\right)e^{-r(t_1+...+t_{k-1})},
\end{aligned}
$$

where $u(x) = \frac{1-e^{-rx}}{r}$ is bidder i's utility function with constant absolute risk aversion parameter $r \geq 0$, where it is understood that $u(x) = x$ when $r = 0$. Note that the dependence of r on i has been suppressed.

From now on we shall proceed as if $r > 0$ because all of the formulae employed here have well-defined limits as r tends to zero that correspond to the risk neutral case $u(x) = x$.

Letting $w_k(b_k,t) = \frac{1}{r}P_k(b_k)\left(1-e^{-r(t_k-b_k)}\right)e^{-r(t_1+...+t_{k-1})}$, we may write,

$$v(b,t) = \sum_{k=1}^{m} e^{r(b_1+...+b_{k-1})}w_k(b_k,t).$$

As shown in (5.2) from subsection 5.1 (and setting $\bar{p} = \underline{p} = 0$ there), for each $k = 2,...,m$,

$$u(t_1+...+t_k) - u(t_1+...+t_{k-1}) = \frac{1}{r}(1-e^{-rt_k})e^{-r(t_1+...+t_{k-1})}, \tag{A.6}$$

is nondecreasing in t according to the partial order $\geq_i$ defined in (5.1). Henceforth, we shall employ the partial order $\geq_i$ on i's type space. We next demonstrate the following facts.

(i) $w_k(b_k, t)$ is nondecreasing in t, and

(ii) $w_k(\bar{b}_k, t) - w_k(\underline{b}_k, t)$ is nondecreasing in t for all $\bar{b}_k \geq \underline{b}_k$,

To see (i), write,

$$\begin{aligned} w_k(b_k, t) &= \frac{1}{r} P_k(b_k)\left(1 - e^{-r(t_k - b_k)}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &= \frac{1}{r} P_k(b_k)\left(1 - e^{-rt_k}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &\quad + \frac{1}{r} P_k(b_k)\left(e^{rb_k} - 1\right)\left(-e^{-r(t_1 + \ldots + t_k)}\right). \end{aligned}$$

The first term in the sum is nondecreasing in t according to $\geq_i$ by (A.6) and the second term, being nondecreasing in the coordinatewise partial order is, a fortiori, nondecreasing in t according to $\geq_i$.

Turning to (ii), if $P_k(\underline{b}_k) = 0$ then $w_k(\underline{b}_k, t) = 0$ and (ii) follows from (i). So, assume $P_k(\underline{b}_k) > 0$. Then,

$$\begin{aligned} w_k(\bar{b}_k, t) - w_k(\underline{b}_k, t) &= \frac{1}{r} P_k(\bar{b}_k)\left(1 - e^{-r(t_k - \bar{b}_k)}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &\quad - \frac{1}{r} P_k(\underline{b}_k)\left(1 - e^{-r(t_k - \underline{b}_k)}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &= \left(\frac{P_k(\bar{b}_k)}{P_k(\underline{b}_k)} - 1\right) w_k(\bar{b}_k, t) \\ &\quad + \frac{1}{r} P_k(\underline{b}_k)\left(e^{r\bar{b}_k} - e^{r\underline{b}_k}\right)\left(-e^{-r(t_1 + \ldots + t_k)}\right). \end{aligned}$$

The first term in the sum is nondecreasing in t according to $\geq_i$ by (i) and the second term, being nondecreasing in the coordinatewise partial order is, a fortiori, nondecreasing in t according to $\geq_i$. This proves (ii).

Suppose now that the vector of bids b is optimal for bidder i when his type vector is t, and that b' is optimal when his type is $t' \geq_i t$. We must argue that $b \vee b'$ is optimal when his type is t'. If $b_k \leq b'_k$ for all k, then $b \vee b' = b'$ and we are done. Hence, we may assume that there exist $j \leq l$ such that $b_k > b'_k$ for $k = j, ..., l$ and $b_{k-1} \leq b'_{k-1}$ and $b_{l+1} \leq b'_{l+1}$, where the first of the latter two inequalities is ignored if $j = 1$ and the second is ignored if $l = m$.

Let $\hat{b}$ be the bid vector obtained from b by replacing its coordinates j through l with the coordinates j through l of b'. Because b is optimal at t and $\hat{b}$ is nonincreasing and therefore feasible, we have

$$
\begin{aligned}
0 &\leq v(b,t) - v(\hat{b},t) \\
&= e^{b_1+\ldots+b_{j-1}} \left[w_j(b_j,t) - w_j(b'_j,t) + \sum_{k=j+1}^{l} e^{b_j+\ldots+b_{k-1}} \left(w_k(b_k,t) - w_k(b'_k,t) \right) \right] \\
&\quad + e^{b_1+\ldots+b_{j-1}} \left(e^{b_j+\ldots+b_l} - e^{b'_j+\ldots+b'_l} \right) \left[w_{l+1}(b_{l+1},t) + e^{b_{l+1}} w_{l+2}(b_{l+2},t) + \ldots + e^{b_{l+1}+\ldots+b_{m-1}} w_m(b_m,t) \right]
\end{aligned}
$$

Consequently, dividing by $e^{b_1+\ldots+b_{j-1}}$ and changing t to $t' \geq_i t$, (i) and (ii) imply that,

$$
\begin{aligned}
0 &\leq \left[w_j(b_j,t') - w_j(b'_j,t') + \sum_{k=j+1}^{l} e^{b_j+\ldots+b_{k-1}} \left(w_k(b_k,t') - w_k(b'_k,t') \right) \right] \\
&\quad + \left(e^{b_j+\ldots+b_l} - e^{b'_j+\ldots+b'_l} \right) \left[w_{l+1}(b_{l+1},t') + e^{b_{l+1}} w_{l+2}(b_{l+2},t') + \ldots + e^{b_{l+1}+\ldots+b_{m-1}} w_m(b_m,t') \right]
\end{aligned}
\tag{A.7}
$$

Focusing on the second term in square brackets in (A.7), we claim that

$$
\begin{aligned}
& w_{l+1}(b_{l+1},t') + e^{b_{l+1}} w_{l+2}(b_{l+2},t') + \ldots + e^{b_{l+1}+\ldots+b_{m-1}} w_m(b_m,t') \\
\leq\ & w_{l+1}(b'_{l+1},t') + e^{b'_{l+1}} w_{l+2}(b'_{l+2},t') + \ldots + e^{b'_{l+1}+\ldots+b'_{m-1}} w_m(b'_m,t')
\end{aligned}
\tag{A.8}
$$

To see this, note that because $b_{l+1} \leq b'_{l+1}$, the bid vector b'' obtained from b' by replacing its coordinates $l+1$ through m with the coordinates $l+1$ through m of b is a feasible (i.e., nonincreasing) bid vector. Consequently, because b' is optimal at t' we must have $0 \leq v(b',t') - v(b'',t')$. But this difference in utilities is precisely the difference between the right-hand and left-hand sides of (A.8) multiplied by $e^{b_1+\ldots+b_l}$, thereby establishing (A.8).

Thus, we may conclude, after making use of (A.8) in (A.7) and multiplying the result by $e^{b'_1+\ldots+b'_{j-1}}$ that,

$$
\begin{aligned}
0 &\leq e^{b'_1+\ldots+b'_{j-1}} \left[w_j(b_j,t') - w_j(b'_j,t') + \sum_{k=j+1}^{l} e^{b_j+\ldots+b_{k-1}} \left(w_k(b_k,t') - w_k(b'_k,t') \right) \right] \\
&\quad + e^{b'_1+\ldots+b'_{j-1}} \left(e^{b_j+\ldots+b_l} - e^{b'_j+\ldots+b'_l} \right) \left[w_{l+1}(b'_{l+1},t') + e^{b'_{l+1}} w_{l+2}(b'_{l+2},t') + \ldots + e^{b'_{l+1}+\ldots+b'_{m-1}} w_m(b'_m,t') \right] \\
&= v(\tilde{b},t') - v(b',t'),
\end{aligned}
$$

where $\tilde{b}$ is the nonincreasing and therefore feasible bid vector obtained from b' by replacing its coordinates j through l with the coordinates j through l of b. Hence, $\tilde{b}$ is optimal at t' because b' is optimal at t'.

Thus, we have shown that whenever $j,\ldots,l$ is a maximal set of consecutive coordinates such that $b_k > b'_k$ for all $k = j,\ldots,l$, replacing in b' the unit-bids $b'_j,\ldots,b'_l$ with the coordinate-by-coordinate larger unit bids $b_j,\ldots,b_l$ results in a bid vector that is optimal at t'. Applying this procedure finitely often leads to the conclusion that $b \vee b'$ is optimal at t', as desired. ■

References

Athey (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 861-889.

Billingsley, P. (1968): *Convergence of Probability Measures*, John Wiley and Sons, New York.

Birkhoff, G. (1967): *Lattice Theory.* American Mathematical Society, Providence, RI.

Borsuk, K. (1966): *Theory of Retracts.* Polish Scientific Publishers, Warsaw, Poland.

Cohn, D. L., (1980): *Measure Theory*, Birkhauser, Boston.

Dugundji, J. (1965): "Locally Equiconnected Spaces and Absolute Neighborhood Retracts," *Fundamenta Mathematicae*, 52, 187-193.

Eilenberg, S., and D. Montgomery (1946): "Fixed point Theorems for Multi-Valued Transformations," *American Journal of Mathematics,* 68, 214-222.

Hart, S., and B. Weiss (2005): "Convergence in a Lattice: A Counterexample," mimeo, Institute of Mathematics, Department of Economics, and Center for the Study of Rationality, The Hebrew University of Jerusalem.

McAdams, D. (2003): "Isotone Equilibrium in Games of Incomplete Information," *Econometrica*, 71, 1191-1214.

McAdams, D. (2004): "Monotone Equilibrium in Multi-Unit Auctions," mimeo, M.I.T.

McAdams, D. (2006): "On the Failure of Monotonicity in Uniform-Price Auctions," mimeo, M.I.T.

Milgrom, P. and J. Roberts (1990): "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58, 1255-1277.

Milgrom, P. and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157-80.

Milgrom, P. and R. Weber (1985): "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, 10, 1985, 619-32.

Reny, P. J., and S. Zamir (2004): "On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions," *Econometrica*, 72, 1105-1126.

Van Zandt, T. and X. Vives (2007): "Monotone Equilibria in Bayesian Games of Strategic Complementarities," *Journal of Economic Theory*, 134, 339-360.

Vives, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19, 305-321.