Competition in Financial Innovation*

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October 19, 2011

Abstract

This paper examines the incentives offered by frictionless markets to innovate asset-backed securities by asset owners who maximize the assets' values. Assuming identical preferences across investors with heterogeneous risk-sharing needs, we characterize economies in which competition provides insufficient incentives to innovate so that, in equilibrium, financial markets are incomplete in all (pure strategy) equilibria—even when innovation is essentially costless. Thus, value maximization generally does not result in complete markets.

JEL Classification: D52, G10

Keywords: Innovation, Efficiency, Endogenously Incomplete Markets

An important economic role of financial innovation is attributed to allowing asset holders to increase the value of the owned assets. Some of the successful innovations in financial markets, such as the practice of tranching and, more generally, asset-backed securities, have been introduced by assets owners to raise capital by benefiting from heterogenous investor demands for hedging and risk sharing.[1] This paper examines the incentives for asset owners to introduce new

* We are grateful to Steven Durlauf, Piero Gottardi, Ferdinando Monte, Herakles Polemarchakis, Bill Sandholm, Dai Zusai, participants at the Macroeconomics Seminar at Wisconsin-Madison, the 2010 NSF/NBER/CEME Conference at NYU, the 2010 Theoretical Economics Conference at Kansas, the 7th Cowles Conference on General Equilibrium, and the XX EWGET in Universidade de Vigo, for helpful comments and suggestions.

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[1] Innovations of this type include mortgage-backed securities (Ginnie Mae first securitized mortgages through passthrough security in 1968; in 1971 Freddie Mac issued its first mortgage passthrough; in 1981 Fannie Mae issued its first mortgage passthrough to increase the money available for new home purchases by securitizing

securities. In particular, we study whether competition in financial innovation among issuers of asset-backed securities, who maximize asset value, provides sufficient incentives to complete markets.

We consider a general model with holders of assets (e.g., entrepreneurs) who strategically choose which securities to issue in *frictionless* markets.[2] The securities are offered to competitive investors who have *identical* utility over consumption, but differ in their pre-existing risks and in their demands for intertemporal consumption smoothing and, hence, exhibit heterogenous risk-sharing needs. We show that under natural assumptions on investors' preferences (i.e., convex marginal utility such as CARA or CRRA), any financial structure with an incomplete set of securities dominates a complete financial structure in terms of the market value of an asset. Consequently, competition in innovation of asset-backed securities does not offer incentives to complete the structure of traded securities, and financial markets are inefficient in providing insurance opportunities to investors. This occurs even if innovation is costless; for any market size (including large markets); for an arbitrary number of states of the world; endowment distributions; possibly idiosyncratic asset returns; and for all monotone preferences of the issuers.

This paper's main economic insight is that frictionless markets give asset owners incentives to introduce a limited range of asset-backed securities. Indeed, when firms raise capital, they tend to issue a small number of securities. Most financial innovations are not introduced by the original asset holders, but by organized financial exchanges, commercial and investment banks and other intermediaries, who can profit from commission fees or bid-ask spreads (Finnerty (1988) and Tufano (2003)). Thus, our results highlight the essential role of intermediaries—who can benefit not only from the mitigation of market frictions, but also by creating value through the risk-sharing, or "spanning", motive itself—for completing the market.[3] If market efficiency is to be improved through innovation of asset-backed securities, then incentives other than maximizing asset value are necessary. We discuss the effectiveness of government regulation of the innovation process.[4]

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mortgages; in 1983 Freddie Mac issued the first collateralized mortgage obligation); Treasury STRIPS; primes and scores issued by firms. Partitioning the anticipated flow of income from the pool into marketable securities that will appeal to particular groups of investors with heterogenous demands for hedging risk has also been applied by firms to many other kinds of credit transactions, including credit card receivables, auto loans, and small business loans.

[2] In our analysis short sales are allowed, and thus value to securitization of an issuer's asset does not derive from short sales constraints. Allen and Gale (1991, 1994), Chen (1995), and Pesendorfer (1995) examine short sales restrictions as a profit source for corporate issuers. In contrast to our model, short sales restrictions introduce limits to arbitrage and thereby create profit opportunities through indirect price discrimination in security design.

[3] See Allen and Santomero (1998), Chen (1995), Pesendorfer (1995) and Bisin (1998).

[4] Missing markets and the resulting exposure to systemic risk have been central to recent discussions of the

The literature has long recognized the spanning motive as a potential determinant of financial innovation.[5] As Duffie and Rahi (1995) emphasized, however, this theory has few concrete normative or predictive results, and these have been demonstrated only in specific numerical examples. Here, we provide sharp predictions on the endogenous financial structure in economies with identical investor utility functions over consumption. The economic mechanism involves (the shape of) the investors' *marginal* utility function, which changes the very nature of competition among asset holders and hence the incentives to innovate: market completion can be seen as a problem of provision of either a public good or a public bad.

Allen and Gale's (1991) seminal paper on the spanning motive suggests an alternative explanation for the limited incentives to issue securities in frictionless markets. Their classic example shows that even if each individual entrepreneur can increase the value of an asset by introducing new securities, in equilibrium, market may be incomplete if issuing securities is costly; with positive probability, innovation may fail to occur due to the entrepreneurs' inability to coordinate their innovation activities to complete the financial structure. Gale (1992)) proposes that the cost of gathering information about unfamiliar securities may lead to gains from standardization. Marin and Rahi (2000) explains market incompleteness in asymmetric information model through the Hirshleifer effect. The mechanism characterized in this paper is different. It operates even if information is symmetric, innovation is essentially costless; does not result from lack of coordination among entrepreneurs; and implies that market incompleteness occurs with probability one.

This paper offers two technical contributions. We characterize the comparative statics of the market value of an asset with respect to the security span. Permitting unlimited short sales, along with the assumed quasi-linearity of the investors' utility function gives tractability to our approach: We recast the maximization of a firm's value over financial structures as an optimization problem over spans. More generally—to the best of our knowledge—this paper is the first to study the class of games in which players' strategy sets are collections of linear subspaces of a common linear space (spans). Apart from the financial application, these types of games arise naturally in competition in bundling commodities or in design of product lines. The results

2008 financial crisis. To monitor financial innovation, in September 2009, the Security and Exchange Commission (SEC) created the Division of Risk, Strategy and Financial Innovation, the first new division the SEC created in 37 years.

[5] Allen and Gale (1994) and Duffie and Rahi (1995) provide surveys. Other strands of the literature attribute innovation to incentives to mitigate frictions: asymmetric information between trading parties or due to imperfect monitoring of performance, short sales restrictions, or transaction costs. While frictions are important for understanding potential benefits from innovation, for many asset-backed securities, such as MBSs, attributes of the underlying assets are largely public information; transaction costs have declined significantly over the past decades (Allen and Santomero (1998); Tufano (2003)).

obtained here directly extend and contribute to these contexts.[6]

The paper is organized as follows: Section 1 reviews Allen and Gale's (1991) classic example; Section 2 presents the model of financial innovation; Section 3 establishes some useful equilibrium properties; Section 4 derives the comparative statics of the firms' market values; Section 5 characterizes the endogenous financial structure of the economy; Section 6 extends results to a more general model; and Section 7 concludes. All proofs appear in the Appendix.

1. The Example of Allen and Gale (1991)

We introduce the problem of competition in financial innovation in the context of the classic example by Allen and Gale (1991; henceforth AG), whose work motivated the role spanning and risk sharing play in financial innovation. Consider a two-period economy with uncertainty in which there are two possible states of the world in the second period, N entrepreneurs, and a continuum of investors. Each entrepreneur is endowed with an asset (a firm), which gives random return $z = (0.5, 2.5)$ in terms of numèraire, contingent on the resulting state of the world.

The entrepreneurs, who only derive utility from consumption in period zero, sell their claims to their future return to two types of investors. As a function of their consumption in period zero, c_0, and their (random) consumption in period one, c_1, one half of the investors have preferences

$$5 + c_0 - \mathrm{E}[\ \exp(-10c_1)],$$

whereas the other half have preferences

$$5 + c_0 + \mathrm{E}[\ \ln(c_1)].$$

The mass of each type is normalized to $N/2$.

To sell their future returns, all entrepreneurs simultaneously choose from two financial structures: each can costlessly issue equity, in which case one market opens and shares of the entrepreneur's firm are traded; or alternatively, the entrepreneur can innovate by issuing, at a cost, two state-contingent claims, in which case, two markets open. There are no other assets in the economy; therefore, if all entrepreneurs choose to issue equity, financial markets are incomplete. However, if one or more entrepreneur innovates, the financial markets are complete.

[6] The problem of an entrepreneur issuing securities to sell a return on an asset is mathematically equivalent to the problem of a producer choosing a portfolio of bundles to sell an inventory of commodities or design of product lines in which consumers have utility over multidimensional characteristics and producers decide what vectors of characteristics to build into their products. The difficulty that stems from strategies being linear subspaces is that collections of all linear subspaces are not convex sets, and payoffs are discontinuous in subspace dimensionality. Thus, standard techniques do not apply.

From the perspective of insurance opportunities available in the market, a key question is whether competition among entrepreneurs gives rise to sufficient innovation to complete large markets. AG demonstrate that, in equilibrium, markets can be incomplete with positive probability for an arbitrary market size: Arbitrage ensures that firms with identical returns have the same market value. In this economy, if we denote by V_C the market value of each entrepreneur's firm when markets are complete and by V_I its value when only equity is issued, AG obtain that $V_C = 0.58603 > 0.58583 = V_I$, so that market value *is greater under complete markets*.[7] Thus, completing the market is essentially a *public good*: All entrepreneurs are better off if at least one pays an innovation cost to introduce contingent claims. AG focus on the symmetric mixed strategy equilibrium, in which each entrepreneur chooses to innovate with positive probability and as a result, all outcomes, including incomplete markets, occur with positive probability.[8]

One lesson from this example is that in the presence of innovation costs, large frictionless markets may be incomplete due to miscoordination among entrepreneurs that results from independent randomization. Clearly, the fact that innovation is costly is necessary for the free riding mechanism to operate, since otherwise markets are complete.

To hint at the economic mechanism presented in this paper, we observe that, if the utility of the first type of investors above is instead given by

$$5 + c_0 + \mathrm{E}[\ln(c_1 + 2)],$$

then the predictions we obtain change: Each firm's market value is maximized in *incomplete* markets, for now $V_C = 2.0952 < 2.3228 = V_I$. Financial innovation is then no longer a public good from the entrepreneurs' point of view. Rather, it becomes a *public "bad"*, as all entrepreneurs are worse off if one or more of them innovate. As a result, issuing equity is a strictly dominant strategy and, in the unique equilibrium, markets are *incomplete with probability* 1*, even if the cost of asset innovation is infinitesimal*.

Both examples describe markets with plausible investor preferences. Yet, the corresponding predictions regarding incentives to introduce new securities and market incompleteness differ markedly. In this paper, we attempt to identify the economic mechanisms that underlie distinct equilibrium predictions. Our primary result is the determination of such a mechanism: We offer sharp predictions about the form of the endogenous financial structure in a general model in which investors value future consumption equally.

[7] See Tables I and II in AG, pp. 1052-1053. All examples in AG share the feature that market value is greater under complete markets.

[8] In fact, with a larger number of entrepreneurs, the free-riding problem becomes more severe: *Ceteris paribus*, for each entrepreneur, the probability that at least one other entrepreneur introduces contingent claims increases. This reduces individual incentives to innovatec and the probability that one or more entrepreneur innovates is bounded away from 1.

2. Investors, Entrepreneurs, and Equilibrium

We first consider a two-period economy ($t = 0, 1$) with uncertainty. In the second period ($t = 1$), there are S states of the world, denoted by $s = 1, \ldots, S$. All the agents in this economy, whom we describe next, agree that the probability of state s occurring in the second period is $\Pr(s) > 0$.

2.1. Investors

Financial securities are demanded by a continuum of investors who derive utility from consumption of numèraire in both periods of the economy (and across states of the world at $t = 1$). There are K types of investors, which we index by $k = 1, \ldots, K$, and these types differ in their endowments of wealth in the second period. In state s, investor k will have wealth $e_{k,s} \geq 0$; the random variable $e_k = (e_{k,1}, \ldots, e_{k,S})$ denotes investor k's future wealth. Types of investors are interpreted as clienteles with heterogeneous demand for risk sharing arising from future income risk. The mass of type k investors is denoted by $\theta_k > 0$, and the mass of all investors is $\theta = \sum_k \theta_k$.

While their endowments of future wealth may differ, all types of investors have the same preferences over consumption, and their utilities are quasilinear and von Neumann-Morgenstern in the second period consumption. For all types, the utility derived from present consumption $c_0 \in \mathbb{R}$ and a state-contingent future consumption $(c_1, \ldots c_S) \in \mathbb{R}^S_+$ is given by $c_0 + U(c_1, \ldots c_S)$, where function $U : \mathbb{R}^S_+ \to \mathbb{R}$ is defined by

$$U(c_1, \ldots c_S) = \mathrm{E}[u(c)] = \sum_s \Pr(s) u(c_s)$$

for a $\mathbb{C}^2$ Bernoulli index $u : \mathbb{R}_+ \to \mathbb{R}$ that satisfies the standard assumptions of strict monotonicity and strict concavity, as well as the Inada condition that $\lim_{x \to 0} u'(x) = \infty$.

2.2. Entrepreneurs

Although investors have common preferences over consumption, they are exposed to distinct endowment risks. Asset holders, who, by issuing asset-backed securities can tailor asset structure to clienteles with different hedging needs, can exploit the resulting heterogeneity in investor demand. Financial securities are issued by a group of entrepreneurs,[9] each of whom has future wealth that may depend on the state of the world, and who wants to "sell" that future wealth in exchange for present consumption. Specifically, suppose that there is a finite number, N, of

[9] "Entrepreneurs" represent any traders who sell future income associated with assets they own by issuing asset-backed securities. This includes firm owners issuing securities backed by firms' cash flows or, banks securitizing the pool of assets they own (e.g., tranching mortgage pools).

strategic entrepreneurs who are indexed by $n = 1, \ldots, N$. Entrepreneur n owns an asset (e.g., a firm) that pays $z_{n,s} > 0$ units of the numèraire in the second period, if state of the world s is realized, but he does not care about future consumption and his utility is given by the present revenue that he can raise from selling the future return on his asset. That is, the random variable $z_n = (z_{n,1}, \ldots, z_{n,S})$ is the return to the asset that entrepreneur n wants to sell in exchange for numèraire in the first period.

Entrepreneurs do not know investors' future endowments and hold probabilistic beliefs over the profile $(e_1, \ldots, e_K)$. These beliefs are common to all entrepreneurs and given by the joint distribution function G, which is defined over $\mathbb{R}_+^{S\times K}$. We assume that distribution G is absolutely continuous with respect to the Lebesgue measure, but no other restrictions are placed on G. In particular, the associated marginal distributions can differ across investor types, and the joint distribution G can feature an arbitrary interdependence of endowments, as long as the correlations are not perfect, which is the case given absolute continuity.

2.3. *Innovation of asset-backed securities*

To sell claims to the return from their assets z_n, entrepreneurs simultaneously[10] issue securities in the first period. In the second period, payments against the issued securities are made and investors consume. Each entrepreneur can choose from various alternative selling strategies. One possibility is opening an equity market to sell shares of his asset. An alternative is to issue S claims, one for each state, paying $z_{n,s}$ units of the numèraire in the corresponding state s and 0 otherwise. More generally, entrepreneur n can issue a portfolio that comprises an arbitrary finite number of securities: A *financial structure* for entrepreneur n is a finite set of securities, $F_n \subseteq \mathbb{R}^S$. Each security $f_n \in F_n$ promises a payment of numèraire $f_{n,s}$, contingent on the realization of the state of the world s, for each $s = 1, \ldots, S$.

Because we only allow finite financial structures, it is convenient to treat financial structures as matrices. We write $F_n = (f_n^1, \ldots, f_n^{|F_n|})$, where $|F_n|$ denotes the cardinality of the structure. Financial structure F_n is required to exhaust the returns to entrepreneur n's asset and, without loss of generality, the supply of each security issued by entrepreneur n is normalized to 1. Formally, let $\mathcal{F}_n$ be the set of all financial structures such that $F_n\mathbf{1} = z_n$, where $\mathbf{1} = (1, \ldots, 1)$; entrepreneur n is restricted to issue $F_n \in \mathcal{F}_n$. We assume that the issuing cost per security is $\gamma > 0$ and hence the cost of issuing financial structure F_n is $\gamma|F_n|$.

[10]All the results from this paper extend to settings in which entrepreneurs innovate sequentially before markets open and the solution concept is Subgame Perfect Nash equilibrium.

2.4. *Equilibrium in financial markets*

Considering all entrepreneurs, $\sum_n |F_n|$ markets open in the present. The securities traded are given by the structure $F = (F_1, \ldots, F_N)$, which we treat as an $S \times (\sum_n |F_n|)$ matrix.

All investors are non-atomic, therefore, the prices at which securities trade are given by the competitive equilibrium prices of the economy under financial structure F, where we assume that investors can sell the issued securities short, but cannot issue any other securities.[11] For entrepreneur n, the relevant prices are those that correspond to his securities, which we denote by p_n. Hence, the market value of his firm is given by $V_n(F) = p_n \cdot \mathbf{1}$. Because the competitive equilibrium prices depend on the investors' profile of future wealth, which is unknown to the entrepreneurs, entrepreneur n's gross payoff is $\mathrm{E}_G[V_n(F)]$.

All entrepreneurs choose their financial structure simultaneously, behaving *à la* Nash, so as to maximize the expected value of their firms, net of issuance costs.

3. Allocation and Market Value

We abstract, momentarily, from the strategic aspects of the problem to study how the market value of the entrepreneurs' assets is determined in financial markets, given a financial structure F. To do this, we first characterize the future allocation of numèraire among investors that results from trading the securities offered in F in competitive financial markets.

3.1. *Market completeness*

The (column) span of F, which is the linear subspace of $\mathbb{R}^S$ defined as

$$\langle F \rangle = \{x \in \mathbb{R}^S \mid Ft = y \text{ for some } t \in \mathbb{R}^{|F|}\},$$

is the set of all transfers of future numèraire that can result from some portfolios of securities in F. A financial structure is said to be *complete* if it spans the entire $\mathbb{R}^S$; or equivalently, if its rank is S. Otherwise, the structure is said to be *incomplete*.

[11] The definition of competitive equilibrium is standard: If the assets issued are structure F, a *competitive equilibrium* comprises security prices $p \in \mathbb{R}^{|F|}$ and an allocation $(t_1, \ldots, t_K) \in \mathbb{R}^{|F| \times K}$ of financial securities across investor types, such that each t_k solves $\max_t\{U(e_k + Ft) - p \cdot t\}$, while $\sum_k \theta_k t_k = \mathbf{1}$. Under our assumptions on preferences, the condition of optimality of t_k can be replaced by the requirement that $F^\mathsf{T} DU(e_k + Ft_k) = p$. We observe in Section 3.3 that equilibrium prices exist and are unique; thus, our reference to *the* equilibrium prices under structure F is justified.

3.2. Characterization of the equilibrium allocation

Our first result asserts that given the financial structure chosen by the entrepreneurs, competitive financial markets allocate future numèraire in the same way a utilitarian planner would, while restricted to allocations that are feasible under the pre-determined financial structure.

Let $\mathcal{L}$ be the set of all linear subspaces $L \subseteq \mathbb{R}^S$ that contain the assets of all entrepreneurs, $\{z_1, \ldots, z_n\} \subseteq L$. Let $X : \mathcal{L} \to \mathbb{R}^{S\times K}_{++}$ be the correspondence that for any L gives a set of allocations of numèraire that can result from transfers in the linear subspace L, namely

$$X(L) = \{x \in \mathbb{R}^{S\times K}_{+} \mid \textstyle\sum_k \theta_k(x_k - e_k) = \sum_n z_n \text{ and } (x_k - e_k) \in L \text{ for all } k\}. \tag{1}$$

Thus, for any financial structure F, $X(\langle F\rangle)$ is a collection of allocations of numèraire that are feasible through the trades of securities in F. Also, for any profile $x = (x_1, \ldots, x_K) \in \mathbb{R}^{S\times K}_{+}$, let $\bar{U}(x) = \sum_k \theta_k U(x_k)$, which aggregates utilities across investor types at allocation $(x_1, \ldots, x_K)$ of future consumption. Given transferable utility, the following characterization of competitive equilibrium allocations of numèraire holds for any financial structure F.

Lemma **1** (Allocative Equivalence)**.** *Fix a financial structure F, let $(t_1, \ldots, t_K)$ be an allocation of the securities F such that $\sum_k \theta_k t_k = \mathbf{1}$, and let $(c_1, \ldots, c_K)$ be the future allocation of numèraire given by $c_k = e_k + Ft_k$. Allocation $(t_1, \ldots, t_K)$ is a competitive equilibrium allocation under structure F if, and only if, $(c_1, \ldots, c_K)$ solves the problem*

$$\max_x \left\{\bar{U}(x) | x \in X(\langle F\rangle)\right\}. \tag{2}$$

The equivalence between the competitive allocation of numèraire and the solution to Problem (2) has useful implications. First, note that for any financial structure F, the numèraire allocation is uniquely determined in the resulting competitive equilibria, even if the securities trades that yield such allocation are not (as is the case, for example, for linearly dependent securities). Moreover, the equilibrium allocation of numèraire depends on the financial structure only up to its span; that is, for any two financial structures F and F', such that $\langle F\rangle = \langle F'\rangle$, the equilibrium numèraire allocations coincide.[12]

[12] From the lemma, the existence of a competitive equilibrium allocation in the markets that open once entrepreneurs choose the financial structure F follows from the compactness of set $X(\langle F\rangle)$ and the continuity of function $\bar{U}(x)$. Its uniqueness holds by the convexity of $X(\langle F\rangle)$ and the strict concavity of $\bar{U}(x)$. In terms of primitives, the uniqueness results from the quasilinearity of the investor utilities, but does not require that utilities are identical. The dependence of the numèraire allocation on the financial structure through span alone is obtained because in Problem (2), structure F enters the constraint only through its span, $\langle F\rangle$.

3.3. *Market value*

Denote by $x : \mathcal{L} \to \mathbb{R}^{S\times K}_{++}$ the (unique) solution to problem

$$x\,(L) \equiv \max_x \{\bar{U}(x) | x \in X(L)\},$$

and let $\kappa : \mathcal{L} \to \mathbb{R}^S_{++}$ be defined as

$$\kappa(L) \equiv \frac{1}{\theta} \sum_k \theta_k DU(x_k(L)). \tag{3}$$

Given Lemma 1, for *any* financial structure F for which $\langle F \rangle = L$, allocation $x(L)$ is the future equilibrium numèraire allocation, while $\kappa(L)$ measures the average marginal utility, across investors in equilibrium.

Function κ determines equilibrium state prices for any financial structure, whether complete or not: Under financial structure F, competitive equilibrium asset prices are characterized by the equality $p^\intercal = \kappa(\langle F \rangle)^\intercal F$. When the financial structure is incomplete, equilibrium consumption vectors and, hence, marginal utilities may differ across investors. However, although state prices are not unique, those defined in (3) can be used to price securities unambiguously.[13] Lemma 2 characterizes the market value of an entrepreneur's asset.

Lemma **2** (Market Value). *The expected market value of an entrepreneur's asset z_n under structure F is given by* $\mathrm{E}_G[V_n(F)] = \mathrm{E}_G[\kappa(\langle F \rangle)] \cdot z_n$.

Two implications of this lemma are immediate. First, note that for any financial structure, the expected market value is defined unambiguously: any two financial structures can be ranked in terms of their profitability for each entrepreneur. In addition, note that just as with the numèraire allocation, market value depends on the financial structure only up to its span. Therefore, financial structures that permit the same numèraire transfers define equivalence classes for market value for each entrepreneur.

4. Financial Structure and Market Value

We now show that within the set of all financial structures, a structure always exists that maximizes the expected market value of an entrepreneur's asset. Then, we characterize this financial

[13] Any vector in the set $\{\kappa(L)\} + L^\perp$ constitutes a valid vector of state prices. In particular, each of the vectors whose average defines $\kappa(L)$ does so; the marginal utilities at equilibrium consumption can differ only in the components that are orthogonal to the security span and their differences are irrelevant for security pricing. Our characterization of $\kappa(L)$ as an average is useful for determining a financial structure that maximizes the entrepreneur's market value.

structure as to whether it is complete or incomplete. Finally, we exploit the equivalence of the equilibrium numèraire allocation and Problem (2) given by Lemma 1, to present a geometric interpretation of our results. This will elucidate the comparative statics of asset span and equilibrium prices (and market value) as well as the impact of asymmetries in investor preferences on predictions about incentives to innovate. To simplify our presentation, we still abstract from the issues of competition in the determination of the financial structure and ignore issuance costs.

4.1. *The Existence of a value-maximizing financial structure*

There are two difficulties with demonstrating the existence of an optimal structure: First, even if one restricts attention to financial structures with a fixed number of securities, the domain over which each entrepreneur optimizes is non-compact. In addition, market value is a discontinuous function because equilibrium state prices change discontinuously with the financial structure when the latter changes rank.[14] To deal with these two problems, we take the following approach. Since any two financial structures with the same span are equivalent in terms of market value (Lemma 2), optimization over financial structures can be recast as the problem of choosing a span—a linear subspace from the set of all linear subspaces of $\mathbb{R}^S$—that maximizes market value rather than optimizing over financial structures directly. The optimization problem over linear subspaces is more tractable: For any dimension $D \leq S$, the set of all D-dimensional linear subspaces of $\mathbb{R}^S$ is a compact manifold known as the Grassmannian, and market value V_n is continuous on it.[15] This allows us to recover the compactness of the domain and the continuity

[14] To see this in an extreme example, consider the following sequence of financial structures with two securities:

$$F^h = \begin{bmatrix} 1/h & 0 \\ 0 & 1/h \end{bmatrix}, \text{ for all } h \in \mathbb{N}.$$

For any finite h, markets are complete under structure F^h and the set of feasible allocations $X(\langle F^h \rangle)$ comprises all allocations. In the limit as $h \to \infty$, security span collapses to a zero-dimensional subspace and $X(\langle \lim_{h\to\infty} F^h \rangle)$ becomes a singleton set that comprises only the autarky point. Consequently, numèraire allocation, and hence the average marginal utility, are discontinuous. AG do not face these difficulties in a general model, since they consider entrepreneurs who choose a financial structure from an exogenously pre-specified, finite set of securities.

[15] Heuristically, suppose that $S = 2$ and an entrepreneur chooses among all one-dimensional linear subspaces. Each subspace is represented by a line passing through the origin and is uniquely identified by a point on a semicircle with a radius 1 (see Figure 1). A bijection that enlarges the distance of any point on the semicircle by a factor of 2 (around the circle) translates a semicircle into a full circle. Given such parameterization of linear subspaces, the entrepreneur effectively chooses a point on a circle, a compact set. In addition, the dimension of any linear subspace in the domain of optimization—each represented by a point on the circle—is, by construction, the same and equal to 1; $X(L)$ is a continuous correspondence defined on the circle. By the Maximum Theorem and Lemma 1, the equilibrium numèraire allocation $x(L)$ is continuous and so are state prices, given by the average marginal utility. The use of a Grassmanian first occurs in the economics literature in Duffie and Shafer's (1985) classic proof of generic existence of a competitive equilibrium for incomplete markets. To the best of our knowledge, we are the first to study the problem of optimization over spans and exploit the compactness of a

of the objective function over subsets of the problem's domain, which we then use to establish the existence of a financial structure that maximizes market value.

LEMMA **3** (EXISTENCE). *For each entrepreneur n, a financial structure $F^* \in \mathcal{F}$ exists such that $E_G[V_n(F^*)] \geq E_G[V_n(F)]$ for all $F \in \mathcal{F}$.*

4.2. Completeness of a value-maximizing financial structure

Proposition 1 asserts that the value maximizing financial structure depends on the shape of the *marginal* utility function, u'. Specifically, any incomplete financial structure is superior or inferior to any complete financial structure, depending on whether the marginal utility is convex or concave, respectively, on the relevant part of the domain.

More formally, let $\mathcal{X} \subseteq \mathbb{R}_+$ be a convex set that contains all the possible values of equilibrium allocations of numèraire, considering all investor types, states of the world, financial structures, and endowment profiles.[16] We say that regarding entrepreneur n's market value, structure F strictly dominates an alternative structure F', if $V_n(F) > V_n(F')$; that F is not dominated by F' if this inequality is weak; and that F and F' are equivalent if $V_n(F) = V_n(F')$.

PROPOSITION **1** (VALUE-MAXIMIZING FINANCIAL STRUCTURE). *Fix any two financial structures F and F', and suppose that F is incomplete and F' is complete. Regarding entrepreneur n's market value,*

(i) *if $u''' > 0$ on $\mathcal{X}$, F strictly dominates F', G-a.s. (and F is not dominated by F', surely);*

(ii) *if $u''' < 0$ on $\mathcal{X}$, F' strictly dominates F G-a.s. (and F' is not dominated by F, surely);*

(iii) *if $u''' = 0$ on $\mathcal{X}$, structures F and F' are equivalent.*

An important implication of Proposition 1 is that, even though investors may differ in their risk-sharing needs, to increase the firm's market value or to raise capital, it is strictly suboptimal for entrepreneurs to introduce asset-backed securities that fully hedge the risks of different investor clienteles, when the investors' marginal utility function is convex. Note that the implication of this proposition holds for almost all realizations of investor endowment profiles in the support of G and not merely in expectation. Furthermore, in an economy with only two states of the world—since effectively there are only two choices of financial structures, complete

Grasmanian to establish the existence of a value maximizing financial structure.

[16] That is, let $\mathcal{X}_k^s(e)$ be the projection, over the consumption set for investors of type k in state s, of the image of function $x(L)$ when the endowment profile is e. Let $\mathcal{X}_k^s$ be the union of all the sets $\mathcal{X}_k^s(e)$ over profiles e in the support of G and let $\mathcal{X} = \cup_{k,s}\mathcal{X}_k^s$.

and equity (incomplete)—Proposition 1 fully characterizes the financial structure that maximizes market value, which we highlight as the following corollary.

Corollary **1** (Two-State Economy). *Suppose that $S = 2$. If $u''' > 0$ ($u''' < 0$) on $\mathcal{X}$, then, G-a.s., a financial structure maximizes an entrepreneur's market value if, and only if, it consists of equity only (respectively, is complete).*

Recall that in the example presented by AG, in which $S = 2$, a complete financial structure maximizes market value. In our model, with identical utility functions across investors, this prediction holds only if the marginal utility function is concave, whereas for the utility functions common in macroeconomics and finance, such as CARA or CRRA, an incomplete financial structure brings higher market value. Next, we provide an example of a two-state economy in which we highlight the key economic intuition behind Proposition 1 and Corollary 1. Given that these results hold almost surely, the example considers deterministic investor endowments for the transparency of the arguments.

Example **1.** *Suppose that $S = 2$, there is one entrepreneur with the riskless asset $z = (1, 1)$, and there are two types of investors of equal mass normalized to 1, whose Bernoulli utility function is $u(x) = 2\ln(x)$. In the second period, the endowments of the investors are $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and the states are equally likely.*

With two states, the entrepreneur is choosing between a complete financial structure and equity. With a complete financial structure, all risk is shared at the equilibrium allocation, $c_1 = c_2 = (1, 1)$; the marginal utility of each investor in each state, given by $1/c_{k,s}$, is the same for both investors and equal to 1; and the market value of the entrepreneur's asset is 2.

If instead only equity is offered, each investor obtains half of the claims to z, which results in equilibrium allocation of $c_1 = (\frac{3}{2}, \frac{1}{2})$ and $c_2 = (\frac{1}{2}, \frac{3}{2})$. The average marginal utility in each state is $1\frac{1}{3}$ and the market value equals $2\frac{2}{3}$.

Hence, in this economy, an incomplete financial structure dominates a complete one in terms of market value. It is clear that, when marginal utility is linear, both complete and incomplete financial structures yield the same market value, while when marginal utility is strictly concave, the complete financial structure maximizes market value.

To understand the economic mechanism behind the example,[17] note first that with a complete

[17] For linear marginal utility (e.g., CAPM), it is well-known that when investors' endowments and riskless asset are in the asset span, markets are effectively complete even if the asset span is not full. Intuitively, in equilibrium, mean-variance traders sell their endowments and purchase the market portfolio and the riskless asset (two-fund separation). Consequently, a larger asset span is irrelevant to attain the first-best outcome. By contrast, we show that for identical quasilinear preferences (but not otherwise), prices of the assets in the span are the same for all financial structures, even if investor endowments are not within the asset span; hence, equilibrium allocation is inefficient (and the two-fund separation does not hold).

financial structure, each investor purchases consumption only in the state for which his initial endowment is 0, and the equilibrium marginal utilities of investors coincide in each state. When only equity is available, for an investor to obtain consumption in the desired state, he must also purchase the security that pays (the same quantity of) numèraire in the other state. Thus, by introducing a wedge in consumption, an incomplete financial structure creates a wedge in marginal utility between the two investors in each state. With convex marginal utility, the wedge increases the willingness to pay of the investor type with lower equilibrium consumption more than it reduces the willingness to pay of the investor who consumes more. Therefore, in each state, an incomplete financial structure induces a higher average marginal utility in equilibrium compared to complete markets. Because the average willingness to pay for consumption in each state remains high after trade, the equilibrium value of equity remains high as well.

Note that in Example 1, the market value of the asset is higher only if the equilibrium allocation of numèraire is inefficient, in the sense that it fails to display full risk-sharing. In general, even with inefficient endowments (which occur G-a.s., given that G is absolutely continuous with respect to the Lebesgue measure), the final allocation under an incomplete financial structure may still be efficient. For any fixed incomplete financial structure, however, the set of endowment realizations that give efficient equilibrium allocations has zero measure; therefore, the equilibrium allocation is inefficient G-a.s.

Since the inequalities in Proposition 1 are strict with G-probability 1, it follows that the result is robust to sufficiently small asymmetries in investor utility functions.[18] However, Proposition 1 does not generalize to arbitrary heterogeneity in utility functions across investors. Indeed, in the AG example, investor marginal utilities are strictly convex, yet a complete financial structure maximizes the market value of an asset. Considered together, Proposition 1 and the AG example suggest that in markets with heterogeneous investor utilities, for convex or concave marginal utility, no general normative predictions based solely on investor preferences can be obtained; the optimality of complete or incomplete financial structures then depends on the details of the economic environment, such as endowment or asset return distributions.

The hypotheses regarding the shape of the marginal utility function in the three claims of Proposition 1 are to hold over some convex subset of the respective domains that is large enough to include all the relevant equilibrium allocations of numèraire. We introduce this qualification because otherwise the class of preferences under consideration, for which the third derivative would have to be strictly negative on the whole domain, may be vacuous.[19] If distribution G has

[18] This is clearly the case for a given incomplete financial structure and, by the compactness argument used in the proof of Lemma 3, extends to all incomplete financial structures.

[19] While a global assumption would not be problematic for claim (i), given the Inada assumption about utility, a strictly concave utility function does not exist wherein marginal utilities are always strictly positive and concave

a bounded support, we can always find a bounded set of outcomes $\mathcal{X}$ to qualify the assumptions on the shape of marginal utilities.

4.3. *Monotonicity of market value*

In general, with more than two states of nature, Proposition 1 asserts that a complete financial structure is almost surely dominated by any incomplete financial structure when the marginal utility is convex. The next example shows that (even under our assumptions of identical, quasi-linear utilities) market value need not decrease "monotonically" with the hedging possibilities financial structures offer to investors. That is: it is *not* true that given a pair of structures F and F', the fact that $\langle F\rangle \subseteq \langle F'\rangle$ implies that $V_n(F) \geq V_n(F')$.

EXAMPLE **2.** *Suppose that* $S = 3$, *there is one entrepreneur with the riskless asset* $z = (1, 1, 1)$, *and there are two types of investors of equal mass normalized to* 1 *whose Bernoulli index is the following* $\mathbb{C}^2$ *function:*

$$u(x) = 3 \times \begin{cases} 2x - \frac{1}{2}x^2 - \frac{3}{2}, & \textit{if } x \leq 1; \\ \ln(x), & \textit{otherwise.} \end{cases} \tag{4}$$

The investor endowments are $e_1 = \big(\frac{1}{2}, 0, 1\big)$ *and* $e_2 = \big(0, \frac{1}{2}, 1\big)$, *and the states are equally likely.*

When only equity is offered, $F = \{(1, 1, 1)\}$, *by symmetry, the equilibrium allocation is given by* $c_1 = (1, \frac{1}{2}, \frac{3}{2})$ *and* $c_2 = (\frac{1}{2}, 1, \frac{3}{2})$, *state prices are* $\kappa(\langle F\rangle) = (\frac{5}{4}, \frac{5}{4}, \frac{2}{3})$, *and the market value of the entrepreneur's asset is* $3\frac{1}{6}$.

Now, consider the following (not necessarily optimal) financial structure with a state-1 contingent claim and a security that pays 1 *in states 2 and 3:*

$$F' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Observe that $\langle F\rangle \subset \langle F'\rangle$. *Because security* $(0, 1, 1)$ *pays in the second state, it is relatively more attractive to type-1 investors and in equilibrium the allocation of securities is* $t_1 \simeq (\frac{1}{4}, \frac{2}{3})$ *and* $t_2 \simeq (\frac{3}{4}, \frac{1}{3})$. *The implied allocation of numèraire is* $c_1 \simeq (\frac{3}{4}, \frac{2}{3}, \frac{5}{3})$ *and* $c_2 \simeq (\frac{3}{4}, \frac{5}{6}, \frac{4}{3})$, *the state prices are* $\kappa(\langle F'\rangle) \simeq (\frac{5}{4}, \frac{5}{4}, \frac{27}{40})$, *and the market value is* $\simeq 3\frac{7}{40} > 3\frac{1}{6}$.

Thus, financial structure F' *strictly dominates* F *in terms of the entrepreneur's market value. Utility function* (4) *can be perturbed so that marginal utility is strictly convex on the entire domain and* F' *still yields a strictly higher market value than* F.

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or linear.

In Example 2, financial structure F introduces a wedge in numèraire consumption in the first two states, whereas the allocation is efficient with respect to the third state. In the first two states, given that consumption takes place in the domain of quadratic utility, distortion brings no increase in market value relative to complete markets; the average marginal utility remains intact. In contrast, while the two-security financial structure F' improves the efficiency of the first two states' allocations, it introduces a wedge in the third state's allocation. Because consumption in this state is in the domain of a logarithmic function with strictly convex marginal utility, the wedge in the third state increases the state price for that state and the firm's market value.

One insight from Example 2 is that in settings in which an incomplete portfolio of asset-backed securities maximizes the issuer's revenue, entrepreneurs may have incentives to offer more sophisticated financial structures than equity and an intermediate degree of incompleteness is optimal. Consequently, our predictions are consistent with both pooling and tranching.

The lack of monotonicity of the market value of an asset in a security span in general extends to strictly concave marginal utility environments.[20] In the important instance of CARA utility and a riskless asset, market value is indeed monotone in the span of the financial structure, and the optimal financial structure involves selling a riskless security (e.g., a bond) only.

Example **3.** *Consider the case of a single entrepreneur with the riskless asset $z = (\lambda, \ldots, \lambda)$, for some $\lambda > 0$, and suppose that all investors have CARA Bernoulli utility $u(x) = -e^{-\alpha x}$, while distribution G is arbitrary. In this case, $u'(x) = -\alpha u(x)$, which implies that*

$$\bar{U}(x(\langle F \rangle)) = -\frac{1}{\alpha} \sum_s \sum_k \theta_k \Pr(s) u'(x_{k,s}(\langle F \rangle)) = -\frac{\theta}{\alpha} \kappa(\langle F \rangle) \cdot \mathbf{1} = -\frac{\theta}{\alpha\lambda} V(F).$$

By Lemma 1, function $\bar{U}(x(\langle F \rangle))$ is increasing in $\langle F \rangle$, thus it follows that the market value of z_1 is monotonically decreasing in the security span. In particular, opening a market for the riskless asset maximizes its market value.

4.4. *A geometric interpretation*

The set of all feasible allocations of numèraire in Example 1 are represented by the Edgeworth box in Figure 2. With a complete financial structure (F), feasible set $X(\langle F \rangle)$ comprises all allocations in the box. If only equity is issued (F'), set $X(\langle F' \rangle)$ is represented by the line segment that connects the endowment points. A planner's welfare function $\bar{U}(x)$ attains its unconstrained maximum at the efficient allocation (where investors consume the same quantities)

[20] As the analysis from Section 5.3 implies, the non-monotonicity of the market value function does not stem from non-monotonicity of the welfare function $\bar{U}$ in asset span.

and decreases for allocations further from the center (Figure 2.A). Thus, if financial markets are complete, the equilibrium allocation is the unconstrained maximum of $\bar{U}$, whereas with only equity, the allocation coincides with its constrained maximum on $X(\langle F' \rangle)$.

Figure 2.B depicts entrepreneur 1's preference map, with each level curve comprising all allocations that give rise to a given firm value $V = [\sum_k \theta_k DU(x_k)] \cdot z_1$. Due to the symmetry of the investor marginal utility, the critical point of the market value function, V, is at the efficient allocation as well. Whether the efficient allocation yields a minimum or a maximum market value, however, depends on whether the marginal utility and hence, the market value function, is convex (as in Example 1 with a logarithmic utility) or concave.[21] In the case of a quadratic Bernoulli utility function, all allocations in the box are equivalent in terms of market value, and entrepreneurs are indifferent to the planner's allocation choice.

In general, the planner preference and market value maps need not overlap, which in economies with more than two states may result in the non-monotonicity of market value in the security span. In Example 2, by offering two securities (F') rather than equity (F), the entrepreneur enlarges the feasible set in the direction for which the welfare function $\bar{U}$ increases, and the planner's new optimum also gives rise to higher market value. For CARA utility with a riskless asset, the two maps coincide (see Example 3). Since the constant of proportionality $(-1/\alpha\lambda)$ is negative, a smaller security span, and hence a smaller choice set in Problem (2), weakly increases market value. On the other hand, with the exception of CARA utility (and its affine transformations), it is apparent that one can specify endowments and an asset return such that increasing the span increases market value.

With heterogeneous investor utilities, predictions regarding the optimality of an incomplete financial structure depend on the environment's details. Considering convex marginal utility, then, the efficient allocation and that which minimizes market value do not necessarily coincide. Indeed, this is the case in the AG example as depicted in Figure 3. With equity only, the equilibrium allocation is the point on the line segment that maximizes $\bar{U}$; with a complete financial structure, it is the unconstrained maximum that yields a higher market value. Thus, with convex investor marginal utility, separation of the efficient and value-minimizing allocations derived from the asymmetry of investor utilities is necessary (but not sufficient) for market

[21] In a two-investor economy, the convexity of market value function in allocation in the Edgeworth box is defined as the convexity of the function

$$\frac{1}{2}(DU(x_1) + DU(e_1 + e_2 + z_1 - x_1)) \cdot z_1$$

in x_1. More generally, convexity is defined with respect to consumption of the first $K-1$ investors and consumption of investor K is the residual of the total resources $\sum_k \theta_k e_k + z_1$. If marginal utilities are convex, then market value is convex in this sense as well.

completion to be profitable for the entrepreneurs. Similarly, one can construct an example with asymmetric strictly concave marginal utilities in which the market value is maximized by an incomplete financial structure.

5. Competition in Security Innovation

The central question of this paper concerns whether competition among asset holders provides sufficient incentives to complete the market. Having determined the comparative statics of market value in financial structures, we now turn to examining the strategic interactions among entrepreneurs when choosing which securities to issue. By a standard argument (e.g., Kreps (1979)), entrepreneurs can affect prices, even in large markets, to the extent that they can impact the span of F. We consider the situation in which all entrepreneurs simultaneously choose structures of issued securities, recalling that there is a per-security innovation cost $\gamma > 0$, so that $\gamma|F_n|$ is the issuance cost of F_n.[22]

5.1. An example

To illustrate how competition among entrepreneurs affects incentives to innovate, we first examine economies with two states, two entrepreneurs, and the riskless assets $z_n = (1, 1)$. By Lemma 1, it suffices to consider that each entrepreneur chooses between equity (E) or two state-contingent claims (C). Markets are incomplete when both entrepreneurs choose equity and are complete for all other strategy pairs.

Under concave marginal utility, competition in asset innovation takes the form of a provision of a public good, as in the heterogeneous-utility example of AG. A complete financial structure maximizes market value of both entrepreneurs, and they both benefit if one innovates. Assuming for simplicity that the market values of the entrepreneurs' assets are 0 when markets are incomplete and 1 when they are complete (by Proposition 1, $V_I < V_C$), it is useful to summarize the entrepreneurs' reduced form net payoffs in Table 1. Let $\gamma < 1$.

One of the insights from the AG example that also holds in our example with concave marginal utility is that the equilibrium financial structure can be (endogenously) incomplete with positive probability, *even if complete markets maximize each entrepreneur's market value.* Considering Table 1, in the mixed strategy Nash equilibrium, all four outcomes, including incomplete markets (E, E), occur with positive probability. The probability of market incompleteness depends

[22] In the absence of this cost, trivial Nash equilibria arise in which each entrepreneur chooses a complete financial structure.

Table 1: Normalized net market values under concave marginal utilities

	E	C
E	$-\gamma, -\gamma$	$1-\gamma, 1-2\gamma$
C	$1-2\gamma, 1-\gamma$	$1-2\gamma, 1-2\gamma$

positively on the innovation cost, and vanishes as costs become negligible. Market incompleteness can be attributed to the entrepreneurs' inability to coordinate on one of the two favorable outcomes ((C, E) or (E, C)) when independently randomizing over two financial structures.[23] The entrepreneurs have *ex post* regret when incomplete markets are realized, each preferring to complete the market, knowing that the other did not.

Importantly, apart from the mixed strategy Nash equilibrium, there are two more equilibria in pure strategies in which one of the two entrepreneurs innovates and markets are complete. Clearly, in a pure strategy equilibrium, the miscoordination that may lead to market incompleteness does not arise, as each entrepreneur best responds to the given financial structure chosen by his opponent.

Next, consider an economy with convex marginal utility. The net values of the entrepreneurs' assets are as presented in Table 2, where we now assume that the market values are 1 when markets are incomplete and 0 when they are complete (by Proposition 1, $V_I > V_C$). With convex marginal utility, innovation is a public "bad"; issuing equity is a strictly dominant strategy, and in the unique Nash equilibrium markets are incomplete.

Table 2: Normalized net market values under convex marginal utilities

	E	C
E	$1-\gamma, 1-\gamma$	$-\gamma, -2\gamma$
C	$-2\gamma, -\gamma$	$-2\gamma, -2\gamma$

These examples demonstrate that predictions regarding the incompleteness of endogenous market structure depend on primitive investor preferences, which qualitatively changes the nature of competition among entrepreneurs and their incentives to innovate.

[23] The inability to coordinate does not stem from randomization over financial structures *per se*, but from the independence of entrepreneurs' strategies (i.e., the independence of mixed strategy distributions). With public (i.e., perfectly correlated) signals, correlated equilibria exist in which one of the events (C, E) or (E, C) is realized, and markets are complete with probability 1.

5.2. Endogenous market completeness

Theorem 1 offers general predictions regarding market (in)completeness. To the extent that miscoordination in financial innovation exhibited by a mixed strategy equilibrium is not a problem, our model provides strong predictions based solely on investor preferences: When the investors' marginal utility function is strictly concave, if the innovation costs are not prohibitively high, then markets are complete in all pure strategy Nash equilibria.[24] With convex investors' marginal utility function, the financial structure is incomplete in all pure strategy Nash equilibria, unless the only feasible structures are complete.

Theorem **1** (Endogenous Market Completeness)**.** *The following statements characterize the equilibrium financial structure.*

1. *If $u''' < 0$ on $\mathcal{X}$, then $\bar{\gamma} > 0$ exists such that, for any $0 < \gamma \leq \bar{\gamma}$, in any pure strategy Nash equilibrium, the resulting financial structure is complete.*
2. *If $u''' \geq 0$ on $\mathcal{X}$ and $\langle\{z_1, \ldots, z_n\}\rangle \neq \mathbb{R}^S$, then, for any $\gamma > 0$, in any pure strategy Nash equilibrium, the resulting financial structure is incomplete.*

The predictions regarding endogenous market (in)completeness are quite robust. They hold (with probability 1) in markets for an arbitrary number of entrepreneurs (that is, regardless of the intensity of competition); any number of states; arbitrary payoff structures of their assets (with common or idiosyncratic risk); and any (absolutely continuous) joint distribution of investor endowments.

Building on the analysis of the monotonicity of entrepreneur n's market value in the (joint) span of F (Section 4.3), and hence in the span of F_n given the financial structures of entrepreneurs $n' \neq n$, the next result provides sufficient conditions under which, in the unique (dominant strategy) equilibrium, no innovation occurs and the resulting financial structure has minimal span. This occurs if there are two states of the world and the investors' marginal utility function is strictly convex, or, for an arbitrary number of states, if the investors' Bernoulli utility is CARA and all entrepreneurs are endowed with riskless assets.

Proposition **2** (Equilibrium in Dominant Strategies)**.** *If any of the following two conditions holds, there is a unique Nash equilibrium, and the resulting financial structure is $F = \{z_1, \ldots, z_n\}$:*

[24] The set of strategies (i.e., the set of linear subspaces) does not have a structure of a vector space, and the existence of a pure strategy Nash equilibrium cannot be established with the standard Brouwer/Kakutani approach. However, it can be shown that equilibrium exists if there are two states only or when all the assets $\{z_1, \ldots, z_n\}$ are "sufficiently close" to collinear. Moreover, when issuance is sequential–a setting to which all our results extend– the subgame-perfect Nash equilibrium exists under general conditions.

1. $S = 2$ *and* $u''' \geq 0$ *on* $\mathcal{X}$*; or*

2. *function* u *is CARA and, for all* k*,* $z_{k,s} = z_{k,s'}$*, for all* s *and* s'*.*

Outside of CARA settings, in a model with convex marginal utility and $S > 2$, issuing equity need not be a dominant strategy, and multiple Nash equilibria may exist. By Theorem 1, markets are then incomplete in all pure strategy equilibria.

Furthermore, when the investors' marginal utility function is convex, in a mixed strategy equilibrium, markets may be *complete* with positive probability even though market value is maximized by an incomplete financial structure and even if innovation is costly. Similar to markets with concave marginal utility (See Section 5.1) or in the economy with heterogeneous utilities studied by AG, equilibrium financial structure then involves a set of securities that are individually suboptimal for each entrepreneur; that is, each has *ex post* regret given the financial structures chosen by the others. The *ex post* regret occurs when entrepreneurs cannot coordinate their activities—unlike pure strategy simultaneous competition. Thus, the miscoordination mechanism identified by AG operates more broadly, even if a complete set of securities is suboptimal: An undesirable outcome (from the entrepreneurs' perspective) occurs due to their inability to coordinate on an optimal financial structure, whether complete or incomplete.

Example **4.** *Suppose that* $S = 3$*, there are two entrepreneurs,* $n = 1, 2$*, both endowed with the riskless asset* $z = (1, 1, 1)$*, and two types of investors whose utility function and endowments are the same as presented in Example 2. Let the mass of each investor type be* 1*, and let the states be equally likely. In Example 2 and Proposition 1, we demonstrate the existence of a financial structure whose span has dimension 2, which strictly dominates equity and (any) complete financial structure. Therefore, the span of a financial structure* F^* *that maximizes the market value of the entrepreneurs' assets, which exists by Lemma 3, has dimension 2. Let* V^* *denote this maximized market value. By continuity of market value on the set of two-dimensional spans, one can find a two-dimensional linear subspace* $L^{**} \neq \langle F^* \rangle$*, with a corresponding two-asset financial structure* F^{**}*, which yields market value* V^{**} *that is arbitrarily close to* V^* *(*$V^{**} \simeq V^*$*). By construction, financial structure* $F = \{F^*, F^{**}\}$ *is complete.*

There is a mixed strategy Nash equilibrium in which entrepreneurs randomize over F^**,* F^{**} *and equity. The equilibrium probabilities of choosing* F^* *and* F^{**} *are*

$$\sigma^* \simeq \sigma^{**} \simeq \frac{1}{3}\left(1 - \frac{\gamma}{V^* - V_C}\right), \tag{5}$$

where V_C *is the market value in a complete market.*[25] *Since* $V^* > V_C$*, for a sufficiently small*

[25] Suppose that entrepreneur n' follows the mixed strategy $(\sigma^*, \sigma^{**}, 1 - \sigma^* - \sigma^{**})$ over structures F^*, F^{**} and

innovation cost γ, probabilities σ^* and σ^{**} are strictly positive. In equilibrium, markets are complete with probability $2\sigma^*\sigma^{**} > 0$.

For the intuition, the market value in the example is not monotonically decreasing in the security span. Each entrepreneur is willing to pay innovation costs in order to partially complete the market—either of the two incomplete financial structures, F^* or F^{**}, gives strictly higher market value than equity. In the described equilibrium, entrepreneurs fail to coordinate on one of F^* and F^{**}, which may result in an undesirable equilibrium outcome of complete financial structure F and $V_C < V^*$.

It is worth noting that except for predictions concerning miscoordination—with concave marginal utility or in the example presented by AG—the innovation costs are not essential for predictions in the following sense.[26] When innovation costs vanish ($\gamma \to 0$), the probability of market incompleteness tends to 0 in mixed strategy equilibria for markets with concave marginal utility and in the example presented by AG. In contrast, in the limit of any pure strategy equilibria of our model, markets with concave (convex) marginal utility remain complete (incomplete).

5.3. *Competition in innovation and welfare*

The ability to alter the security span and hence the allocation of future consumption among investors allows entrepreneurs to affect prices even in markets with large numbers of entrepreneurs. A question naturally arises regarding how the power of entrepreneurs to create markets impacts welfare.

Our model has the following implications for the welfare appraisal of asset innovation. Assuming negligible innovation costs, $\gamma \simeq 0$, to achieve *ex ante* (and, generically, *ex post*) efficiency of market outcomes, a policy must induce a full-span portfolio of securities. As suggested by Lemma 1, this recommendation can be strengthened: Introducing an additional security is never

$\{(1,1,1)\}$. The expected profits of entrepreneur n, under F^*, F^{**} and $\{(1,1,1)\}$ are, respectively, $(1-\sigma^{**})V^* + \sigma^{**}V_C - 2\gamma$, $(1-\sigma^*)V^{**} + \sigma^* V_C - 2\gamma$ and $(1-\sigma^*-\sigma^{**})V_C + \sigma^*V^* + \sigma^{**}V^{**} - \gamma$, where we used that under $\{(1,1,1)\}$, market value coincides with V_C (in Example 2, under equity, there is no distortion in the third state consumption and market value is V_C). Equating the three net expected payoffs and taking the limit as $V^{**} \to V^*$ gives $\sigma^* = \sigma^{**}$ as in (5). In the example, with a sufficiently small innovation cost γ, when entrepreneur $n' \neq n$ issues equity $\{(1,1,1)\}$, it is optimal for entrepreneur n to choose $F_n = F^*$, in which case the market value equals V^*; it is marginally less profitable to chose F^{**} and obtain V^{**}. If entrepreneur n' chooses $F_{n'} = F^*$ or $F_{n'} = F^{**}$, however, then, given costly innovation, issuing equity alone maximizes the entrepreneur's profit.

[26] Innovation costs eliminate the (trivial) multiplicity of Nash equilibria, which would be present in the model with costless innovation in which entrepreneurs simultaneously choose financial structures. If one entrepreneur chooses a complete financial structure, it is a weak best response for all other entrepreneurs to issue complete financial structures as well, regardless of market primitives (by changing F_n, an entrepreneur has no impact on financial structure F).

detrimental to welfare, even if asset innovation does not fully complete the financial structure. Given quasi-linearity of utilities in present consumption, for both investors and entrepreneurs, utility is transferable and monetary transfers in period one are irrelevant for the overall welfare in two periods. For any pair of structures F and F' such that $\langle F\rangle \subseteq \langle F'\rangle$, by Lemma 1, the change in deadweight loss is equal to

$$\max_x\{\bar{U}(x) \mid x \in X(\langle F'\rangle)\} - \max_x\{\bar{U}(x) \mid x \in X(\langle F\rangle)\}.$$

Because $X(\langle F\rangle) \subseteq X(\langle F'\rangle)$, it follows that the deadweight loss is (weakly) decreasing in the span of a financial structure.

By our results, the equilibrium financial structure F necessarily distorts allocation in markets in which investor marginal utility is convex: maximizing the market value of an asset by all entrepreneurs requires market incompleteness, which (G-a.s.) introduces a wedge in investors' marginal utilities in equilibrium. Indeed, the very mechanism through which market incompleteness provides an effective means to increase entrepreneurs' market values involves introducing inefficiency in the allocation of numèraire among investors and the incompleteness of the equilibrium set of securities is always in conflict with the socially optimal innovation. This holds for an arbitrary number of states and investors' endowments. As we demonstrate in Section 6, it also holds for general preferences of entrepreneurs and investors.[27]

As a more general insight from our analysis, unlike competition in quantities such as the Cournot or Stackelberg models, the market power exercised by choosing asset innovation (spans) and the market failure of competition among entrepreneurs do not depend on the number of innovators or timing of strategies. Rather, convexity of investors' marginal utility is the key determinant of the completeness, and hence allocative efficiency, of financial markets.

6. Model Generalizations

The model analyzed so far is quite stylized. To highlight the paper's main insights to the economics of financial innovation, in this section we discuss the relevance of some of the assumptions for our predictions. In particular, we examine assumptions regarding the entrepreneurs' preferences and available alternatives for securitization and the investors' preferences. Dynamic aspects of innovation are also considered.

[27] With linear marginal utilities, market value is invariant to financial structure, but among all such financial structures, only those with a full span yield an efficient allocation.

6.1. *Entrepreneurs' choice sets and preferences*

Our assumptions on issuers' preferences and their available financial structures abstract from important aspects of financial innovation. For example, issuing institutions choose the asset portfolios to be securitized, which determines z_n. Moreover, it may not be revenue maximizing to sell the entire return z_n, because less than full monetization may yield higher state prices and, hence, the value of an asset. Furthermore, entrepreneurs or investment banks issuing securities are often concerned not only about the expectation, but also the riskiness of revenue from selling securities. Issuers derive utility from present *and future* returns. On the other hand, not all choices of financial structures may be available to issuers. For instance, entrepreneurs may be restricted by limited liability, or it might be cost-efficient to use only standardized securities. We next demonstrate that our main result (Theorem 1) encompasses these aspects.

6.1.1. Exogenous intermediation

Along with entrepreneurs, financial markets may include other types of agents with different objectives to innovate. For example, the incentives to set up an exchange for a new stock option or futures contracts comes from the trade commissions or bid-ask spread. To allow such objectives in our model, we introduce a "noise" innovator with portfolio of securities F_0. For simplicity, we assume that the bid-ask spread is negligible. Hence, structure F_0 represents securities that are exogenous to our model.

6.1.2. Entrepreneurs' choice sets

Suppose that each entrepreneur chooses the asset to be sold and the financial structure with which he will sell that asset. For each asset $z \in \mathcal{Z}_n$ where $\mathcal{Z}_n \subset \mathbb{R}^S_{++}$ is compact, let $\mathcal{F}_n(z) \neq \varnothing$ be the compact set of financial structures feasible for entrepreneur n, should he choose to sell that asset. To allow for a large class of environments, we impose little structure on the (exogenously given) correspondence $\mathcal{F}_n$: we only require that for all $F_n \in \mathcal{F}_n(z)$, the following conditions hold: (i) $0 \ll F_n\mathbf{1} \leq z$; (ii) there exists a complete $F'_n \in \mathcal{F}_n(z)$ such that $F_n\mathbf{1} = F'_n\mathbf{1}$; and (iii) $\{F_n\mathbf{1}\} \in \mathcal{F}_n(z)$.

Assumption (i) ensures that the promised payment associated with each feasible financial structure is strictly positive and does not exceed the return from the asset, so that the entrepreneur is solvent in all future states.[28] Given the possibly restricted choice of securities,

[28] In previous sections, we assumed $F_n\mathbf{1} = z_n \gg 0$; hence, strictly positive returns in each state to the portfolio sold. To allow for less than full monetization, we now relax this assumption by allowing $F_n\mathbf{1} \leq z_n$. The assumption of strictly positive payoffs in all states is technical. It makes all states "relevant" in the sense that entrepreneurs have incentives to increase state price in a given state. Our result on market incompleteness (part 2 in Corollary

assumptions (ii) and (iii) make the entrepreneur's choice of financial structure non-trivial. Any payoff that can be sold by issuing some financial structure can also be sold by issuing a complete financial structure or a single security.

In this setting, we assume that entrepreneur n chooses a pair (z_n, F_n) subject to the constraint that $z_n \in \mathcal{Z}_n$ and $F_n \in \mathcal{F}_n(z_n)$. Stated this way, the model accommodates important financial environments beyond those analyzed in previous sections, such as markets in which entrepreneurs do not fully monetize the return; markets in which entrepreneurs with limited liability can only issue securities with non-negative payoffs; markets in which the entrepreneurs can securitize by issuing only options (assuming that the asset yields different payoffs in different states). In the most restrictive set of alternatives, an entrepreneur's choice set comprises two securities for any given return: equity and the corresponding complete financial structure.

6.1.3. Entrepreneurs' preferences

Let entrepreneur n's cost of obtaining return $z_n \in \mathcal{Z}_n$, be given by $C_n(z_n)$. This is interpreted as the cost of inputs required to generate the future return z_n or, if an "entrepreneur" is an institutional investor, the cost of buying the portfolio to be securitized, which can be heterogenous across entrepreneurs.

Now, given F_0 and the profile of choices of $\{(z_1, F_1) \ldots, (z_N, F_N)\}$, entrepreneur n's revenue in the first period is the random variable $r_{n,0} = V_n(F) - C_n(z_n) - \gamma|F_n|$, where $V_n(F)$ is the market value of portfolio F_n, given that the market financial structure is $F = \{F_0, F_1, \ldots, F_n\}$. Also, future consumption is the net asset return $r_{n,1} = z_n - F_n\mathbf{1}$.

Unlike the previous analysis, we now assume that each entrepreneur derives utility from present and future consumption. That is, given $R = (r_0, r_1)$, where r_0 is a random variable and $r_1 \in \mathbb{R}^S_+$, entrepreneur n's utility is $U_n(R)$. Function U_n is assumed to be continuous and strictly increasing in r_0, in the sense that for any r_1, if r_0 first-order stochastically dominates r_0', then $U_n(r_0, r_1) > U_n(r_0', r_1)$.

By the argument analogous to that in Section 3.3, market values are well defined for all profiles $\{(z_1, F_1), \ldots, (z_N, F_N)\}$, and, therefore, so are the entrepreneurs' preferences.[29]

2) straighforwardly extends to settings in which financial structures satisfy only weak inequality, $F_n\mathbf{1} \geq \mathbf{0}$. For the complete market result (part 1 of Corollary 2), however, the completeness of the financial structure must be defined with respect to "relevant" states; that is, states for which payoffs of all traders are strictly positive.

[29] Note that we assume that the entrepreneurs do not participate in trading the assets in the sense that they do not buy (or sell) the assets issued by the other traders.

6.1.4. Equilibrium

In the first period, all entrepreneurs simultaneously choose the pairs (z_n, F_n) to maximize their utilities over consumption in the two periods. Our next result asserts that the predictions about endogenous market incompleteness from Section 5.2 carry over to this setting.

Corollary **2** (Robustness: Entrepreneurs). *The following statements characterize the equilibrium financial structure:*

1. *If $u''' < 0$ on $\mathcal{X}$, then $\bar{\gamma} > 0$ exists such that, for any $0 < \gamma \leq \bar{\gamma}$, in any pure strategy Nash equilibrium, the resulting financial structure is complete.*
2. *Suppose that $u''' \geq 0$ on $\mathcal{X}$. For any $\gamma > 0$, if $\{(z_1^*, F_1^*), \ldots, (z_N^*, F_N^*)\}$ is a pure strategy Nash equilibrium and*
$$\langle F_0 \cup \{F_1^* \mathbf{1}, \ldots, F_N^* \mathbf{1}\} \rangle \neq \mathbb{R}^S, \tag{6}$$
then the financial structure $\{F_0^*, F_1^*, \ldots, F_N^*\}$ is incomplete.

Note that whenever there are more states than entrepreneurs (or in a symmetric equilibrium, in which $F_n \mathbf{1}$ is the same for all n) and $F_0 = \varnothing$, condition (6) is automatically satisfied. In such a case, an immediate implication of Corollary 2 is that under convex marginal utility, markets are incomplete in all pure strategy Nash equilibria.[30]

In general, Corollary 2 demonstrates that with convex marginal utility, entrepreneurs have incentives to innovate in a way that leaves investors away from the Pareto efficient allocation. Offering investors limited mutual insurance opportunities is optimal even if it requires that entrepreneurs retain a risky part of the firm. The predictions hold under mild assumptions on entrepreneurs' preferences over present and future consumption. The class of preferences includes those under risk, uncertainty, or ambiguity, such as the standard expected utility with arbitrary risk attitudes, non-expected utility models, or models with multiple priors, (assuming entrepreneurs' appropriate knowledge of beliefs about distributions of endowments). The profitability rankings of complete and incomplete financial structures established in Section 4.2 hold *ex post*. Essentially, the entrepreneurs' risk (or ambiguity) preferences affect which part of the risky portfolio they securitize, but not *how* they do it.

[30] Markets can be trivially complete if the cost structure gives a "premium" for diversified returns. Consider an example with two states and two entrepreneurs and suppose that each entrepreneur can generate payoff in one state for free, whereas the cost of payoff in the other state is prohibitively high. In equilibrium, each entrepreneur will produce one contingent claim; thus, the financial structure will be complete, even if investor marginal utility is convex.

6.2. *Investors' preferences*

The model introduced in Section 2 assumes that the investors' utility function is linear in the consumption of the first period. With quasi-linear utility functions, the investors' demands for assets are well-behaved and the market values of all assets are uniquely defined for each financial structure. Without quasi-linearity, income effects may lead to non-trivial multiplicity of competitive equilibria, so that for any given financial structure, a firm may have a different market value depending on which particular equilibrium is realized. In this case, without a selection criterion, the entrepreneurs' preferences over financial structures and, hence the game of competition among entrepreneurs, are not well-defined.

The assumption of quasi-linearity is still restrictive in terms of which financial environments it admits. In particular, it makes the marginal rates of substitution and the state prices independent from consumption in the first period. Abusing notation slightly, assume now that the utility derived by the investors is measured by $U(c_0, c_1)$, which need not be quasi-linear. Then, the vector of state prices is given by the average marginal rates of substitution between present and future consumption in different states: for each $s = 1, \dots, S$,

$$\kappa_s = \frac{1}{\theta} \sum_k \theta_k \frac{\partial U(c_k)/\partial c_{k,s}}{\partial U(c_k)/\partial c_{k,0}}.$$

With convex marginal utility, market incompleteness has an additional effect on state prices through the marginal utility of present consumption: Investors postpone consumption to hedge uninsurable risk (precautionary saving). The overall impact of market incompleteness on asset value is determined by the two countervailing effects.

Corollary 3 shows that in symmetric economy investors, the increase in future average marginal utility dominates the precautionary savings effect and in the unique Nash equilibrium (in dominant strategies), markets are incomplete. Say that an economy is a *two-state symmetric economy* if the following assumptions hold: (i) there are two equally likely states, $s = 1, 2$; (ii) there are two types of investors with equal mass and both have the same present endowment, whereas future endowments are symmetric with respect to the two states, in the sense that $e_1 = (a, b)$ and $e_2 = (b, a)$ for two absolutely continuous random variables $a, b > 0$; (iii) the investors' utility function is

$$U(c_0, c_1, c_2) = u(c_0) + \frac{\beta}{2}[u(c_1) + u(c_2)],$$

where the discount factor is $\beta > 0$, and the Bernoulli utility index u is $\mathbb{C}^2$, strictly concave, strictly increasing, and satisfies the Inada conditions $\lim_{x \to 0} u'(x) = \infty$ and $\lim_{x \to \infty} u'(x) = 0$; (iv) each entrepreneur must sell a riskless asset z_n; and (v) a unique competitive equilibrium exists for each financial structure. For simplicity, suppose there are no noise innovators, and that the entrepreneurs are concerned with only their present revenue.

Corollary **3** (Robustness: Investors). *For the case of two-state symmetric economies, the following statements characterize the equilibrium financial structure:*

1. *If $u''' < 0$ on $\mathcal{X}$, then $\bar{\gamma} > 0$ exists such that, for any $0 < \gamma \leq \bar{\gamma}$, in any pure strategy Nash equilibrium, the resulting financial structure is complete.*

2. *If $u''' \geq 0$ on $\mathcal{X}$, then, for any $\gamma > 0$, in the unique pure strategy Nash equilibrium, the resulting financial structure is incomplete.*

The corollary holds also if the entrepreneurs have preferences over present and future consumption and if financial structures are restricted to some correspondence $\mathcal{F}_n(z_n)$, as described in Section 6.1. In fact, the result can be extended to markets with S states and $K = S$ investors with symmetric future endowments and a minimal $\mathcal{F}_n(z_n)$ that consists of a complete financial structure and equity z_n. The following example explains Corollary 3 in an economy with Cobb-Douglass preferences.

Example **5.** *Suppose that $S = 2$ and that there is one entrepreneur with the riskless asset $z_1 = (1, 1)$. The two states are equally likely. There are two types of investors of equal mass, with utility function*

$$U(c_0, c_1, c_2) = u(c_0) + \frac{1}{2}[u(c_1) + u(c_2)],$$

where $u(x) = 2\ln(x)$. The investors' endowments are all equal to 3 in the first period, while in the future they are $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

With Cobb-Douglas preferences, competitive equilibrium is unique both for complete and incomplete financial structures. Moreover, by the construction of the economy, the equilibrium is symmetric. Equilibrium state prices are identical in the two states, $\kappa_1 = \kappa_2 = \kappa$, and present consumption is the same for both investors: $c_{1,0} = c_{2,0} = c_0$. These variables are jointly determined by two conditions: the budget constraint and the equalization of the (average) marginal rate of substitution with the state price in each state.

Under complete markets, the budget constraint of an investor is $c_0 + \kappa \times 1 = 3$. The second period allocation is Pareto efficient, the average marginal utility is 1, and the average marginal rate of substitution, becomes $\kappa(c_0) = \frac{1}{2}c_0$. Thus, the two conditions give $(c_0, \kappa) = (2, 1)$ (see Figure 4.A) and the market value of the riskless asset equals 2.

When markets are incomplete, the budget constraint is $c_0 + \kappa \times (\frac{1}{2} + \frac{1}{2}) = 3$. The wedge in future consumption across investors increases the average marginal utility, and the average *marginal rate of substitution* $\kappa(c_0) = \frac{2}{3}c_0$ *shifts upward for any* c_0*. The shift results in an endogenous adjustment of savings, and in equilibrium* $(c_0, \kappa) = (1\frac{4}{9}, 1\frac{1}{5})$*, and the asset value becomes* $2\frac{2}{5}$*. Thus distorting the second period consumption benefits the entrepreneur.*

Decreasing marginal utility in period zero introduces the following new effect. Under both complete and incomplete financial structures, the average marginal utility in each of the two future states is the same in Examples 1 and 5. Moreover, with complete markets, the marginal utility in period zero also coincides in the two examples. Yet, with Cobb-Douglas utility, the entrepreneur's benefit from distorting future consumption is smaller than in the quasi-linear case ($\frac{2}{5} < \frac{2}{3}$). With quasi-linear utility, the marginal rate of substitution is independent from present consumption and is affected only by the future average marginal utility. Thus, the endogenous adjustment of present consumption resulting from market incompleteness has no impact on state price in the quasi-linear environment. In the Cobb-Douglas example (or in any economy with decreasing marginal utility of the first period consumption), however, such an adjustment adversely affects state prices (see Figure 4).

Our result need not hold beyond symmetric economies. Specifically, one can find examples of markets with strictly convex marginal utility in which the overall effect of consumption distortion on market value is negative. With asymmetric endowments, investors operate on different subsets of a domain of a utility function, which, effectively results in heterogenous utilities over consumption profiles and, as in the case of heterogeneous quasilinear utility, the revenue rankings of financial structures need not hold.[31]

6.3. *Multiperiod economies*

The assumption of a two-period economy precludes important aspects of trades and innovation in financial markets: presence of long-lived securities, possibility of re-trade in spot markets and dynamic innovation plans. It is well known that in dynamic economies, by adopting (re)trading strategies of long-lived securities in spot markets, traders might be able to perfectly hedge the risky returns even if the number of long-lived assets is smaller than the number of states. Thus, then economies behave as if financial markets were complete (*effectively complete markets*). We demonstrate that our result holds in the strong sense; with convex marginal utility spot markets are effectively incomplete. Moreover, in many markets it is common in financing production activity, to borrow first and then sell part of the firm's equity to repay. We show that the result holds when entrepreneurs innovate dynamically by introducing various securities in different periods (or states), as long as entrepreneurs pre-commit to issuing securities before period zero:

[31] We need to assume riskless returns to preserve the symmetry of the investors for any financial structure.

Suppose that the economy evolves over a finite date-event tree $\mathcal{S}$, whose root we denote by $s = 0$. For any date-event $s \neq 0$, denote by $b(s)$ the date-event that comes immediately before; we refer to this date-event as the *immediate predecessor of* s. For any s, denote by $a(s)$ the set of date-events that come immediately after s, a set that we refer to as the *immediate successors of* s; and let $A(s)$ be the set of all date-events that may occur after s, which we call its *successors*. Date-event s is said to be *terminal* if $A(s) = \varnothing$.

Entrepreneur n is endowed with a future return $z_n : \mathcal{S} \setminus \{0\} \to \mathbb{R}_{++}$, whereas each investor of type k receives a future wealth given by $e_k : \mathcal{S} \setminus \{0\} \to \mathbb{R}_+$. As previously explained, we maintain that the profile $(e_1, \dots, e_K)$ is not known by the entrepreneurs, who hold common probabilistic beliefs G over it; this function is assumed to be absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^{(|\mathcal{S}|-1)K}$.

An *asset issued at date-event* s is a function $f : A(s) \to \mathbb{R}$. A *financial structure* is a collection of assets; for convenience, we write financial structures as $\mathbf{F} = \cup_s F_s$, where each F_s comprises the assets issued at s. We assume that trade occurs at all non-terminal date-events for all assets newly issued there, along with the re-trade of any assets previously issued. That is, denote by $\bar{F}_s$ the collection of all assets issued at either s or one of its predecessors.[32] An *investment plan* is a collection of functions $\mathbf{t} = \{t_s : \bar{F}_s \to \mathbb{R} \mid s \in \mathcal{S}\}$, so that $t_s(f)$ represents the holdings of asset f after trade at date-event s.[33] For any investment plan t, the resulting consumption plan $c : \mathcal{S} \to \mathbb{R}$ is as follows: at the root node, $c_k(0) = -\sum_{f \in F_0} p_0(f) t_{k,0}(f)$; at any non-terminal $s \neq 0$,

$$c_k(s) = e_k(s) + \sum_{f \in \bar{F}_{b(s)}} t_{k,b(s)}(f) f(s) - \sum_{f \in F_s} p_s(f) t_{k,s}(f) - \sum_{f \in \bar{F}_{b(s)}} p_s(f)[t_{k,s}(f) - t_{k,b(s)}(f)];$$

and at any terminal s, $c_k(s) = e_k(s) + \sum_{f \in \bar{F}_{b(s)}} t_{k,b(s)}(f) f(s)$. Given a consumption plan c, for all types of investors, utility is, $c(0) + \sum_{s \neq 0} \Pr(s) \beta(s) u[c(s)]$, where $\beta(s)$ represents the discount factor applied to date-event s.

We assume that all entrepreneurs choose their financial structures simultaneously and commit to them. If each entrepreneur n chooses a structure $\mathbf{F}_n$, the overall structure is $\mathbf{F} = \cup_n \mathbf{F}_n$. Asset prices are given by a collection of functions $p = \{p_s : \bar{F}_s \to \mathbb{R} \mid s \in \mathcal{S}\}$, where $p_s(f)$ represents the price of asset f at date-event s. We shall assume that the economy is *determinate*, in the sense that it has a unique equilibrium for each financial structure.

As discussed in Section 2, we assume that the entrepreneurs care only about present revenues and that they sell all of their future income, subject to remaining solvent in all date-events. As

[32] Formally, $\bar{F}_0 = F_0$ and $\bar{F}_s = F_s \cup \bar{F}_{b(s)}$.

[33] Strictly speaking, t_s is not defined for terminal date-event s. We shall keep notation light, by not being explicit about this or similar details.

such, this requires that, given $\mathbf{F}_{-n} = \{\mathbf{F}_{n'} \mid n' \neq n\}$, entrepreneur n chooses $\mathbf{F}_n = \cup_s F_{n,s}$, subject to the constraint that, for the equilibrium prices p resulting under $\mathbf{F} = \mathbf{F}_n \cup \mathbf{F}_{-n}$,

$$\sum_{f \in \bar{F}_{n,b(s)}} f(s) = z_n(s) + \sum_{f \in F_{n,s}} p_s(f)$$

in every future date-event s. As a result, the entrepreneur's present revenues are the sum of the prices of the assets he issues for the first instance of trade—namely, $V_n(F) = \sum_{f \in F_{n,0}} p_0(f)$.

Financial structure $\mathbf{F}$ is said to be *dynamically complete* if, for any non-terminal s and any $\hat{s} \in a(s)$, there exists a function $t : \bar{F}_s \to \mathbb{R}$ such that

$$\sum_{f \in \bar{F}_s} t(f) f(s') = \begin{cases} 1, & \text{if } s' = \hat{s}; \\ 0, & \text{if } s' \in A(s) \setminus \{\hat{s}\}. \end{cases}$$

Our next result is that our previous characterizations extend to this more general setting.

Corollary **4** (Robustness: Spot Markets). *For the case of determinate economies, the following statements characterize the equilibrium financial structure:*

1. *If $u''' < 0$ on $\mathcal{X}$, then $\bar{\gamma} > 0$ exists such that, for any $0 < \gamma \leq \bar{\gamma}$, in any pure strategy Nash equilibrium, the resulting financial structure is dynamically complete.*

2. *If $u''' \geq 0$ on $\mathcal{X}$ and structure $\{z_1, \ldots, z_n\}$ is not dynamically complete, then, for any $\gamma > 0$, in any pure strategy Nash equilibrium, the resulting financial structure is not dynamically complete.*

One of the implications of Corollary 4 is that the mechanism identified in this paper operates even if there are no risk sharing needs, in the perfect foresight models in which entrepreneurs issue securities to take advantage of heterogeneity of investors' demands for consumption smoothing over time; hence, the model applies to fixed income securities.

7. Concluding Remarks

To summarize, we have considered a game in which asset holders strategically choose securities to issue in frictionless markets with short selling. Our results show that with competitive investors who have identical quasi-linear preferences over consumption, the outcome of competition in innovation of asset-backed securities among issuers critically depends on the convexity or concavity of the investors' *marginal* utility function. Under convexity, this paper's main message is that frictionless markets fail to give asset holders incentives to introduce asset-backed securities to offer complete risk-sharing opportunities, even if innovation is essentially costless, under general conditions.

Assessing whether convex or concave marginal utility is more plausible requires a theory of the third derivative of the utility functions of investors. Empirical evidence developed for theories that recognize the importance of the third derivative, for example, precautionary savings, provides some support in favor of convex marginal utility ("prudence").[34] Insofar as such preferences (including the standard logarithmic, CARA and CRRA utility functions) describe markets well, in equilibrium, asset owners will offer a structure of asset-backed securities that allows investors to achieve less-than-perfect insurance opportunities. In economies with convex marginal utility in which market value is monotone in security span, such as markets with two states and economies with CARA utilities and riskless assets, issuing a single security (equity) is a strictly dominant strategy. Consequently, in the unique (dominant strategy) equilibrium, the financial structure has minimal span.

As a normative implication, the following welfare and regulation recommendations emerge: A policy to encourage innovation to complete markets by reducing innovation costs might be effective if marginal utility is concave, but it is ineffective in markets with convex marginal utility.

Appendix

Proof of Lemma 1: To prove necessity, suppose that p and $(t_1, \ldots, t_K)$ are the prices and the allocation of securities in a competitive equilibrium under structure F, and let x be any allocation of future consumption in $X(\langle F \rangle)$. Let t'_k be such that $x_k - e_k = Ft'_k$ and $\sum_k F\theta_k t'_k = \sum_n z_n$. Because (p, t) is a competitive equilibrium, we have that $p = F^{\mathsf{T}} DU(c_1)$ and, hence,

$$p \cdot \sum_k \theta_k t'_k = DU(c_1)^{\mathsf{T}} \sum_k \theta_k F t'_k = DU(c_1)^{\mathsf{T}} \sum_n z_n = DU(c_1)^{\mathsf{T}} \sum_k \theta_k F t_k = p \cdot \sum_k \theta_k t_k.$$

In addition, by the optimality of each investor's choice,

$$U(e_k + Ft'_k) - p \cdot t'_k \leq U(e_k + Ft_k) - p \cdot t_k. \tag{7}$$

[34] Loosely speaking, the mechanism behind the theory of precautionary savings shares the implication of convex marginal utility that lowering consumption increases an agent's marginal utility more than increasing consumption reduces it. The precautionary savings effect is present in a single-agent problem, however, whereas ours crucially operates as an *equilibrium* mechanism through heterogeneity across agents. Furthermore, while the precautionary savings phenomenon concerns differences in marginal utilities (and transferring consumption) across states, the conditions for optimality of (in)complete financial structures involve differences in marginal utilities and consumption across agents *within* states.

Aggregating across types and using the equality above, we have that

$$\bar{U}(x) = \sum_k \theta_k U(e_k + Ft'_k) \leq \sum_k \theta_k U(e_k + Ft_k) = \bar{U}(c).$$

Since $c_k - e_k = Ft_k$, it follows that the resulting transfers of numèraire lie in $\langle F \rangle$ and that

$$\sum_k \theta_k c_k = \sum_k \theta_k e_k + F\mathbf{1} = \sum_k \theta_k e_k + \sum_n z_n,$$

which in turn implies that $(c_1, \ldots, c_K) \in X(\langle F \rangle)$. This observation and equation (7) imply that c indeed solves Problem (2).

For sufficiency, note first that set $X(\langle F \rangle)$, defined in (1), can be alternatively written as

$$X(\langle F \rangle) = \{(e_1 + Ft'_1, \ldots, e_K + Ft'_K) \mid \textstyle\sum_k \theta_k t'_k = \mathbf{1}\},$$

while Problem (2) can be equivalently written as

$$\max_{(t'_1,\ldots,t'_K)} \left\{ \textstyle\sum_k \theta_k U(e_k + Ft'_k) \mid \textstyle\sum_k \theta_k t'_k = \mathbf{1} \right\},$$

and, by assumption, t_k is its solution. By the assumption that index u satisfies the Inada condition and using the fact that $\sum_n z_n$ is strictly positive in all states and lies in $\langle F \rangle$, we have that c_k is strictly positive in all components and for all k. Then, multipliers p must exist such that for all k, $F^\top DU(e_k + Ft_k) = p$. Since function u is strictly concave, the latter suffices to imply that t_k solves problem $\max_{t'_k} \{U(e_k + Ft'_k) - p \cdot t'_k\}$ and, since, $\sum_k \theta_k t_k = \mathbf{1}$, securities allocation $(t_1, \ldots, t_K)$ and prices p constitute a competitive equilibrium under structure F. *Q.E.D.*

Proof of Lemma 2: Recall that $x(L)$ is the competitive allocation of numèraire for any F such that $\langle F \rangle = L$. Since equilibrium prices satisfy $p = F^\top DU(x_k(\langle F \rangle))$ for each investor type k, if we take the average across all investors, we obtain

$$p^T = \frac{1}{\theta} \sum_k \theta_k DU(x_k(\langle F \rangle))^\top F = \kappa(\langle F \rangle)^\top F.$$

In addition,

$$V_n(F) = p_n \cdot \mathbf{1} = \kappa(\langle F \rangle)^\top F_n \mathbf{1} = \kappa(\langle F \rangle) \cdot z_n.$$

The expected market value of entrepreneur n is, then, $E_G[V_n(F)] = E[\kappa(\langle F \rangle)] \cdot z_n$. *Q.E.D.*

Proof of Lemma 3: Take any linear space L, such that $\{z_1, \ldots, z_n\} \subseteq L$. If $\{z_1, \ldots, z_n\}$ contains M linearly independent assets, then the orthogonal complement of $\langle \{z_1, \ldots, z_n\} \rangle$ is a linear subspace of dimension $S - M$ and a basis for L can be constructed by taking the M linearly independent assets and $\dim(L) - M$ linearly independent vectors in $\langle\{z_1, \ldots, z_n\}\rangle^\perp$. It follows that, for any $M \leq D \leq S$, the space of D-dimensional spaces of trades that contain $\{z_1, \ldots, z_n\}$ is topologically equivalent to the set of $(D - M)$-dimensional linear subspaces of $\mathbb{R}^{S-M}$. This Grassmannian is a compact manifold (of dimension $(D - M) \times (S - D)$).

Consider the set of all structures F for which the dimension of span $\langle F\rangle$ is D. Over this set, correspondence $X(\langle F\rangle)$ is upper- and lower-semicontinuous. Since function $\bar{U}$ is continuous on $X(\langle F\rangle)$ for any F, it follows by the Theorem of the Maximum that the allocation function, $x(L)$, is continuous on the Grassmannian. Further, it also follows that the expected market value, $\mathrm{E}_G[V_n(L)] = \mathrm{E}_G[\kappa(L)] \cdot z_n$, is continuous as well and, therefore, that a linear space L^* exists that maximizes it over the set of all linear subspaces of dimension D.

Denote by V_n^D the maximized expected market value over the set of structures that span D-dimensional spaces of numèraire transfers. Since S is finite, the entrepreneur's program is reduced to finding the maximum of $\{V_n^M, \ldots, V_n^S\}$. *Q.E.D.*

Proof of Proposition 1: With the complete financial structure F', by Lemma 1, the allocation of numèraire is such that all investors consume the same in the second period:

$$x_k(\langle F'\rangle) = \frac{1}{\theta}\left(\sum_n z_n + \sum_k \theta_k e_k\right),$$

The resulting market value for entrepreneur n equals, therefore,

$$V_n(F') = \kappa(\langle F'^{\mathsf{T}} z_n = DU\left(\frac{1}{\theta}\sum_n z_n + \frac{1}{\theta}\sum_k \theta_k e_k\right) \cdot z_n.$$

Next, consider a feasible financial structure F for which $\langle F\rangle \neq \mathbb{R}^S$. Consider the linear subspace of endowment profiles defined by

$$E = \{(e_1, \ldots, e_K) \in \mathbb{R}^{S\times K} \mid \ (\textstyle\sum_n z_n + \sum_k \theta_k e_k) \in \langle F\rangle\}.$$

Since, by feasibility, $\sum_n z_n \in \langle F\rangle$, it follows that E has dimension lower than $S \times K$ and, hence, has zero Lebesgue measure.[35] Since G is absolutely continuous with respect to the Lebesgue measure for $\mathbb{R}^{S\times K}$, it follows that, G-a.s., $x_k(\langle F\rangle) \neq x_{k'}(\langle F\rangle)$ for at least two types of investor, k and k'.

[35] To see this, suppose by way of contradiction that $E = \mathbb{R}^{S\times K}$. Let ι_s be the s-th canonical vector in $\mathbb{R}^S$ and construct the following profile of endowments: $e_1 = \frac{1}{\theta_1}\iota_s$ and $e_k = (0, \ldots, 0)$ for every $k \geq 2$. Since this profile lies in E, we have that $\sum_n z_n + \iota_s = \sum_n z_n + \sum_k \theta_k e_k \in \langle F\rangle$; then, using $\sum_n z_n \in \langle F\rangle$, we have that $\iota_s \in \langle F\rangle$. But since this is true for all $s = 1, \ldots, S$, we have that $\langle F\rangle = \mathbb{R}^S$.

For claim (i), note that since function u' is strictly convex over the relevant domain and since, for each state s, $\sum_k \theta_k x_{k,s}(\langle F\rangle) = \sum_n z_{n,s} + \sum_k \theta_k e_{k,s}$, one has that

$$\kappa_s(\langle F\rangle) = \frac{1}{\theta}\sum_k \theta_k u'\left(x_{k,s}(\langle F\rangle)\right) > u'\left(\frac{1}{\theta}\sum_n z_{n,s} + \frac{1}{\theta}\sum_k \theta_k e_{k,s}\right) = \kappa_s(\langle F'\rangle),$$

and hence, $\kappa(\langle F\rangle) \gg \kappa(\langle F'\rangle)$. It follows that

$$V_n(F) = \kappa(\langle F\rangle)^\top z_n > \kappa(\langle F'\rangle)^\top z_n = V_n(F'),$$

G-a.s. (In the G-null set where all investors equate second period consumption, the two levels of market value are equal.)

The arguments for claims (ii) and (iii) are analogous and are hence omitted. *Q.E.D.*

Proof of Theorem 1: For the first claim, define the cost threshold

$$\bar{\gamma} = \frac{1}{2}\min\left\{\frac{V_n^S - V_n^D}{S} \mid D = M, \ldots, S-1 \text{ and } n = 1, \ldots N\right\},$$

where M and V_n^D for all $n = 1, \ldots N$ and all $D = M, \ldots, S$ are defined as in the proof of Lemma 3; by Proposition 1, $\bar{\gamma} > 0$. Now, given any $\gamma \le \bar{\gamma}$, if entrepreneurs other than n chose an incomplete financial structure $F_{-n} = \{F_1, ..., F_{n-1}, F_{n+1}, ..., F_N\}$, then it is a best response for entrepreneur n to choose F_n such that the resulting structure $\{F_n, F_{-n}\}$ is complete, again by Proposition 1.

For the second claim, let $F = \{F_1, \ldots, F_N\}$ be the profile of financial structures chosen at a pure strategy Nash equilibrium, and suppose that F is complete. Because by assumption, the payoff structure of assets constitutes an incomplete financial structure, at least one entrepreneur exists for whom $F_n \neq \{z_n\}$. For such entrepreneur, $F_n' = \{z_n\}$ gives strictly higher payoff: it is less costly and his asset's market value is at least as high as with F_n. *Q.E.D.*

Proof of Proposition 2: Consider entrepreneur n and suppose that the strategies chosen by his competitors are F_{-n}. If V_n is decreasing in the span of the financial structure, then, regardless of the rank of F_{-n}, financial structure $F_n = \{z_n\}$ maximizes the market value of z_n and is also the least expensive structure to issue. Thus, it is a unique best response to the arbitrary choices of financial structures by other entrepreneurs.

Since market value is monotone in the security span in economies with two states or CARA utility function and riskless assets, issuing equity is a strictly dominant strategy. The game has a unique (pure strategy) Nash equilibrium, with $F = \{z_1, \ldots, z_n\}$. *Q.E.D.*

Proof of Corollary 2: Consider the case when $u''' < 0$, and fix that $\{(z_1, F_1), \ldots, (z_N, F_N)\}$ is such that $\langle \cup_{n=0}^{N} F_n \rangle \neq \mathbb{R}^S$. Then, it must be true that $\langle F_N \rangle \neq \mathbb{R}^S$. Suppose that entrepreneur N unilaterally deviates to (z_N, F_N'), such that $F_N' \mathbf{1} = F_N \mathbf{1}$ and F_N' is complete. By assumption (ii) on $\mathcal{F}_N$, such a deviation is feasible for him, while, since F_N and F_N' are associated with the same future payoff, his future revenue, $r_{N,1} = z_N - F_N' \mathbf{1}$, remains unchanged. The change in his present consumption is given by

$$\Delta r_{N,0} = V_N(F') - V_N(F) - \gamma(|F_N'| - |F_N|),$$

where $F' = \{F_0, \ldots, F_{N-1}, F_N'\}$ is complete. By previous arguments, $V_N(F') - V_N(F) > 0$, *G*-a.s. (with weak inequality surely), which implies that, as $\gamma \to 0$, $\Delta r_{N,0}$ converges to a random variable that first-order stochastically dominates 0. By continuity and strict monotonicity of U_N, there exist $\tilde{\gamma} > 0$ such that for all $\gamma < \tilde{\gamma}$, strategy (z_N, F_N') is strictly preferred to (z_N, F_N), so that $\{(z_1, F_1), \ldots, (z_N, F_N)\}$ cannot be a Nash equilibrium. Define

$$\Omega = \left\{ ((z_n, F_n), \ldots, (z_n, F_n)) \mid z_n \in \mathcal{Z}_n \text{ and } F_n \in \mathcal{F}_n(z_n) \text{ for all } n \text{ and } \langle \cup_{n=0}^{N} F_n \rangle \neq \mathbb{R}^S \right\},$$

and observe that, since all the sets $\mathcal{Z}_n$ are compact, then set Ω is also compact compact, given compactness of the Grasmannian manifolds (as in the proof of Lemma 3). Let

$$\bar{\gamma} = \frac{1}{2S} \inf\{\tilde{\gamma} \mid ((z_n, F_n), \ldots, (z_n, F_n)) \in \Omega\}.$$

By compactness of Ω, $\bar{\gamma} > 0$. Then for any $\gamma < \bar{\gamma}$ and any arbitrary profile $((z_n, F_n), \ldots, (z_n, F_n))$ such that $z_n \in \mathcal{Z}_n$ and $F_n \in \mathcal{F}_n(z_n)$ for all n, that gives rise to incomplete financial structure, entrepreneur N has incentives to unilaterally deviate and hence profile is not a Nash equilibrium.

Now, suppose that $u''' \geq 0$, and consider any profile $\{(z_1, F_1), \ldots, (z_N, F_N)\}$ such that $\langle \cup_{n=0}^{N} F_n \rangle = \mathbb{R}^S$ while $\langle F_0 \cup \{F_1 \mathbf{1}, \ldots, F_N \mathbf{1}\} \rangle \neq \mathbb{R}^S$. It must be true that $F_n \neq \{F_n \mathbf{1}\}$ for some n. Consider n's deviation $(z_n, \{F_n \mathbf{1}\})$, which is feasible for him by assumption (iii) on $\mathcal{F}_n$. The future consumption of entrepreneur n, $r_{n,1} = z_n^* - F_n \mathbf{1}$, remains unchanged, whereas the change in his present consumption is

$$\Delta r_{n,0} = V_n(F') - V_n(F) - \gamma(1 - |F_n|),$$

for $F' = \{F_0, \ldots, F_{n-1}, \{F_n \mathbf{1}\}, F_{n+1}, \ldots, F_N\}$. By previous results, $V_n(F') - V_n(F) \geq 0$ surely, while $|F_n| > 1$, by construction. By strict monotonicity of U_n, strategy $(z_n, \{F_n \mathbf{1}\})$ is strictly preferred to (z_n, F_n), so $\{(z_1, F_1), \ldots, (z_N, F_N)\}$ cannot be a Nash equilibrium. *Q.E.D.*

Proof of Corollary 3: Let $(\bar{z}, \bar{z}) = \frac{1}{\theta} \sum_n z_n$. We first construct symmetric equilibria under a complete and an incomplete financial structure. For a complete financial structure, all investors' consumption will be the same, $\frac{a+b}{2} + \bar{z}$ in both states. Under an incomplete financial structure, each type of investor will obtain one half of the available riskless assets, so that in both states one half of the investors will consume $a+\bar{z}$ while the other will consume $b+\bar{z}$. Also by symmetry, $c_{1,0} = c_{2,0} = c_1$.

Future consumption is thus expressed in terms of primitives, and the state prices (for both states) become functions of c_1 given by

$$\kappa_C(c_1) = \frac{\beta}{2} \frac{u'(\frac{a+b}{2} + \bar{z})}{u'(c_1)} \tag{8}$$

if markets are complete, whereas if they are incomplete they are given by

$$\kappa_I(c_1) = \frac{\beta}{4} \frac{u'(a+\bar{z}) + u'(b+\bar{z})}{u'(c_1)}. \tag{9}$$

Both of these functions are increasing in c_1, from 0 (as $c_1 \to 0$) to ∞ (as $c_1 \to \infty$).

Consider now a second equation, resulting from the investors' budget constraint, $c_1 + 2\kappa\bar{z} = e$, where e is their endowment in the first period. By direct computation

$$\kappa = \frac{1}{2\bar{z}}(e - c_1), \tag{10}$$

which is linear, has a positive intercept and is decreasing in c_1. Each of the two schedules (8) and (9) cross the budget line (10) precisely once (see Figure 4), which gives precisely one solution for the complete financial structure (κ_C, c_C) and one for the incomplete financial structure (κ_I, c_I). It is straightforward to verify that the implied prices of securities and the corresponding allocation define a competitive equilibrium under the complete and incomplete financial structures, respectively. Moreover, by assumption, the competitive equilibria are unique and, hence, the constructed symmetric equilibria are globally unique.

If $u''' \geq 0$, the wedge in future consumption under incomplete markets implies that schedule (9) is strictly above schedule (8); therefore, these two solutions satisfy $\kappa_I \geq \kappa_C$. It follows that the value of the riskless assets are at least as high under incomplete markets, and, by the standard argument, that issuing equity is a strictly dominant strategy for each entrepreneur.

Conversely, with $u''' < 0$, one has that $\kappa_I < \kappa_C$ and market value is maximized under complete markets. For sufficiently small innovation costs in pure strategy Nash equilibrium, one of the entrepreneurs completes the market. *Q.E.D.*

Proof of Corollary 4: For the simplicity of presentation, we provide the argument after two intermediate steps: First, we recast the dynamic setting as a two-period economy in which asset trading occurs only once. Second, we invoke some previous results on two-period economies to characterize the value-maximizing structures in the general setting.

Step 1: Consider any profile $\{\mathbf{F}_1, \ldots, \mathbf{F}_n\}$. Let the *unique* competitive equilibrium prices and investment plans be p and $\{\mathbf{t}_1, \ldots, \mathbf{t}_K\}$. Equilibrium prices are characterized by market-clearing conditions: for all $f \in \bar{F}_s$, in all non-terminal date-event $s' \in A(s)$,

$$\sum_k \theta_k t_{k,s'}(f) = |\{n \mid f \in \mathbf{F}_n\}|, \tag{11}$$

and by the first-order conditions that for all k, at all non-terminal $s \neq 0$, and for all $f \in \bar{F}_s$,

$$p_s(f) = \sum_{s' \in A(s)} \Pr(s'|s) \frac{\beta(s')}{\beta(s)} \frac{u'[c_k(s')]}{u'[c_k(s)]} f(s'), \tag{12}$$

whereas, if $f \in F_0$,

$$p_0(f) = \sum_{s \neq 0} \Pr(s)\beta(s)u'[c_k(s)]f(s). \tag{13}$$

Now, we define an alternative collection of assets, all of which are issued at date-event 0. For each $f \in F_{n,0}$, let $\phi_f = f$. For each $s \neq 0$ and $f \in \bar{F}_{n,s}$, let

$$\phi_{f,s}(s') = \begin{cases} -p_s(f), & \text{if } s' = s; \\ f(s'), & \text{if } s' \in A(s); \\ 0, & \text{otherwise.} \end{cases}$$

Finally, for each $s \neq 0$ and $f \in \bar{F}_{n,s} \setminus F_{n,s}$, let $\phi^-_{f,s} = -\phi_{f,s}$. (Introduction of $\phi^-_{f,s}$ will allow us to assume that the supply of each security is equal to 1, even though in the dynamic economy re-traded assets are in zero net supply). The collection of all these assets is denoted by Φ_n. Consider an economy in which all trade has to take place at date-event 0, using the assets in $\Phi = \cup_n \Phi_n$. Denote by π the equilibrium prices of this economy. It can be verified, using equations (11) to (13), that the equilibrium prices of this economy are $\pi(\phi_f) = p_0(f)$ for each $f \in F_{n,0}$, while every other asset is traded at $\pi(\phi) = 0$.[36]

Note that the economy in which trade occurs only at date-event 0 (in the assets contained in Φ) is equivalent to the two-period economy as in Section 2 in which there are $|\mathcal{S}| - 1$ future states of the world. Thus, we can invoke the results derived in Section 4.

Step 2: Fix any two financial structures $\mathbf{F}$ and $\mathbf{F}'$, with the associated equilibrium prices p and p'. Suppose that $\mathbf{F}$ is dynamically incomplete, while $\mathbf{F}'$ is not. Let Φ and Φ' be, respectively, the associated structures constructed in Step 1, and let π and π' be the equilibrium prices for the two-period economies. By construction,

$$V_n(\mathbf{F}) = \sum_{f \in F_{n,0}} p_0(f) = \sum_{f \in F_{n,0}} \pi(\phi_f)$$

[36] Also, that equilibrium demand for assets is as follows: for each $f \in F_{n,0}$, $\tau_k(\phi_f) = t_{k,0}(f)$; for each $f \in F_{n,s}$, $s \neq 0$, $\tau_k(\phi_{f,s}) = t_{k,s}(f)$; and for each $f \in \bar{F}_{n,s} \setminus F_{n,s}$, $\tau_k(\phi_{f,s}) = [t_{k,s}(f) - t_{k,b(s)}(f)]_+$ and $\tau_k(\phi^-_{f,s}) = [t_{k,s}(f) - t_{k,b(s)}(f)]_-$.

and

$$V_n(\mathbf{F}') = \sum_{f \in F'_{n,0}} p'_0(f) = \sum_{f \in F'_{n,0}} \pi'(\phi'_f).$$

Using backward induction over $\mathcal{S}$, note that a financial structure is dynamically complete if, and only if, the associated structure constructed in Step 1 is complete (in the two-period economy). Thus according to Proposition 1, it follows that:
(i) if $u''' > 0$, then, G-a.s., $V_n(\mathbf{F}) > V_n(\mathbf{F}')$ while $V_n(\mathbf{F}) \geq V_n(\mathbf{F}')$ surely;
(ii) if $u''' < 0$, then, G-a.s., $V_n(\mathbf{F}') > V_n(\mathbf{F})$ while $V_n(\mathbf{F}') \geq V_n(\mathbf{F})$ surely; and
(iii) if $u''' = 0$, then $V_n(\mathbf{F}') = V_n(\mathbf{F})$.

Step 3: Based on characterization of Step 2, the argument follows according to the proof of Theorem 1. For the first claim, suppose that issuance costs are sufficiently low. If entrepreneurs other than n chose an incomplete $\mathbf{F}_{-n}$, then it is a best response for entrepreneur n to choose $\mathbf{F}_n$ such that the resulting $\{\mathbf{F}_n, \mathbf{F}_{-n}\}$ is complete, as indicated in Step 2.

For the second claim, let $\mathbf{F} = \{\mathbf{F}_1, \ldots, \mathbf{F}_N\}$ be a pure strategy Nash equilibrium; suppose that financial plan is dynamically complete. Since $\{z_1, \ldots, z_n\}$ is not dynamically complete, it must be true that $\mathbf{F}_n \neq \{z_n\}$ for some n. But this is a contradiction, as $\mathbf{F}'_n = \{z_n\}$ is less costly and—according to Step 2—V_n is at least as high with $\mathbf{F}'_n$ as with $\mathbf{F}_n$. *Q.E.D.*

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