Sequential Search and Choice from Lists*

JOB MARKET PAPER

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Abstract

Decision-makers frequently encounter choice alternatives presented in the form of a *list*. A wealth of evidence shows that decision-making in the list environment is influenced by the order of the alternatives. The prevailing view in psychology and marketing is that these *order effects* in choice result from cognitive bias. In this paper, I offer a standard economic rationale for order effects.

Taking an axiomatic approach, I model choice from lists as a process of sequential search (with and without recall). The characterization of these models provides choice-theoretic foundations for sequential search and recall. The list-structure of the environment permits a natural definition of search and preference in terms of choice. For a decision-maker whose behavior can be represented as the outcome of sequential search, the search strategy can be determined uniquely.

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1 Introduction

There is considerable evidence showing that decision-makers are influenced by the order of the alternatives when choosing from a list. In psychology and marketing, the prevailing view is that these *order effects* in choice result from cognitive bias. Motivated by the evidence, I provide an economic rationale for order effects by modeling choice from lists as a process of sequential search. Taking an axiomatic approach, I study models of search with and without recall. The characterization of these models provides choice-theoretic foundations for sequential search and recall. The list structure of the choice environment permits a natural identification of search and preference. For choice behavior that can be represented in terms of sequential search, the search strategy is uniquely identified.

Although the standard model in economics studies choice from sets, decision-makers frequently choose from lists. In some situations, the list is temporal in that the menu is revealed only gradually. Job applicants, for instance, generally receive employment offers one at a time. In other situations, the list more literally describes the organization of the menu. Just as the wine selection at a restaurant is usually presented in the form of a wine list, internet search engines provide a list of search results.

Much evidence suggests that decision-makers are systematically influenced by the order of the alternatives when choosing from a list. In the political economy literature, for instance, studies consistently show that a candidate's chance of being elected increases when she is listed first on the ballot.[1] Evidence from a variety of other choice settings reflects a similar *primacy bias*. Whether confronted with the task of choosing among the results of a *Google* query (see e.g. Joachims et al. [2005]), answering a multiple-choice survey (see e.g. Galesic et al. [2008]), or picking a florist from the *Yellow Pages* (Lohse [1997]), decision-makers generally select alternatives located near the beginning of the list. Conversely, some evidence reflects a *recency bias* towards options at the end of the list. While this effect is more common when the decision-maker is required to examine the entire list[2], there is some evidence of a recency bias for internet search behavior (Murphy, Hofacker and Mizerski [2006]).

In both psychology and marketing, order effects, like the primacy and recency biases, are generally explained as manifestations of cognitive bias. The idea is that the list-structure of the menu frames choice in a way that leads to systematic errors in decision-making. The *serial position effect*, for instance, suggests that decision-makers find it easier to remember *and* choose alternatives at either end of the list (see e.g. Glanzer and Cunitz [1966]). Other biases, like *anchoring* and the *confirmation bias*, suggest that the first item induces a mind-set that makes it more difficult for subsequent items to be chosen (see e.g. Russo, Carlson and Meloy [2006]).

A wealth of evidence from eye-tracking experiments, discussed in Section 1.1 below, suggests that decision-makers generally examine lists sequentially and frequently fail to examine the whole list.[3] Motivated by this evidence, I provide a standard economic rationale for order effects by modeling choice from

[1] Meredith and Salant [2009] provide a general overview of this vast literature.

[2] This case is not dealt with by my models. In some sports competitions, contestants who appear later in the competition receive higher scores. See Salant [2009] and Meredith-Salant [2009] for the appropriate references. Similarly, some decision-makers choose later options when the list is read out loud. See e.g. Mantonakis et al. [2009].

[3] Some recent work in economics also exploits eye-tracking data. See e.g. Arieli, Ben-Ami & Rubinstein [2009].

lists as a process of sequential search. The seeds of this approach may be traced back to the work of Krosnick [1991] in marketing. His work suggests that the primacy bias in multiple-choice survey data might be explained by Simon's [1955] model of satisficing. The contribution of this paper is to provide choice-theoretic foundations for search and to provide a unique characterization of the decision-maker's strategy when her choice behavior can be represented in terms of sequential search.

My approach builds on the list-choice framework developed by Rubinstein and Salant [2006]. Faced with a list of feasible alternatives, the decision-maker examines the list sequentially (one item at a time) following a *search procedure* (s, r). Intuitively, the *search strategy* s determines the depth of search while the *choice rule* r determines what to choose from among the items considered. Based on the list of items already considered, the strategy specifies whether to stop searching or to continue and examine the next item in the list. Once the decision-maker stops or reaches the end of the list, the choice rule then determines which item to select from among those considered.

I focus on the two choice rules most frequently studied in the search literature: search with (perfect) recall and search without recall. When the decision-maker searches with recall, she chooses the best item that she encountered according to her preference. As such, a *search procedure with recall* $(s, \succ)$ can be succinctly described in terms of a search strategy s and a strict preference $\succ$. In other situations, like job search, the decision-maker searches without recall. In particular, she cannot *return to* previous items and can only pick the item that she is currently considering. Formally, a *search procedure without recall* can be simply described in terms of a search strategy s that determines both the depth of search and which alternative is chosen.

These models provide a rich framework to study search behavior. Beyond the requirement that the search decision only depends on the items that have already been considered, the models impose no *a priori* restriction on the strategy. In other words, search procedures are flexible enough to describe any *optimal stopping rule* in circumstances where the decision-maker encounters a random sequence of alternatives and faces an incremental cost of search.[4] For some specifications of search costs and beliefs, the optimal stopping rule displays an intricate dependence on the items previously considered (see e.g. Kohn-Shavell [1974], Rothschild [1974] and Rosenfield-Shapiro [1981] for examples). At the same time, search procedures also describe simple search heuristics, like satisficing, where the search strategy depends on relatively little information.

These two models also accommodate the choice biases most frequently observed in the data. For search with recall, order effects arise only when the decision-maker chooses before considering the entire list. When she examines the entire list, the decision-maker selects the best item available and her choice is order-independent. Depending on the extent of search, choice behavior exhibits a primacy bias that is more or less pronounced. For search without recall, the presence of order effects does not require limited consideration of the list. Rather, the extent of search determines *which* type of order effect results. When the decision-maker searches the entire list without recall, she chooses the last list-item, thus exhibiting an

[4] In the baseline models, the alternatives must be distinct and the sequence of unknown length. Both requirements are relaxed in the extensions.

extreme form of recency bias. Conversely, choice behavior exhibits a strong form of primacy bias when the decision-maker rarely examines more than the first few items.

The paper provides four results on the relationship between choice and sequential search.

First, I show how to identify search and preferences from behavior in circumstances when the decision-maker can be modeled in terms of sequential search. To identify the extent of search associated with a particular list, one need only consider the lists obtained by modifying the *tail* of the list from the n^{th} item onward. Provided her behavior is consistent with sequential search, the decision-maker's choice of the n^{th} item (or perhaps some later item) on one of the modified lists reveals that she must have searched at least up to the n^{th} item of the original list. Otherwise, the decision-maker could not have chosen as she did on the modified list. This approach uniquely identifies the decision-maker's strategy for both models of sequential search.[5] For search with recall, it is then straightforward to define a revealed preference. Intuitively, an item x is preferred to another item y if there exists a list where the decision-maker examines y but chooses x.

Second, I provide an axiomatic characterization of choice behavior that can be represented as the outcome of sequential search. For both models of search, choice behavior satisfies a weak recursivity requirement that may be interpreted as a relaxation of *Independence of Irrelevant Alternatives* (IIA) in the list environment. Formally, this requirement describes the choice implications of sequential search. In particular, incremental search can only affect choice by causing the decision-maker to choose an item *further down the list*. For behavior that can be represented in terms of search with recall, choice satisfies an additional property that reflects a weak form of the primacy bias. If the decision-maker chooses the last item in the list, that item can be *moved up* in the list without affecting choice. For behavior that can be represented in terms of search without recall, choice satisfies a different invariance property. If the decision-maker chooses an item from the initial portion of the list, the tail of the list can be modified without affecting her choice.

The paper provides two further results related to applications and extensions of the baseline models. Before discussing these, I comment briefly on the practical implications of the first two results.

In terms of empirical work, my results provide a means to identify search behavior at the individual level. Although the optimal search literature is vast, there has been relatively little effort to investigate which search strategies are used in practice. Instead, most empirical work estimates parameters for a *particular* model of optimal search at the aggregate level (see e.g. Schotter and Braunstein [1981] and Zwick et al. [2003]). The few papers which study search behavior at the individual level "benchmark" choice data against a limited set of candidate strategies (see Hey [1981], [1982] and Moon and Martin [1996]). As a related matter, my results provide choice-based tests for features generally treated as *inputs* in search theory models. In particular, the weak recursivity property provides a simple test for sequential search that requires very little data and does not rely on direct observation (like eye-tracking). Moreover, the two invariance properties discussed provide a way to test the standard recall assumptions against choice behavior.

[5] For search with recall, an additional (but very natural) limitation is required to ensure uniqueness.

The first two results also have practical implications for internet search. Historically, search engines have used keyword density measures and link analysis to rank-order search results. Recently, the industry has shown an interest in leveraging user feedback to improve their rankings. Although most efforts have focused on explicit user feedback,[6] increasing attention is being devoted to the inference of user judgements about relevance from click-through behavior.[7] Provided that "clicks" can be interpreted as "choices", the problem is analogous to the problem of identifying preference for search procedures with recall. Since search engine users tend to examine search results sequentially, the extent of the user's search can be inferred from her click-through choices on a chain of related queries. In that case, the revealed preference ranking may be interpreted as a revealed relevance ranking.

My third result provides choice-theoretic foundations for several simple search heuristics discussed in the literature: Simon's [1955] model of satisficing, a model of Markov search, and Selten's [1998] model of aspiration adaptation. Although the baseline models impose no restrictions on the form of the decision-maker's search strategy, these three heuristics involve *cutoff rules* where the decision-maker only conditions further search on the best option encountered so far.[8] In a search setting with recall, these simple heuristics are characterized by stringent choice recursivity requirements and an axiom that reflects a strong primacy bias.

The axiomatic foundations for these simple heuristics show how my model is related to the previous work on choice from lists due to Salant [2003] and Rubinstein-Salant [2006]. Moreover, the axioms provide a natural test for cutoff search rules. Because these rules are optimal in a variety of search environments, the axioms have implications for empirical work on optimal search.[9] Finally, the straightforward characterization of these heuristics suggests that the list-choice framework is particularly well-suited to study bounded rationality. If the decision-maker faces memory or computational constraints, she may be inclined to adopt a *simple heuristic* that does not depend on all of the information available. Since search is uniquely identified in the baseline models, it is possible to characterize a variety of simple search strategies beyond the ones discussed.

Finally, I show that the approach used to identify search in the baseline models can be extended to more general models of sequential search. In particular, the same approach identifies search for a wide variety of choice rules different from recall and no recall. Moreover, the baseline approach has natural extensions to lists with duplication, choice correspondences, and a model where the decision-maker conditions her search on a signal that accurately describes some attribute of the list (such as its length).

The remainder of the paper is structured as follows. After reviewing the experimental evidence and related literature below, I formalize the notions of choice from lists and list-search in section 2. Section 3 provides an axiomatic characterization of choice behavior that can be represented as sequential search and gives identification results. In section 4, I formally characterize some simple search heuristics discussed in the literature. Section 5 establishes that the baseline models can be adapted to cope with several natural

[6] *Google Stars*, for example, allows users to give feedback about the relevance of a particular website by assigning it a star.
[7] See Joachims and Radlinski [2007] for an overview of this literature.
[8] In the case of aspiration adaptation, the decision-maker also conditions on calendar time.
[9] See e.g. Rothschild [1974], Kohn and Shavell [1974], Rosenfield and Shapiro [1981], and Seierstad [1992].

extensions.

1.1 Evidence of Limited Sequential Search

As discussed in the Introduction, my approach assumes that decision-makers proceed sequentially through the list and frequently do not examine the entire list. There is a great deal of experimental evidence supporting these two hypotheses.

Sequential Search: As the example of the job applicant shows, there are circumstances where the decision-maker cannot help but proceed sequentially. In other circumstances, the decision-maker may be somewhat less restricted. Consider, for instance, the task of choosing from a wine list at a restaurant. Even in these circumstances however, the evidence suggests that decision-makers tend to examine lists sequentially.

Experiments frequently use eye-tracking data to observe search order. These experiments establish that subjects generally scan the items of the list in sequence. In Joachims et al. [2005], for example, subjects were asked to pick the most relevant result from a list of ten *Google* search results.[10] As one might expect, the subjects followed the list sequence when the results were listed in the order provided by *Google*. Perhaps more surprising is that the sequential pattern persisted in the treatment where the results were listed in reverse-order. While certainly not determinative, this suggests that the subjects did not use any information they gleaned from search (namely that the results were generally becoming increasingly relevant) as a basis to skip ahead.

Eye-tracking studies in a variety of list-choice settings support these findings. In the context of multiple-choice surveys, for instance, Galesic et al. [2008] find that reversing the list-order had no real impact on search behavior. In both treatments, subjects generally followed the sequence dictated by the list. In another study, Lohse [1997] finds that subjects scanned *Yellow Pages* listings in alphabetical order (even though there is no clear relationship between alphabetical order and the quality of advertisers).

A less common approach is to ask the subjects directly. By and large, the verbal protocols from experiments are consistent with the eye-tracking data. In one study by Duffy [2003], for instance, 74% of subjects claimed that they read the survey options from *top to bottom* while 13% claimed to have read from *bottom to top*.

From an economic standpoint, the approach taken by Caplin, Dean and Martin [2010] is arguably more appealing. In a recent set of experiments, they used list-choice data to infer search order. Recording subjects' interim choices over time, they find that later choices generally correspond to items further down the list. Across all treatments, 75% of subjects displayed behavior consistent with sequential consideration of the list.

Limited Search: Decision-makers frequently fail to examine the entire list. In some list-choice settings, the effect is pronounced. In a study of *Google* click-through behavior by Lorigo et al. [2006], for instance, 96% of subjects failed to consider results beyond the first page (of ten links) and no subject looked beyond

[10] Granka, Feusner and Lorigo [2008] provide a comprehensive survey of related studies.

the third page. In other settings, the effect is less pronounced. In the study of Galesic et al. [2008], for instance, 90% of subjects in one trial scanned all twelve multiple-choice options. In another trial, only 54% scanned all of the options from a shorter list of five.

1.2 Related Literature

Most importantly, my paper contributes to the literature of search. While this literature is vast, it is primarily concerned with a different question—that of determining *optimal search* given certain assumptions about the data generating process, the decision-maker's utility, and her search costs. Very few papers examine the choice-theoretic foundations of search.

Other than the work of Horan [2009] and Papi [2010] discussed below, the only notable exception is Caplin and Dean's [2010] model of *alternative-based search* (ABS). In a recent paper, they characterize choice in terms of search by considering *choice process data* that tracks how the decision-maker's provisional choices change with contemplation time. The main difference from my approach is that the ABS model is largely silent about how much search takes place on any set of alternatives and, consequently, does not provide unique identification of the decision-maker's search strategy. As discussed at greater length in Appendix 7.1, their model also accommodates search behavior where the decision-maker can be interpreted as having a sophisticated understanding of the feasible set *before* she examines any items. In contrast, my model contemplates search behavior where the decision-maker *only* conditions further search on the list-items she has already considered.

My paper also contributes to the growing literature on framing effects. For any list, choice may depend on the feasible alternatives *as well as* the order of those alternatives. In other words, the structure of the feasible set *frames* the decision-maker's choice.[11] In the theory literature, this idea has also been explored by Salant [2003], Rubinstein and Salant [2007], Horan [2009], and Papi [2010].

Although Salant and Rubinstein-Salant also consider choice from lists, the list-order plays a different role in their models. In particular, the decision-maker considers all of the feasible alternatives and the list-order only serves to *break ties* when she cannot strictly rank two items. At the same time, there is some connection, discussed in Appendix 7.2, with the simple search heuristics studied in this paper.

By way of contrast, Horan and Papi consider the impact of different menu structures on choice. Whereas Horan studies menus that are sub-divided into categories, Papi studies generalized lists where each list "item" consists of a set of alternatives. Although they focus on different choice environments, both pursue the same basic goal as my paper: to characterize choice as a process of search that depends on the structure of the menu. (Similarly, Masatlioglu and Nakajima [2009] model choice with a reference point as a process of iterative search *that depends on* the reference point.) In terms of search possibilities, the choice environments considered by Horan and Papi are less restrictive than lists. At the same time, their focus is narrower than mine. Both only characterize models where the decision-maker uses a cutoff (or

[11] Rubinstein and Salant [2008] define framing effects in these broad terms. In the marketing literature, Schkade and Kleinmuntz [1993] provide a general discussion of the impact of information display on choice.

satisficing) search strategy. While I also provide an axiomatization of satisficing, my broader contribution is to characterize a general model of sequential search in the list environment.

More broadly, my paper also contributes to the *procedural rationality* literature initiated by Simon [1955]. Not only do I characterize a variety of boundedly-rational heuristics (like satisficing), but I also provide a characterization of choice in terms of a general procedure. The model describes a *two-stage choice procedure* where the decision-maker first filters the feasible set of alternatives before making a final selection. Several recent papers study related two-stage procedures.[12] Of these, my work is most closely related to the model of *limited attention* due to Masatlioglu, Nakajima, and Ozbay [2009].[13] In my framework, sequential search determines which alternatives attract the decision-maker's attention. In their model, the decision-maker's first-stage choices can also be interpreted as the alternatives that attract attention. At the same time, their framework accommodates limited attention that does not result from sequential search. The basic idea is that the consideration sets in their model need not have the structure that sequential search requires. This point is examined at greater length in Appendix 7.1.

Finally, my paper is also related to Salant's [2010] recent work on choice complexity. Using the *state measure* of complexity developed in game theory (see e.g. Kalai [1990]), he characterizes the complexity of various list-choice rules.[14] While my paper does not address the issue of choice complexity directly, it provides choice-theoretic foundations for some of the list-choice rules studied by Salant.

2 Preliminaries

2.1 Choice From Lists

Let X be a countable grand set of alternatives. A *list* L_n is a sequence of n distinct alternatives in X

$$< l_1, ..., l_i, ..., l_n >$$

where l_i denotes the i^{th} item of L_n. Let $\mathcal{L}_n \equiv \{< l_1 ..., l_n > \in X^n : l_i = l_j \text{ iff } i = j\}$ be the collection of lists of length n and let $\mathcal{L} \equiv \cup_{n=1}^{\infty} \mathcal{L}_n$ be the collection of finite lists. It is worth noting that this definition excludes lists with repetition. In Section 5 below, I consider a more general setting where the list may contain duplicate items. I use L_n to refer to a typical list of length n (i.e. a list in $\mathcal{L}_n$) and L to refer to a typical element of $\mathcal{L}$. If the list L contains the item a, I abbreviate by writing $a \in L$. Moreover, I denote the unordered set of items $\{l_i : 1 \leq i \leq n\}$ in L_n by $S(L_n)$.

I use four natural operations on lists. First, any list L can be *trimmed* by deleting the items in some set $A \subset X$ while maintaining the list-order of the remaining items. The resulting list, denoted $L - A$, is a

[12] Many of these papers, such as Horan [2008], are related to the model of sequentially rationalizable choice studied by Manzini and Mariotti [2007]. The interested reader should consult Manzini and Mariotti [2010] for the appropriate references. It appears that the first to study two-stage choice were Soviet economists. See Aizerman and Aleskerov [1995] for an overview of Soviet contributions to choice theory.

[13] To a lesser degree, it is related to the companion paper they co-wrote with Lleras [2010].

[14] In earlier work, Futia [1977] studies the complexity of list-choice using the algebraic (or Krohn-Rhodes) measure.

sub-list of L. Second, any two *disjoint* lists L' and L'' (such that $S(L) \cap S(L') = \emptyset$) can be *concatenated* by appending L'' after the last item of L' to form the longer list $L' \oplus L''$. When I write $L' \oplus L''$, I explicitly assume that the two lists L' and L'' are disjoint. When L' is a sub-list of L such that $L = L' \oplus L''$ for some list L'', L' is an *initial segment* of L. Where it causes no confusion, I denote the length i initial segment of L_n by L_i. Likewise, I denote the sub-list beginning with the i^{th} item of L_n and ending with the later j^{th} item by L^i_j. Third, any list $L' \oplus L''$ can be *expanded* by inserting an item $a \notin L' \oplus L''$ to form the list $L' \oplus a \oplus L''$.[15] Finally, any list L can be *permuted* into another list σL by a permutation σ over X (i.e. $\sigma : X \to X$ is a bijection). I denote by $\sigma_{ab}L$ the simple permutation that "swaps" the items a and b in L. Naturally, $\sigma_{ab}L$ is defined even when $a \notin L$ or $b \notin L$.

Rubinstein and Salant [2006] extend the conventional notion of choice to the list environment as follows:

Definition 1 *A **list-choice function** D is mapping $\mathcal{L} \to X$ such that $D(L) \in L$ for any list $L \in \mathcal{L}$.*

For any list-choice function D, the i^{th} *provisional choice* from L_n is the choice $D(L_i)$ on the length i initial segment of L_n. Likewise, $D(L)$ is the *provisional choice* on the initial segment L of the list $L \oplus L'$.

2.2 Search Procedures

The two baseline models focus on search with recall and search without recall. In Section 5 below, I extend the analysis to consider a variety of other choice rules.

Formally, a *search procedure with recall* is a pair $(s, \succ)$ where $s : \mathcal{L} \to \{\texttt{stop}, \texttt{continue}\}$ is a search strategy and $\succ$ is a linear order over X. Based on the list-segment of items already considered, the search strategy s specifies whether to stop searching or to continue and examine the next item of the list (if any). Once the decision-maker stops searching, the preference $\succ$ then determines which item to select from among those considered. Stated more formally:

Definition 2 *The search procedure with recall $(s, \succ)$ determines a list-choice function $D_{s,\succ}$ where, for any list $L_n = < l_1, ..., l_i, ..., l_n > \in \mathcal{L}$, the choice $D_{s,\succ}(L_n)$ is given recursively by $D_{s,\succ}(L_1) \equiv l_1$ and*

$$D_{s,\succ}(L_{j+1}) \equiv \begin{cases} l_{j+1} & \text{if } s(L_i) = \texttt{continue} \text{ for all } i \leq j \text{ and } l_{j+1} \succ D_{s,\succ}(L_j) \\ D_{s,\succ}(L_j) & \text{otherwise} \end{cases}$$

*The search procedure $(s, \succ)$ **represents** the list-choice function D if $D(L) = D_{s,\succ}(L)$ for any list $L \in \mathcal{L}$. Moreover, $(s, \succ)$ and $(s', \succ')$ are **equivalent** if $D_{s,\succ}(L) = D_{s',\succ'}(L)$ for any list $L \in \mathcal{L}$.*

When choice from lists can be represented by a search procedure $(s, \succ)$, the decision-maker behaves *as if* she ignores the items that appear after the point where the strategy s specifies to stop searching. In this sense, only the items that appear before this point attract her attention. Borrowing from the psychology and marketing literature, the items that attract the decision-maker's attention determine her *consideration*

[15] Technically, one should write $L' \oplus < a > \oplus L''$ where $< a >$ denotes the list whose only member is a. For convenience, I'll drop the ordered-set brackets when the omission causes no confusion.

set. Formally, the consideration set $\mathcal{A}^s(L_n)$ associated with the list L_n and the strategy s is the set of items in L_i where i is the smallest index such that $s(L_i) = \texttt{stop}$:

$$\mathcal{A}^s(L_n) \equiv \{l_i \in L_n : s(L_j) = \texttt{continue} \text{ for all } j < i\}$$

Given the strategy s, the item a *attracts attention* on L if $a \in \mathcal{A}^s(L)$. Stated in terms of the consideration set $\mathcal{A}^s(L)$, the choice from L can then be represented by:

$$D_{s,\succ}(L) = \max_{\succ} \mathcal{A}^s(L)$$

where $\max_{\succ} A$ denotes the maximal element in A according to the preference $\succ$. This way of rewriting the representation emphasizes the interpretation that a decision-maker who follows a search procedure with recall chooses by maximizing a preference over the items that attract her attention. In other words, list-search can be represented as a *two-stage choice procedure* where the decision-maker first determines which items to consider before choosing the best item among those considered.

For search with recall, the decision-maker is able to choose the *best item* she encounters even when she continues on to consider subsequent list-items. The recall assumption may be interpreted in terms of search costs. Informally, the decision-maker finds it relatively cheap to "return to" any list-item previously considered and easy to "remember" the best item she encountered. While the assumption about the search environment is strong, the assumption about the decision-maker's memory capacity is relatively modest.

There are other circumstances where the decision-maker cannot return to list-items previously considered. In this case, she faces the restricted choice between continuing her search and choosing the *last item* considered. Formally, a *search procedure without recall* is a search strategy $s : \mathcal{L} \rightarrow \{\texttt{stop}, \texttt{continue}\}$ that defines a list-choice function D_s such that

$$D_s(L_n) \equiv \texttt{last}(\mathcal{A}^s(L_n))$$

for any list $L_n \in \mathcal{L}$. In this expression, $\mathcal{A}^s(L_n)$ denotes the consideration set associated with L_n and the strategy s and $\texttt{last}(\{a_i\}_{i \in I})$ denotes the element of $\{a_i\}_{i \in I}$ with the highest index $i \in \mathbb{N}$. Moreover, the search procedure s represents D if $D(L) = D_s(L)$ for any list $L \in \mathcal{L}$.

Of course, one might also consider "intermediate" versions of recall like *bounded recall.* As discussed in Section 5 below, my analysis is easily extended to such models.

2.3 Examples

Some examples help to motivate the analysis to follow. To simplify the presentation, I first discuss a variety of heuristics in the context of search with recall before discussing search without recall. One particularly simple heuristic is the satisficing rule proposed by Simon [1955]:

Example 1 (Best-Satisficing) *The decision-maker has a cutoff item $c^* \in X$ and a strict preference $\succ$. She stops searching when she reaches an item ranked higher than c^* according to $\succ$. Once she stops (or reaches the end of the list), she chooses the best item among those considered.*

Note that the special case where the cutoff item c^* is the decision-maker's favorite item in X coincides with *rational choice on lists*. While best-satisficing is straightforward to implement and generally leads to desirable choices, it requires extensive search in the worst case. In order to limit the extent of her search, a decision-maker could follow a strategy like the one proposed by Salant [2008]:

Example 2 (Markov Search) *The decision-maker has a cutoff set $C^* \subset X$ and a strict preference $\succ$. She stops searching if her provisional choice is in C^*. Once she stops (or reaches the end of the list), she chooses the best item among those considered.*

The strategy is Markovian in the sense that the search decision which determines the $(i+1)^{st}$ provisional choice only depends on the i^{th} provisional choice. There are two ways to interpret this feature. On the one hand, the provisional choice might serve as a sufficient statistic indicating that subsequent list-items are not likely to be worth considering. On the other hand, the limited dependence of the strategy on the list may also reflect limited memory and computational resources of the decision-maker.

Formally, the only difference from best-satisficing is that Markov search does not correlate the search decision with preference. In particular, a decision-maker using a Markov strategy may stop searching when her provisional choice is a even though she would have continued to search if her provisional choice were $a' \succ a$.

A related possibility is that the decision-maker conditions her search decision on her provisional choice *and* calendar time. In other words, the decision to stop after n items depends on whether her provisional choice is in the cutoff set C_n^*. Selten's [1998] model of aspiration adaptation is the special case of this strategy where the cutoff sets are nested so that $C_n^* \subset C_{n+1}^*$.[16] The basic idea is that the decision-maker starts with limited decision-making resources and aspirations about what is achievable with a certain amount of search. Calendar time serves as a proxy for the resources she has depleted. As the decision-maker expends more decision resources and learns about what items are available, she retreats to more modest aspirations.

Example 3 (Aspiration Adaptation) *The decision-maker has a collection of nested cutoff sets $\{C_i^*\}$ and a strict preference $\succ$. She stops searching after considering n items if her provisional choice is in C_n^*. Once she stops (or reaches the end of the list), she chooses the best item among those considered.*

In the no recall context, satisficing has been studied by Rubinstein-Salant [2006] and Salant [2010].

Example 4 (Last-Satisficing) *The decision-maker has a cutoff set $C^* \subset X$. She stops searching if her provisional choice is in C^*. Once she stops (or reaches the end of the list), she chooses the last item among those considered.*

[16] Sauermann and Selten's [1962] original paper on aspiration adaptation was published in German. The model was subsequently studied by Futia [1977].

The conventional interpretation is that C^* is a set of satisfactory items $\{x \in X : x \succ c^*\}$ above a cutoff c^* according to the (unmodeled) preference $\succ$. However, the way of writing the example makes clear that, for no recall, there is no difference between satisficing and Markov search strategies in terms of choice behavior. Given the cutoff C^*, one can simply *define* the preference $a \succ b$ if $a \in C^*$ and $b \notin C^*$. Since the decision-maker never chooses based on her preference, this definition poses no difficulty.

Even when the set of satisfactory items C^* is the same, best-satisficing and last-satisficing may induce different choice behavior. Although the two procedures coincide whenever the list contains a satisfactory item, they may differ when the list contains no satisfactory items. In that case, best-satisficing does not lead to the last item of the list being chosen unless it is also the most preferred.

3 Characterization of List-Search

In this section, I first provide choice-theoretic foundations for search with recall before considering the related characterization of search without recall.

3.1 Search with Recall

Before providing choice-theoretic foundations, I characterize the extent to which a representation for search with recall is identified (when it exists). For choice behavior that can be represented in terms of search with recall, straightforward behavioral definitions of attention and preference provide a basis to construct a canonical representation of search behavior.

3.1.1 Behavioral Definitions

For the moment, suppose that the list-choice function D results from search with recall. How can the decision-maker's search strategy and preference be inferred from D? My approach is to provide a behavioral definition of attention and construct a revealed preference from revealed attention. In turn, I use revealed preference to define revealed inattention. The notions of revealed attention and revealed inattention can then be used to provide bounds on the depth of the decision-maker's search. It goes without saying that these definitions depend critically on the assumption that D is consistent with sequential search. When this is not the case, the definitions are nonsensical.

Beginning with revealed attention, notice that the item l_i attracts attention on L_n when the decision-maker chooses l_i or a subsequent item on L_n. More generally, choice on a list that does not contain l_i may also reveal that l_i attracts attention on L_n. In particular, l_i attracts attention on L_n if there is a continuation $a \oplus L$ of the initial segment L_{i-1} where $D(L_{i-1} \oplus a \oplus L) \in a \oplus L$. Since $D(L_{i-1} \oplus a \oplus L)$ attracts attention on $L_{i-1} \oplus a \oplus L$, any list-item *weakly* before it also attracts attention. In particular, a attracts attention on $L_{i-1} \oplus a \oplus L$. Since the decision to examine a only depends on L_{i-1}, l_i attracts attention on L_n. Accordingly, the **minimal revealed attention set** $\mathcal{A}_R^-(L_n)$ is defined by

$$\mathcal{A}_R^-(L_n) \equiv \{l_i \in L_n : \exists \text{ list } L \in \mathcal{L} \text{ such that } D(L_{i-1} \oplus L) \in L\}$$

where L_{i-1} is the length $i-1$ initial segment of L_n. Based on this definition, the chosen item attracts attention (by setting $L = L_n^i$).

Given the behavioral notion of attention, the definition of a revealed preference is straightforward. In particular, an item a is revealed preferred to another item a' if there exists a list L where a is chosen and a' is revealed to attract attention. As such, the **direct revealed preference** P is defined by:

$$aPa' \text{ if there exists a list } L \in \mathcal{L} \text{ such that } a' \in \mathcal{A}_R^-(L) \text{ and } D(L) = a$$

Naturally, the **indirect revealed preference** P_R is defined as the transitive closure of P.

Given the revealed preference P_R, revealed inattention may be defined as follows. Suppose that there is some list $L_i \oplus a$ where a is not chosen. If a is indirectly revealed preferred to $D(L_i \oplus a)$, the decision-maker's strategy must have specified for her to stop before examining a. Otherwise, she would have chosen a. In other words, a cannot attract attention on $L_i \oplus a$. Since the choice to examine a only depends on the initial segment L_i however, l_{i+1} cannot attract attention on L_n. Following the same reasoning, none of the list-items after l_{i+1} attract attention on L_n. Formalizing these observations, the **revealed inattention set** $\mathcal{I}_R(L)$ is defined by

$$\mathcal{I}_R(L_n) \equiv \{l_j \in L_n : \exists \text{ item } a \in X \text{ such } aP_R D(L_i \oplus a) \text{ for some } i < j\}$$

where L_i is the length i initial segment of L_n. From $\mathcal{I}_R(L)$, one can then define the **maximal revealed attention set** $\mathcal{A}_R^+(L) \equiv S(L) \backslash \mathcal{I}_R(L)$.

In order to see that these behavioral definitions capture the extent to which preference and (in)attention are identified from choice, consider the set $R(s, \succ) \equiv \{(s', \succ') : D_{s,\succ}(L) = D_{s',\succ'}(L) \text{ for any list } L \in \mathcal{L}\}$ consisting of *all* representations equivalent to $(s, \succ)$. Certainly, the features common to the elements of $R(s, \succ)$ provide an "upper bound" on what might be identified from $D_{s,\succ}$. The following shows that the behavioral definitions capture *all* of the features common to the equivalent representations of $D_{s,\succ}$.

Proposition 1 (Identification) *Suppose $R(s, \succ) = \{(s_j, \succ_j)\}_{j \in J}$ is the class of all representations equivalent to the search procedure with recall $(s, \succ)$. Then:*

(I) *a is indirectly revealed preferred to a' iff a is $\succ_j$-preferred to a' for every $\succ_j$:*

$$aP_R a' \text{ iff } a \succ_j a' \text{ for all } j \in J.$$

(II) *a is in the minimal revealed attention set $\mathcal{A}_R^-(L)$ iff it attracts attention on L for every s_j:*

$$\mathcal{A}_R^-(L) = \cap_{j \in J} \mathcal{A}^{s_j}(L).$$

(III) *a is in the revealed inattention set $\mathcal{I}_R(L)$ iff it does not attract attention on L for any $(s_j, \succ_j)$:*

$$\mathcal{A}_R^+(L) = \cup_{j \in J} \mathcal{A}^{s_j}(L).$$

While part (I) describes the extent to which preference may be identified from behavior, parts (II) and (III) establish well-defined bounds on the extent of search consistent with the behavior in question. Since $\cap_{j \in J} \mathcal{A}^{s_j}(L) \subset \mathcal{A}^{s_i}(L) \subset \cup_{j \in J} \mathcal{A}^{s_j}(L)$, it follows that:

$$\mathcal{A}_R^-(L) \subset \mathcal{A}^{s_j}(L) \subset \mathcal{A}_R^+(L) \text{ for any strategy-preference pair } (s_j, \succ_j) \text{ that represents } D$$

The lower bound $\mathcal{A}_R^-(L)$ has a very natural interpretation. It describes the longest initial segment of the list L where choice does not change with further search. Stated more informally, it describes the point on the list where the marginal benefit of search is zero. Provided that any amount of incremental search carries an ϵ cost (for any $\epsilon > 0$), the lower bound may be interpreted as the point where a "sensible" decision-maker stops searching.

Moreover, the lower bound does not depend critically on which *choice rule* the decision-maker uses to pick from her consideration set $\mathcal{A}^s(L)$ once she stops searching. In Section 5 below, I show that part (II) of Proposition 1 applies equally to any model of search where the decision-maker's choice rule satisfies the mild requirement that additional search can only affect her choice by causing her to pick an item *further down the list.*[17] As such, the lower bound $\mathcal{A}_R^-$ also applies to search without recall.

Corollary 1 (Identification) *For any search procedure without recall s, $\mathcal{A}_R^-(L) = \mathcal{A}^s(L)$.*

The result follows directly from the proof of part (II) of Proposition 1 and the observation that the consideration sets $\mathcal{A}^s$ and $\mathcal{A}^{s'}$ coincide for search procedures s and s' that are equivalent. Since search without recall requires the decision-maker to choose the last item she examined, $\texttt{last}(\mathcal{A}^s(L_n)) = \texttt{last}(\mathcal{A}^{s'}(L_n))$ so that $\mathcal{A}^s(L_n) = \mathcal{A}^{s'}(L_n)$ for any list $L_n \in \mathcal{L}$.

3.1.2 Canonical Representation

These behavioral definitions can be used to construct a canonical representation of search behavior.

Definition 3 *For any search strategy s, the* ***canonical search strategy*** *$s_R(L)$ is defined by:*

$$s_R(L) = \texttt{continue} \textit{ iff } a \in \mathcal{A}_R^-(L \oplus a) \textit{ for some } a \in X$$

The consideration sets induced by this strategy coincide with the minimal revealed attention sets.

$$\mathcal{A}^{s_R}(L_n) \equiv \{l_i \in L_n : s_R(L_j) = \texttt{continue} \text{ for all } j < i\} = \mathcal{A}_R^-(L_n)$$

[17] The upper bound $\mathcal{A}_R^+$ does not have the same appealing feature. By definition, $\mathcal{A}_R^+(L)$ depends on the revealed preference relation P_R (and, consequently, the assumption that the decision-maker maximizes a preference in the second stage).

In this sense, the canonical strategy reflects the *most conservative* search strategy consistent with behavior. Provided that the behavior can be represented by a search procedure with recall, the following proposition establishes that the canonical search strategy may be used to construct a representation of choice behavior. To state the result, define $\succ_R$ to be a (generic) *completion* of P_R.

Proposition 2 (Canonical Representation) *If D can be represented by a search procedure with recall, it can be represented by $(s_R, \succ_R)$.*

This establishes that there is a family of *canonical representations* $\{(s_R, \succ_R)\}$ for any search procedure with recall. The remarks following Proposition 1 motivate the following definition:

Definition 4 *$(s, \succ)$ is **sensible** if $s(L) = \texttt{stop}$ for any list L such that $\max_{\succ} X \in L$.*

In general, there are representations of choice (see Example 5 below) where the decision-maker continues to search after she encounters her most preferred item. The sensibility criterion rules out such representations. The next proposition shows that sensible search procedures and canonical representations are equivalent.

Proposition 3 (Uniqueness) *A search procedure with recall is sensible iff it is canonical.*

Necessity is straightforward. To show sufficiency, suppose that the decision-maker uses a sensible search procedure but searches to the end of the list L (beyond her minimal revealed attention set) and picks some item $D(L) \neq \max_{\succ} X$. From the definition of revealed attention, $D(L \oplus a) = D(L)$ for any $a \notin L$. In particular, $D(L \oplus a) = D(L)$ even if a is the preference-maximizing item in X. This is a contradiction: either the decision-maker must choose a in this case or she must have stopped searching before reaching the end of the list L.

This sharpens the result obtained in Proposition 2. Provided the decision-maker searches sensibly, Proposition 3 establishes that her consideration sets are uniquely identified by $\mathcal{A}_R^-$.[18] Since every search procedure with recall admits an equivalent sensible representation, the restriction to sensible search is without loss of generality.

One might be curious whether the maximal revealed attention sets can be used to represent behavior. An example illustrates why this is not possible in general:

Example 5 *For $X = \{a_1, a_2\}$, $D(a_1, a_2) = a_1$ and $D(a_2, a_1) = a_2$.*

By definition, $\mathcal{A}_R^-(a_i, a_{-i}) = \{a_i\}$ and $\mathcal{A}_R^+(a_i, a_{-i}) = X$ for $i = 1, 2$. While the behavior reveals that the decision-maker pays attention to the first item of each list, it does not reveal that she ignores the last item of either list. It is straightforward to show that no linear order $\succ$ rationalizes these choices when the decision-maker pays attention to both items on $< a_1, a_2 >$ and $< a_2, a_1 >$. As the example suggests, the sets $\mathcal{A}_R^+$ are generically too inclusive to yield a representation. Roughly stated, the reason is that maximal

[18] Proposition 8 in the Appendix establishes a slightly weaker uniqueness result without the sensibility assumption.

revealed attention *mixes up* different representations $(s_i, \succ_i)$ that display countervailing dependencies of the strategy s_i on the preference $\succ_i$. To see this, note that the pairs $(s_1, \succ_1)$ and $(s_2, \succ_2)$ defined by

$$s_i(a_i) = \texttt{continue},\ s_i(a_j) = \texttt{stop} \text{ and } a_i \succ_i a_j \text{ where } i \neq j$$

both represent D.

3.1.3 Existence

I now turn to the existence of a representation for search with recall. The characterization relies on four simple axioms. The first two are related to the main biases associated with choice from lists. The first axiom captures the behavioral content of sequential search while the second reflects a weak form of the primacy bias. The other two axioms ensure that choice behavior is consistent across lists.

Axiom 1 (Sequential Choice) $D(L \oplus a) \in \{D(L), a\}$.

Lemma 3 of the Appendix shows that Sequential Choice is equivalent to $D(L \oplus L') \in \{D(L)\} \cup S(L')$. Intuitively, Sequential Choice imposes a kind of weak recursivity requirement on choice. For any list $L \oplus L'$, the decision-maker chooses between her provisional choice $D(L)$ on L and the subsequent items in L'. Sequential Choice may be viewed as a relaxation of the List-IIA property proposed by Rubinstein and Salant [2006] which states that:

$$D(L_n) = l_i \ \Rightarrow \ D(L_n - \{l_j\}) = l_i \text{ for any } i \neq j$$

Like List-IIA, Sequential Choice ensures that the last item in the list may be deleted without affecting choice (provided that it is not chosen). More precisely, $D(L_n - \{l_n\}) = D(L_n)$ if $n \neq i$. However, it says little about choice when a different unchosen item l_j is deleted. It merely requires that $D(L_n - \{l_j\}) \in \{D(L_{j-1})\} \cup S(L_n^{j+1})$.

The next axiom states that an improvement in the list-position of the chosen item has no impact on choice *unless* it prolongs search.

Axiom 2 (Weak Indifference to Improvement) *If $D(L \oplus b \oplus a) = a$, then $D(L \oplus a \oplus b) = a$.*

Since $D(L \oplus b \oplus a) = a$, the decision-maker considers all of the items in $L \oplus b \oplus a$ and prefers a. As such, her choice is not be affected by swapping the items a and b. Although the swap may cause her to ignore b, she continues to choose a.

The third axiom ensures that the decision-maker's preference does not depend on the list-order.

Axiom 3 (Preference Consistency) *If $D(L \oplus c) = c$ and $D(\bar{L} \oplus \bar{c}) = \bar{c}$, then for any $a \neq b$*

$$D(L \oplus b \oplus L') = a \textit{ implies } D(\bar{L} \oplus a \oplus \bar{L}') \neq b$$

Effectively, this axiom rules out framing effects not related to search. Formally, it is similar to the classical choice requirement that b cannot be chosen when a *is available* if a is chosen when b *is available*. For lists, this translates to the requirement that b is not chosen when a *attracts attention* if a is chosen when b *attracts attention*. The choice $D(L \oplus c) = c$ indicates that the decision-maker examines all of the items in L and continues to search. As such, b attracts attention on $L \oplus b \oplus L'$. The same reasoning establishes that a attracts attention on $\bar{L} \oplus a \oplus \bar{L}'$. Since $D(L \oplus b \oplus L') = a$ so that a is chosen when b attracts attention, a is revealed preferred to b. Accordingly, the requirement that $D(\bar{L} \oplus a \oplus \bar{L}') \neq b$ simply reflects the fact that b is not be chosen if a attracts attention and a is revealed preferred to b.

The final axiom guarantees that the decision-maker searches consistently.

Axiom 4 (Search Consistency) *If $D(L \oplus c) = c$ and $D(L \oplus b \oplus L') = a$, then*

$$D(\bar{L} \oplus b) = b \text{ implies } D(\bar{L} \oplus a) = a$$

If the chosen item is replaced with a preferable alternative (and the decision-maker's search remains unchanged), the replacement item is chosen. By the same reasoning as in the discussion above, the choices $D(L \oplus b \oplus L') = a$ and $D(L \oplus c) = c$ establish that a is revealed preferred to b and the choice $D(\bar{L} \oplus b) = b$ indicates that b is revealed preferred to every item in $\bar{L}$. Since the decision-maker examines all the items in $\bar{L}$ and continues to search, the choice $D(\bar{L} \oplus a) = a$ simply states that she chooses the item a *indirectly preferred* to every item in $\bar{L}$. Combined with Sequential Choice and Preference Consistency, Search Consistency ensures that the decision-maker's choice *must* improve when her choice is replaced by a more attractive alternative. Either the decision-maker chooses the replacement item or the list modification encourages further search and she ends up with a better choice.

Example 5 illustrates that choice behavior may reveal little or nothing about preference. However, the representation theorem establishes that the incompleteness of P_R has a straightforward interpretation in terms of choice behavior. In particular, it shows that two items are unranked by P_R if and only if they are *symmetric* in terms of choice. Formally:

Definition 5 *Items a and b are **choice-symmetric** with respect to D if, for every L such that $a \in L$:*

$$D(L) = a \text{ if and only if } D(\sigma_{ab} L) = b$$

For search with recall, any feasible alternative can play two roles. Just as in the standard choice setting, the alternative could be chosen. Moreover, the alternative could affect the extent of the decision-maker's search. Choice-symmetry teases apart these two roles. Roughly stated, two items are choice-symmetric when they cannot be distinguished by choice behavior alone.

Theorem 1 *D can be represented by the search procedure with recall $(s_R, \succ_R)$ iff it satisfies Sequential Choice, Indifference to Improvement, Preference Consistency, and Search Consistency.* ***(Uniqueness)*** *Among sensible search strategies, the canonical search strategy s_R is unique. Moreover, the linear order $\succ_R$ (which completes P_R) is unique up to the ordering of items that are choice-symmetric.*

Because the existence proofs in the paper all adopt the same approach, it is helpful to provide a brief overview. The bulk of the proof focuses on showing that the direct revealed preference P is asymmetric and acyclic. Asymmetry follows from Preference Consistency. Combining this axiom with Sequential Choice, Weak Indifference to Improvement, and Search Consistency, it can be shown that aP_Rb if and only if aPb or $aPa'Pb$ for some $a' \in X$. Since it can be shown that P is triple-acyclic, it follows that P is acyclic. The same result also establishes that two items are unranked by P_R if and only if they are choice-symmetric.

3.2 Search without Recall

I now turn to the existence of a representation for search without recall. It is clear that Sequential Choice is a necessary property for search without recall. Since the decision-maker invariably chooses the last item she considers, the following property is also necessary:

Axiom 5 (No Recall) *If $D(L \oplus L') \in L$, then $D(L \oplus L'') \in L$.*

Intuitively, this property states that choice is unaffected by modifying the tail of the list after the decision-maker's choice. It turns out that Sequential Choice and No Recall are sufficient to characterize search procedures without recall.

Theorem 2 *D can be represented by the search procedure without recall s_R iff it satisfies Sequential Choice and No Recall.* ***(Uniqueness)*** *Moreover, the canonical search strategy s_R is unique.*

Although the canonical search strategy was defined in the context of search with recall, it applies equally to search without recall. This follows directly from Corollary 1 and the definition of the canonical search strategy. For search without recall, there is no need to qualify uniqueness (in terms of sensible search or any other criterion). Since the decision-maker chooses the last item that she examines, any decision to continue searching may be observed directly from choice behavior.

4 Applications: Simple Search Heuristics

The search heuristics discussed in Examples 1-4 are simple *cutoff rules* where the search decision on the list L_i only depends on the decision-maker's provisional choice $D(L_i)$ and a cutoff set $C_i^* \subset X$ so that:

$$s(L_i) = \texttt{stop} \text{ iff } D(L_i) \in C_i^*$$

Each is succinctly characterized by strengthening the weak recursivity requirement of Sequential Choice. For the three heuristics with recall, choice also satisfies a property that strengthens Weak Indifference to Improvement and captures a strong form of the primacy bias.

4.1 Satisficing

For satisficing, the cutoff sets do not depend on how long the decision-maker searches.[19] Accordingly, the *last-satisficing procedure* in Example 4 is simply described in terms of a cutoff set C^*. For the *best-satisficing procedure* discussed in Example 1, the cutoff C^* corresponds to a set of *satisfactory items* $\{a \in X : a \succ c^*\}$ above a cutoff item c^*. As such, this procedure may be described in terms of a cutoff-preference pair $(c^*, \succ)$.

Both heuristics satisfy a form of path-independence for lists due to Rubinstein and Salant [2006].

Axiom 6 (Partition Independence) $D(L \oplus L') = D(D(L) \oplus D(L'))$.

Intuitively, this may be understood as a particularly strong form of choice recursivity. To see this, note that Partition Independence is equivalent to the List-IIA property discussed in Section 3.1.3 above (as shown by Rubinstein and Salant in their paper).

To see that both satisficing heuristics satisfy Partition Independence, suppose $D(L_n) = l_i$ and consider the sub-list $L_n - \{l_j\}$ obtained by removing an unchosen item l_j (i.e. $j \neq i$). First, suppose L_n contains a satisfactory item (so that the satisficing heuristics coincide). In this case, l_i is the first satisfactory item (i.e. $l_i \in C^*$) that the decision-maker encounters on L_n. Since it remains the first satisfactory item on $L_n - \{l_j\}$, her choice is unchanged. Next, suppose L_n contains no satisfactory items so that the decision-maker examines the whole list. In this case, the two heuristics differ. For best-satisficing (respectively last-satisficing), l_i is the best (respectively last) item in L_n. Since l_i remains the best (respectively last) item in the sub-list $L_n - \{l_j\}$, choice is unaffected. This establishes that both satisficing heuristics satisfy List-IIA and hence Partition Independence.

Combined with No Recall, Partition Independence is sufficient to characterize last-satisficing. In order to state the uniqueness portion of the theorem, define the *canonical cutoff set* C^*_R in terms of the canonical search strategy s_R:

$$C^*_R \equiv \{a : s_R(L) = \texttt{stop} \text{ and } D(L) = a \text{ for some } L \in \mathcal{L}\}$$

Theorem 3 *D can be represented by the last-satisficing procedure C^*_R iff it satisfies Partition Independence and No Recall.* ***(Uniqueness)*** *Moreover, the canonical cutoff set C^*_R is unique.*

The characterization of best-satisficing relies on two additional properties. The first strengthens Weak Indifference to Improvement while the second is an analogue of Preference Consistency for two-item lists.

Axiom 7 (Indifference to Improvement) *If $D(L \oplus b \oplus a \oplus L') = a$, then $D(L \oplus a \oplus b \oplus L') = a$.*

This property captures a strong form of the primacy bias by stating that *no* improvement in the list-position of the chosen item affects choice. While this property imposes strong limitations on choice, it finds support in the prevalence of *paid inclusion* and *search engine optimization.* Effectively, both of these

[19] Formally, $C^*_i = C^*_j$ for any $i \neq j$.

practices may be understood as investments aimed at improving internet search engine rankings. Paid inclusion refers to direct payments to the search engine while search engine optimization is an investment in "reverse engineering" the search engine's ranking scheme.[20] It is unlikely that commercial web sites would be willing to make these kinds of investments unless search engine users satisfied Indifference to Improvement on average.

Axiom 8 (Binary Preference Consistency) *If $D(a,b) \neq D(b,a)$, then*

$$D(a,c) = c \text{ implies } D(b,d) = b \text{ for any item } d \in X.$$

Formally, this property weakens Preference Consistency.[21] Intuitively, it ensures that order effects on two-item lists may be attributed to sequential search. In order to see this, note that the choice $D(a,c) = c$ indicates that the decision-maker continues to search after examining a. As such, b attracts attention on $< a, b >$. In order for $D(a,b) \neq D(b,a)$ to be consistent with sequential search, the decision-maker cannot consider a on $< b, a >$ since she would then choose the better of a and b from both lists. Hence, she stops searching after examining b. Stated in terms of choice, this is requirement that $D(b,d) = b$ for any item $d \in X$.

For behavior that can be represented as last-satisficing, Proposition 2 ensures that C_R^* is unique among cutoff representations where the best item is satisfactory. Following the approach taken for search, I define a cutoff set C^* to be *sensible* if $\max_{\succ} X \in C^*$.

Theorem 4 *D can be represented by the best-satisficing procedure $(c_R^*, \succ_R)$ iff it satisfies Partition Independence, Indifference to Improvement and Binary Preference Consistency.* ***(Uniqueness)*** *Among sensible cutoff representations, the canonical cutoff set C_R^* and the cutoff item $c_R^* = \max_P X \backslash C_R^*$ are unique. Moreover, $\succ_R$ is unique up to the ranking of the satisfactory items in $C_R^* = \{a : \text{no } b \in X \text{ such that } bPa\}$ and coincides with the revealed preference P on $X \backslash C_R^*$.*

The theorem establishes that one cannot identify a preference among the satisfactory items in C_R^*. This is a natural consequence of last-satisficing. Whenever the decision-maker encounters a satisfactory item, she stops searching and chooses that item.

As noted, *rational choice on lists* is as a form of best-satisficing where $c^* = \max_{\succ} X$. Since rational choice on lists displays no order effects, it may be characterized by strengthening Binary Preference Consistency in a way that rules out this behavior. In particular:

Axiom 9 (Binary Order Independence) $D(a,b) = D(b,a)$.

Theorem 5 *D can be represented as rational choice on lists iff it satisfies Partition Independence and Binary Order Independence. Moreover, the direct revealed preference P is a linear order.*

[20] Of course, this kind of behavior is not limited to cyberspace. Long before the internet, businesses chose strange and unappealing names in order to appear ahead of their competitors in the Yellow Pages.

[21] If Weak Indifference to Improvement holds, then $D(a,b) = a$ and $D(b,a) = b$ when $D(a,b) \neq D(b,a)$. Since $D(a,c) = c$, Preference Consistency implies $D(b,d) = b$ for any $d \in X$.

When combined with Partition Independence, Binary Order Independence implies Indifference to Improvement. The result then follows directly from Theorem 4 above.

4.2 Markov Search

The *Markov search procedure* in Example 2 can also be described in terms of a cutoff-preference pair $(C^*, \succ)$. Unlike best-satisficing however, the cutoff set C^* need not correspond to a set of satisfactory items. Intuitively, Markov search requires the decision-maker to evaluate the list successively by comparing her provisional choice with the next item in the list. Formally, this requirement is captured by the following choice property due to Salant [2003]:

Axiom 10 (Successive Choice) $D(L \oplus a) = D(D(L) \oplus a)$.

While similar in spirit to Sequential Choice, Successive Choice imposes a much stronger recursive structure on choice. As shown in Lemma 12 in the Appendix, it is equivalent to $D(L \oplus L') = D(D(L) \oplus L')$. As such, it may also be understood as a *partial* version of Partition Independence.

Theorem 1 establishes that the revealed preference P_R is identified up to choice-symmetry for search with recall. For the special case of Markov search, a stronger form of behavioral symmetry holds for any two items that are unranked by P_R:

Definition 6 *Two items a and b are **symmetric** with respect to D if, for every L:*

$$\sigma_{ab} D(L) = D(\sigma_{ab} L)$$

This definition strengthens the concept of choice-symmetry. Not only does it require that symmetric items be chosen symmetrically, but it also requires that choice coincide even when symmetric items are unchosen. Formally, it requires $D(\sigma_{ab} L) = b$ when $D(L) = a$ (as in the case of choice-symmetry) *and* $D(L) = D(\sigma_{ab} L)$ when $D(L) \notin \{a, b\}$. The added requirement captures the fact that symmetric items are indistinguishable in terms of choice *and* search. To see this, suppose that a and b are symmetric and suppose that $D(L \oplus c) = c$ so that the decision-maker continues searching after considering L. By symmetry, $D(\sigma_{ab}[L \oplus c]) = c$ so that the decision-maker continues searching after considering the swapped list $\sigma_{ab}[L \oplus c]$.

Theorem 6 *D can be represented by the Markov Search procedure $(C^*_R, \succ_R)$ iff it satisfies Successive Choice, Indifference to Improvement and Binary Preference Consistency. **(Uniqueness)** The canonical cutoff set C^*_R is unique among sensible cutoff representations and the linear order $\succ_R$ is unique up to the ordering of any two items that are symmetric.*

Theorem 6 formalizes the intuition that symmetry is the list-analogue of revealed indifference. This follows from the fact that it defines an equivalence on X. To see that symmetry is transitive, suppose that a and b are symmetric while b and c are symmetric. Intuitively, swapping a and b on L induces a "symmetric" choice on $\sigma_{ab}L$. The same holds true when b and c are swapped on $\sigma_{ab}L$ and a and b are re-swapped on $\sigma_{bc}\sigma_{ab}L$. Since $\sigma_{ac} = \sigma_{ab}\sigma_{bc}\sigma_{ab}$ holds by definition of permutations, $\sigma_{ac}D(L) = D(\sigma_{ac}L)$.[22]

Because symmetry defines an equivalence on X, Theorem 6 shows that the indirect revealed preference P_R is a weak order (with $a \sim_R b$ when neither aP_Rb nor bP_Ra). In other words, it establishes that $\succ_R$ is unique up to the ordering of items that are revealed indifferent.

In the discussion of Example 4, I mentioned that satisficing and Markov search strategies coincide for no recall. From the standpoint of behavior, this is born out by the fact that Successive Choice is equivalent to Partition Independence in the presence of No Recall (as established in Lemma 7 of the Appendix). Given Theorem 3, it follows that:

Corollary 2 *D can be represented by the last-satisficing procedure C_R^* iff it satisfies Successive Choice and No Recall.* ***(Uniqueness)*** *Moreover, the canonical cutoff set C_R^* is unique.*

4.3 Aspiration Adaptation

The *aspiration adaptation procedure* in Example 3 can be described as a pair $(\{C_i^*\}, \succ)$ consisting of a collection of nested cutoff sets $\{C_i^*\}$ (i.e. $C_i^* \subset C_{i+1}^*$) and a preference $\succ$. To characterize aspiration adaptation, two additional properties are required. The first weakens Successive Choice while the second is an analogue of Search Consistency for two-item lists.

Axiom 11 (Aspiration Successive Choice) *If $D(L'_m \oplus c) = c$ and $D(L'_m) = D(L_n)$ for some L'_m such that $n \leq m$ then:*

$$D(L_n \oplus a) = D(D(L_n) \oplus a)$$

Intuitively, this property ensures that the decision-maker chooses successively in a way that is consistent with aspiration adaptation. From $D(L'_m \oplus c) = c$, the decision-maker continues to search on L'_m when her provisional choice is $D(L'_m)$. Since her provisional choice is the same, she also continues to search on the shorter list L_n. In this case, her choice on $L_n \oplus a$ comes down to her choice between her provisional choice $D(L_n)$ on L_n and the last item of the list a.

Axiom 12 (Binary Search Consistency) *If $D(b, a) = a$ and $D(b, c) = b$, then $D(a, c) = a$.*

Formally, this property weakens Search Consistency.[23] Intuitively, it also weakens the choice consistency requirements associated with best-satisficing and Markov search.[24] At the same time, it is sufficiently strong to help ensure the acyclicity of the revealed preference P when combined with Sequential Choice and Indifference to Improvement. From $D(b, a) = a$ and $D(b, c) = b$, it follows that aPb and bPc. From $D(a, c) = a$, it follows that cPa cannot hold.

[22] Formally: $D(\sigma_{ab}\sigma_{bc}\sigma_{ab}L) = \sigma_{ab}D(\sigma_{bc}\sigma_{ab}L) = \sigma_{bc}\sigma_{ab}D(\sigma_{ab}L) = \sigma_{ab}\sigma_{bc}\sigma_{ab}D(L)$. For choice-symmetry, this line of reasoning fails at the first step since $D(\sigma_{ab}\sigma_{bc}\sigma_{ab}L) = c$ need not imply that $\sigma_{ab}D(\sigma_{bc}\sigma_{ab}L) = c$.

[23] To see this, suppose that $D(b, a) = a$ and $D(a, c) = c$. By Search Consistency, it follows that $D(b, c) = c$.

[24] In Lemma 8 of the Appendix, I show that Successive Choice must fail when Binary Preference Consistency fails.

For aspiration adaptation, it is natural to define a (nested) *collection of canonical cutoff sets* $\{C^*_{iR}\}$ by:

$$C^*_{iR} \equiv \{a : s_R(L_k) = \texttt{stop} \text{ and } D(L_k) = a \text{ for some } L_k \in \mathcal{L}_k \text{ such that } k \leq i\}$$

Following the approach above, I define a cutoff collection $\{C^*_i\}$ to be *sensible* if $\max_{\succ} X \in C^*_1$.

Theorem 7 *D can be represented by the Aspiration Adaptation procedure $(\{C^*_{iR}\}, \succ_R)$ iff it satisfies Sequential Choice, Aspiration Successive Choice, Indifference to Improvement, Binary Preference Consistency, and Binary Search Consistency.* ***(Uniqueness)*** *The canonical collection $\{C^*_{iR}\}$ is unique among sensible cutoff collections and the linear order $\succ_R$ is unique up to the ordering of any two items that are choice-symmetric.*

5 Extensions

In this section, I examine four natural extensions of the baseline models and show that the basic approach used to identify search can be extended these settings. For brevity, the text contains only a brief discussion of how to modify the baseline axioms in order to provide choice-theoretic foundations for these extensions. For the interested reader, a formal treatment is given in Appendix 9.

5.1 Choice Rules

The baseline models consider the two choice rules most frequently studied in search theory. Whereas *recall* requires that the best item considered be chosen, *no recall* requires that the last item considered be chosen. Clearly, there are plausible choice rules that do not coincide with either of these possibilities:

Example 6 (N-Recall Satisficing) *The decision-maker has a cutoff c^* and a strict preference $\succ$. She stops searching when she reaches an item ranked higher than c^* according to $\succ$. Provided that she stops before reaching the end of the list, she chooses the most recent (and highest ranked) item considered. Otherwise, she chooses the best item* ***among*** *the last N items considered.*

The example describes a *bounded recall* rule where the decision-maker can only choose one of the last N items considered.[25] Intuitively, bounded recall is *part-way* between recall and no recall. To see this, notice that N-recall satisficing induces behavior distinct from the baseline satisficing heuristics when the list contains no satisfactory items. If the best item is not one of the last N items, choice differs from best-satisficing. If the last item is not the best among the last N items, choice differs from last-satisficing.

While distinct from the baseline rules, bounded recall shares the feature that additional search can only affect choice by causing the decision-maker to choose an item *further down the list.* In other words, the bounded recall rule satisfies Sequential Choice. In a sense, this is a minimal requirement to represent

[25] This rule has a particularly straightforward interpretation in the context of internet search. If the decision-maker exhausts the list of results without finding a satisfactory result, she clicks the best link on the last page.

list-choice behavior in terms of search. Absent the restriction imposed by Sequential Choice, any D can be represented as a search procedure (s, D) where s is the trivial strategy that continues on every list.

In order to capture choice rules like bounded recall, let (s, r) define a *generalized search procedure* where s is a search strategy and $r : \mathcal{L} \to X$ is a *choice rule* such that $r(L \oplus a) \in r(L) \cup \{a\}$. As in the baseline case, the strategy s determines the extent of search on any list L and the choice rule r determines which item is chosen from the list-segment considered. It is straightforward to see that the baseline identification results carry through to generalized search procedures.

Proposition 4 *For any generalized procedure (s, r) such that $R(s, r) = \{(s_j, r_j)\}_{j \in J}$, $\mathcal{A}_R^-(L) = \cap_{j \in J} \mathcal{A}^{s_j}(L)$.*

5.2 Foreknowledge of the List

The baseline approach assumes that the decision-maker is *unaware* that an item is available until it is examined. One can relax this requirement somewhat by assuming that the decision-maker receives a coarse but informative signal about the list *before* she starts searching. In a variety of search environments, for instance, the decision-maker knows the list length before searching.[26] Just as search engines estimate the number of query results, Chinese restaurants number the dishes available on the menu. In these circumstances, the decision-maker can follow a strategy that conditions on the length of the list. Consider the following example (related to the simple search heuristics studied in the previous section):

Example 7 (Simple Length-Dependent Search) *The decision-maker has a collection of cutoff sets $\{C_i^*\}$. She stops searching on a list of length n after $j \leq n$ items if her provisional choice is in C_{n-j}^*.*

In order to model strategies that depend on a signal $\omega \in \Omega \subset \mathbb{N}$, let $\hat{\omega} : \mathcal{L} \to \Omega$ define a surjection so that $\mathcal{L}_\omega \equiv \{L \in \mathcal{L} : \hat{\omega}(L) = \omega\}$ is the sub-collection of lists that generate the signal ω. From the decision-maker's point of view, the collection of initial segments that might be associated with the signal ω is $\mathcal{L}^\omega \equiv \{L : L \oplus L' \in \mathcal{L}_\omega \text{ for some } L' \in \mathcal{L}\}$. To better understand these definitions, suppose that the signal $\omega = n$ reflects the length of the list. In that case, $\mathcal{L}_\omega$ is the collection $\mathcal{L}_n$ of lists with n items and $\mathcal{L}^\omega$ is the collection $\cup_{i=1}^n \mathcal{L}_i$ of lists with n items or less (i.e. the initial segments that the decision-maker *might* encounter on a list of n items). Given a signal $\omega \in \Omega$, the search strategy s_ω defines a mapping $\mathcal{L}^\omega \to \{\texttt{stop}, \texttt{continue}\}$. As such, $(\{s_\omega\}, r)$ defines a *generalized search procedure with a signal.*

Provided the signal is observable by the analyst, the only difference from the baseline models is that the signal imposes an additional measurability requirement on search. Intuitively, this requirement limits the *test lists* that can be used to identify the extent of search. In particular, l_i is revealed to attract attention on L_n when there is a list L' such that $D(L_{i-1} \oplus L') \in L'$ *and* the signal $\hat{\omega}(L_{i-1} \oplus L')$ coincides with $\hat{\omega}(L_n)$. Formally, minimal revealed attention is defined by:

$$\mathcal{A}_R^{\omega-}(L_n) \equiv \{l_i \in L_n : \exists \text{ list } L' \text{ such that } D(L_{i-1} \oplus L') \in L' \textbf{ and } \hat{\omega}(L_{i-1} \oplus L') = \hat{\omega}(L_n)\}$$

[26] Another possibility would be to model the decision-maker as having limited visibility of items that are *not too far* down the list from the item being considered.

Given the definition of revealed attention, the definition of the revealed preference P is straightforward. The baseline identification results for minimal attention and preference carry through to the setting of search with a signal.

Proposition 5 ***(I)*** *For any procedure with recall* $(\{s_\omega\}, \succ)$ *such that* $R(\{s_\omega\}, \succ) = \{(\{s_\omega\}_j, \succ_j)\}_{j \in J}$*:*

$$aP_R a' \text{ iff } a \succ_j a' \text{ for all } j \in J.$$

(II) *For any generalized procedure* $(\{s_\omega\}, r)$ *such that* $R(\{s_\omega\}, r) = \{(\{s_\omega\}_j, r_j)\}_{j \in J}$*,* $\mathcal{A}_R^{\omega-}(L) = \cap_{j \in J} \mathcal{A}^{\{s_\omega\}_j}(L)$*.*

To see the basic intuition, note that the collections $\mathcal{L}^\omega$ can be treated individually for the purpose of minimal revealed attention. The identification result for minimal attention can then be established in a manner analogous to the baseline case.

In general, there is no natural way to extend the baseline axioms to this setting. Intuitively, the problem is that Sequential Choice has no obvious analogue for search with a signal. In particular, the inclusion $D(L \oplus a) \in \{D(L), a\}$ need only hold when the signals $\hat{\omega}(L \oplus a)$ and $\hat{\omega}(L)$ coincide. At the same time, it is straightforward to extend the baseline axioms when the signal reflects the length of the list and the decision-maker is *more* inclined to stop searching (after examining an initial segment) when the list is longer. Formally, $s_n(L) = \texttt{stop}$ implies $s_{n+1}(L) = \texttt{stop}$ for any $L \in \mathcal{L}^n$.

In that case, Sequential Choice and Weak Indifference to Improvement can be stated as in the baseline case. Moreover, Preference Consistency, Search Consistency, and No Recall are easily reformulated. The only difference is that $D^*(L \oplus c) = c$ need not establish that a attracts attention on $L \oplus a \oplus L'_n$. To draw this inference, choice must satisfy the stronger requirement that $D^*(L \oplus L'_m) \in L'_m$ for some $m > n$.

5.3 Lists with Duplication

The baseline models study the impact of the list-order on the depth of search. In some situations, search may also be affected by the number of times that an item appears. Consider a decision-maker who uses a search engine (like *Google Shopping*) to make a purchase online. In this case, her query is likely to return a number of results that differ only on choice-irrelevant dimensions (i.e. the identity of the seller). Moreover, it is conceivable that the duplicates might affect the extent of her search by providing information about the results that are likely to appear further down the list.

Allowing for duplication, any finite sequence of elements in X defines a list. I denote the collection of lists (with duplication) by $\mathcal{L}^*$ and a choice function over $\mathcal{L}^*$ by D^*. In this setting, the search strategy can be extended to a mapping $s^* : \mathcal{L}^* \to \{\texttt{stop}, \texttt{continue}\}$. As such, (s^*, r) defines a *generalized search procedure (with duplication)* that induces a choice function on $\mathcal{L}^*$.

Duplication poses some challenges for identification. To determine that l_i attracts attention on L_n, it is no longer sufficient to consider the lists obtained by modifying the tail L_n^i. The problem is that $D^*(L_{i-1} \oplus a) = a$ is not enough to determine whether l_i attracts attention on L_n.[27] When $a \in L_{i-1}$, the

[27] Naturally, I relax the definition of $\oplus$ to allow for concatenation of lists that are not disjoint.

choice a may correspond to the last item in $L_{i-1} \oplus a$ but it could equally represent a duplicate in L_{i-1}. To distinguish between these possibilities, it is sufficient that $D^*(L_{i-1}) \neq a$. The basic intuition is that additional search only affects choice by causing the decision-maker to choose an item further down the list.[28] In order for her choice to change from L_{i-1} to $L_{i-1} \oplus a$, the decision-maker must have continued searching up to the end of the list L_{i-1}. Accordingly, minimal revealed attention is defined by:

$$\mathcal{A}_R^{*-}(L_n) \equiv \{l_i \in L_n : \exists \text{ list } L' \text{ such that } D^*(L_{i-1} \oplus L') \neq D^*(L_{i-1})\}$$

It is worth pointing out that the preceding definition would have worked equally well for lists without duplication (though it is more general than needed). Given the definition of revealed attention, the revealed preference P can be defined as in the baseline case. Using these definitions, the baseline identification results can be extended to lists with duplication.

Proposition 6 ***(I)*** *For any procedure with recall $(s^*, \succ)$ such that $R(s^*, \succ) = \{(s_j^*, \succ_j)\}_{j \in J}$:*

$$aP_R a' \textit{ iff } a \succ_j a' \textit{ for all } j \in J.$$

(II) *For any generalized procedure (s^*, r) such that $R(s^*, r) = \{(s_j^*, r_j)\}_{j \in J}$, $\mathcal{A}_R^{*-}(L) = \cap_{j \in J} \mathcal{A}^{s_j^*}(L)$.*

It is not difficult to extend the baseline axioms to lists with duplication. Formally, the only difference is that $D^*(L \oplus c) = c$ is no longer sufficient to establish that c attracts attention on $L \oplus c$. In order to draw this inference, choice must satisfy the stronger requirement that $D^*(L \oplus c) \neq D^*(L)$. While this change has no impact on Sequential Choice and Weak Indifference to Improvement, it requires a natural reformulation of Preference Consistency, Search Consistency, and No Recall.

5.4 List-Choice Correspondences

It is natural to extend the model of search with recall to list-choice correspondences $\bar{D} : \mathcal{L} \to 2^X$ where $\bar{D}(L) \subset S(L)$ by modeling preference as a weak order $\succeq$ over the items in X. Given a search procedure $(s, \succeq)$, the search strategy s determines the depth of search as in the baseline model. Extending the baseline model, the choice correspondence $\bar{D}_{s,\succeq}$ reflects *all* of the $\succeq$-maximizing items considered.

This extension poses no difficulty for identification. In particular, l_i is revealed to attract attention on L_n only when there is some list L' such that $\bar{D}(L_{i-1} \oplus L')$ *contains* an item in L'. Formally, the minimal revealed attention is defined as:

$$\bar{\mathcal{A}}_R^{-}(L_n) \equiv \{l_i \in L_n : \exists \text{ list } L \text{ such that } \bar{D}(L_{i-1} \oplus L) \cap L \neq \emptyset\}$$

[28]Formally, generalized choice rules satisfy Sequential Choice.

It is then straightforward to define direct revealed preference relations by:

$$aRa' \text{ if there exists a list } L \text{ such that } a' \in \bar{\mathcal{A}}_R^-(L) \text{ and } a \in \bar{D}(L); \text{ and}$$
$$aPa' \text{ if there exists a list } L \text{ such that } a' \in \bar{\mathcal{A}}_R^-(L) \backslash \bar{D}(L) \text{ and } a \in \bar{D}(L).$$

Here, R defines a *weak* preference while P defines a *strict* preference. The revealed indifference relation I_R is defined as the symmetric part of the transitive closure of R. Likewise, the indirect revealed preference P_R is defined by aP_Ra' if there is a chain $a = a_1R...Ra_n = a'$ such that a_iPa_{i+1} for some $1 \leq i \leq n$. Using these definitions, the baseline identification results for minimal attention and preference carry through to choice correspondences.

Proposition 7 *For any search procedure with recall* $(s, \succeq)$ *such that* $R(s, \succeq) = \{(s_j, \succeq_j)\}_{j \in J}$*:*
(I) aP_Ra' *(respectively* aI_Ra'*) iff* $a \succ_j a$ *(respectively* $a \sim_j a'$*) for all* $j \in J$*; and*
(II) $\bar{\mathcal{A}}_R^-(L) = \cap_{j \in J} \bar{\mathcal{A}}^{s_j}(L)$.

It is fairly straightforward to extend the baseline axioms to this setting. Broadly, it is sufficient to replace any "=" in the axioms with "∈" and to replace "∈" with "⊂". In addition, Preference Consistency must be modified to capture the asymmetry of the *strict* revealed preference. Provided b attracts attention on L and a attracts attention on $\bar{L}$, $a \in D(L)$ does not rule out the possibility that $b \in D(\bar{L})$ when a and b are indifferent. In other words, it is necessary to impose the additional requirement that $b \notin D(L)$.

6 Conclusion

Motivated by a wealth of empirical evidence, I model choice from lists in terms of sequential search. Taking an axiomatic approach, I study models of search with and without recall. The axiomatization of these models provides choice-theoretic foundations for sequential search and recall. The structure of the choice environment permits a natural identification of search and preference. For behavior that can be represented in terms of sequential search, the search strategy can be uniquely determined from choice.

As discussed, my results have practical implications for empirical work and internet search. The paper also suggests two avenues of further research. First, the natural extensions I consider show that the basic approach used to identify search is robust and can be easily extended to a variety of settings. Another extension that might be worth considering is a random choice model where the decision-maker uses a mixed strategy stopping rule. Second, the simple search heuristics characterized in this paper suggest that lists provide a natural framework to study bounded-rationality heuristics based on search. Future work might explore the choice-theoretic foundations for some other heuristics discussed in the literature.[29]

[29] Hey ([1981] and [1982]), Moon-Martin [1996], Rubinstein-Salant [2006] and Salant [2010] discuss a variety of possibilities.

7 Appendix: Related Literature

7.1 Search Procedures with Recall

Search procedures with recall are related to Caplin and Dean's [2010] model of alternative-based search and the model of choice with limited attention due to Masatlioglu, Nakajima and Ozbay [2009] (MNO).

7.1.1 Alternative-Based Search

Caplin and Dean study how choice evolves over time. Informally, the idea of their model is that the decision-maker searches through the feasible set and, at any point in time, her provisional choice reflects the set of preference-maximizing items among those considered so far. To present the model more formally, let $\Pi_{t=1}^{\infty}2^X$ define the sequences of subsets in X. The decision-maker's sequence of provisional choices then defines a *choice process* $C : 2^X \to \Pi 2^X$ such that $\cup_{t=1}^{\infty}C_t(A) \subset A$ for any $A \subset X$. Caplin and Dean use this framework to study *alternative-based search* (ABS). Formally, an ABS procedure is a pair $(\bar{s}, \succeq)$ that defines a choice process

$$C_{\bar{s},\succeq}(A) \equiv \prod_{t=1}^{\infty} \max_{\succeq} \bar{s}_t(A)$$

where the *search process* $\bar{s}$ is a non-decreasing choice process $\bar{s} : 2^X \to \Pi 2^X$ such that $\bar{s}_t(A) \subset \bar{s}_{t+1}(A)$ and the preference $\succeq$ is a weak order. Intuitively, the search process describes how the set of alternatives considered "grows" over time. At any time t, the provisional choice $\max_{\succeq} \bar{s}_t(A)$ can be interpreted as the set of optimal items among those considered so far.

Any list-choice function D can be used to construct a choice process. For every menu $A \subset X$, fix a list $\bar{L}(A) \in \mathcal{L}$ whose items coincide with A. This defines a *unique collection* $\bar{\mathcal{L}} \subset \mathcal{L}$ that contains exactly one list-ordering $\bar{L}(A)$ for every subset A of X. Given any pair $(D, \bar{\mathcal{L}})$ where $\bar{\mathcal{L}}$ is a unique collection, the sequence of provisional choices on $\bar{L}(A)$ defines a choice process on A

$$C_{D,\bar{\mathcal{L}}}(A) \equiv D(\bar{L}_1(A)), ..., D(\bar{L}_t(A)), ..., D(\bar{L}_{|A|}(A)), ...$$

where $\bar{L}_t(A)$ is the initial segment consisting of the first t items in $\bar{L}(A)$. Taking this approach, any search procedure with recall naturally generates an ABS.

Remark 1 *For any D representable by $(s, \succ)$ and any unique collection $\bar{\mathcal{L}}$, $C_{D,\bar{\mathcal{L}}}$ is ABS-representable by $(\bar{s}, \succ)$ with $\bar{s}_t(A) \equiv \mathcal{A}^s(\bar{L}_t(A))$.*

However, there are ABS procedures (whose components are singletons) that do not correspond with any pair $(D_{s,\succ}, \bar{\mathcal{L}})$. A simple example serves to illustrate.

Example 8 $C(\{x, y\}) = (\{x\}, \{y\}, ...)$, $C(\{x, z\}) = (\{x\}, ...)$, *and* $C(\{y, z\}) = (\{y\}, \{z\}, ...)$

To simplify the presentation, I have condensed the portions of the process where choice is unchanged. This pattern of choice is uniquely represented as an ABS procedure with:

$$z \succ y \succ x \text{ ; } \bar{s}(x,y) = (\{x\},\{x,y\},...); \ \bar{s}(x,z) = (\{x\},...); \text{ and } \bar{s}(y,z) = (\{y\},\{y,z\},...)$$

To see why, consider the menu $\{x, y\}$. Since x is the first provisional choice but is ultimately displaced by y, it follows that $y \succ x$ and that $\bar{s}(x,y) = (\{x\},\{x,y\},...)$. Similar reasoning applies to $\{y, z\}$. For $\{x, z\}$, the search process $\bar{s}(x, z)$ is identified by the revealed preference $z \succ x$ and the fact that z is never provisionally chosen.

However, the behavior does not correspond to any $(D_{s,\succ}, \bar{\mathcal{L}})$. To see this, notice that the decision-maker must use the lists $< x, y >$ and $< x, z >$ to choose from $\{x, y\}$ and $\{x, z\}$. Otherwise, she cannot provisionally choose x first from either set. The problem is that no search strategy induces the desired choice process on both sets: if $s(x) = \texttt{continue}$, then $\bar{s}_2(x, z) = \{x, z\}$; and, if $s(x) = \texttt{stop}$, then $\bar{s}_2(x, y) = \{y\}$.

The example shows that ABS procedures cannot be reconciled with the naïve search behavior contemplated by my model. Given the menus $\{x, y\}$ and $\{x, z\}$, the decision-maker arrives at conflicting search decisions after considering x. This suggests that ABS procedures accommodate search behavior where the decision-maker has a sophisticated understanding of the feasible set before considering any of its items. Since $|\{x, y\}| = |\{x, z\}|$, this understanding goes beyond the size of the menu.

7.1.2 Limited Attention

MNO study two-stage procedures $(\Gamma, \succ)$ (without lists) where the consideration sets $\Gamma(A)$ satisfy the *attention filter* property:

$$\Gamma(A) = \Gamma(A \backslash \{a\}) \text{ for any } a \notin \Gamma(A)$$

The basic rationale for this property is that the decision-maker's attention should not be affected by removing an item that she ignores. In their paper, a *limited attention* procedure $(\Gamma, \succ)$ defines a standard choice function $2^X \to X$ such that $c_{\Gamma,\succ}(A) = \max_{\succ} \Gamma(A)$ for any $A \subset X$.

For any pair $(D, \bar{\mathcal{L}})$, the list-choices on $\bar{\mathcal{L}}$ define a standard choice function where:

$$c_{D,\bar{\mathcal{L}}}(A) \equiv D(\bar{L}(A))$$

for any $A \subset X$. Any standard choice function can be generated by $(D_{s,\succ}, \bar{\mathcal{L}})$ when $(s, \succ)$ is the trivial search procedure that picks the first item on every list and the unique collection $\bar{\mathcal{L}}$ is chosen appropriately. To draw a more meaningful connection, consider the restrictive class of *consistent collections* where the list associated with every menu $A \subset X$ reflects the *same* search order. In order to obtain a consistent collection $\hat{\mathcal{L}}$, fix a list $\hat{L}(X) \in \mathcal{L}$ whose items coincide with the grand set X. For any subset $A \subset X$, let $\hat{L}(A) \equiv \hat{L}(X) - X \backslash A$ be the list obtained by trimming the items not in A.

Remark 2 *For any D representable by $(s, \succ)$ and any consistent collection $\hat{\mathcal{L}}$, $c_{D,\hat{\mathcal{L}}}$ is limited attention-representable by $(\hat{\Gamma}^s, \succ)$ with $\hat{\Gamma}^s(A) \equiv \mathcal{A}^s(\hat{L}(A))$.*

However, there are limited attention choice functions that do not correspond to any $(D_{s,\succ}, \hat{\mathcal{L}})$.

Example 9 *$c(\{x, z\}) = x$, $c(\{y, z\}) = y$ and $c(\{x, y, z\}) = z$*

This behavior is uniquely represented in terms of choice with limited attention by:

$$z \succ x;\ z \succ y;\ \Gamma(x, z) = \{x\};\ \Gamma(y, z) = \{y\};\ \text{and}\ \Gamma(x, y, z) = \{x, y, z\}$$

To see why, consider the menu $\{x, y, z\}$. Since she chooses z, the decision-maker pays attention to z. However, she also pays attention to x and y because her choice changes when each of these items is removed. This establishes $\Gamma(x, y, z) = \{x, y, z\}$. Since the decision-maker pays attention to all the items on $\{x, y, z\}$ and chooses z, it must be that z is preferred to x and y so that $z \succ x$ and $z \succ y$. These preferences imply that the decision-maker cannot pay attention to z on $\{x, z\}$ or $\{y, z\}$. Otherwise, she would choose z.

At the same time, the behavior does not correspond to any $(D_{s,\succ}, \hat{\mathcal{L}})$. Regardless of how the grand list $\hat{L}(x, y, z)$ is specified, no search strategy induces these choices. To see this, suppose that z is the first item in $\hat{L}(x, y, z)$. If $s(z) = \texttt{continue}$, $c(x, z) = x$ requires $x \succ z$. Since $\{x, z\} \subset \hat{\Gamma}^s(x, y, z)$, it follows that $c_{D_{s,\succ},\hat{\mathcal{L}}}(x, y, z) \neq z$. If, on the other hand, $s(z) = \texttt{stop}$, then $\hat{\Gamma}^s(x, z) = \{z\}$ so that $c_{D_{s,\succ},\hat{\mathcal{L}}}(x, z) = z$. By similar reasoning, it follows that x and y cannot be the first items in the grand list $\hat{L}(x, y, z)$.

The example suggests that MNO's framework allows for limited attention that does not result from search. Since the consideration sets are uniquely identified in the example, this may be inferred directly from the representation. Intuitively, the consideration sets do not have the structure that search requires. For $\Gamma(x, y, z) = \{x, y, z\}$ to be consistent with sequential search, the decision-maker must continue after considering the first item of the grand list. Regardless of how the grand list is specified, it must be that $\Gamma(x, z) = \{x, z\}$ or $\Gamma(x, y) = \{x, y\}$.

7.2 Simple Search Heuristics

The simple heuristics studied in this paper are related to Rubinstein and Salant's [2006] model of *priority choice*, their recent paper on welfare preferences [2010], and Salant's model of *successive choice* [2003].

7.2.1 Satisficing

Rubinstein and Salant [2006] propose a model of *priority choice* from lists where the decision-maker uses the list order to break ties when she is indifferent among two or more alternatives. Formally, the model can be described in terms of a pair $(\succeq, \delta)$ where $\succeq$ is a weak order on X and $\delta : X \to \{1, 2\}$ is a *priority indicator* such that $\delta(a) = \delta(b)$ whenever $a \sim b$. For any list L, the priority list-choice function $D_{\succeq,\delta}$ picks the first (respectively last) $\succeq$-maximal item in L when the priority δ of the $\succeq$-maximal indifference class is 1 (respectively 2).

In their paper, Rubinstein and Salant show that Partition Independence is both necessary and sufficient to characterize the model of priority choice. From Theorems 3 and 4, it follows that:

Remark 3 *Both satisficing procedures can be represented in terms of priority choice. For best-satisficing, the priority indicator is defined by $\delta(a) = 1$ for any $a \in X$ and the weak order $\succeq$ is defined by the revealed preference P. For last-satisficing, the indicator is defined by $\delta(a) = 1$ for any $a \in C_R^*$ and $\delta(a) = 2$ for any $a \in X \backslash C_R^*$. The weak order $\succeq$ is defined by $a \succ b$ for any $a \in C_R^*$ and $b \in X \backslash C_R^*$ and $a \sim b$ otherwise.*

Although Rubinstein and Salant acknowledge this possibility (in the case of last-satisficing at least), Theorems 3 and 4 formalize their intuition by providing choice-theoretic foundations.

In a recent paper, Rubinstein and Salant [2010] provide an alternate characterization of best-satisficing and last-satisficing. However, there are several key differences from my approach worth noting. For one, they characterize both heuristics in terms of conditions (such as acyclicity and asymmetry) on revealed preference relations. Theorems 3 and 4 directly connect these revealed preference conditions to choice behavior. Moreover, Rubinstein and Salant focus on satisficing when the cutoff may vary across lists. In contrast, I assume that the cutoff is fixed across lists.[30]

7.2.2 Markov Search

In Salant's model of *successive choice* from lists, the decision-maker chooses progressively with an anchoring bias. Starting with the first list-item, she examines *the entire list* in sequence. At any point, the most recent item she examines displaces her provisional choice when she finds "good reason", as described by an asymmetric ranking relation R, to prefer it. Formally, a successive choice function D_R is defined by:

$$D_R(b, a) \equiv a \text{ if and only if } aRb; \text{ and } D_R(L_n) \equiv D_R(D_R(L_2) \oplus L_n^3) \text{ for } n \geq 3$$

Shu [2009] establishes that D can be represented as a successive choice function iff it satisfies Successive Choice and Indifference to Improvement *as restricted to* two-item lists (i.e. $D(a, b) = b \Rightarrow D(b, a) = b$). In the representation of D, the ranking relation is defined by aRb if $D(b, a) = a$.

In Lemma 1 of the next section, I establish that the ranking R is transitive if and only if D satisfies Indifference to Improvement. Combined with Theorem 6, it follows that:

Remark 4 *Any Markov search procedure $(C^*, \succ)$ can be represented in terms of successive choice with a transitive ranking relation R such that aRb if $b \notin C_R^*$ and aPb.*

Accordingly, Binary Preference Consistency provides a means to distinguish successive choice from search in terms of choice behavior. Although Binary Preference Consistency does not rule out a successive

[30] Another minor difference is that they study linear orders $\succ_{L(X)}$ on grand lists $L(X)$ induced by $a \succ_{L(X)} b$ iff $D(L(a, b)) = a$. It is easy to see that the revealed preference $\succ_{L(X)}$ on any grand list is a linear order iff D satisfies List-IIA.

choice representation, it imposes a strong restriction on the ranking relation R that has no obvious motivation in Salant's model. Restated in terms of R, this property requires that the decision-maker cannot be indifferent between a and b (i.e. neither aRb nor bRa) when there is an item c that she finds good reason to prefer over both a and b.

7.2.3 Proof

Shu [2009] defines Insertion Stability as the list-choice property that:

$$\text{If } D(L_n) = l_i, \text{ then } D(L_{k-1} \oplus b \oplus L_n^k) \neq l_j \text{ for any } i < k \leq j \text{ or } j < k \leq i.$$

In her paper, she shows that: *D can be represented in terms of successive choice with a transitive ranking relation iff it satisfies Successive Choice, Binary Indifference to Improvement (i.e. Indifference to Improvement restricted to two-item lists), and Insertion Stability.*

I show that Shu's axiomatization is equivalent to the one given in the text.

Lemma 1 ***(I)*** *Successive Choice, Binary Indifference to Improvement, and Insertion Stability imply Indifference to Improvement.* ***(II)*** *Indifference to Improvement and Successive Choice imply Insertion Stability.*

Proof. **(I)** Suppose $D(L_n) = l_i$. By Successive Choice, $D(L_i) = l_i$. By Insertion Stability, $D(L_{i-2} \oplus l_i) = l_i$. By Successive Choice, $D(L_{i-2} \oplus l_i \oplus L_n^{i+1}) = D(D(L_{i-2} \oplus l_i) \oplus L_n^{i+1}) = D(l_i \oplus L_n^{i+1}) = D(D(L_i) \oplus L_n^{i+1}) = D(L_n) = l_i$. By Insertion Stability, it follows that $D(L_{i-2} \oplus l_i \oplus l_{i-1} \oplus L_n^{i+1}) \notin L_n^{i+1}$. Then, $D(L_{i-2} \oplus l_i \oplus l_{i-1} \oplus L_n^{i+1}) = D(L_{i-2} \oplus l_i \oplus l_{i-1}) = D(D(L_{i-2} \oplus l_i), l_{i-1}) = D(l_i, l_{i-1})$ by Successive Choice. Now, suppose (by way of contradiction) that $D(l_i, l_{i-1}) = l_{i-1}$. Then, $D(L_{i-2} \oplus l_i \oplus l_{i-1}) = l_{i-1}$ so that $D(L_{i-1}) = l_{i-1}$ by Insertion Stability. By Successive Choice, it follows that $D(L_i) = D(D(L_{i-1}), l_i) = D(l_{i-1}, l_i) = l_i$. By Binary Indifference to Improvement, it follows that $D(l_i, l_{i-1}) = l_i$ which establishes the desired contradiction. Thus, $D(L_{i-2} \oplus l_i \oplus l_{i-1} \oplus L_n^{i+1}) = l_i$.

(II) Suppose $D(L_n) = l_i$. There are two cases: (i) $i < k$; and (ii) $k \leq i$. Case **(i)**: Suppose $D(L_{k-1} \oplus b \oplus L_n^k) = l_j \in S(L_n^k)$. Then, by Indifference to Improvement, $D(L_{k-1} \oplus l_j \oplus b \oplus L_{j-1}^k \oplus L_n^{j+1}) = l_j$. By Successive Choice, $D(L_{k-1} \oplus l_j) = D(D(L_{k-1}), l_j) = l_j$. By Successive Choice, it also follows that $D(L_{k-1}) = D(L_{j-1}) = D(L_j) = l_i$ so that $D(D(L_{k-1}), l_j) = D(l_i, l_j) = D(D(L_{j-1}), l_j) = D(L_j)$. This is the desired contradiction. Case **(ii)**: Suppose $D(L_{k-1} \oplus b \oplus L_n^k) = l_j \in L_{k-1}$. By Successive Choice, $D(D(L_{k-1} \oplus b) \oplus L_n^k) = l_j$ so that $D(L_{k-1} \oplus b) = l_j$. By Successive Choice, it follows that $D(L_{k-1}) = l_j$. Thus, $D(L_{k-1} \oplus b \oplus L_n^k) = D(l_j \oplus L_n^k) = D(D(L_{k-1}) \oplus L_n^k) = D(L_n)$. This is the desired contradiction. ■

8 Appendix: Proofs

8.1 Search with Recall

8.1.1 Identification and Uniqueness

Proof of Proposition 1 (Identification-Recall). The proof of (I) relies on (II) and Proposition 2 below. The proof of (III) relies on the proof of Proposition 8 below. The reader may want to follow in this order.

(I) Revealed Preference: ($\Longrightarrow$) First, observe that aPa' implies $a \succ_j a'$ for any $(s_j, \succ_j)$ that represents D. To see this, suppose that aPa'. Since aPa', there must be a list L such that $D(L) = a$ and $a' \in \mathcal{A}_R^-(L)$. By the equality established in part (II) below, it follows that $\mathcal{A}_R^-(L) \subset \mathcal{A}^{s_j}(L)$ so that $\{a, a'\} \subset \mathcal{A}^{s_j}(L)$. Since $D_{s_j,\succ_j}(L) = D(L) = a$, it must be that $a \succ_j a'$. Now, more generally, suppose that aP_Ra'. Since aP_Ra', it follows that:

$$aPa_1P...Pa_mPa'$$

for some choice of $\{a_i\}_{i=1}^m$. From the observation that aPa' implies $a \succ_j a'$ for any $(s_j, \succ_j)$ that represent D, it follows that:

$$a \succ_j a_1 \succ_j ... \succ_j a_m \succ_j a'$$

so that $a \succ_j a'$ by transitivity. ($\Longleftarrow$) Suppose $a \succ_j a'$ for all $(s_j, \succ_j)$ that represent D but that $\neg(aP_Ra')$. Define the pair $(s_R, \succ_R)$ where s_R is the canonical search strategy and $\succ_R$ is a completion of P_R such that $a' \succ_R a$. Because $\neg(aP_Ra')$, one can construct such a completion. By Proposition 2 below, $(s_R, \succ_R)$ is a canonical representation of D. Since $a' \succ_R a$ by construction, it follows that $a \succ_j a'$ cannot hold for all $(s_j, \succ_j)$ that represent D. This is the desired contradiction.

(II) Revealed Attention: ($\Longrightarrow$) Suppose $l_i \in \cap_{j\in J}\mathcal{A}^{s_j}(L_m)$ where $\{(s_j, \succ_j)\}_{j\in J}$ is the non-empty collection of pairs that represent D. By way of contradiction, suppose that $l_i \notin \mathcal{A}_R^-(L_m)$. By definition, $D(L_{i-1} \oplus L) \notin L$ for all possible list extensions L. Thus, $D(L_{i-1} \oplus L) = D(L_{i-1})$ for any list extension L. Now, pick any $(s_j, \succ_j)$ that represents D. It must be that $s_j(L_k) = \texttt{continue}$ for all $k < i$. Using the strategy s_j, define the search strategy s'_j so that $s'_j(L_{i-1}) \equiv \texttt{stop}$ and $s'_j(L') \equiv s_j(L')$ for all $L' \neq L_{i-1}$. By construction, $(s'_j, \succ_j)$ represents D since $D_{s'_j,\succ_j}(L_{i-1} \oplus L) = D_{s_j,\succ_j}(L_{i-1}) = D(L_{i-1}) = D(L_{i-1} \oplus L)$ for any list extension L (and $D_{s'_j,\succ_j}(L') = D_{s_j,\succ_j}(L')$ for any list L' different from $L_{i-1} \oplus L$). Moreover, $l_i \notin \mathcal{A}^{s'_j}(L_n)$ which contradicts the assumption that $l_i \in \cap_{j\in J}\mathcal{A}^{s_j}(L_m)$. ($\Longleftarrow$) Suppose $l_i \in \mathcal{A}_R^-(L_m)$. By way of contradiction, suppose there is some $(s_j, \succ_j)$ that represents D such that $l_i \notin \mathcal{A}^{s_j}(L_m)$. By definition, it must be that $s_j(L_k) = \texttt{stop}$ for some $k < i$. By construction, it follows that $D_{s_j,\succ_j}(L_{i-1} \oplus L) \notin L$ for all extensions L. Since $D_{s_j,\succ_j} = D$, it follows that $D(L_{i-1} \oplus L) \notin L$ for all extensions L. By definition, this contradicts the fact that $l_i \in \mathcal{A}_R^-(L_m)$.

(III) Revealed Inattention: ($\Longrightarrow$) Suppose $l_k \notin \cup_{j\in J}\mathcal{A}^{s_j}(L_m)$ where $\{(s_j, \succ_j)\}_{j\in J}$ is the non-empty collection of pairs that represent D. By way of contradiction, suppose $l_k \notin \mathcal{I}_R(L_m)$. By construction, there is no $a \neq D(L_i \oplus a)$ such that $aP_RD(L_i \oplus a)$ for any $i < k$. Now, consider $D(L_{k-1})$. By definition,

$D(L_{k-1})Pa$ for all $a \in L_{k-1} - \{D(L_{k-1})\}$. Since $l_k \notin \cup_{j\in J}\mathcal{A}^{s_j}(L_m)$, $D(L_{k-1} \oplus a) \neq a$ for any $a \notin L_{k-1}$. Thus, $D(L_{k-1} \oplus a) = D(L_{k-1})$. Since there is no there is no $a \in X \backslash [S(L_{k-1}) \backslash \{D(L_{k-1})\}]$ such that $aP_R D(L_{k-1} \oplus a)$, it follows that there is no $a \in X \backslash [S(L_{k-1}) \backslash \{D(L_{k-1})\}]$ such that $aP_R D(L_{k-1})$. By asymmetry of P_R, $D(L_{k-1})$ is revealed undominated in X. Now, consider a representation $(s, \succ)$ where $\succ$ is a completion of $P_R \cup \{(D(L_{k-1}), a) : a \in X\}$ and $\mathcal{A}^s(L) = \mathcal{A}_R^-(L)$ for any L such that $D(L) \neq D(L_{k-1})$ and $\mathcal{A}^s(L) = S(L)$ otherwise. By Proposition 8 below, $(s, \succ)$ represents D. Since $\mathcal{A}^s(L_k) = S(L_k)$, this contradicts the fact that $l_k \notin \cup_{j\in J}\mathcal{A}^{s_j}(L_m)$. It follows that $l_k \in \mathcal{I}_R(L_m)$. ($\Longleftarrow$) Suppose that $l_k \in \mathcal{I}_R(L_n)$. By way of contradiction, suppose that there is some $(s, \succ)$ that represents D such that $l_j \in \mathcal{A}^s(L_m)$ for $k \leq j$. By definition of $l_k \in \mathcal{I}_R(L_m)$, there is some $a' \neq D(L_i \oplus a')$ such that $a' P_R D(L_i \oplus a')$ for some $i < k$. From the proof of part (I), it follows that $a' \succ D(L_i \oplus a')$ so that $a' \succ a$ for any $a \in L_i$. Since $l_k \in \mathcal{A}^s(L_n)$, it follows that $a' \in \mathcal{A}^s(L_i \oplus a')$. By construction, it then follows that $D_{s,\succ}(L_i \oplus a') = a' \neq D(L_i \oplus a')$ which is the desired contradiction. ■

Proof of Corollary 1 (Identification-No Recall). By the argument outlined in the text, $\mathcal{A}^s(L_m) = \mathcal{A}^{s'}(L_m)$ for any s and s' that represent D. Thus, $\cap_{j=1}^n \mathcal{A}^{s_j}(L_m) = \mathcal{A}^s(L_m)$ where $\{s_j\}_{j=1}^n$ is the non-empty collection of pairs that represent D. The argument given for Part (II) ($\Longleftarrow$) of Proposition 1 establishes that $\mathcal{A}_R^-(L_m) \subset \mathcal{A}^s(L_m)$. To see $\mathcal{A}^s(L_m) \subset \mathcal{A}_R^-(L_m)$, suppose $l_i \in \mathcal{A}^{s_j}(L_m)$ for some s that represents D. By definition of D_s, $D_s(L_i) = l_i$. Since $D_s = D$, $D(L_i) = l_i$ so that $l_i \in \mathcal{A}_R^-(L_m)$ by definition. ■

Lemma 2 *For any list $L \in \mathcal{L}$, $\mathcal{A}^{s_R}(L) = \mathcal{A}_R^-(L)$.*

Proof. This follows directly from the definitions of s_R, $\mathcal{A}^s$ and $\mathcal{A}_R^-$. Combining and simplifying gives:

$$\begin{aligned}
\mathcal{A}^{s_R}(L_m) &= \{l_i \in S(L_m) : \text{for all } j < i,\ a \in \mathcal{A}_R^-(L_j \oplus a) \text{ for some } a \notin L_j\} \\
&= \{l_i \in S(L_m) : \text{for all } j < i,\ D(L_j \oplus L) \in L \text{ for some list } L\} \\
&= \{l_i \in S(L_m) : \exists \text{ list } L \text{ such that } D(L_{i-1} \oplus L) \in L\} = \mathcal{A}_R^-(L_m)
\end{aligned}$$

■

Proof of Propostion 2 (Canonical Representation). First, notice that P_R is acyclic when D can be represented by $(s, \succ)$. Otherwise $aP_R a' P_R a$ implies $a \succ a' \succ a$ by Part (II) ($\Longrightarrow$) of Proposition 1. Let s_R be the canonical search strategy and $\succ_R$ is any completion of P_R. Suppose $D(L_m) = l_i \neq l_j = D_{s_R,\succ_R}(L_m)$ for some L_m. By Lemma 2, $\mathcal{A}^{s_R}(L_m) = \mathcal{A}_R^-(L_m)$. Since $l_j \in \mathcal{A}^{s_R}(L_m)$ and $l_i \in \mathcal{A}_R^-(L_m)$, $\{l_i, l_j\} \subset \mathcal{A}^{s_R}(L_m) = \mathcal{A}_R^-(L_m)$. By definition of the revealed preference P, $l_i P l_j$. By definition of $D_{s_R,\succ_R}$, $l_j \succ_R l_i$. This is a contradiction since $l_j \succ_R l_i$ cannot hold in any completion of P_R where $l_i P l_j$. ■

Proof of Proposition 3 (Uniqueness). ($\Longrightarrow$) Suppose that there is some canonical representation $(s_R, \succ_R)$ of D such that $S(L_i) \subsetneq \mathcal{A}^{s_R}(L_m)$ for some list L_m where l_i is the $\succ_R$-maximal element. By definition, it follows that $\mathcal{A}_R^-(L_m) = S(L_i)$ since there is no extension $L_{i-1} \oplus L$ such that $D(L_{i-1} \oplus L) \in L$. Hence, $\mathcal{A}_R^-(L_m) \subsetneq \mathcal{A}^{s_R}(L_m)$. This establishes the result.

($\Longleftarrow$) Suppose that there is a sensible representation $(s, \succ)$ of D such that $\mathcal{A}_R^-(L_m) \subsetneq \mathcal{A}^s(L_m)$ for some list L_m. Suppose that $D(L_m) = l_i$, $\mathcal{A}_R^-(L_m) = S(L_j)$ and $\mathcal{A}^s(L_m) = S(L_k)$ with $i \leq j < k$. [This is without loss of generality: $j, k \geq i$ follows from the fact that $S(L_i) \subset \mathcal{A}_R^-(L_m)$; and, $k > j$ from the assumption that $\mathcal{A}_R^-(L_m) \subsetneq \mathcal{A}^s(L_m)$.]

It follows that $\max_{\succ} X = l_i$. Since $(s, \succ)$ represents D and $D(L_m) = l_i$, l_i must be preferred (according to $\succ$) to any list item in $L_k - \{l_i\}$. For any item $a \notin L_k$, consider the list $L_{k-1} \oplus a$. By construction, $a \in \mathcal{A}^s(L_{k-1} \oplus a)$. Suppose, instead that $a \succ l_i$ for some a. Then $D(L_{k-1} \oplus a) = D_{s,\succ}(L_{k-1} \oplus a) = a$ by construction. By definition, $a \in \mathcal{A}_R^-(L_{k-1} \oplus a)$ so that $l_k \in \mathcal{A}_R^-(L_k)$ and, consequently, $S(L_k) \subset \mathcal{A}_R^-(L_m)$. Since $\mathcal{A}^s(L_m) = S(L_k)$, this contradicts the assumption that $\mathcal{A}_R^-(L_m) \subsetneq \mathcal{A}^s(L_m)$. So, $l_i \succ a$ for all $a \notin L_k$. Together with the fact that $l_i \succ a$ for all $a \in L_k - \{l_i\}$, this establishes that $\max_{\succ} X = l_i$. Since $(s, \succ)$ is sensible, it follows that $\mathcal{A}^s(L_m) = S(L_i) \subset S(L_j) = \mathcal{A}_R^-(L_m)$. But, this contradicts the assumption $\mathcal{A}_R^-(L_m) \subsetneq \mathcal{A}^s(L_m)$. The result follows. ■

Together with Proposition 2, Proposition 3 implies the following:

Corollary 3 *For every search procedure $(s, \succ)$, $(s_R, \succ_R)$ is an equivalent sensible search procedure.*

8.1.2 Uniqueness without the Sensible Search Assumption

Even without the sensible search assumption, a high degree of uniqueness is achieved.

Proposition 8 (Uniqueness) *If $(s, \succ)$ represents D, then:*

(I) $\mathcal{A}_R^-(L) = \mathcal{A}^s(L)$ for all L where $aP_R D(L)$ for some $a \in X$; and

(II) If $\mathcal{A}_R^-(L) \subsetneq \mathcal{A}^s(L)$ for some L,

(i) The preference $\succ$ is a completion of $P_R \cup \{(D(L), x) : x \in X \backslash D(L)\}$ and

(ii) For any list $L' \neq L$:

$$\begin{aligned} \mathcal{A}^s(L') &= \mathcal{A}_R^-(L') \text{ if } D(L') \neq D(L)\text{; and} \\ \mathcal{A}_R^-(L') &\subset \mathcal{A}^s(L') \subset S(L') \text{ otherwise.} \end{aligned}$$

Before showing the result, I explain how this limits the scope of possible consideration sets for behavior D that can be represented in terms of search with recall. Part (I) establishes that $\mathcal{A}_R^-(L) = \mathcal{A}^s(L)$ if $D(L)$ is revealed preferred by any item in the grand set X. This is an application of revealed inattention. For an item a not revealed preferred by any other item, it may be that $\mathcal{A}_R^-(L) \subsetneq \mathcal{A}^s(L)$ when $D(L) = a$. Part (II) shows there is at most one undominated a item where this is the case. For any other undominated item $a' \neq a$, $\mathcal{A}^s(L') = \mathcal{A}_R^-(L')$ when $D(L') = a'$. For the undominated item a, very little can be said. For any list L' such that $D(L') = a$, only the weakest set inclusion holds: $\mathcal{A}_R^-(L') \subset \mathcal{A}^s(L') \subset S(L')$.

Summarizing: (i) for any list L, $\mathcal{A}_R^-(L) = \mathcal{A}^s(L)$ if $D(L)$ is revealed preferred by some other item or there is some item $a = D(L') \neq D(L)$ such that $\mathcal{A}_R^-(L') \subsetneq \mathcal{A}^s(L')$; and (ii) for any list L such that $D(L) = a$, only the weak inclusion $\mathcal{A}_R^-(L') \subset \mathcal{A}^s(L') \subset S(L')$ holds.

Proof. The argument given in the proof of Proposition 2 actually establishes a somewhat broader point: when $\mathcal{A}_R^-(L) \subsetneq \mathcal{A}^s(L)$ for some L, then $D(L)$ must be maximal according to $\succ$. This, in turn, establishes point (II)(i) that the preference $\succ$ must be a completion of $P_R \cup \{(D(L), a) : a \in X\}$ (and not just P_R as Proposition 1 stipulates). It also establishes point (I), which states that the preference cannot contradict the revealed preference P_R.

The first part of (II)(ii), namely that $\mathcal{A}^s(L') = \mathcal{A}^-_R(L')$ whenever $D(L') \neq D(L)$, also follows readily from the fact that $D(L)$ is maximal. Suppose that $\mathcal{A}^-_R(L') \subsetneq \mathcal{A}^s(L')$ for some L' such that $D(L') \neq D(L)$. Then, by the same maximality argument, it follows that $D(L') \succ D(L)$. This contradicts the fact that $D(L)$ is maximal. It then follows that $\mathcal{A}^s(L') = \mathcal{A}^-_R(L')$ whenever $D(L') \neq D(L)$.

The second part of (II)(ii) can be established straight from the definition of $\mathcal{I}_R(L')$. In particular, the fact that there is no $aP_R D(L')$ establishes that $\mathcal{I}_R(L') = \emptyset$ and hence $\mathcal{A}^+_R(L') = S(L')$. As such, the set inclusion $\mathcal{A}^-_R(L') \subset \mathcal{A}^s(L') \subset \mathcal{A}^+_R(L')$ established in Proposition 1 can be rewritten as $\mathcal{A}^-_R(L') \subset \mathcal{A}^s(L') \subset S(L')$. ■

8.1.3 Characterization

First, I establish a lemma which proves a claim mentioned in the text.

Lemma 3 *Sequential Choice is equivalent to $D(L) \in \{D(L')\} \cup S(L'')$ for any list $L = L' \oplus L''$.*

Proof. ($\Longleftarrow$) This is by definition. ($\Longrightarrow$) The proof is by induction on the length of the list. The base case where $n = 2$ is by definition. For the induction step, suppose that $D(L'_i \oplus L''_j) \in \{D(L'_i)\} \cup S(L''_j)$ for any list $L_N = L'_i \oplus L''_j$ of length N. Now, consider the list $L_{N+1} = L'_i \oplus L''_j$ of length $n = N + 1$. By Sequential Choice, it follows that $\{D(L_{N+1})\} \subset \{D(L'_i \oplus L''_{j-1})\} \cup \{l''_j\}$. By the induction step, $\{D(L'_i \oplus L''_{j-1})\} \subset \{D(L'_i)\} \cup S(L''_{j-1})$. Combining these results:

$$\{D(L_{N+1})\} \subset \{D(L'_i \oplus L''_{j-1})\} \cup \{l''_j\} \subset \{D(L'_i)\} \cup S(L''_{j-1}) \cup \{l''_j\} = \{D(L'_i)\} \cup S(L''_j)$$

It follows that $D(L'_i \oplus L''_j) \in \{D(L'_i)\} \cup S(L''_j)$. ■

Next, I establish a lemma that simplifies the expression of search and preference.

Lemma 4 ***(I)*** *If D satisfies Sequential Choice and Weak Indifference to Improvement:*

$$s_R(L) = \texttt{continue} \textit{ iff } D(L \oplus d) = d \textit{ for some } d \notin L.$$

(II) *If D **also** satisfies Preference Consistency: aPb iff aAb or aBb where A and B are defined by*

$$\begin{aligned} aAb \text{ if } D(L \oplus b \oplus a) &= a \\ aBb \text{ if } D(L \oplus a \oplus b) &= a \text{ and } D(L \oplus a \oplus c) = c \text{ for some } c. \end{aligned}$$

Proof. **(I)** ($\Longleftarrow$) By definition, $s_R(L) = \texttt{continue}$ if $D(L \oplus a) = a$ for some $a \notin L$. ($\Longrightarrow$) By definition, $s_R(L) = \texttt{continue}$ if there exists some b such that $b \in \mathcal{A}^-_R(L \oplus b)$. It follows that $b \in \mathcal{A}^-_R(L \oplus b)$ if there exists some $L' \oplus d \oplus L''$ such that $D(L \oplus L' \oplus d \oplus L'') = d$. By Sequential Choice, $D(L \oplus L' \oplus d) = d$. By repeated application of Weak Indifference to Improvement and Sequential Choice, $D(L \oplus d) = d$.

(II) ($\Longleftarrow$) By definition, aPb if aAb or aBb. ($\Longrightarrow$) Suppose aPb. There are two possibilities.

(i) Suppose $D(L \oplus b \oplus L' \oplus a \oplus L'') = a$ [Note: $b \in \mathcal{A}_R^-(L \oplus b \oplus L' \oplus a \oplus L'')$ follows from $D(L \oplus b \oplus L' \oplus a \oplus L'') = a$]. By repeated application of Weak Indifference to Improvement and Sequential Choice, $D(L \oplus b \oplus a) = a$ establishing aAb. **(ii)** Suppose $D(L \oplus a \oplus L' \oplus b \oplus L'') = a$ and $D(L \oplus a \oplus L' \oplus c \oplus L''') = c \in L'''$ [so that $b \in \mathcal{A}_R^-(L \oplus a \oplus L' \oplus b \oplus L'')$]. Applying Weak Indifference to Improvement and Sequential Choice to the second identity, $D(L \oplus a \oplus c) = c$. From Preference Consistency, it follows that $D(L \oplus a \oplus b) \neq b$. By Sequential Choice, $D(L \oplus a \oplus b) = D(L \oplus a)$. From $D(L \oplus a \oplus L' \oplus b \oplus L'') = a$, Sequential Choice implies that $D(L \oplus a) = a$. Thus, $D(L \oplus a \oplus b) = a$ which establishes that aBb. ■

Proof of Theorem 1 (Search with Recall). ($\Longrightarrow$) *Sequential Choice* holds by definition of $(s, \succ)$.

Weak Indifference to Improvement: Suppose $D_{s,\succ}(L \oplus b \oplus a) = a$. There are two cases: $s(L \oplus a) = \texttt{stop}$; and $s(L \oplus a) = \texttt{continue}$. In the first case, $\mathcal{A}^s(L \oplus a \oplus b) = S(L \oplus a)$ and $\mathcal{A}^s(L \oplus b \oplus a) = S(L \oplus b \oplus a)$ so:

$$D_{s,\succ}(L \oplus a \oplus b) = \max_{\succ} \mathcal{A}^s(L \oplus a \oplus b) = \max_{\succ} S(L \oplus a) = \max_{\succ}(\mathcal{A}^s(L \oplus b \oplus a) \backslash \{b\})$$

Since $D_{s,\succ}(L \oplus b \oplus a) = \max_{\succ} \mathcal{A}^s(L \oplus b \oplus a) = a$, it follows that $D_{s,\succ}(L \oplus a \oplus b) = a$. In the second case, $\mathcal{A}^s(L \oplus a \oplus b) = S(L \oplus b \oplus a) = \mathcal{A}^s(L \oplus b \oplus a)$ so:

$$D_{s,\succ}(L \oplus a \oplus b) = \max_{\succ} \mathcal{A}^s(L \oplus a \oplus b) = \max_{\succ} \mathcal{A}^s(L \oplus b \oplus a) = D_{s,\succ}(L \oplus b \oplus a) = a$$

Preference Consistency: Suppose $D_{s,\succ}(L \oplus c) = c$, $D_{s,\succ}(\bar{L} \oplus \bar{c}) = \bar{c}$ and $D_{s,\succ}(L \oplus b \oplus L') = a$. By construction of $D_{s,\succ}$, $c \in \mathcal{A}^s(L \oplus c)$ so that $b \in \mathcal{A}^s(L \oplus b \oplus L')$. Since $D_{s,\succ}(L \oplus b \oplus L') = a$, $a \succ b$. To obtain a contradiction, suppose $D_{s,\succ}(\bar{L} \oplus a \oplus \bar{L}') = b$. By the same kind of reasoning, $b \succ a$. This is a contradiction since $\succ$ is a linear order (and hence asymmetric).

Search Consistency: Suppose $D_{s,\succ}(L \oplus b \oplus L') = a$ and $D_{s,\succ}(L \oplus c) = c$. By the same reasoning as above, it follows that $a \succ b$. By construction of $D_{s,\succ}$, $\mathcal{A}^s(\bar{L} \oplus b) = S(\bar{L} \oplus b)$ so that $\mathcal{A}^s(\bar{L} \oplus a) = S(\bar{L} \oplus a)$. Now, suppose $D_{s,\succ}(\bar{L} \oplus a) = d \neq a$. It follows that $d \succ a$. Since $d \in S(\bar{L} \oplus b)$ and $D_{s,\succ}(\bar{L} \oplus b) = b$, it follows that $b \succ d$. By the transitivity of $\succ$, $b \succ a$. This is the desired contradiction since $\succ$ is asymmetric.

($\Longleftarrow$) The result follows by: **(I)** establishing that P is irreflexive, asymmetric and acyclic; **(II)** establishing that two items a and b are unrelated by P_R if and only if they are choice-symmetric; and **(III)** showing that $D = D_{s,\succ}$ where $s = s_R$ is the canonical strategy and $\succ$ is a completion of P_R.

(I) By definition, P is irreflexive. Moreover, P is also asymmetric.

The proof is by contradiction. There are three possibilities: (i) aBb and bBa; (ii) aAb and bBa; and (iii) aAb and bAa. **(i)** Preference Consistency rules out this possibility. By definition, aBb implies that $D(L \oplus a \oplus b) = a$ and $D(L \oplus a \oplus d) = d$ for some list L. If $D(L' \oplus b \oplus d') = d'$, Preference Consistency implies $D(L' \oplus b \oplus a) \neq b$. Consequently, bBa cannot obtain. **(ii)** By definition, aAb implies $D(L \oplus b \oplus a) = a$ for some list L. By Weak Indifference to Improvement and Sequential Choice, $D(L \oplus b \oplus a) = a$ implies $D(L \oplus a) = a$. If $D(L' \oplus b \oplus d') = d'$ for some list L', Preference Consistency implies that $D(L' \oplus b \oplus a) \neq b$.

Thus, bBa cannot obtain. **(iii)** By definition, aAb requires $D(L \oplus b \oplus a) = a$ for some list L (so that $D(L \oplus a) = a$) and bAa requires $D(L' \oplus a \oplus b) = b$ for some list L' (so that $D(L' \oplus b) = b$). But this contradicts Preference Consistency.

To establish the acyclicity of P, consider the following claims:

Claim 1. aPb and bAc imply aAc;

Claim 2. aAb and bBc imply cPa is impossible; and

Claim 3. aBb and bBc imply aPc.

Claim 1: By definition, bAc implies that $D(L' \oplus c \oplus b) = b$ for some list L'. By asymmetry of P, it follows that $a \notin L'$. Suppose otherwise that $L' = L'' \oplus a \oplus L'''$. Then, repeated application of Weak Indifference to Improvement and Sequential Choice give $D(L'' \oplus a \oplus b) = b$ so that bAa (which contradicts the asymmetry of A). Since aPb and $D(L \oplus c \oplus b) = b$, Search Consistency implies that $D(L \oplus c \oplus a) = a$. By definition, aAc.

Claim 2: Suppose cPa. By Claim 1, cAb. Since bBc, this contradicts the asymmetry of P.

Claim 3: By definition, aBb implies that $D(L \oplus a \oplus b) = a$ and $D(L \oplus a \oplus d) = d$ for some list L. There are two cases to consider. First, suppose $c \in L$ so that $L = L' \oplus c \oplus L''$. By repeated application of Weak Indifference to Improvement and Sequential Choice, it follows that $D(L' \oplus c \oplus a) = a$ so that aAc. Next, suppose $c \notin L$ and consider $L \oplus a \oplus c$. By Sequential Choice, $D(L \oplus a \oplus c) \in \{D(L \oplus a), c\}$ and $D(L \oplus a \oplus b) = a \in \{D(L \oplus a), b\}$. Thus, $D(L \oplus a) = a$ so that $D(L \oplus a \oplus c) \in \{a, c\}$. By Search Consistency, bBc and $D(L \oplus a \oplus b) \neq b$ implies that $D(L \oplus a \oplus c) \neq c$. Thus, $D(L \oplus a \oplus c) = a$. By definition, $D(L \oplus a \oplus c) = a$ and $D(L \oplus a \oplus d) = d$ establish that aBc.

These claims ensure that P is acyclic. In fact, a stronger claim (used in (II) below) holds:

$$a_1 P a_2 P ... P a_{n-1} P a_n \text{ imply } a_1 P a_n \text{ or } a_1 A a' B a_n \text{ (for } a' \in \{a_i\}_2^{n-1}) \qquad (*)$$

for any n. Property $(*)$ guarantees that P is acyclic. If $a_1 P a_n$, it follows by asymmetry of P that $a_n P a_1$ cannot obtain. If $a_1 A a' B a_n$, Claim 2 establishes that $a_n P a_1$ cannot obtain. The proof of property $(*)$ is by induction on the length of the chain. Claims 1 and 3 establish the case where $n = 3$. So, assume that:

$$a_1 P a_2 P ... P a_{N-1} P a_N \text{ imply } a_1 P a_N \text{ or } a_1 A a' B a_N \text{ (for } a' \in \{a_i\}_2^{N-1})$$

for any chain of length $n = N$. Now, consider a chain of length $N + 1$ such that:

$$a_1 P a_2 P ... P a_N P a_{N+1}$$

Using the induction hypothesis, there are four possibilities: (i) $a_1 P a_N A a_{N+1}$; (ii) $a_1 B a_N B a_{N+1}$; (iii) $a_1 A a' B a_N A a_{N+1}$; and (iv) $a_1 A a' B a_N B a_{N+1}$. The remaining case, $a_1 A a_N B a_{N+1}$, establishes the desired conclusion directly. By Claim 1, $a_1 A a_{N+1}$ in case (i). By Claim 3, $a_1 P a_{N+1}$ in case (ii). By Claim 1,

$a_1 A a_{N+1}$ in case (iii). In case (iv), Claim 3 establishes that $a_1 A a' B a_{N+1}$ (in case $a' B a_N B a_{N+1}$ implies that $a' B a_{N+1}$); or $a_1 A a' A a_{N+1}$ (in case $a' B a_N B a_{N+1}$ implies $a' A a_{N+1}$). In the latter case, Claim 1 establishes that $a_1 A a_{N+1}$. Collecting these conclusions establishes the property $(*)$ when $n = N + 1$.

(II) Property $(*)$ establishes that $a P_R b$ implies aPb or $aAa'Bb$ for some a'. By definition, aPb and $aAa'Bb$ both imply $aP_R b$. It follows that:

$$aP_R b \text{ if and only if } aPb \text{ or } aAa'Bb \text{ for some } a'$$

($\Longrightarrow$) Given two items a and b unrelated by P_R, suppose that they are not choice-symmetric. Without loss of generality, suppose that $D(L \oplus a \oplus L') = a$ and $D(\sigma_{ab}[L \oplus a \oplus L']) \neq b$. There are three possibilities to consider: **(a)** $b \notin L \oplus L'$; **(b)** $b \in L$; and **(c)** $b \in L'$.

(a) The proof is by contradiction. There are two cases to consider: **(i)** $D(L \oplus b \oplus L') \in L$; and **(ii)** $D(L \oplus b \oplus L') \in L'$. **(i)** Suppose that $L = \bar{L} \oplus c \oplus \bar{L}'$ and that $D(L \oplus b \oplus L') = c$. By repeated application of Sequential Choice and Weak Indifference to Improvement, $D(\bar{L} \oplus c \oplus a) = a$ so that aAc. By Sequential Choice, $D(L \oplus b \oplus L') = c \in \{D(\bar{L} \oplus c)\} \cup S(\bar{L}' \oplus b \oplus L')$ so that $D(\bar{L} \oplus c) = c$. Again by Sequential Choice, $D(\bar{L} \oplus c \oplus b) \in \{D(\bar{L} \oplus c), b\} = \{c, b\}$. It can be shown that $D(\bar{L} \oplus c \oplus b) = c$. To see why, suppose instead that $D(\bar{L} \oplus c \oplus b) = b$. By Weak Indifference to Improvement and Sequential Choice, $D(\bar{L} \oplus c \oplus b) = b$ implies that $D(\bar{L} \oplus b) = b$. Since $D(L \oplus a \oplus L') = a$, it also follows by Sequential Choice that $D(L \oplus a) = a$. Given $D(L \oplus a) = a$ and $D(\bar{L} \oplus b) = b$, $D(L \oplus b \oplus L') = c$ implies that $D(\bar{L} \oplus c \oplus b) \neq b$ by Preference Consistency. This contradiction establishes that $D(\bar{L} \oplus c \oplus b) = c$. Since $D(\bar{L} \oplus c \oplus a) = a$ and $D(\bar{L} \oplus c \oplus b) = c$, it follows that aAc and cBb. Thus, $aAcBb$ so that a and b are related by P_R. This is the desired contradiction which establishes that $D(L \oplus b \oplus L') \notin L$. **(ii)** Suppose instead that $D(L \oplus b \oplus L') = c' \in L'$. By repeated application of Sequential Choice and Weak Indifference to Improvement, $D(L \oplus b \oplus c') = c'$. Now consider the list $L \oplus b \oplus a$. By Sequential Choice, $D(L \oplus b \oplus a) \in \{D(L), a, b\}$. It follows that $D(L \oplus b \oplus a) \neq D(L)$. In order to see why, suppose that $L = \bar{L} \oplus c \oplus \bar{L}'$ and $D(L) = c$. Since $D(L \oplus a \oplus L') = a$, repeated application of Sequential Choice and Weak Indifference to Improvement implies $D(\bar{L} \oplus c \oplus a) = a$ and $D(\bar{L} \oplus a) = a$. Given that $D(L \oplus b \oplus c') = c'$ and $D(\bar{L} \oplus a) = a$, $D(\bar{L} \oplus c \oplus a) = a$ implies that $D(L \oplus b \oplus a) \neq c$ by Preference Consistency. This is the desired contradiction. Thus, $D(L \oplus b \oplus a) \in \{a, b\}$. If $D(L \oplus b \oplus a) = b$, then $D(L \oplus b \oplus c') = c'$ implies that bBa which is a contradiction. If $D(L \oplus b \oplus a) = a$, then aAb which is again a contradiction. This establishes that $D(L \oplus b \oplus L') \notin L'$.

(b) Repeated application of Sequential Choice and Weak Indifference to Improvement to $L \oplus a \oplus L'$ imply aAb, which contradicts the assumption that a and b are unrelated by P_R.

(c) If $D(\sigma_{ab}[L \oplus a \oplus L']) = a$, repeated application of Sequential Choice and Weak Indifference to Improvement implies aAb. Otherwise, simply apply the same reasoning as in case (a)(ii) above.

($\Longleftarrow$) The proof is by contradiction. Given two items a and b that are choice-symmetric, suppose a and b are related by P_R. Without loss of generality, there are three possibilities: **(i)** aAb; **(ii)** aBb; and **(iii)** $aAa'Bb$ for some a'.

(i) By definition, $D(L \oplus b \oplus a) = a$ for some L. Since a and b are choice-symmetric, $D(L \oplus a \oplus b) = b$. Since this contradicts Weak Indifference to Improvement, aAb cannot obtain. **(ii)** By definition, $D(L \oplus a \oplus b) = a$ and $D(L \oplus a \oplus d) = d$ for some L and d. Since a and b are choice-symmetric, $D(L \oplus b \oplus a) = b$ and $D(L \oplus b \oplus d) \neq b$. By Sequential Choice, it follows that $D(L \oplus b \oplus a) = b \in \{D(L \oplus b), a\}$ so that $D(L \oplus b) = b$. Applying Sequential Choice, it follows that $D(L \oplus b \oplus d) \in \{D(L \oplus b), d\} = \{b, d\}$. Since $D(L \oplus b \oplus d) \neq b$, it follows that $D(L \oplus b \oplus d) = d$. Since $D(L \oplus b \oplus a) = b$ and $D(L \oplus b \oplus d) = d$, it follows that bBa. Since this contradicts the asymmetry of B, aBb cannot obtain. **(iii)** By definition, $D(L \oplus a' \oplus a) = a$ for some L. Since a and b are choice-symmetric, $D(L \oplus a' \oplus b) = b$ so that bAa'. Since this contradicts the asymmetry of P, $aAa'Bb$ cannot obtain.

(III) The proof is by contradiction. Suppose $D(L_n) = l_i \neq l_j = D_{s,\succ}(L_n)$ for some L_n. By Lemma 2, $\mathcal{A}^s(L_m) = \mathcal{A}^-_R(L_m)$. Since $l_j \in \mathcal{A}^s(L_m)$ (by construction) and $l_i \in \mathcal{A}^-_R(L_m)$ (by definition), $\{l_i, l_j\} \subset \mathcal{A}^s(L_m) = \mathcal{A}^-_R(L_m)$. By definition of the revealed preference P, $l_i P l_j$. By definition of $D_{s,\succ}$, $l_j \succ l_i$. This is a contradiction since $l_j \succ l_i$ cannot hold in any completion of P_R where $l_i P l_j$. ■

8.2 Search without Recall

Lemma 5 *If D satisfies No Recall, then:*

$$s_R(L) = \texttt{continue} \textit{ iff } D(L \oplus d) = d \textit{ for some } d \notin L$$

Proof. ($\Longrightarrow$) By definition, $s_R(L) = \texttt{continue}$ if $D(L \oplus a) = a$ for some $a \notin L$. ($\Longleftarrow$) By definition, $s_R(L) = \texttt{continue}$ if there exists some b such that $b \in \mathcal{A}^-_R(L \oplus b)$. It follows that $b \in \mathcal{A}^-_R(L \oplus b)$ if there exists some $L' \oplus d \oplus L''$ such that $D(L \oplus L' \oplus d \oplus L'') = d$. By No Recall, $D(L \oplus d) = d$. ■

Proof of Theorem 2 (Search Without Recall). ($\Longrightarrow$) *Sequential Choice* holds by definition. In order to show *No Recall*, suppose $L_n \oplus a \oplus \bar{L} \equiv L$ and $D_s(L \oplus L') = a$. Then $s(L_n \oplus a) = \texttt{stop}$ (while $s(L_i) = \texttt{continue}$ for any $i \leq n$) so $D_s(L \oplus L'') = a$. Thus, D_s satisfies No Recall.

($\Longleftarrow$) Let $s = s_R$ be the canonical search strategy. The proof is by contradiction. Suppose $D(L_n) = l_i \neq l_j = D_s(L_n)$ for some L_n. By Lemma 2, $\mathcal{A}^s(L_m) = \mathcal{A}^-_R(L_m)$. Since $l_j \in \mathcal{A}^s(L_m)$ (by construction) and $l_i \in \mathcal{A}^-_R(L_m)$ (by definition), $\{l_i, l_j\} \subset \mathcal{A}^s(L_m) = \mathcal{A}^-_R(L_m)$. By definition of D_s, l_j is the last item considered so that $j > i$. Since $D(L_i \oplus L_n^{i+1}) = l_i \in L_i$, then $D(L_i \oplus L') \in L_i$ for any L' by No Recall. By definition, this contradicts the assumption that $D_s(L_n) = l_j$ (so that $l_j \in \mathcal{A}^s(L_m)$). ■

8.3 Simple Search Heuristics

8.3.1 Search without Recall: Last-Satisficing

Lemma 6 *If D that satisfies Successive Choice and No Recall:*

$$s_R(L) = \texttt{continue} \textit{ iff } D(D(L) \oplus d) = d \textit{ for some } d \notin L$$

Proof. Since No Recall holds, Lemma 5 applies. By Lemma 5, $s_R(L) = \texttt{continue}$ iff $D(L \oplus d) = d$ for some d. By Successive Choice, $D(L \oplus d) = d$ iff $D(D(L) \oplus d) = d$. The result follows. ■

Proof of Theorem 3 (Last-Satisficing). ($\Longrightarrow$) Theorem 2 establishes *No Recall. Partition Independence* was established in the text. ($\Longleftarrow$) Define the cutoff set C_R^* as in the text of the paper. By Theorem 2, $D = D_{s_R}$. By Lemma 6, it follows that:

$$D(L) \notin C^* \iff s_R(L') = \texttt{continue} \text{ for some } L' \text{ such that } D(L) = D(L') \iff s_R(L) = \texttt{continue}$$

As such, $D_{s_R} = D_{C_R^*}$ so that $D = D_{C_R^*}$. ■

Lemma 7 *Successive Choice and No Recall imply Partition Independence.*

Proof. Without loss of generality, there are two possibilities: $D(L \oplus L') \in L$; and $D(L \oplus L') \in L'$. In the first case, Successive Choice implies $D(L \oplus L') = D(D(L) \oplus L') = D(L)$. By No Recall, $D(D(L) \oplus D(L')) \neq D(L')$ so that $D(L \oplus L') = D(D(L) \oplus D(L'))$. In the second case, No Recall implies that $D(L \oplus D(L')) = D(L')$ and Successive Choice implies $D(L \oplus D(L')) = D(D(L) \oplus D(L'))$. In order to see that $D(L \oplus L') = D(L')$, let $L' = L'_n$. By No Recall, $D(L \oplus l'_1) = l'_1$. By Successive Choice, $D(L \oplus L'_n) = D(D(L \oplus l'_1) \oplus [L'_n - \{l'_1\}])$ so that $D(L \oplus L'_n) = D(l'_1 \oplus [L'_n - \{l'_1\}]) = D(L'_n)$. Thus, $D(L \oplus L'_n) = D(L'_n) = D(D(L) \oplus D(L'_n))$. ■

8.3.2 Search with Recall

The key to the existence result for the aspiration adaptation heuristic is to establish that the axioms imply Preference Consistency and Search Consistency. Then, the result follows more or less directly from Theorem 1 (since Sequential Choice holds and Indifference to Improvement strengthens Weak Indifference to Improvement). By the following lemma, the axioms for Markov search and best-satisficing *also* imply Preference Consistency and Search Consistency. As a result, the existence results for these heuristics can also be established from Theorem 1.

Lemma 8 *Successive Choice and Indifference to Improvement imply Binary Search Consistency.*

Proof. Suppose $D(b, a) = a$ and $D(b, c) = b$ and consider $< b, a, c >$. It can be shown that $D(b, a, c) = a$. The result then follows by Successive Choice. Since $D(b, a, c) = D(D(b, a), c) = D(a, c)$, it follows that $D(a, c) = a$.

The proof that $D(b, a, c) = a$ is by contradiction. First, suppose $D(b, a, c) = b$. By Successive Choice, it then follows that implies $D(b, a, c) = D(D(b, a), c) = D(a, c) \neq b$. Next, suppose $D(b, a, c) = c$. Applying Indifference to Improvement, $D(b, a, c) = D(b, c, a)$. By Successive Choice, $D(b, c, a) = D(D(b, c), a) = D(b, a) = a \neq c$. Since $D(b, a, c) \in \{a, b, c\}$, it follows that $D(b, a, c) = a$. ■

In order to show that the axioms for aspiration adaptation imply Preference Consistency and Search Consistency, I establish a series of lemmas. First, I prove an analogue of Lemma 4 showing that the revealed preference only depends on two-item lists.

Lemma 9 *If D satisfies Sequential Choice, Indifference to Improvement and Aspiration Successive Choice, then aPb implies aA_2b, aB_2b or aA_2dB_2b for some d where A_2 and B_2 are defined by:*

$$\begin{aligned} aA_2b \text{ if } D(b,a) &= a \\ aB_2b \text{ if } D(a,b) &= a \text{ and } D(a,c) = c \text{ for some } c \end{aligned}$$

Proof. By the same kind of reasoning given in Lemma 4, there are two possibilities: **(i)** $D(L \oplus b \oplus a) = a$ for some list L; or **(ii)** $D(L \oplus a \oplus b) = a$ and $D(L \oplus a \oplus c) = c$ for some list L and some c. [Formally, the only difference is in case (ii). By Sequential Choice, $D(L \oplus a \oplus L' \oplus b) = a$ implies $D(L \oplus a) = D(L \oplus a \oplus L') = a$. Since $D(L \oplus a \oplus L' \oplus c) = c$, then Aspiration Successive Choice implies $D(L \oplus a \oplus L' \oplus b) = D(D(L \oplus a \oplus L') \oplus b) = D(D(L \oplus a) \oplus b) = D(L \oplus a \oplus b) = a$].

(i) By Aspiration Successive Choice, $D(D(L \oplus b) \oplus a) = a$ so that $aA_2D(L \oplus b)$. By Sequential Choice, it follows that $D(L \oplus b) \in \{D(L), b\}$. If $D(L \oplus b) = b$, then aA_2b. If $D(L \oplus b) = D(L)$, Aspiration Successive Choice ensures that $D(D(L) \oplus a) = a$ and $D(D(L) \oplus b) = D(L)$. Thus, $D(L)B_2b$ so that aA_2dB_2b for some $d \in L$. **(ii)** Applying Sequential Choice to the first identity, $D(L \oplus a) = a$. By Aspiration Successive Choice, it follows that $D(D(L \oplus a) \oplus b) = a$ and $D(D(L \oplus a) \oplus c) = c$. Since $D(L \oplus a) = a$, it follows that $D(a,b) = a$ and $D(a,c) = c$ so that aB_2b. ■

Let P_2 be defined as aA_2b, aB_2b or aA_2dB_2b for some d. The next lemma establishes that the axioms for aspiration adaptation ensure that P_2 is asymmetric.

Lemma 10 *If D satisfies Indifference to Improvement, Binary Preference Consistency, and Binary Search Consistency, then P_2 is asymmetric.*

Proof. To establish the result, first consider the following claims:

Claim 1. aA_2b and bA_2c imply aA_2c (and hence aP_2c); and

Claim 2. aB_2b and bA_2c imply aA_2c (and hence aP_2c).

Claim 1: By way of contradiction, suppose that $D(c,a) = c$. Since $D(c,b) = b$, Binary Search Consistency implies $D(b,a) = b$. This contradiction establishes $D(c,a) = a$. It follows by definition that aA_2c (and hence aP_2c).

Claim 2: By way of contradiction, suppose that $D(a,c) = c$. Since $D(a,b) = b$, Binary Search Consistency implies that $D(c,b) = c$. This contradiction establishes $D(a,c) = a$. Now, suppose that $D(c,a) = c$. Since $D(a,c) = a \neq c = D(c,a)$ and $D(a,c) = c$, Binary Preference Consistency implies $D(c,d) = c$ for all items d. This contradicts the fact that $D(c,b) = b$. Since $D(c,a) = a$, then aA_2c (and hence aP_2c).

To see that P_2 is asymmetric, suppose $a \neq b$. The proof is by contradiction. There are six possibilities:

(i) aA_2b and bA_2a: By Indifference to Improvement, $D(b,a) = a$ implies $D(a,b) = a$. This contradicts the assumption that $D(a,b) = b$ (i.e. that bA_2a). **(ii)** aA_2b and bB_2a: By definition of D, $D(b,a) = a$

implies $D(b, a) \neq b$. This contradicts the assumption that $D(b, a) = b$ (i.e. that bB_2a). **(iii)** aA_2b and bA_2dB_2a: By Binary Search Consistency, $D(d, b) = b$ and $D(d, a) = d$ imply $D(b, a) = b$. This contradicts the assumption that $D(b, a) = a$ (i.e. aA_2b). **(iv)** aB_2b and bB_2a: Since $D(a, c) = c$ and $D(a, b) = a \neq b = D(b, a)$, then $D(b, d) = b$ for all $d \in X$ (by Binary Preference Consistency). This contradicts the assumption that $D(b, c) = c$ for some $c \in X$ (i.e. that bB_2a). **(v)** aB_2b and bA_2dB_2a: By Claim 2 above, aB_2bA_2d implies aA_2d. Then, by (ii) above, aA_2d and dB_2a cannot obtain. **(vi)** aA_2cB_2b and bA_2dB_2a: By Claim 2 above, cB_2bA_2d implies cA_2d. By Claim 1, aA_2cA_2d implies aA_2d. Then, by (ii) above, aA_2d and dB_2a cannot obtain. ■

Corollary 4 *If D satisfies Sequential Choice, Aspiration Successive Choice, Indifference to Improvement, Binary Preference Consistency, and Binary Search Consistency, then D satisfies Preference Consistency and Search Consistency.*

Proof. To establish *Preference Consistency*, suppose that aPb and bPa. By Lemma 9, it follows that aP_2b and bP_2a. By Lemma 10, this is a contradiction (which establishes Preference Consistency).

To establish *Search Consistency*, suppose that aPb, $D(\bar{L} \oplus b) = b$ and $D(\bar{L} \oplus a) \neq a$. By Aspiration Successive Choice, $D(\bar{L} \oplus b) = D(D(\bar{L}) \oplus b)$ and $D(\bar{L} \oplus a) = D(D(\bar{L}) \oplus a)$. Since $bA_2D(\bar{L})B_2a$, bP_2a by definition. By Lemma 9, aPb implies aP_2b. By Lemma 10, bP_2a and aP_2b is a contradiction (which establishes Search Consistency). ■

8.3.3 Aspiration Adaptation

First, I establish that the canonical search strategy has a simple expression when D satisfies Sequential Choice, Indifference to Improvement and Aspiration Successive Choice.

Lemma 11 *If D satisfies Sequential Choice, Indifference to Improvement and Aspiration Successive Choice:*

$$s_R(L_n) = \texttt{continue} \textit{ iff } D(L_m \oplus d) = d \textit{ for some } L_m \textit{ with } m \geq n \textit{ such that } D(L_m) = D(L_n)$$

Proof. Part (I) of Lemma 4 applies because Indifference to Improvement strengthens Weak Indifference to Improvement. ($\Longrightarrow$) By Lemma 4, $s_R(L_n) = \texttt{continue}$ implies $D(L_n \oplus d) = d$ for some d. Set $L_n = L_m$. It follows that $D(L_m \oplus d) = d$ for some L_m with $m \geq n$ such that $D(L_m) = D(L_n)$. ($\Longleftarrow$) From Aspiration Successive Choice, it follows that $D(L_n \oplus d) = D(D(L_n) \oplus d)$. Since Aspiration Successive Choice also implies $D(L_m \oplus d) = D(D(L_m) \oplus d) = d$ and $D(L_m) = D(L_n)$, it follows that $D(L_n \oplus d) = D(D(L_n) \oplus d) = D(D(L_m) \oplus d) = d$. By Lemma 4, the result follows. ■

Proof of Theorem 7 (Aspiration Adaptation). ($\Longrightarrow$) Suppose that the list-choice function D is induced by the aspiration adaptation procedure $(\{C_i^*\}, \succ)$. Clearly, *Sequential Choice* follows by definition. To establish the other axioms:

Indifference to Improvement: Suppose $D(L_n \oplus b \oplus a \oplus L') = a$. There are two cases to consider: **(i)** $a \in C_{n+1}^*$; and **(ii)** $a \notin C_{n+1}^*$. In the first case, it follows that $\mathcal{A}^{\{C_i^*\}}(L \oplus a \oplus b \oplus L') = S(L \oplus a)$ and

$\mathcal{A}^{\{C_i^*\}}(L \oplus b \oplus a \oplus L') = S(L \oplus b \oplus a)$ so that:

$$D(L \oplus a \oplus b \oplus L') = \max_{\succ} \mathcal{A}^{\{C_i^*\}}(L \oplus a \oplus b \oplus L') = \max_{\succ} S(L \oplus a) = \max_{\succ}(\mathcal{A}^{\{C_i^*\}}(L \oplus b \oplus a \oplus L') \backslash \{b\})$$

From the fact that $D(L \oplus b \oplus a \oplus L') = \max_{\succ} \mathcal{A}^{\{C_i^*\}}(L \oplus b \oplus a \oplus L') = a$, it follows that $D(L \oplus a \oplus b \oplus L') = a$. In the second case, it follows that $\mathcal{A}^{\{C_i^*\}}(L \oplus a \oplus b \oplus L') = S(L \oplus b \oplus a \oplus L') = \mathcal{A}^{\{C_i^*\}}(L \oplus b \oplus a \oplus L')$ so:

$$D(L \oplus a \oplus b \oplus L') = \max_{\succ} \mathcal{A}^{\{C_i^*\}}(L \oplus a \oplus b \oplus L') = \max_{\succ} \mathcal{A}^{\{C_i^*\}}(L \oplus b \oplus a \oplus L') = D(L \oplus b \oplus a \oplus L') = a$$

Aspiration Successive Choice: Suppose $D(L_m \oplus c) = c$. Then, $D(L_m) \notin C_m^*$ by construction. Now, consider a list L_n such that $D(L_n) = D(L_m)$ and $n \leq m$. Since $D(L_m) \notin C_m^*$ and the cutoff sets are nested, $D(L_m) \notin C_n^*$ and $D(L_m) \notin C_1^*$. It then follows that $\mathcal{A}^{\{C_i^*\}}(L_n \oplus a) = S(L_n \oplus a)$ and $\mathcal{A}^{\{C_i^*\}}(D(L_n), a) = \{D(L_n), a\}$ so:

$$D(L_n \oplus a) = \max_{\succ}(S(L_n) \cup \{a\}) = \max_{\succ}\{\max_{\succ} S(L_n), a\} = \max_{\succ}\{D(L_n), a\} = D(D(L_n), a)$$

Binary Search Consistency: Suppose that $D(b, a) = a$ and $D(b, c) = b$. By construction of D, $b \notin C_1^*$. In order to obtain the desired choices, $a \succ b \succ c$ must hold. Then, $D(a, c) = a$ regardless of whether $a \notin C_1^*$.

Binary Preference Consistency: Suppose that $D(a, c) = c$, $D(a, b) = a$, $D(b, a) = b$. By the first identity, it follows that $a \notin C_1^*$. As such, the second identity requires $a \succ b$ and the third requires $b \in C_1^*$. By construction, it follows that $D(b, d) = b$ for any d.

($\Longleftarrow$) Let C_i^* be the revealed cutoff set C_{iR}^* defined in the text and let $\succ$ be any completion of P_R. That $D = D_{\{C_i^*\},\succ}$ follows from Theorem 1 and Lemma 6. To see that Theorem 1 applies, note that Indifference to Improvement strengthens Weak Indifference to Improvement. By Corollary 4, Preference Consistency and Search Consistency are satisfied. This establishes that $D = D_{s_R,\succ}$. By Lemma 11, it follows that:

$$D(L_i) \notin C_i^* \iff s_R(L_j) = \texttt{continue} \text{ for some } L_j \text{ s.t. } D(L_j) = D(L_i) \text{ and } j \geq i \iff s_R(L_i) = \texttt{continue}$$

As such, $D_{s_R,\succ} = D_{\{C_i^*\},\succ}$ so that $D = D_{\{C_i^*\},\succ}$. ■

8.3.4 Markov Search

First, I establish a lemma which proves a claim mentioned in the text.

Lemma 12 *Successive Choice is equivalent to $D(L) = D(D(L') \oplus L'')$ for any list $L = L' \oplus L''$.*

Proof. ($\Longleftarrow$) By definition. ($\Longrightarrow$) The proof is by induction on the length of the list. The base case where $n = 2$ is by definition. To show the induction step, suppose that $D(L_i' \oplus L_j'') = D(D(L_i') \oplus L_j'')$ for any list $L_N = L_i' \oplus L_j''$ of length N. Now, consider the list $L_{N+1} = L_i' \oplus L_j''$ of length $n = N + 1$. Using Successive

Choice and the induction step, it follows that:

$$D(L_{N+1}) = D(L'_i \oplus L''_j) = D(D(L'_i \oplus L''_{j-1}) \oplus l''_j) = D(D(D(L_i) \oplus L''_{j-1}) \oplus l'_j) = D(D(L_i) \oplus L''_{j-1} \oplus l'_j)$$

Thus,$D(L_{N+1}) = D(D(L_i) \oplus L''_j)$ as required. ■

Next, I establish that the canonical search strategy has a simple expression.

Lemma 13 *If D satisfies Successive Choice and Indifference to Improvement:*

$$s_R(L) = \texttt{continue} \textit{ iff } D(D(L) \oplus d) = d \textit{ for some } d.$$

Proof. Since Indifference to Improvement strengthens Weak Indifference to Improvement and Successive Choice strengthens Sequential Choice, Lemma 4 applies. By Lemma 4, $s_R(L) = \texttt{continue}$ iff $D(L \oplus d) = d$ for some d. By Successive Choice, $D(L \oplus d) = d$ iff $D(D(L) \oplus d) = d$. The result follows. ■

Finally, I establish that choice-symmetry coincides with symmetry when D satisfies Successive Choice.

Lemma 14 *If D satisfies Successive Choice, then any two items that are 2-symmetric (i.e. $D(\sigma_{ab}L_2) = \sigma_{ab}D(L_2)$ for any $L_2 \in \mathcal{L}_2$) are symmetric.*

Proof. Suppose a and b are 2-symmetric. There are three cases to consider: **(i)** $a, b \notin S(L)$; **(ii)** $a \in S(L)$ and $b \notin S(L)$; and **(iii)** $a, b \in S(L)$.

(i) By definition, $\sigma_{ab}L = L$ and $\sigma_{ab}D(L) = D(L)$ so that $D(\sigma_{ab}L) = \sigma_{ab}D(L)$.

(ii) Without loss of generality, suppose that $L = L' \oplus a \oplus L''_n$. By Successive Choice, $D(L) = D(D(D(L'), a) \oplus L''_n)$. If $D(L) = a$, then $D(D(L'), a) = a$ and $D(a, l''_i) = a$ for every $l''_i \in S(L''_n)$. Since a and b are 2-symmetric, it follows that $D(D(L'), b) = b$ and $D(b, l''_i) = b$. Using Successive Choice:

$$\begin{aligned} D(\sigma_{ab}L) &= D(L' \oplus b \oplus L''_n) = D(D(D(L'), b) \oplus L''_n) \\ &= D(b \oplus L''_n) = D(D(...(D(b, l''_1)...), l''_n)) \\ &= b = \sigma_{ab}a = \sigma_{ab}D(L) \end{aligned}$$

If $D(L) \neq a$, Successive Choice implies that: **(a)** $D(D(L'), a) = D(L')$; or **(b)** $D(D(L'), a) = a$ and there is a minimal i such that $D(a, l''_i) = l''_i$ (and $D(a, l''_j) = a$ for all $j < i$).

(a) $D(D(L'), b) = D(L')$ since a and b are 2-symmetric so that:

$$\begin{aligned} D(\sigma_{ab}L) &= D(L' \oplus b \oplus L''_n) = D(D(D(L'), b) \oplus L''_n) \\ &= D(D(L') \oplus L''_n) = D(D(D(L'), a) \oplus L''_n) = \sigma_{ab}D(L) \end{aligned}$$

(b) $D(D(L'), b) = b$, $D(b, l''_i) = l''_i$ and $D(b, l''_j) = b$ for all $j < i$ since a and b are 2-symmetric so that:

$$\begin{aligned} D(\sigma_{ab}L) &= D(L' \oplus b \oplus L''_i \oplus L''^{i+1}_n) = D(D(L' \oplus b \oplus L''_i) \oplus L''^{i+1}_n) \\ &= D(l''_i \oplus L''^{i+1}_n) = D(D(L' \oplus a \oplus L''_i) \oplus L''^{i+1}_n) = \sigma_{ab}D(L) \end{aligned}$$

(iii) Without loss of generality, suppose $L = L' \oplus a \oplus L'' \oplus b \oplus L'''$. By Successive Choice:

$$D(\sigma_{ab}L) = D(L' \oplus b \oplus L'' \oplus a \oplus L''') = D(D(L' \oplus b \oplus L'') \oplus a \oplus L''')$$

There are two cases to consider: (a) $D(L' \oplus b \oplus L'') = b$ and (b) $D(L' \oplus b \oplus L'') = c \neq b$. In case (a):

$$\begin{aligned} D(\sigma_{ab}L) &= D(L' \oplus b \oplus L'' \oplus a \oplus L''') = D(D(L' \oplus b \oplus L'') \oplus a \oplus L''') \\ &= D(b \oplus a \oplus L''') = D(D(b,a) \oplus L''') = \sigma_{ab}D(D(a,b) \oplus L''') \end{aligned}$$

where the last equality follows from 2-symmetry and part (ii) above. Since $D(L' \oplus b \oplus L'') = b$, part (ii) also implies $D(L' \oplus a \oplus L'') = a$ so that:

$$D(\sigma_{ab}L) = \sigma_{ab}D(D(D(L' \oplus a \oplus L''), b) \oplus L''') = \sigma_{ab}D(L' \oplus a \oplus L'' \oplus b \oplus L''') = \sigma_{ab}D(L)$$

In case (b):

$$\begin{aligned} D(\sigma_{ab}L) &= D(L' \oplus b \oplus L'' \oplus a \oplus L''') = D(D(L' \oplus b \oplus L'') \oplus a \oplus L''') \\ &= D(c \oplus a \oplus L''') = D(D(c,a) \oplus L''') = \sigma_{ab}D(D(c,b) \oplus L''') \end{aligned}$$

where the last equality follows from 2-symmetry and part (ii) above. Since $D(L' \oplus b \oplus L'') = c$, part (ii) also implies $D(L' \oplus a \oplus L'') = c$ so that:

$$D(\sigma_{ab}L) = \sigma_{ab}D(D(D(L' \oplus a \oplus L''), b) \oplus L''') = \sigma_{ab}D(L' \oplus a \oplus L'' \oplus b \oplus L''') = \sigma_{ab}D(L)$$

The three cases establish the claim. ■

Proof of Theorem 6 (Markov Search). ($\Longrightarrow$) Suppose that D is induced by the Markov search procedure $(C^*, \succ)$. The proof of *Binary Preference Consistency* is the same as Theorem 7. The proof of *Indifference to Improvement* follows from the proof in Theorem 7 (since Markov search is a special case of aspiration adaptation).

To establish *Successive Choice*, there are two cases to consider: (i) $D(L) \in C^*$; and (ii) $D(L) \notin C^*$. In the first case, $\mathcal{A}^{C^*}(L \oplus a) = \mathcal{A}^{C^*}(L) = S(L)$ and $\mathcal{A}^{C^*}(D(L) \oplus a) = \{D(L)\}$ so that:

$$D(L \oplus a) = \max_{\succ} S(L) = D(L) = \max_{\succ}\{D(L)\} = D(D(L) \oplus a)$$

In the second case, $\mathcal{A}^{C^*}(L \oplus a) = S(L \oplus a)$ so that:

$$D(L \oplus a) = \max_{\succ} S(L \oplus a) = \max_{\succ}(S(L) \cup \{a\}) = \max_{\succ}\{\max_{\succ} S(L), a\} = \max_{\succ}\{D(L), a\}$$

Since $\mathcal{A}^{C^*}(D(L) \oplus a) = \{D(L), a\}$, $D(L \oplus a) = D(D(L) \oplus a)$.

($\Longleftarrow$) Let C^* be the revealed cutoff set C_R^* defined in the text and let $\succ$ be any completion of P_R. The fact that $D = D_{C^*,\succ}$ follows from Theorem 1 and Lemma 13. The fact that Theorem 1 applies follows from the same reasoning as in Theorem 7 (given Lemma 8). This establishes that $D = D_{s_R,\succ}$. By Lemma 13, it follows that:

$$D(L) \notin C^* \iff s_R(L') = \texttt{continue} \text{ for some } L' \text{ such that } D(L) = D(L') \iff s_R(L) = \texttt{continue}$$

As such, $D_{s_R,\succ} = D_{C^*,\succ}$ so that $D = D_{C^*,\succ}$.

From Theorem 1, items are unranked by P_R iff they are choice-symmetric. By Lemma 14, it follows that items are unranked by P_R iff they are symmetric. (By definition, two items are choice-symmetric when they are symmetric. Since choice symmetric items are 2-symmetric, Lemma 14 establishes that choice-symmetric items are symmetric) ■

8.3.5 Best-Satisficing

Proof of Theorem 4 (Best-Satisficing).

($\Longrightarrow$) Partition Independence was established in the text of the paper. *Binary Preference Consistency* follows from Theorem 6 (since best-satisficing is a special case of Markov search) and *Indifference to Improvement* from Theorem 7 (since best-satisficing is a special case of aspiration adaptation).

($\Longleftarrow$) Let C^* be the revealed cutoff set C_R^* defined in the text and let $\succ$ be any completion of P_R. The fact that $D = D_{C^*,\succ}$ follows from Theorem 6 (given that Partition Independence strengthens Successive Choice). To complete the proof, I show:

(i) $C^* = \{a : \text{no } x \in X \text{ such that } xPa\}$; and **(ii)** P is a linear order when restricted $X \backslash C^*$.

(i) By Theorem 6, $C^* = \{a : D(a,x) = a \text{ for all } x \in X \backslash \{a\}\}$. Thus, C^* consists of the items such that $\neg(xA_2a)$ for all $x \in X\backslash\{a\}$. Now, suppose that cPa for some $a \in C^*$ and some $c \in X$. By Lemma 9, it follows that cA_2a, cB_2a or cA_2dB_2a for some $d \in X$. Since the first case is impossible (i.e. $\neg(xA_2a)$ for all $x \in X\backslash\{a\}$), it is sufficient to establish that cB_2a leads to a contradiction to show that cPa cannot hold. From cB_2a, $D(c,a) = c$ and $D(c,c') = c'$ for some $c' \in X\backslash\{a,c\}$. Thus, $c'A_2cB_2a$. Now consider $< c,a,c' >$. By Partition Independence, $D(c,a,c') = D(D(c,a),c') = D(c,c') = c'$ and $D(c,a,c') = D(c,D(a,c')) = D(c,a) = c$. This contradiction ensures that cB_2a cannot hold. Thus, $C^* = \{a : \text{no } x \in X \text{ such that } xPa\}$.

(ii) To see this, first notice that

$$D(x,c^*) = c^* \text{ for any } x \in X\backslash C^* \text{ and } c^* \in C^* \tag{$*$}$$

Otherwise, $D(x,d) = d$ for some $d \in X$ and $D(x,c^*) = x$. It follows that xB_2c^* which contradicts part (i). Without loss of generality, there are two possibilities. Either **(a)** $C^* = \{c^*\}$ is a singleton or **(b)** C^* contains multiple elements.

(a) Since $D(x,c^*) = c^*$ for any $x \in X\backslash\{c^*\}$, it follows that $x'A_2x$ or xB_2x' for any $x,x' \in X\backslash\{c^*\}$.

The first possibility arises when $D(x, x') = x'$ and the second when $D(x, x') = x$. Since $c^* A_2 x$ for any $x \in X \backslash \{c^*\}$, P is complete on X. So, it is a linear order.

(b) I claim that C^* consists of all the symmetric items in X. By the reasoning given in case (a), it then follows that P is complete on $X \backslash C^*$ so that P defines a linear order on $X \backslash C^*$.

Proof of Claim: First note that the symmetry classes respect the partition $\{C^*, X \backslash C^*\}$. To see this, suppose $a \in C^*$ and $b \in X \backslash C^*$. By part (i) above, $D(a, x) = a$ for all $x \in X \backslash \{a\}$ and $D(b, x) = x$ for some $x \in X \backslash \{b\}$. Then, a and b are not 2-symmetric (and cannot be symmetric). It is easy to see that any $a, b \in C^*$ are symmetric. By part (i), $D(\sigma_{ab} < a, x >) = \sigma_{ab} D(a, x) = b$ for any $x \in X \backslash \{a\}$. Moreover, $D(x, a) = a$ for any $x \in X \backslash C^*$ (by the identity $(*)$) and $D(x, a) = x$ for any $x \in C^*$ (by definition of C^*). As such, a and b are 2-symmetric. From Lemma 14, a and b are symmetric. To see that $a, b \in X \backslash C^*$ cannot be symmetric, suppose otherwise. By symmetry, $D(a, b) = \sigma_{ab} D(b, a)$. By Indifference to Improvement, $D(a, b) = a$ and $D(b, a) = b$ (since $D(a, b) = b$ implies $D(b, a) = b$). Now consider some $c^* \in C^*$. By the identity $(*)$, $D(a, c^*) = D(b, c^*) = c^*$. Since $D(a, b) \neq D(b, a)$ and $D(a, c^*) = c^*$, Binary Preference Consistency implies $D(b, c^*) = b$ (since $D(b, d) = b$ for all items $d \in X$). This is the desired contradiction. ■

Proof of Theorem 5 (Rational Choice). ($\Longrightarrow$) *Partition Independence* was established in the text. To show *Binary Order Independence*, note $c^* = \max_{\succ} X$ so that $\mathcal{A}^{c^*}(a, b) = \mathcal{A}^{c^*}(b, a) = \{a, b\}$. Then:

$$D_{c^*,\succ}(a, b) = \max_{\succ}\{a, b\} = D_{c^*,\succ}(b, a)$$

($\Longleftarrow$) The fact that $D = D_{c^*,\succ}$ follows from Theorem 4 since Partition Independence and Binary Order Independence imply Indifference to Improvement. (The fact that Binary Preference Consistency holds follows trivially from Binary Order Independence). To see that one can dispense with Indifference to Improvement, note that Partition Independence implies:

$$D(L \oplus a \oplus a' \oplus L') = D(D(L) \oplus D(a \oplus a' \oplus L')) = D(D(L) \oplus D(D(a, a') \oplus D(L')))$$

Since Binary Order Independence implies that $D(a, a') = D(a', a)$, it follows that:

$$D(D(L) \oplus D(D(a, a') \oplus D(L'))) = D(D(L) \oplus D(D(a', a) \oplus D(L'))) = D(D(L) \oplus D(a' \oplus a \oplus L')) = D(L \oplus a' \oplus a \oplus L')$$

so $D(L \oplus a \oplus a' \oplus L') = D(L \oplus a' \oplus a \oplus L')$ for any a and a' (and not just when $D(L \oplus a \oplus a' \oplus L') = a'$).

To see that P defines a linear order, suppose that a and b are symmetric. Then $D(a, b) = \sigma_{ab} D(b, a)$. First, suppose $D(a, b) = a$. Then, Binary Order Independence requires $D(b, a) = a$. Next, suppose $D(a, b) = b$. Then, Binary Order Independence requires $D(b, a) = b$. As such, no two items are symmetric. From Theorem 4, it follows that P defines a linear order. ■

8.3.6 Extensions

Proof of Proposition 4 (Generalized Search). ($\Longrightarrow$) The proof is similar to Proposition 1. Since D satisfies Sequential Choice, $D(L_{i-1} \oplus L) \notin L$ implies $D(L_{i-1} \oplus L) = D(L_{i-1})$. ($\Longleftarrow$) The proof is similar to Proposition 1. ■

Proof of Proposition 5 (Search with a Signal). The proof is similar to Proposition 1. ■

Proof of Proposition 6 (Search with Duplication). (I) Revealed Preference: ($\Longrightarrow$) The proof is similar to Proposition 1. ($\Longleftarrow$) As in Lemma 2, $\mathcal{A}^{*s_R}(L) = \mathcal{A}_R^-(L)$ for any L. Following the same reasoning as Proposition 2, $(s_R, \succ_R)$ is a canonical representation of D^* (when D^* is representable in terms of search with recall). The same reasoning as in Proposition 1 then establishes the desired result.

(II) Revealed Attention: ($\Longrightarrow$) Suppose $l_i \in \cap_{j \in J} \mathcal{A}^{*s_j}(L_m)$ where $\{(s_j, r_j)\}_{j=1}^n$ is the non-empty collection of pairs that represent D. By way of contradiction, suppose that $l_i \notin \mathcal{A}_R^{*-}(L_m)$. By definition, $D^*(L_{i-1} \oplus L) = D^*(L_{i-1})$ for all possible list extensions L (as in Proposition 4). The rest of the proof is similar to Proposition 1. ($\Longleftarrow$) Suppose $l_i \in \mathcal{A}_R^-(L_m)$. By way of contradiction, suppose there is some (s_j, r_j) that represents D such that $l_i \notin \mathcal{A}^{s_j}(L_m)$. By definition, it must be that $s_j(L_k) = \texttt{stop}$ for some $k < i$. By construction, it follows that $D^*_{s_j,r_j}(L_{i-1} \oplus L) = D^*_{s_j,r_j}(L_{i-1})$ for all extensions L. Since $D^*_{s_j,r_j} = D^*$, $D^*(L_{i-1} \oplus L) = D^*(L_{i-1})$ for all extensions L. By definition, this contradicts the fact that $l_i \in \mathcal{A}_R^-(L_m)$. ■

Proof of Proposition 7 (List-Choice Correspondences). (I) Revealed Preference: ($\Longrightarrow$) By reasoning similar to Proposition 1, aPa' (resp. aRa') implies $a \succ_j a'$ (resp. $aR \succeq_j a'$) for any $(s_j, \succeq_j)$ that represents $\bar{D}$. Now, suppose that

$$a = a_1 R ... R a_m = a'$$

for some choice of $\{a_i\}_{i=1}^m$. From the observation that aRa' implies $a \succeq_j a'$ for any $(s_j, \succeq_j)$ that represent $\bar{D}$, it follows that:

$$a \succeq_j a_1 \succeq_j ... \succeq_j a_m \succeq_j a' \tag{$*$}$$

so that $a \succeq_j a'$ by transitivity. By definition of revealed indifference, $aI_R a'$ if $a = a_1 R ... R a_m = a'$ and $a' = a'_1 R ... R a'_{m'} = a$. By identity ($*$), it follows that $a \succeq_j a'$ and $a' \succeq_j a$ so that $a \sim_j a'$ by definition. By definition of the strict revealed preference, $aP_R a'$ if $a = a_1 R ... R a_m = a'$ and $a_i P a_{i+1}$ for some $1 \leq i \leq m$. By identity ($*$), it follows that $a \succeq_j a_i P a_{i+1} \succeq_j a'$. Since $a_i P a_{i+1}$ implies $a_i \succ_j a_{i+1}$, $a \succeq_j a_i \succ_j a_{i+1} \succeq_j a'$ so that $a \succ_j a'$ by transitivity.

($\Longleftarrow$) From ($\Longrightarrow$) above, P_R is acyclic. Otherwise, $\bar{D}$ is not representable in terms of search with recall. Now, define $\succ_R^I$ to be any completion of P_R on $X/(I_R)$ (i.e. the set where the indifference classes in X induced by I_R have been collapsed into single elements $a/(I_R)$). Let $a \succ_R a'$ if $\neg(aI_R a')$ and $a/(I_R) \succ_R^I a'/(I_R)$; and let $a \sim_R a'$ if $aI_R a'$. To see that $(s_R, \succeq_R)$ is a canonical representation of $\bar{D}$ (when $\bar{D}$ is representable in terms of search with recall), suppose $\bar{D}(L) \neq \bar{D}_{s_R, \succeq_R}(L)$ for some L. As in Lemma 2, $\bar{\mathcal{A}}^{s_R}(L) = \bar{\mathcal{A}}_R^-(L)$ for any L. Since $\bar{D}_{s_R, \succeq_R}(L) \subset \bar{\mathcal{A}}^{s_R}(L)$ and $\bar{D}(L) \subset \bar{\mathcal{A}}_R^-(L)$, $\bar{D}(L) \cup \bar{D}_{s_R, \succeq_R}(L) \subset \bar{\mathcal{A}}^{s_R}(L) = \bar{\mathcal{A}}_R^-(L)$. There are four separate cases to consider: **(i)** $\bar{D}(L) \cap \bar{D}_{s_R, \succeq_R}(L) = \emptyset$; **(ii)** $\bar{D}(L) \subset \bar{D}_{s_R, \succeq_R}(L)$; **(iii)** $\bar{D}_{s_R, \succeq_R}(L) \subset \bar{D}(L)$; and **(iv)** there is an $a \in \bar{D}(L) \backslash \bar{D}_{s_R, \succeq_R}(L)$ and $a' \in \bar{D}_{s_R, \succeq_R}(L) \backslash \bar{D}(L)$.

(i) aPa' and $a' \succ_R a$ for some $a \in \bar{D}(L)$ and $a' \in \bar{D}_{s_R,\succeq_R}(L)$. This is a contradiction since $a' \succ_R a$ cannot hold in any $\succeq_R$ where aPa'. **(ii)** aPa' and $a' \sim_R a$ for some $a \in \bar{D}(L)$ and $a' \in \bar{D}_{s_R,\succeq_R}(L)$. Since $a' \sim_R a$ implies $a'I_Ra$ so that aP_Ra, this is a contradiction. **(iii)** aIa' and $a' \succ_R a$ for some $a \in \bar{D}(L)$ and $a' \in \bar{D}_{s_R,\succeq_R}(L)$. By the construction of $\succeq_R$, this is a contradiction. **(iv)** The proof for this case reduces to any one of the other three.

In order to complete the proof, it suffices to show that the following lead to contradictions: **(a)** $a \sim_j a'$ for all $(s_j, \succeq_j)$ that represent $\bar{D}$ but that $\neg(aI_Ra')$; and, **(b)** $a \succ_j a'$ for all $(s_j, \succeq_j)$ that represent $\bar{D}$ but that $\neg(aP_Ra')$. In case **(a)**, $(s_R, \succeq_R)$ is a representation of $\bar{D}$ such that $a \succ_j a'$ (or $a' \succ_j a$). This is the desired contradiction. In case **(b)**, there is some completion $\succeq_R$ such that $a' \succeq_R a$ where $(s_R, \succeq_R)$ represents $\bar{D}$. (If aI_Ra', then $a \sim_R a'$ by construction. If $\neg(aP_Ra')$ and $\neg(aI_Ra')$, then there exists an $\succ_R^I$ such that $a' \succ_R^I a$.) This is the desired contradiction.

(II) Revealed Attention: ($\Longrightarrow$) Suppose $l_i \in \cap_{j\in J}\bar{\mathcal{A}}^{s_j}(L_m)$ where $\{(s_j, \succeq_j)\}_{j=1}^n$ is the non-empty collection of pairs that represent $\bar{D}$. By way of contradiction, suppose that $l_i \notin \bar{\mathcal{A}}_R^-(L_m)$. By definition, $\bar{D}(L_{i-1} \oplus L) \cap L = \emptyset$ for all possible list extensions L. Thus, $\bar{D}(L_{i-1} \oplus L) = \bar{D}(L_{i-1})$. The rest of the proof is similar to Proposition 1. ($\Longleftarrow$) Suppose that $l_i \in \bar{\mathcal{A}}_R^-(L_m)$. By way of contradiction, suppose there is some $(s_j, \succeq_j)$ that represents $\bar{D}$ such that $l_i \notin \bar{\mathcal{A}}^{s_j}(L_m)$. By definition, it must be that $s_j(L_k) = \texttt{stop}$ for some $k < i$. By construction, it follows that $\bar{D}_{s_j,\succeq_j}(L_{i-1} \oplus L) \cap L = \emptyset$. Since $\bar{D}_{s_j,\succeq_j} = \bar{D}$, $\bar{D}(L_{i-1} \oplus L) \cap L = \emptyset$ for all extensions L. By definition, this contradicts the fact that $l_i \in \bar{\mathcal{A}}_R^-(L_m)$. ■

9 Appendix: Axiomatic Foundations for Extensions

9.1 Choice Rules

If the choice rule satisfies Sequential Choice, this property is inherited by the choice function D. Trivially, any choice function D that satisfies Sequential Choice can be represented in terms of a choice rule that satisfies Sequential Choice. As such:

Theorem 8 (Generalized Search) *D can be represented by the generalized search procedure (s_R, D) iff it satisfies Sequential Choice.*

9.2 Awareness of the List

In this section, I describe the axiomatic foundations for the special case discussed in the text. Formally, let *search monotonicity* describe the following restriction on the search strategies $\{s_n\}$:

$$\text{For any } L \in \mathcal{L}^n \text{ and any } n \in \mathbb{N},\ s_n(L) = \texttt{stop} \text{ implies } s_{n+1}(L) = \texttt{stop}$$

Define any search procedure $(\{s_n\}, r)$ that satisfies search monotonicity to be a *monotonic search procedure.* Based on the discussion in the text, the modifications required to characterize monotonic search procedures (with and without recall) are:

3_n. **(Preference Consistency)** *If* $D(L \oplus L''_m) \in L''_m$ *and* $D(\bar{L} \oplus \bar{L}''_{\bar{m}}) \in \bar{L}''_{\bar{m}}$ *for* $m > n$ *and* $\bar{m} > \bar{n}$*, then* $D(L \oplus b \oplus L'_n) = a$ implies $\bar{D}(\bar{L} \oplus a \oplus \bar{L}'_{\bar{n}}) \neq b$ *for any* $a \neq b$.

4_n. **(Search Consistency)** *If* $D(L \oplus b \oplus L'_n) = a$ *and* $D(L \oplus L''_m) \in L''_m$ *for* $m > n$*, then*

$$D(\bar{L} \oplus b) = b \text{ implies } D(\bar{L} \oplus a) = a$$

5_n. **(No Recall)** If $D(L \oplus L'_n) \in L$, then $D(L \oplus L''_m) \in L$ for $m \geq n$.

Theorem 9 (Signal of List Length) ***(I)*** *D can be represented by the monotonic search procedure* $(\{s_n\}_R, \succ_R)$ *iff it satisfies Sequential Choice, Weak Indifference to Improvement, Axiom* 3_n*, and Axiom* 4_n*.* ***(II)*** *D can be represented by the monotonic search procedure* $\{s_n\}_R$ *iff it satisfies Sequential Choice and Axiom* 5_n*.*

Proof. (I) ($\Longrightarrow$) This is straightforward. ($\Longleftarrow$) It can be shown that aPb iff aAb or aBb where A and B are defined as in Lemma 4. Moreover:

$$s_{nR}(L_i) = \texttt{continue} \text{ iff } D(L_i \oplus L'_{m-i}) \in L'_{m-i} \text{ for some } L'_{m-i} \text{ and } m \geq n$$

It suffices to establish that: **(a)** P is irreflexive, asymmetric and acyclic; and **(b)** $D = D_{s_n,\succ}$ where $s_n = s_{nR}$ is the canonical strategy and $\succ$ is a completion of P_R. **(a)** This can be established as in Theorem 1. **(b)** The proof is similar to Theorem 1.

(II) ($\Longrightarrow$) This is straightforward. ($\Longleftarrow$) The proof is similar to Theorem 2. ■

9.3 Lists with Duplication

Based on the discussion in the text, the required modifications to the baseline axioms are:

3^*. **(Preference Consistency)** *If* $D^*(L \oplus c) \neq D^*(L)$ *for some* c *and* $D^*(\bar{L} \oplus \bar{c}) \neq D^*(\bar{L})$ *for some* $\bar{c}$*, then* $D^*(L \oplus b \oplus L') = a$ implies $D^*(\bar{L} \oplus a \oplus \bar{L}') \neq b$ *for* $a \neq b$.

4^*. **(Search Consistency)** *If* $D^*(L \oplus b \oplus L') = a$ *and* $D^*(L \oplus c) \neq D^*(L)$ *for some* c*, then*

$$D^*(\bar{L} \oplus b) \neq D(\bar{L}) \text{ implies } D^*(\bar{L} \oplus a) = a$$

5^*. **(No Recall)** *If* $D^*(L \oplus L') \neq D^*(L)$ *for some* L'*, then* $D^*(L \oplus L'') \in L''$.

Theorem 10 (Lists with Duplication) ***(I)*** *D^* can be represented by the search procedure* $(s^*_R, \succ_R)$ *iff it satisfies Sequential Choice, Weak Indifference to Improvement, Axiom* 3^**, and Axiom* 4^**.* ***(II)*** *D^* can be represented by the search procedure* s^*_R *iff it satisfies Sequential Choice and Axiom* 5^**.*

Proof. **(I)** ($\Longrightarrow$) It is straightforward to verify the axioms. ($\Longleftarrow$) As in Lemma 4, it can be shown that aPb iff aAb or aBb where A and B are defined by:

$$\begin{aligned} aAb \text{ if } D^*(L \oplus b \oplus a) &\neq D^*(L \oplus b) \\ aBb \text{ if } D^*(L \oplus a \oplus b) &= a \neq b \text{ and } D^*(L \oplus a \oplus c) \neq D^*(L \oplus a) \text{ for some } c \end{aligned}$$

Moreover, it can be shown that:

$$s_R^*(L) = \texttt{continue} \text{ iff } D^*(L \oplus d) \neq D^*(L) \text{ for some } d \notin L$$

So, it suffices to establish that: **(a)** P is irreflexive, asymmetric and acyclic; and **(b)** $D^* = D^*_{s,\succ}$ where $s = s_R^*$ is the canonical strategy and $\succ$ is a completion of P_R. **(a)** By definition, P is irreflexive. Asymmetry and acyclicity can be established as in Theorem 1. **(b)** The proof given in Theorem 1 must be modified (since it relies on the index of $D(L_n)$). First, suppose $\mathcal{A}^{*s}(L_n) = S(L_i)$ and $D^*(L_n) \notin L_i$. By repeated application of Weak Indifference to Improvement and Successive Choice, $D^*(L_i \oplus D^*(L_n)) = D^*(L_n)$. Since $D^*(L_n) \notin L_i$, $D^*(L_i \oplus D^*(L_n)) \neq D^*(L_i)$. By definition of s, $s(L_i) = \texttt{continue}$ so that $\mathcal{A}^{*s}(L_n) \neq S(L_i)$. Thus, $D^*(L_n) \in \mathcal{A}^{*s}(L_n)$. Now, suppose $D^*(L_n) \neq D^*_{s,\succ}(L_n)$. Then, $D^*_{s,\succ}(L_n) \succ D^*(L_n)$. Moreover, $D^*(L_n) P D^*_{s,\succ}(L_n)$ by definition of P. Since P is acyclic (by part (a) above) and $\succ$ completes P, this is the desired contradiction.

(II) ($\Longrightarrow$) It is straightforward to verify the axioms. ($\Longleftarrow$) It can be shown that the canonical strategy $s = s_R^*$ can be stated as in part (I). The proof that $D^* = D^*_s$ in Theorem 2 must be modified (since it relies on the index of $D(L_n)$). First, suppose that $\mathcal{A}^{*s}(L_n) = S(L_i)$. By a similar argument as that given in part (I), $D^*(L_n) \in L_i$. Now, suppose $D^*(L_n) \neq D^*_s(L_n)$. Then, by definition of D^*_s, $D^*(L_n) \neq l_i$ (since D^*_s picks out the last item in $\mathcal{A}^{*s}(L_n)$). In other words, $D^*(L_n) \notin L_n^i$. By No Recall, it follows that $D^*(L_{i-1}) = D^*(L_{i-1} \oplus L')$ for any L'. This is the desired contradiction since $\mathcal{A}^{*s}(L_n) = S(L_i)$ requires $D^*(L_{i-1} \oplus d) \neq D^*(L_{i-1})$ for some $d \notin L_{i-1}$. ■

9.4 List-Choice Correspondences

Based on the discussion in the text, the required modifications to the baseline axioms are:

$\bar{1}$. **(Sequential Search)** $\bar{D}(L \oplus a) \subset \bar{D}(L) \cup \{a\}$.

$\bar{2}$. **(Indifference to Improvement)** *If* $a \in \bar{D}(L \oplus b \oplus a)$, *then* $a \in \bar{D}(L \oplus a \oplus b)$.

$\bar{3}$. **(Preference Consistency)** *If* $c \in \bar{D}(L \oplus c)$ *and* $\bar{c} \in \bar{D}(\bar{L} \oplus \bar{c})$, *then for any* $a \neq b$

$$a \in \bar{D}(L \oplus b \oplus L') \text{ and } b \notin \bar{D}(L \oplus b \oplus L') \text{ implies } b \notin \bar{D}(\bar{L} \oplus a \oplus \bar{L}')$$

$\bar{4}$. **(Search Consistency)** *If* $a \in \bar{D}(L \oplus b \oplus L')$ *and* $c \in \bar{D}(L \oplus c)$, *then* $b \in \bar{D}(\bar{L} \oplus b)$ implies $a \in \bar{D}(\bar{L} \oplus a) \subset \sigma_{ab}\bar{D}(\bar{L} \oplus b)$.

Theorem 11 (List-Choice Correspondences) *$\bar{D}$ can be represented by the search procedure $(s_R, \succeq_R)$ iff it satisfies Axioms $\bar{1}$ to $\bar{4}$.*

Proof. ($\Longrightarrow$) It is straightforward to verify the axioms. ($\Longleftarrow$) As in Lemma 4, it can be shown that aRb (resp. aPb) iff $a\bar{A}b$ or $a\bar{B}b$ (resp. aAb or aBb) where $\bar{A}$ and $\bar{B}$ (resp. A and B) are defined by:

$$\begin{aligned} a\bar{A}b \text{ (resp. } aAb\text{) if } a &\in \bar{D}(L \oplus b \oplus a) \text{ (resp. ...and } b \notin \bar{D}(L \oplus b \oplus a)) \\ a\bar{B}b \text{ (resp. } aBb\text{) if } a &\in \bar{D}(L \oplus a \oplus b),\ c \in \bar{D}(L \oplus a \oplus c) \text{ for some } c \text{ (resp. ...and } b \notin \bar{D}(L \oplus a \oplus b)) \end{aligned}$$

Moreover, it can be shown that $\bar{s}_R(L) = \texttt{continue}$ iff $d \in \bar{D}(L \oplus d)$ for some $d \notin L$.

So, it suffices to establish that: **(I)** R only contains "weak" cycles; and **(II)** $\bar{D} = \bar{D}_{s,\succeq}$ where $s = \bar{s}_R$ is the canonical strategy and $\succeq$ is a completion of P_R (as defined in Proposition 7).

(I) To establish that R only contains "weak" cycles, I show that:

$$a = a_1 R ... R a_n = a \text{ implies } \neg(a_i P a_{i+1}) \text{ for all } 1 \leq i < n \tag{WC}$$

where the a_i are all distinct. By definition, P is irreflexive. Then, by Preference Consistency, aRb implies $\neg(bPa)$. To establish property (WC), the proof proceeds by induction as in Theorem 1. First, consider the following claims:

Claim 1(a).	aRb and $b\bar{A}c$ imply aRc;
Claim 1(b).	aPb and $b\bar{A}c$ imply aAc;
Claim 2.	$a\bar{A}b$ and $b\bar{B}c$ imply cPa is impossible; and
Claim 3.	$a\bar{B}b$ and $b\bar{B}c$ imply aRc.

These can be established in a manner similar to Theorem 1 with some modifications. In order to prove these claims, I first show that $\bar{D}(L \oplus a) \in \{\bar{D}(L) \cup \{a\}, \bar{D}(L), \{a\}\}$. By way of contradiction, suppose that $c \in \bar{D}(L \oplus a)$ for $c \in L$, $c' \notin \bar{D}(L \oplus a)$ and $c' \in \bar{D}(L)$. By Sequential Choice, $c \in \bar{D}(L \oplus a)$ implies $c \in \bar{D}(L)$. As such, $c'Rc$ and cPc'. By Preference Consistency, this is a contradiction.

Claim 1(a): By definition, $b\bar{A}c$ implies $b \in \bar{D}(L' \oplus c \oplus b)$ for some list L'. If $a \in L'$, then $a \in \bar{D}(L' \oplus c \oplus b)$ so that $a\bar{B}c$. If $a \notin L'$, Search Consistency implies $a \in \bar{D}(L \oplus c \oplus a)$ so that $a\bar{A}c$.

Claim 1(b): By definition, $b\bar{A}c$ implies $b \in \bar{D}(L' \oplus c \oplus b)$ for some list L'. By Preference Consistency, $a \notin L'$. Then, by Search Consistency $a \in \bar{D}(L' \oplus c \oplus a)$. By way of contradiction, suppose $c \in \bar{D}(L' \oplus c \oplus a)$ so that $c\bar{B}a$. Since aPb, it follows that aAb or aBb. In the first case, there is a list L such that $a \in \bar{D}(L \oplus b \oplus a)$ and $b \notin \bar{D}(L \oplus b \oplus a)$. If $c \in L$, Preference Consistency requires that $c \notin \bar{D}(L \oplus b \oplus a)$ so that aAc. If $c \notin L$, $c \in \bar{D}(L \oplus b \oplus c)$ by Search Consistency. By Preference Consistency, $b \in \bar{D}(L \oplus b \oplus c)$ so that $b\bar{B}c$. By Claim 3 below, $b\bar{B}c\bar{B}a$ implies bRa. But, this contradicts Preference Consistency so $c \notin \bar{D}(L' \oplus c \oplus a)$. In the second case, there is a list L such that $a \in \bar{D}(L \oplus a \oplus b)$, $b \notin \bar{D}(L \oplus a \oplus b)$ and $c' \in \bar{D}(L \oplus a \oplus c')$. By Search Consistency, $c \notin \bar{D}(L \oplus a \oplus c)$ so that aBc. By Preference Consistency, this is a contradiction so $c \notin \bar{D}(L' \oplus c \oplus a)$. Since $c \notin \bar{D}(L' \oplus a \oplus c)$ in either case, it follows that aAc.

Claim 2: Suppose cPa. By Claim 1(b), it follows that cAb. Since $b\bar{B}c$, this contradicts Preference Consistency.

Claim 3: By definition, $a\bar{B}b$ implies $a \in \bar{D}(L \oplus a \oplus b)$ and $d \in \bar{D}(L \oplus a \oplus d)$ for some list L. If $c \in L$, $a\bar{A}c$. If $c \notin L$, consider $L \oplus a \oplus c$ and suppose $a \notin \bar{D}(L \oplus a \oplus c)$. By Sequential Choice, $a \in \bar{D}(L \oplus a)$. As such, $a \notin \bar{D}(L \oplus a \oplus c)$ implies that $\bar{D}(L \oplus a \oplus c) = c$. By Search Consistency, $\bar{D}(L \oplus a \oplus b) = b$. This is the desired contradiction.

As in Theorem 1, these claims establish that:

$$a_1 R a_2 R ... R a_{n-1} R a_n \text{ imply } a_1 R a_n \text{ or } a_1 \bar{A} a' \bar{B} a_n \text{ (for } a' \in \{a_i\}_2^{n-1}) \quad (*)$$

for any n. The proof of property $(*)$ is by induction on the length of the chain. Claims 1(a) and 3 establish property $(*)$ when $n = 3$. The induction case $n = N + 1$ then follows by an argument similar to that given in Theorem 1. Property $(*)$ guarantees property (WC). If $a_1 R a_n$, it follows that $\neg(a_n P a_1)$. If $a_1 \bar{A} a' \bar{B} a_n$, Claim 2 establishes that $a_n P a_1$ cannot obtain.

(II) If $l_i \in \bar{D}_{s,\succeq}(L_n)$ then $l_i \in \bar{\mathcal{A}}^s(L_n)$. Now, suppose $l_i \notin \bar{D}(L_n)$ and consider any $l_j \in \bar{D}(L_n)$. Since $l_j \in \bar{D}(L_j)$, it follows that $l_j \in \bar{\mathcal{A}}_R^-(L_n)$. Consequently, $l_j P l_i$. Since $\bar{\mathcal{A}}_R^-(L_n) = \bar{\mathcal{A}}^s(L_n)$, it follows that $l_i \notin \bar{D}_{s,\succeq}(L_n)$. This is the contradiction which establishes that $l_i \in \bar{D}(L_n)$. Thus, $\bar{D}_{s,\succeq}(L_n) \subset \bar{D}(L_n)$. If $l_j \in \bar{D}(L_n)$, $l_j \in \bar{\mathcal{A}}_R^-(L_n) = \bar{\mathcal{A}}^s(L_n)$. Now, suppose $l_j \notin \bar{D}_{s,\succeq}(L_n)$ and consider any $l_i \in \bar{D}_{s,\succeq}(L_n)$. By construction, $l_i \succ l_j$. Moreover, $l_j R l_i$ since $l_i \in \bar{\mathcal{A}}_R^-(L_n) = \bar{\mathcal{A}}^s(L_n)$. By the argument above, $l_i \in \bar{D}(L_n)$ so that $l_i R l_j$ and consequently $l_i I_R l_j$. By construction, this is a contradiction so that $l_j \notin \bar{D}_{s,\succeq}(L_n)$. Thus, $\bar{D}(L_n) \subset \bar{D}_{s,\succeq}(L_n)$. Since $\bar{D}_{s,\succeq}(L_n) \subset \bar{D}(L_n)$ and $\bar{D}(L_n) \subset \bar{D}_{s,\succeq}(L_n)$, it follows that $\bar{D}(L_n) = \bar{D}_{s,\succeq}(L_n)$. ■

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