SMOOTH STABLE MATCHING

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Abstract. We analyze a continuous version of the Gale-Shapley matching problem. Men and women are represented by a d-dimensional vector of characteristics (such as intelligence, beauty, wealth, etc.) and their preferences over matches with the opposite sex depend only on the respective characteristics. We assume that preferences are monotonic. We show that each differentiable and pairwise stable matching has to satisfy a system of partial differential equations. For generic values of parameters, there exists at most one smooth (i.e., analytic) stable matching.

1. Introduction

In the traditional marriage problem, a finite set of men is to be matched with a finite set of women so that the resulting matching is pairwise stable: there exists no pair of a man and a woman that would prefer to marry each other rather than stay with their current spouses.[12] A stable matching always exists (Gale and Shapley (1962)), but it is typically not unique. There are many reasons why to worry about non-uniqueness. First, empirical analysis is difficult without unique prediction. Second, one cannot do comparative statics. Third, non-uniqueness is very closely related to incentive issues. For example, in men-optimal matching, women may have incentives to misrepresent their preferences in order to manipulate their marriage outcome (Roth (1982), and Dubins and Freedman (1981)). If women act strategically, then the stable prediction of the model is no longer valid. Arguably, the first two issues are not so

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[2]Typically, a good matching also has to satisfy an individual rationality constraint: no man or woman prefers to remain single rather than stay married to his or her current spouse. In this paper, we do not discuss individual rationality. For a review of matching theory, see Roth and Sotomayor (1992).

important in markets in which a central algorithm implements the designed solution (for example, residency allocation), but they are important in decentralized markets (for example, such as those studied in Echenique and Yariv (2011)). The last issue remains problematic in decentralized as well as centralized markets.

Many real-world matching markets are large. Drawing on intuition from other economic markets, one may suspect that the incentive to manipulate is related to a type of monopoly power held by a discrete individual, and that this monopoly power disappears when the number of agents becomes large. This intuition seems to be confirmed by simulations (for example, Roth and Peranson (1999)), as well as recent theoretical results that show that incentive problems become small in large random markets. For example, Immorlica and Mahdian (2005) show that when preferences are chosen randomly from uniform distribution, then for all but a diminishingly small (in the size of the market) number of agents, truthful reporting of their preferences is ε-best response, and ε converges to 0 as the size of the market grows. A similar result is presented by Kojima and Pathak (2009) regarding the allocation of students to colleges. The current literature usually makes restrictive assumptions about the preference domains: for example, Kojima and Pathak (2009) assumes that there are finitely many (stochastic) preference types of students; Lee (2011) studies the marriage problem with common dirstribution for each side of the market. The finiteness of the preference domain is a strong assumption when the number of agents converges to infinity.[3]

This paper analyzes stable matching in the limit of large markets with a continuum of preference types. We assume that men and women are drawn from a smooth (i.e., analytic) distribution over domains, respectively, E_M and E_W, of d-dimensional space of characteristics, where $d \geq 2$. The dimensions correspond to various characteristics that are relevant for a particular problem (for example, intelligence, beauty, wealth, etc.), and each agent is represented by a vector of intensities of all characteristics. The preferences over matching partners depend only on the respective characteristics of a

[3]A recent paper by Azevedo and Leshno (2011) analyzes stable matching in the college-student problem with finitely many colleges and continuum of students. Similarly to here, the authors show that stable matching is generically unique.

man and a woman, and they are represented by smooth utility functions. We assume that utilities are monotonic: partners with higher characteristics are preferable to partners with lower characteristics.[4] A smooth stable matching is an analytic function $\mu : E_M \to E_W$ that preserves masses of matched men and women and that satisfies the pairwise stability property.

We show that any continuous and pairwise stable matching must satisfy partial differential equations of three types. The first type of equation comes from the requirement that the matching has to preserve masses of matched men and women. The second type of equation comes from a local characterization of stability. A key observation is that any stable matching maps hyperplanes that are tangent to indifference curves of the agents from one side of the market onto hyperplanes that are tangent to the indifference curves of the agents from the other side. The third type of equation comes from the fact that any smooth matching maps the boundary of the domain of men's characteristic E_M onto the boundary of women's domain E_W.

Given the assumption that the domains E_M and E_W have a cone shape, we show that for generic preferences and boundary conditions, the smooth stable matching is unique. The cone shape assumption together with monotonicity implies that the man and the woman with characteristics at the apices of the respective cones form a top match: they have strict preferences to match with each other. These two are matched together by any stable matching. We use the differential characterization around the top match to show that stable matching is uniquely determined by preferences and the domain assumptions.

The matching literature discusses some sufficient assumptions for the uniqueness. For example, if all men have the same common (and strict) preferences over all women and all women have the same common (and strict) preferences over all men, then unique stable matching assigns men and women assortatively according to the common rankings. Other conditions mentioned in the literature are also very strong (for example, see Eeckhout (1999), Clark (2006), Yariv and Niederle (2009), or Pycia

[4]When $d = 1$, the monotonicity implies that the assortative matching is the unique stable matching.

(2011)). Clark (2006) shows that there exists a unique stable matching in every submarket of a given problem if and only if the preferences satisfy *alpha-reducibility*: for every subsets of men and women, there exists a man and a woman who are the mutually most prefered choices from these subsets. (This property was introduced in Alcalde (1994), and it is also known as the top-top match property. See also the comment after the second example in Section 2.)

There are many open questions resulting from the current paper. The most important one concerns existence: Although we know that smooth stable matchings exist in some cases (and whenever they exist, our result implies that they are unique), we do not know whether they exist in general. The second question concerns convergence: Arguably the continuum limit is interesting mostly if it is a limit of finite markets. At this moment, we do not know whether the (possibly non-unique) stable matchings of finite markets converge to the unique matching in the continuum limit. We discuss these issues further in section 7.

The next section uses examples to illustrate some ideas of this paper. Section 3 describes the model. Section 4 presents a local characterization of a stable matching. In section 5, we define the cone-shaped domains and state our main result, Theorem 1. Section 6 contains the first step of the proof of Theorem 1: we show that the first derivative of a stable matching around the top match is determined by the fundamentals of the model. Finally, section 7 comments on existence, convergence, and alternative domain assumptions. The appendix contains the rest of the proof of Theorem 1 and computations for example of Section 2.

2. Examples

In this section, we use a series of examples of matching situations to illustrate some of the properties of smooth stable matching. In each of the examples, we plot an approximation to the unique smooth stable matching. The computations are postponed untill Appendix C.

Example 1. In all of the examples but the last one, we assume that men and women are fully described by a two-dimensional vector (i, b). To focus attention, we refer to the first dimension, i, as intelligence, and the second one, b, as beauty. The men and women belong to the same domain:

$$E = \{(i,b) : i, b \leq 0\}.$$

and they are drawn from the same probability distribution with non-disappearing Lesbegue density on E. For simplicity, we assume that the Lesbegue density is constant in some neighborhood of $(0,0)$. In particular, the choice of domains implies that man $\hat{m} = (0,0)$ is (weakly) smarter and more beautiful than any other man in the domain. Similarly, woman $\hat{w} = (0,0)$ is the smartest and the most beautiful among all women.

Each man m derives utility

$$\mathcal{M}(m_i, m_b, w_i, w_b) = 2w_i + w_b + m_i w_i + m_b w_b$$

from the match with woman w. Likewise, each woman w derives utility

$$\mathcal{W}(m_i, m_b, w_i, w_b) = m_i + m_b + m_i w_i + m_b w_b$$

from the match with man m.

We discuss important properties of the preferences. First, in this and all the examples below, the preferences of both men and women are strictly monotonic, i.e., all men and women would strictly prefer to marry smarter and/or more attractive spouses. Because of monotonicity, man $\hat{m}$ and woman $\hat{w}$ are mutual top matches, and must be matched with each other by any stable matching. We use this starting point to describe the matching in some neighborhood of the top match.

Second, up to first-order approximations, in a small neighborhood of the top man $\hat{m}$, men rank women according to the quality index equal to $2w_i + w_b$. Similarly, in a neighborhood of the top woman $\hat{w}$, women rank men according to the quality index equal to $m_i + m_b$. Because the indices are common for all men and women in some small neighborhood of the top match, one can think about them as common preference rankings. The well-known results about common ranking suggests that the unique stable matching will match the men and women assortatively by quality.

Sorting by qualities is closely related to the key property of stable matching: that the tangent lines to the indifference curves at the point of the match are locally mapped onto each other (see Lemma 1 below). Here, the tangent line to the women's

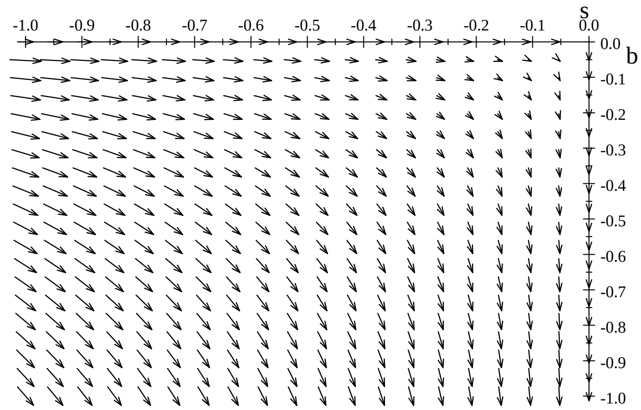


FIGURE 1.

indifference curve in the neighborhood of the top match is equal to $m_i + m_b = \text{const}$, i.e., the set of men with the same quality. Similarly, the tangent line to the men's indifference curve is equal to $2w_i + w_b = \text{const}$, i.e., the set of women with the same quality. Stable matching maps the former onto the latter.

Finally, the preferences contain the second-order component which implies that, other things equal, smarter men prefer smarter women and smarter women prefer smarter men. Because the preferences of men and women agree, one may expect that stable matching to marry smart men (axis $m_i = 0$) with smart women (axis $w_i = 0$) and beautiful men (axis $m_b = 0$) with beautiful women (axis $w_b = 0$). We say that the matching preserves orientation.

We compute (the second-order) approximation of the unique smooth stable matching μ and plot it on Figure 1. As it is expected, the matching preserves orientation. The figure plots the difference between μ and the identity matching, i.e., matching that assigns each man to the woman with identical characteristics. Each arrow originates at some m and points towards woman $\mu(m)$. For example, an arrow pointing in the NE direction means that the man at the beginning of the arrow is matched with a woman that is smarter and more attractive than he. The length of the arrow is proportional to the distance between the man and his woman.

Notice that smart men (i.e., men that lie on the b-axis on the Figure) are matched typically with women that are equally smart but less beautiful. At the same time,

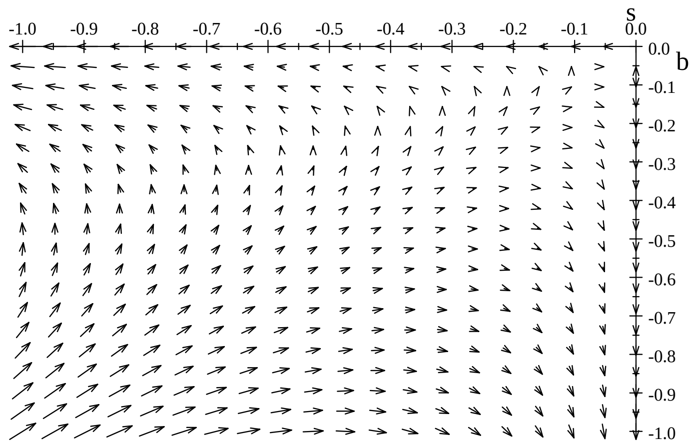


FIGURE 2.

beautiful men are married to women that are equally beautiful but smarter. This observation follows from an asymmetry between the comparative value of being smart among women and men. Men value smart women relatively more than they value beautiful women, whereas women value both intelligence and beauty equally. Because more men are attracted to smart women than the other way, smart women get better matches than smart men.

Example 2. Assume that the domains and density of men and women are the same as in the previous example and the preferences are given by the following functions:

$$\mathcal{M}(m_i, m_b, w_i, w_b) = w_i + w_b + 2(m_i w_i + m_b w_b),$$
$$\mathcal{W}(m_i, m_b, w_i, w_b) = m_i + m_b + m_i w_i + m_b w_b.$$

The preferences are monotonic, which implies that man $\hat{m}$ and woman $\hat{w}$ must be matched with each other. Figure 2 plots (the second-order approximation to) stable matching.

At the first-order approximation, both men and women value rank their potential partners according to the quality index equal to the sum of the two characteristics, intelligence and beauty. In a small neighborhood of the top match, stable matching should match man and women assortatively according to their quality ranking.

At the second-order approximation, the preferences include cross-derivative terms that imply that smarter men prefer smarter women and smarter women prefer smarter men. Similarly to the first example, this implies that stable matching preserves orientation.

Finally, men's preferences put relatively more weight on the cross-derivative term. This means that, choosing among women of certain fixed quality, men typically value women with more extreme characteristics (i.e., that are close to the i- and b-axes), whereas women (as compared to men) prefer more balanced characteristics (i.e., men that are located near the $i = b$ axis). Thus, there is relatively more demand for women with extreme characteristics and relatively less demand for women with balanced characteristics. In the same vein, there is more demand for men with balanced characteristics.

These differences between men and women are reflected by stable matching. Relative to the identity matching, men with balanced characteristics are matched with women of higher quality. Additionally, men with extreme characteristics are matched with women of lower quality and, more specifically, with a lower value of the feature about which the men care less. For example, very smart men are matched with women who are somehow less beautiful then themselves.

One can show that the preferences in the first two examples exhibit a continuous version of alpha-reducibility introduced in Clark (2006) (also called top–top match property): for every (compact) subset of men and women $E'_M \subseteq E_M$ and $E'_W \subseteq E_W$, there exists $m' \in E'_M$ and $w' \in E'_W$ such that m' is the most preferred man of woman w' among all men in set E'_M and w' is the most preferred woman of man m' among all members of set E'_W. Clark (2006) shows that this property is a sufficient condition for the uniqueness of stable matching in the discrete markets. Next, we discuss two examples without alpha-reducibility.

Example 3. Assume that the domains and density of men and women are the same as in the first example and the preferences are given by the following functions:

$$\mathcal{M}\left(m_i, m_b, w_i, w_b\right) = w_i + w_b - 2\left(m_i w_i + m_b w_b\right),$$
$$\mathcal{W}\left(m_i, m_b, w_i, w_b\right) = m_i + m_b + m_i w_i + m_b w_b.$$

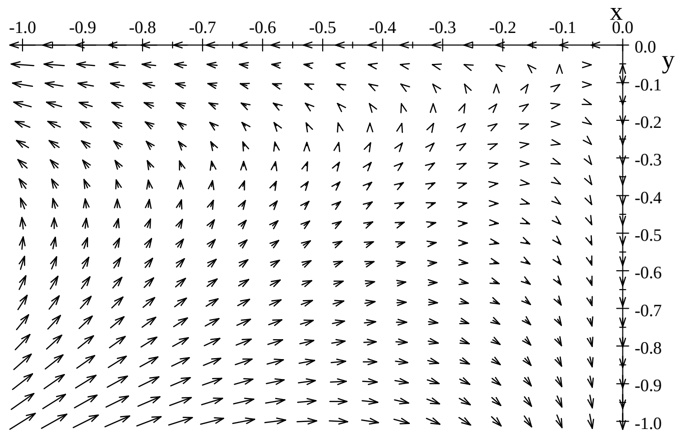


FIGURE 3.

Thus, at the first-order approximation, men and women rank each other with respect to the quality index that is equal to the sum of intelligence and beauty. As in the previous two examples, stable matching matches men and women assortatively by quality.

Additionally, the second-order coefficients imply that smart women prefer to match with smart men, but smart men prefer (relatively) beautiful women. Thus, women would prefer a matching that preserves orientation and men would like a matching that reverses the orientation (i.e., that matches the i-axis of men with the b-axis of women).

In section 6.5, we show that the orientation depends on the weighted average of the cross-derivative matrices of men and women. The weight depends, among other things, on the relative densities of men and women around the top match. We interpret this weight as a measure of a local bargaining power. In this example, the absolute value of the cross-derivative coefficient in men's preferences is larger than the corresponding coefficient in women's preferences, which implies that stable matching should reverse orientation.

Figure 3 plots the difference between (the second-order approximation to) the unique stable matching and the reverse matching, i.e., a matching that maps man (i, b) to woman (b, i). So, for example an arrow that originates at (i, b) points towards the direction of a match of man (b, i). Notice that Figure 3 resembles Figure 2. This is not surprising: because of the reverse orientation, women prefer to match with relatively balanced partners and men prefer relatively more extreme women. This creates an excess demand for women with extreme characteristics and a lower demand for balanced women. The difference in the relative demand pushes balanced men to matches with higher quality women and extreme men to matches with relatively lower quality partners.

Example 4. In the last example, we assume that the space of characteristics has three dimensions: intelligence, beauty, and wealth. We assume that distribution of the characteristics of men and women are the same and the domains are equal to

$$E_M = E_W = \left\{(i, b, w) : i^2 + b^2 + w^2 - 2\left(sb + sw + bw\right) \leq 0\right\}.$$

Thus, the domains have the shape of a symmetric cone with the apex at $(0, 0, 0)$ and the central axis equal to $x = y = z$.

Preferences are given by

$$\mathcal{M}\left(\left(m_i, m_b, m_w\right), \left(w_i, w_b, w_w\right)\right) = w_i + w_b + w_w + m_i w_b + m_b w_w + m_w w_i,$$
$$\mathcal{W}\left(\left(m_i, m_b, m_w\right), \left(w_i, w_b, w_w\right)\right) = m_i + m_b + m_w + m_i w_i + m_b w_b + m_b w_b.$$

Thus, at the first-order approximation, both men and women rank their partners according to the quality index equal to the sum of all three characteristics. Additionally, the second-order component implies that smarter women would like to match with smarter men, more beautiful women want more handsome men, and wealthier women want wealthier men. At the same time, smart men would like to match with beautiful women, handsome men would like wealthy women, and wealthy men would like smart women.

As in the above examples, up to first-order approximation, stable matching sorts men and women by their qualities. In particular, the hyperplanes that consist of men with the same quality, $m_i + m_b + m_w = \text{const}$, are mapped onto hyperplanes that consist of women of the same quality, $w_i + w_b + w_w = \text{const}$.

Sorting by qualities does not explain the orientation of the matching of the hyperplanes of the same qualities. In order to shed some light on the orientations, we

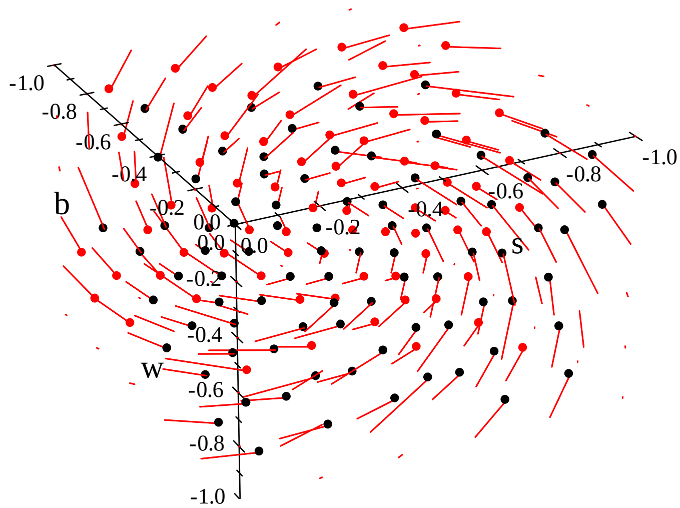


FIGURE 4.

consider a couple of hypothetical alternatives. First, suppose that instead of $\mathcal{M}$, the second order terms of men's preferences are equal to the second-order terms in women preferences $\mathcal{W}$. In such a case, both men and women want to match intelligence with intelligence, beauty with beauty, and wealth with wealth. The unique stable matching is the identity matching, or the women's preferred matching. Second, if instead, women's utility function included the second-order terms from the men' utility $\mathcal{M}$, then both men and women would want to match smart men with beautiful women, beautiful men with wealthy women, and wealthy men with smart women. The unique stable matching would be a rotation around the axis $x = y = z$ that replaces coordinate i with b, b with w, and w with i. Such matching is the preferred matching for men.

Given utility functions $\mathcal{M}$ and $\mathcal{W}$, the unique stable matching is somewhere between two extreme cases. It is a rotation that goes in the same direction but not as far as the rotation from the second case described above. The first-order approximation to the unique smooth stable matching is shown in Figure 4. Each arrow originates at some man m and its end (denoted with a big dot) points towards woman $\mu(m)$. The arrows are arranged counter-clockwise around the central axis of the conic domain $x = y = z$. For example, the beautiful and wealthy men (the i-axis, which corresponds to line $b = w = 0$) are matched with slightly wealthier and smarter but somehow less beautiful women.

3. Model

For any two vectors v and u, we write $v \cdot u$ or $v'u$ to denote the scalar product. For any vector v or matrix A, we denote the transpose as v' and A'. For any two vectors $v, u \in R^d$, we write $v \leq u$ (or $v < u$) if $v_i \leq u_i'$ (or $v_i < u_i'$) for each i. We write e^j for the unit vector with 1 on its jth coordinate and 0 otherwise.

There are two types of agents, men and women. Men and women are represented as a d-dimensional vector of characteristics, $m, w \in R^d$. Each of the dimensions corresponds to a characteristic that is relevant in the particular problem (like intelligence, beauty, wealth, etc.).

The mass of men is given by a distribution $G_M \in \Delta E_M$ with a closed support $E_M \subseteq R^d$. Similarly, the mass of women is given by a distribution $G_M \in \Delta E_M$ with a closed support $E_M \subseteq R^d$. We assume that distributions G_M and G_W have Lesbegue densities, respectively, g_M and g_W, that these densities are analytic, and that function

$$g(m, w) = \frac{g_M(m, w)}{g_W(m, w)} \text{ for } m \in E_M, w \in E_W$$

can be extended as an analytic and strictly positive function to some neighborhood of $E_M \times E_W$.

Let $\mathcal{M}(m, w)$ denote the utility of man m from the match with woman w. Let $\mathcal{W}(w, m)$ denote the match utility of woman w with man m. We assume that the utility functions are analytic, i.e., locally, they have a representation by an infinite Taylor series. We denote the first- and second-order derivatives at (m_0, w_0) by, respectively,

$$\mathcal{M}(m_0, w_0), \mathcal{M}_w(m_0, w_0), \mathcal{M}_{mw}(m_0, w_0), \mathcal{M}_{mm}(m_0, w_0), \text{ and } \mathcal{M}_{ww}(m_0, w_0).$$

For example, $\mathcal{M}_w(m_0, w_0)$ is a normal vector to man m_0's indifference curve at his match with woman w_0. We use similar notation for the derivatives of function $\mathcal{W}$. If man m_0 and woman w_0 are clear from the context, we write $\mathcal{M}_w$ instead of $\mathcal{M}_w(m_0, w_0)$ with a similar convention for other pieces of notation. All the results and definitions depend only on the ordinal properties of the utility function and are not affected by any (analytic) monotone transformation.

We assume that preferences are *strictly monotonic*: $\mathcal{M}_w(m_0, w_0), \mathcal{W}_m(m_0, w_0) > \mathbf{0}$ for each m_0, w_0.

A *matching* is a measurable function with a measurable inverse $\mu : E_M \to E_W$ such that for all measurable subsets $E \subseteq E_M$ and $E' \subseteq E_W$,

$$F_W(\mu(E)) = \mu(E) \text{ and } F_M\left(\mu^{-1}(E')\right) = \mu^{-1}(E'). \tag{3.1}$$

Matching μ is *stable* if for each $m, m' \in E_M$, either $\mathcal{M}(m, \mu(m)) \geq \mathcal{M}(m, \mu(m'))$, or $\mathcal{W}(m', \mu(m')) \geq \mathcal{W}(m, \mu(m'))$.

Matching μ is *differentiable (or continuously differentiable, smooth)* if function μ is differentiable (or continuously differentiable, analytical). For any differentiable matching μ (not necessarily smooth), equation (3.1) is equivalent to

$$|\det D_\mu(m_0)| = g(m_0, \mu(m_0)) \text{ for each } m_0, \tag{3.2}$$

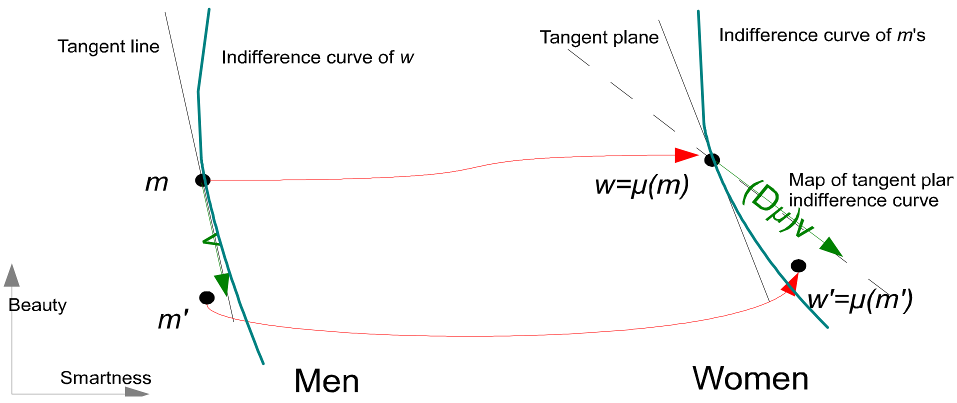
where $D_\mu(m_0) = \frac{d\mu}{dm}|m_0$ is the derivative matrix of μ computed at m_0.

4. Local characterization

Next, we describe a key local property of differentiable stable matching. We show that any stable matching must (locally) match hyperplanes that are tangent to the indifference curves for each matched pair of man m and woman $w = \mu(w)$ and the indifference curves increase in the direction for both men and women. For each man m, let $\mathcal{M}_w$ and $\mathcal{W}_m$ denote the normal vectors to the indifference curves of man m and woman $w = \mu(w)$ that are computed at match $(m, \mu(w))$. Let $D\mu$ denote the derivative of the matching function at m.

Lemma 1. *Suppose that μ is twice differentiable stable matching. For each m, $\mathcal{M}'_w(D\mu)\mathcal{W}_m > 0$ and for each vector v, if $\mathcal{W}'_m v = 0$, then $\mathcal{M}'_w(D\mu)v = 0$.*

Proof. The argument relies on the first-order approximations to the utility functions. On the contrary, suppose that the Lemma is not true. There are two possible cases: (a) either $\mathcal{M}'_w(D\mu)\mathcal{W}_m < 0$, in which case assume that $v = 0$, or (b)$\mathcal{M}'_w(D\mu)\mathcal{W}_m \geq$



0 and there exists a small vector v such that $\mathcal{M}'_w (D\mu) v > 0$ (see Figure 4 for case (b)). In each case, we can find $\varepsilon > 0$ sufficiently small, so that

$$\begin{aligned}
&\mathcal{M} (m, \mu (m + v - \varepsilon \mathcal{W}_m)) - \mathcal{M} (m, w) \\
&= \mathcal{M}'_w (D\mu) v - \varepsilon \mathcal{M}'_w (D\mu) \mathcal{W}_m + O \left(\|v\|^2 \right) > 0,
\end{aligned}$$

and man m prefers to match with woman $w' = \mu (m + v - \varepsilon \mathcal{W}_m)$, instead of his current match w. At the same time, woman w' strictly prefers to match with m instead of her current match, $m' = m + v - \varepsilon \mathcal{W}_m$,

$$\begin{aligned}
&\mathcal{W} (m, \mu (m + v - \varepsilon \mathcal{W}_m)) - \mathcal{W} (m + v - \varepsilon \mathcal{W}_m, \mu (m + v - \varepsilon \mathcal{W}_m)) \\
&= -\mathcal{W}'_m v + \varepsilon \|\mathcal{W}_m\|^2 + O \left(\|v\|^2 \right) = \varepsilon \|\mathcal{W}_m\|^2 + O \left(\|v\|^2 \right) > 0.
\end{aligned}$$

The existence of blocking pair $(m, \mu (m + v - \varepsilon \mathcal{W}_m))$ violates stability. □

The Lemma has two important implications. The first implication is formal. For each $j < d$, let

$$y^{*j} (m, w) = - \frac{(\mathcal{W}_m (m, w))_d}{(\mathcal{W}_m (m, w))_j} \tag{4.1}$$

be the (minus) ratio of dth and jth coordinate of the normal vector $\mathcal{W}_m (m, w)$. Then, vector $y^{*j} (m, w) e^j + e^d$ is tangent to woman w's indifference curve at the point of her match with man m, $\left(y^{*j} (m, w) e^j + e^d \right) \cdot \mathcal{W}_m (m, w) = 0$. Lemma 1 implies that for each m,

$$\mathcal{M}_w(m,\mu(m))\cdot(D\mu(m))\left(y^{*j}(m,\mu(m))\,e_j+e_d\right)=0. \tag{4.2}$$

The above equality is a system of partial differential equations in μ that must be satisfied at each point of the match. The equality plays a central role in the characterization of the unique stable matching.

The second implication is that, locally, any stable matching looks like an assortative matching. To explain this claim, imagine that the space of men's characteristics around man m is sliced by hyperplanes that are orthogonal to the normal vector $\mathcal{W}_m$ to woman w's indifference curve. These hyperplanes are approximately tangent to the indifference curves of women in a neighborhood of woman w. It is natural to think about the hyperplanes as collections of men with (approximately) the same quality, where the quality is measured by the distance from m with respect to vector $\mathcal{W}_m$. Locally, all women in the neighborhood of w have the same preferences and they rank men in a neighborhood of m in approximately the same way, by their quality. In the same way, imagine that the space of women's characteristics is sliced by hyperplanes that are orthogonal to the normal vector $\mathcal{M}_w$ to man m's indifference curves. Locally, all men in the neighborhood of man m rank women in the neighborhood of w in approximately the same way, by their quality.

The second part of Lemma 1 implies that slices from the space of characteristics of men are matched with slices of the space of characteristics of women. The first part of Lemma 1 implies that the matching is assortative: higher quality slices are matched with slices of lower quality.

5. Main result

In this section, we state the main result of this paper.

We assume that the domains of men's and women's characteristics have the shape of a cone. Define a *standard cone* with apex at $\mathbf{0}$ as

$$\begin{aligned}E^* &= \left\{m : m_1^2+...+m_{d-1}^2-m_d^2\leq 0 \text{ and } m_d\leq 0\right\}\\ &= \left\{m : m'I^*m\leq 0 \text{ and } m_d\leq 0\right\},\end{aligned}$$

where I^* is a d-dimensional matrix obtained from the identity matrix but with the last cell in the diagonal switched from 1 to -1.

We focus on domains that are cones obtained as linear transformations of standard cones and with an apex that lies above any other element of the domain. Say that linear operator $\phi : R^d \to R^d$ is *proper* if $\phi^{-1}(m) < \mathbf{0}$ for each $m \in E^* \setminus \{\mathbf{0}\}$. Define symmetric d-dimensional matrix

$$\Phi = \phi' I^* \phi,$$

where ϕ' is a transpose of matrix ϕ. We say that Φ is a *proper cone matrix.* We say that cone $E = \phi^{-1}(E^*) = \{m : m'\Phi m \leq 0 \text{ and } m_d \leq 0\}$ is generated by ϕ (or, equivalently, by Φ).

We assume that E_M and E_W are generated by proper cone matrices, respectively, Φ and Ψ. Because continuous matching μ maps the boundary of E_M onto the boundary of E_W, the boundary conditions imply that

$$m'\Phi m = 0 \Longrightarrow \mu(m') \Psi \mu(m) = 0 \text{ for each } m. \tag{5.1}$$

Domain E_M contains the top man $\hat{m} = \mathbf{0} \in \mathbf{E}_M$: any other man in the domain $m \in E_M$ has strictly worse characteristics than $\hat{m}$. Similarly, domain E_W contains the top woman $\hat{w} = \mathbf{0}$. The monotonicity assumption implies that $\hat{m}$ is the top match of woman $\hat{w}$ and vice versa. It follows that each stable matching matches them with each other, $\mu(\hat{m}) = \hat{w}$.

From now on, we write $\hat{f}$ instead of $f(\hat{m})$ for any function f of men's characteristics evaluated at $\hat{m}$ and instead of $f(\hat{m}, \hat{w})$ for any function of men' and women's characteristics. For example, $\hat{\mathcal{M}}_{mw}$ is the cross-derivative matrix evaluated at the point of top matching $(\hat{m}, \hat{w})$.

For each d, let $M_d = R^{d^2}$ be the space of d-dimensional matrices, and let $P_d \subseteq M_d$ be the subspace of proper linear operators. Let $\mathcal{P} = \left(R^d\right)^2 \times M_d^2 \times P_d^2 \times R_+$ be the space of parameters $\left(\hat{\mathcal{M}}_w, \hat{\mathcal{W}}_m, \hat{\mathcal{M}}_{mw}, \hat{\mathcal{W}}_{mw}, \phi, \psi, \hat{g}\right) \in \mathcal{P}$. Space $\mathcal{P}$ is a convex subset of a Euclidean space; hence it can be equipped with a Lesbegue measure $\lambda_{\mathcal{P}} \in \Delta\mathcal{P}$. We say that a claim holds for the generic values of the parameters if there exists subset $\mathcal{P}_0 \subseteq \mathcal{P}$ such that $\lambda_{\mathcal{P}}(\mathcal{P} \setminus \mathcal{P}_0) = 0$, and that the claim holds whenever $\left(\hat{\mathcal{M}}_w, \hat{\mathcal{W}}_m, \hat{\mathcal{M}}_{mw}, \hat{\mathcal{W}}_{mw}, \phi, \psi, \hat{g}\right) \in \mathcal{P}_0$.

Theorem 1. *For the generic values of the parameters, there exists at most one smooth stable matching.*

The proof relies on two observations. First, the discussion above implies that each smooth stable matching must be a solution to a system of differential equations (3.2) and (4.2) subject to boundary conditions (5.1). Second, any smooth (i.e., analytical) function over an open connected domain is determined by its first- and higher-order derivatives computed at a single point. Thus, it is enough to show that all the $\hat{m}$-derivatives of a solution to the system of differential equations μ are determined by the fundamentals of the model. The idea is to differentiate equations (3.2), (4.2), and (5.1) and evaluate the differentials at the top match. We proceed by induction on the order of derivatives. At each step, we obtain a system of finitely many linear equations with unknowns being equal to the derivatives of μ computed at the top match. At each step, the number of equations is the same as the number of unknowns. We show that, for the generic values of the parameters, the equations have unique solutions.

In the next section, we show how to derive the first-order derivatives. The higher-order derivatives are determined in Appendix B (with some mathematical preliminaries contained in Appendix A).

6. First-order derivatives of stable matching

In this section, we show that the first-order derivatives of smooth stable matching at the top match are determined by the fundamentals of the model. We begin with a discussion of local characterization of stable matching that expands on Lemma 1 by taking into account second-order approximations of preferences. We use this approximation to introduce a notion of local bargaining power that measures how one side of the matching can trade off between her first- and second-order gains in the matching. We show that the local bargaining power at the apex is determined by the density of men and women at the top match together with the shapes of men's and women's cones. We use this information to find all the first-order derivatives. In the end of the section, we illustrate some of the issues in two separate examples for case $d = 2$ and $d \geq 3$.

6.1. **Second-order approximations.** Although Lemma 1 provides some information about the derivative of the matching function $D\mu$, the information is quite limited. For example, the lemma does not explain how the tangent hyperplanes are matched with each other. The reason is that Lemma 1 relies on the first-order approximations of preferences and from the point of view of the first-order approximations, men (or women) on the tangent hyperplanes are indifferent across all women (or men) on the tangent hyperplane.

Additional information about $D\mu$ can be obtained from more careful second-order approximations to the utility functions. We begin with two definitions. Let

$$\theta(m) = \frac{\mathcal{M}_w}{\|\mathcal{M}_w\|} \cdot (D\mu) \frac{\mathcal{W}_m}{\|\mathcal{W}_m\|} \tag{6.1}$$

and

$$W(m) = \frac{\mathcal{M}_{wm}}{\|\mathcal{M}_w\|} + \theta \frac{\mathcal{W}_{mw}}{\|\mathcal{W}_m\|}. \tag{6.2}$$

The values of parameter θ and matrix W depend on the stable matching μ and the matched man m. We suppress the reference to man m whenever his identity is clear from the context.

We interpret θ as a measure of *local bargaining power* of woman $w = \mu(m)$ matched with man m. Suppose that the utilities are normalized so that the normal vectors have unit length, $\|\mathcal{M}_w\| = \|\mathcal{W}_m\| = 1$, and consider the following hypothetical scenario. Assume that woman w gives up her current match m and marries man $m - \varepsilon\mathcal{W}_m$ instead. This leads to her first-order utility loss of $\varepsilon \|\mathcal{W}_m\|^2 = \varepsilon$. At the same time, her new partner, man $m - \varepsilon\mathcal{W}_m$, experiences utility gain of $\varepsilon\mathcal{M}_w \cdot (D\mu)\mathcal{W}_m = \theta\varepsilon$. Thus, θ is equal to the ratio of the first order utility gain of the man relative to the first-order utility loss of the woman. If θ is large, woman w has an opportunity to provide a substantial first-order utility gain to her partner with relatively little cost to herself. She can leverage this opportunity to gain second-order improvement, or, more precisely, an improvement in directions that are tangent to her indifference curve.

Cross-derivative matrix $\mathcal{M}_{mw}$ describes the behavior of the normal vector to men's indifference curves in a neighborhood of m : the normal vector to the indifference curve of man $m + v$ is approximately equal to $\mathcal{M}_w + \mathcal{M}'_{mw}v$. Let $\mathcal{P}_M\mathcal{M}'_{mw}v$ denote the projection of such vector on the hyperplane tangent to m's indifference curve. We interpret $\mathcal{P}_M\mathcal{M}'_{mw}v$ as the locally favorite rematch direction of man $m+v$ among all vectors that lie on the tangent hyperplane. Similarly, we interpret $\mathcal{P}_W\mathcal{W}_{mw}u$ as the locally favorite rematch direction of woman $w+u$ among all vectors that lie on the tangent hyperplane to woman w's indifference cures.

Matrix W, defined in (6.2), is a weighted average of the "favorite direction" matrices of men and women. The larger local bargaining power of women θ, the more W depends on the women's preferences. We have two results:

Lemma 2. *Suppose that μ is a twice continuously differentiable stable matching. Then, for each man m, any vector v such that $v\cdot\mathcal{W}_m=0$, $v'W(D\mu)v\geq 0$.*

Proof. Take any vector v that is tangent to woman w's indifference curve. Choose ε so that woman $\mu(m+v)$ is indifferent between her current match and man $m'=m-v-\varepsilon\mathcal{W}_m$. Due to the second-order approximations, we have

$$\begin{aligned}0&=\mathcal{W}(m',\mu(m+v))-\mathcal{W}(m+v,\mu(m+v))\\&=\left[\begin{array}{c}\mathcal{W}(m,w)-v'\mathcal{W}_m-\varepsilon\|\mathcal{W}_m\|^2+\mathcal{W}_w(D\mu)v\\-v'\mathcal{W}_{mw}(D\mu)v+v'\mathcal{W}_{mm}v+((D\mu)v)'\mathcal{W}_{ww}((D\mu)v)\end{array}\right]\\&-\left[\begin{array}{c}\mathcal{W}(m,w)+v'\mathcal{W}_m+\mathcal{W}_w(D\mu)v\\+v'\mathcal{W}_{mw}(D\mu)v+v'\mathcal{W}_{mm}v+((D\mu)v)'\mathcal{W}_{ww}((D\mu)v)\end{array}\right]\\&+O\left(\|v\|^3,\varepsilon^2,\varepsilon\|v\|\right)\\&=-2v'\mathcal{W}_{mw}(D\mu)v-\varepsilon\|\mathcal{W}_m\|^2+O\left(\|v\|^3,\varepsilon^2,\varepsilon\|v\|\right).\end{aligned}$$

In the last equality, we used the fact that $v'\mathcal{W}_m=0$ by the choice of vector v. It follows that

$$\varepsilon=-\frac{1}{\|\mathcal{W}_m\|^2}2v'\mathcal{W}_{mw}(D\mu)v+O\left(\|v\|^3\right). \tag{6.3}$$

If the matching is stable, then man m' (weakly) prefers his current match to woman $\mu(m+v)$,

$$\begin{aligned}0&\leq\mathcal{M}(m',\mu(m'))-\mathcal{M}(m',\mu(m+v))\\&=2v'\mathcal{M}_{mw}(D\mu)v-\varepsilon\mathcal{M}'_w(D\mu)\mathcal{W}_m+O\left(\|v\|^3,\varepsilon^2,\varepsilon\|v\|\right).\end{aligned} \tag{6.4}$$

In the last equality, we use the fact that, by Lemma 1, $\mathcal{M}'_w (D\mu) v = 0$. After substituting (6.3) into (6.4) and dropping the higher-order terms, we obtain

$$\begin{aligned} 0 &\leq 2v' \left[\mathcal{M}_{mw} + \left(\mathcal{M}'_w (D\mu) \mathcal{W}_m\right) \mathcal{W}_{mw}\right] (D\mu) v \\ &= 2 \left\|\mathcal{M}_w\right\| \left(v' W (D\mu) v\right). \end{aligned}$$

□

Lemma 3. *Suppose that μ is a twice continuously differentiable stable matching. Then, for each m, any two vectors v and u such that $v \cdot \mathcal{W}_m = u \cdot \mathcal{W}_m = 0$,*

$$v' W (D\mu) u = u' W (D\mu) v. \tag{6.5}$$

Proof. Take any two vectors v and u that are tangent to woman w's indifference curve and such that the length of vector u is small relative to v, $\|u\| = \|v\|^{3/2}$. Choose ε so that woman $\mu(m+v)$ is indifferent between her current match and man $m' = m + v + u - \varepsilon \mathcal{W}_m$. Due to the second-order approximations, we have

$$\begin{aligned} 0 &= \mathcal{W}(m', \mu(m+v)) - \mathcal{W}(m+v, \mu(m+v)) \\ &= u' \mathcal{W}_{mm} v + u' \mathcal{W}_{mw} (D\mu) v - \varepsilon \left\|\mathcal{W}_m\right\|^2 + \varepsilon O(\|v\|) + O\left(\|v\|^3, \varepsilon^2, \varepsilon \|v\|\right). \end{aligned} \tag{6.6}$$

In the last equality, we used the fact that $v' \mathcal{W}_m = u' \mathcal{W}_m = 0$ and the choice of vectors u and v and that $u' A u$ is of order $\|v\|^3$ for any matrix A.

At the same time, if the matching is stable, then man m' (weakly) prefers his current match to woman $\mu(m+v)$,

$$\begin{aligned} 0 &\leq \mathcal{M}(m', \mu(m')) - \mathcal{M}(m', \mu(m+v)) \\ &= ((D\mu) u)' \mathcal{M}_{ww} (D\mu) v + v' \mathcal{M}_{mw} (D\mu) u - \varepsilon \mathcal{M}'_w (D\mu) \mathcal{W}_m + O\left(\|v\|^3, \varepsilon^2, \varepsilon \|v\|\right). \end{aligned} \tag{6.7}$$

After substituting the solution of ε from (6.6) into (6.7), we obtain

$$\begin{aligned} 0 \leq{}& ((D\mu) u)' \frac{\mathcal{M}_{ww}}{\|\mathcal{M}_w\|} (D\mu) v - \theta u' \frac{\mathcal{W}_{mm}}{\|\mathcal{W}_m\|} (D\mu) v \\ &+ v' \frac{\mathcal{M}_{wm}}{\|\mathcal{M}_w\|} (D\mu) u - \theta u' \frac{\mathcal{W}_{mw}}{\|\mathcal{W}_m\|} (D\mu) v. \end{aligned} \tag{6.8}$$

Inequality (6.8) holds for all vectors u and v that are tangent to woman w's indifference curve. In particular, it holds when u is replaced by $-u$. Together with

(6.8), the latter implies that the inequality sign in (6.8) can be replaced by equality. Additionally, (6.8) holds when the roles of u and v are replaced. Because matrices $\mathcal{M}_{ww}$ and $\mathcal{W}_{mm}$ are symmetric, we obtain,

$$v'\frac{\mathcal{M}_{wm}}{\|\mathcal{M}_w\|}(D\mu)\,u-\theta u'\frac{\mathcal{W}_{mw}}{\|\mathcal{W}_m\|}(D\mu)\,v=u'\frac{\mathcal{M}_{wm}}{\|\mathcal{M}_w\|}(D\mu)\,v-\theta v'\frac{\mathcal{W}_{mw}}{\|\mathcal{W}_m\|}(D\mu)\,u, \quad (6.9)$$

After some rearrangement of terms, we get (6.5). □

Lemma 3 holds trivially when $d=2$. In such a case, any two vectors v and u that are tangent to woman w's indifference curve are colinear. The result is non-trivial when $d\geq 3$.

Notice that the theses of the above results imply that for each man m, any vectors v and u,

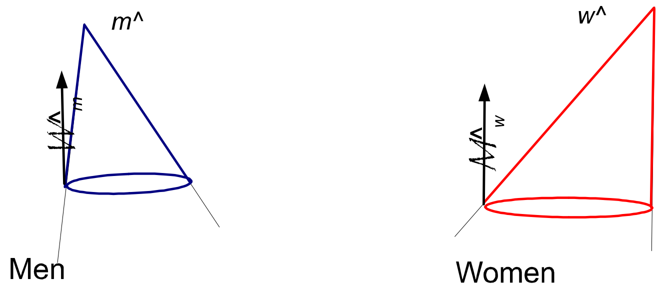
$$v'\mathcal{P}_W'W\,(D\mu)\,\mathcal{P}_W v\geq 0 \text{ and } v'\mathcal{P}_W'W\,(D\mu)\,\mathcal{P}_W u=u'\mathcal{P}_W'W\,(D\mu)\,\mathcal{P}_W v."$$

It follows that matrix $\mathcal{P}_W'W\,(D\mu)\,\mathcal{P}_W$ is symmetric and positive-definite. We use this observation in the sequel.

6.2. **Normalization and change of coordinates.** From now on, we assume that preferences are normalized so that the normal vectors at the top match have unit length, $\|\mathcal{M}_{\hat{w}}\|=\|\mathcal{W}_{\hat{m}}\|=1$. The assumption is without loss of generality, because we can always multiply the utility function by a constant without changing any of its properties.

It is convenient to rotate the coordinate systems in the space of both men's and women's characteristics, so that the normal vectors at the top match have 0 on all coordinates except for the last, $\mathcal{M}_{\hat{w}}=\mathcal{W}_{\hat{m}}=(0,...,0,1)$. We use this convention because it is easier to express all the relevant objects in the rotated coordinates. In order to go back to the original coordinates that correspond to features like intelligence or beauty, we need an inverse rotation.

For any d-dimensional matrix A, we write A^P to denote the d-dimensional matrix obtained by crossing out the last column and the last row. It is easy to check that if Φ is a proper cone matrix, then Φ^P is symmetric and positive-definite.



6.3. **Local bargaining power $\hat{\theta}$.** We compute the local bargaining power $\hat{\theta}$. Notice that any continuous matching μ maps the boundary of set E_M onto the boundary of set E_W. Moreover, Lemma 1 implies that hyperplanes orthogonal to $\mathcal{W}_m^{\hat{}}$ are approximately mapped with hyperplanes that are orthogonal to $\mathcal{M}_w^{\hat{}}$. Because of the choice of coordinates, it means that hyperplane $m_d = -\varepsilon$ is (approximately) mapped onto plane $m_d = -\mathcal{M}_w^{\hat{}} \cdot \left(\hat{D}\mu\right) \mathcal{W}_m^{\hat{}} \varepsilon = -\hat{\theta}\varepsilon$. It also means that cone

$$E_M(\varepsilon) = \{m \in E_M : m_d \geq -\varepsilon\}$$

is mapped onto cone

$$E_W\left(\hat{\theta}\varepsilon\right) = \left\{w \in E_W : w_d \geq -\hat{\theta}\varepsilon\right\}.$$

(See Figure 6.3.)

The matching equation (3.1) implies that the mass of cone $E_M(\varepsilon)$ must be (approximately) equal to the mass of cone $E_W\left(\left(\mathcal{M}_w^{\hat{}} \cdot \left(\hat{D}\mu\right) \mathcal{W}_m^{\hat{}}\right) \varepsilon\right)$. The two masses can easily be computed using standard geometric methods (see Appendix A.4):

$$\frac{\text{mass of } E_W\left(\hat{\theta}\varepsilon\right)}{\text{mass of } E_M(\varepsilon)} \approx \frac{\hat{g}_W \operatorname{vol}_d E_W\left(\hat{\theta}\varepsilon\right)}{\hat{g}_M \operatorname{vol}_d E_M(\varepsilon)} = \frac{1}{\hat{g}}\hat{\theta}^d \left(\frac{\det \Psi^P}{\det \Phi^P}\right)^{\frac{1}{2}(d-2)} \left(\frac{\det \Phi}{\det \Psi}\right)^{\frac{1}{2}(d-1)},$$

where $\det \Phi^P$ is a determinant of $(d-1)$-dimensional matrix Φ^P obtained by crossing out the last row and the last column from matrix Φ, $\det \Psi^P$ is obtained in a similar way from matrix Ψ, and vol_d is the volume in R^d. The above equation leads to formula that expresses $\hat{\theta}$ in terms of matrices Φ, Ψ, and the ratio of density functions

$\hat{g}$ computed at $\hat{m}$,

$$\hat{\theta} = \hat{g}^{\frac{1}{d}} \left(\frac{\det \Phi^P}{\det \Psi^P} \right)^{\frac{1}{2}\frac{d-2}{d}} \left(\frac{\det \Psi}{\det \Phi} \right)^{\frac{1}{2}\frac{d-1}{d}}. \tag{6.10}$$

Local bargaining power $\hat{\theta}$ has natural comparative statics. The smaller mass of the women in the neighborhood of the top match, i.e., the smaller value of $\hat{g}_W$, the larger density ratio $\hat{g}$ and the larger woman $\hat{w}$'s bargaining power.

6.4. **First-order derivative $\hat{D}\mu$.** We are ready to compute the first-order derivative matrix $\hat{D}\mu$ of the matching function at the top match. We begin with three preliminary observations about $\hat{D}\mu$. First, together with the choice of coordinates, Lemma 1 implies that for any vector $\left(\left(\hat{D}\mu \right) v \right)_d = 0$ for any vector v with the last coordinate equal to $v_d = 0$. Moreover, the definition of bargaining power and the unit length of normal vectors implies that $\left(\left(\hat{D}\mu \right) (0, ..., 0, 1) \right)_d = \hat{\theta}$. Thus, the derivative matrix can be written as

$$\hat{D}\mu = \begin{bmatrix} \left(\hat{D}\mu \right)^P & \left(\hat{D}\mu \right)^y \\ \mathbf{0} & \hat{\theta} \end{bmatrix}, \tag{6.11}$$

where $\left(\hat{D}\mu \right)^P$ is a $(d-1)$-dimensional matrix and $\left(\hat{D}\mu \right)^y$ is a $(d-1)$-vector. The last row of matrix $\hat{D}\mu$ consists of 0s in all places but the last column.

Second, Lemmas 2 and 3 imply that for each pair of vectors v and u with the last coordinate 0, $v'\hat{W} \left(\hat{D}\mu \right) u = u'\hat{W} \left(\hat{D}\mu \right) v$ and $v'\hat{W} \left(\hat{D}\mu \right) v \geq 0$, where $\hat{W}$ is the matrix defined (6.2) as the weighted sum of cross-derivative matrices and evaluated at $\hat{m}$.Therefore, matrix

$$\hat{S} = \left(\hat{W} \left(\hat{D}\mu \right) \right)^P = \hat{W}^P \left(\hat{D}\mu \right)^P \tag{6.12}$$

is symmetric and non-negatively definite (the second equality follows from representation (6.11)). We make a generic assumptions that matrix $\hat{W}^P$ is invertible.

Third, notice that equation (3.2) implies that

$$\det \left| \hat{D}\mu \right| = \hat{g}. \tag{6.13}$$

Next, we consider a first-order approximation to the boundary conditions equation (5.1) using a first-order approximation to matching μ, $\mu(m') = \left(\hat{D}\mu \right) m + O \left(\|m\|^2 \right)$.

After dropping the higher-order terms, it follows that

$$m'\Phi m = 0 \Longrightarrow m'\left(\hat{D}\mu\right)'\Psi\left(\hat{D}\mu\right)m = 0 \text{ for each } m.$$

The two equations on both sides of the above equality are irreducible polynomials of the same order. The above implication is satisfied if and only if the former polynomial is equal to the latter polynomial multiplied by a constant, or if and only if the two polynomials have the same coefficients. Therefore,

$$\left(\hat{D}\mu\right)'\Psi\left(\hat{D}\mu\right) = c\Phi \text{ for some } c. \tag{6.14}$$

Because the determinants of the matrices on both sides of (6.14) must be equal, it must be that $c = \left(\left(\det\hat{D}\mu\right)^2\frac{\det\Psi}{\det\Phi}\right)^{\frac{1}{d}} = \left(\hat{g}^2\frac{\det\Psi}{\det\Phi}\right)^{\frac{1}{d}}$, with the last equality due to (6.13).

Because of (6.11) and (6.12), equation (6.14) implies that

$$\begin{aligned}&\left(\left(\hat{D}\mu\right)^P\right)'\Psi^P\left(\hat{D}\mu\right)^P \\ &= \hat{S}\left(\hat{W}^{P\prime}\right)^{-1}\Psi^P\left(\hat{W}^P\right)^{-1}\hat{S} = \left(\hat{g}^2\frac{\det\Psi}{\det\Phi}\right)^{\frac{1}{d}}\Phi^P.\end{aligned} \tag{6.15}$$

Equation (6.15) is a quadratic matrix equation with unknown square matrix $\hat{S}$. By the discussion above, matrix $\hat{S}$ is symmetric and non-negatively definite. It is well-known that equation (6.15) has a unique symmetric and non-negatively definite solution. (For the sake of completeness, we present the argument in Lemma 4 Appendix A.3). Together with (6.12), this determines $\left(\hat{D}\mu\right)^P = \left(\hat{W}^P\right)^{-1}\hat{S}$.

Finally, vector $\left(\hat{D}\mu\right)^y$ is determined from equation (6.14) after substituting all the other elements of matrix (6.11). This completes the derivation of matrix $\hat{D}\mu$.

We illustrate the above derivations and the role of preferences and local bargaining power in determining the matching around $\hat{m}$ and $\hat{w}$. In order to focus attention, we assume that, given the choice of coordinates described earlier in this section, the boundary conditions are given by the standard cones. In other words,

$$\Phi = \Psi = I^*, \tag{6.16}$$

which implies that $\Phi^P = \Psi^P = I_{d-1}$, where I_{d-1} is $(d-1)$-dimensional identity matrix. We consider two cases: $d = 2$ and $d > 2$ separately.

6.5. **Example $d = 2$.** When the space of the characteristics has only two dimensions, $d = 2$, the standard boundary conditions (6.16) imply that sets E_M and E_W are contained between two diagonal lines:

$$E_M = \{(m_1, m_2) : m_2 \leq 0 \text{ and } m_2 \leq m_1 \leq -m_2\},$$
$$E_M = \{(w_1, w_2) : w_2 \leq 0 \text{ and } w_2 \leq w_1 \leq -w_2\}$$

There are essentially two ways of matching the boundary of E_M with the boundary of E_W: either the matching *preserves orientation* and maps $m_1 = m_2$ onto $w_1 = w_2$ (and $m_1 = -m_2$ onto $w_1 = -w_2$), or the matching *reverses the orientation* and maps $m_1 = m_2$ onto $w_1 = -w_2$ (and $m_1 = -m_2$ onto $w_1 = w_2$). The orientation depends on the sign of $\left(\hat{D}\mu\right)^P$ (notice that in the two-dimensional case, $\hat{D}\mu$ is a 2×2-matrix and $\left(\hat{D}\mu\right)^P$ is a real number). The value of $\left(\hat{D}\mu\right)^P$ is positive if the orientation is preserved, and negative if it is reversed. Because $\hat{S}$ is defined as the positive-definite solution to (6.15), its sign is positive. Therefore, it follows from equation (6.12) that the sign of $\left(\hat{D}\mu\right)^P$ is equal to the sign of the weighted average $\hat{W}^P$ of the cross-derivatives $\hat{\mathcal{M}}^P_{mw}$ and $\hat{\mathcal{W}}^P_{mw}$.

It is instructive to describe an interpretation of the signs of cross-derivatives $\hat{\mathcal{M}}^P_{mw}$ and $\hat{\mathcal{W}}^P_{mw}$. We do it in few steps. First, recall that in the original coordinate system of section 3, men and women are represented by two characteristics. In order to focus attention, say that the first coordinate is called intelligence and the second is called beauty. (Because of the above normalizations, the original coordinates are not the same as the coordinates used in this section. However, we can easily translate the current coordinates into the original ones through a rotation.) The tangent line to man $\hat{m}$'s indifference curve at his match with woman $\hat{w}$ represents a local trade-off between these two characteristics. So, women to the right of woman $\hat{w}$, i.e., $w_1 > 0$ in the current coordinates, are smarter (and uglier) than woman $\hat{w}$, and the women to the left of $\hat{w}$ are prettier (and less smart) than woman $\hat{w}$.Similarly, we say that men to the right of $\hat{m}$, i.e., $m_1 > 0$ in the current coordinates, are smarter (and uglier) than $\hat{m}$, and the men to the left of $\hat{m}$, are prettier (and less smart) than $\hat{m}$.

Notice that for each man m sufficiently close to $\hat{m}$, $\mathcal{M}_w(m) \approx \mathcal{M}_w^{\hat{}} + m'\mathcal{M}_{mw}^{\hat{}}$. In particular, cross-derivative $\mathcal{M}_{mw}^{\hat{}P}$ describes how the men's preferences change with the their characteristics. There are two cases: If $\mathcal{M}_{mw}^{\hat{}P} > 0$, than the first coordinate of vector $\mathcal{M}_w(m)$ has the same sign as the first coordinate of m. In such a case, we say that men that are smarter than $\hat{m}$ (and that lie along the line that is tangent to woman $\hat{w}$'s indifference curve) prefer women that are smarter than $\hat{w}$ (and among all women that lie along the tangent line to $\hat{m}$'s indifference curve). Similarly, prettier men prefer prettier women. If $\mathcal{M}_{mw}^{\hat{}P} < 0$, then smarter men prefer prettier women and prettier men prefer smarter women. A similar interpretation holds for cross-derivative $\mathcal{W}_{mw}^{\hat{}P}$.

We go back to the analysis of the orientation of stable matching. Above, we show that the sign of $\left(\hat{D}\mu\right)^P$ is equal to the sign of $\hat{W}^P = \mathcal{M}_{mw}^{\hat{}P} + \hat{\theta}\mathcal{W}_{mw}^{\hat{}P}$. This means that the orientation is preserved in one of these three cases:

(a) the preferences are aligned so that both smarter men prefer smarter women (i.e., $\mathcal{M}_{mw}^{\hat{}P} > 0$) and smarter women prefer smarter men (i.e., $\mathcal{W}_{mw}^{\hat{}P} > 0$), or

(b) the preferences are misaligned, smarter men prefer smarter women, smarter women prefer prettier men, and the local bargaining power of women is low, or

(c) the preferences are misaligned, smarter men prefer prettier women, smarter women prefer smarter men, and the local bargaining power of women is high.

We can easily find a similar description of situations for which the orientation of stable matching is reversed.

6.6. **Example $d \geq 3$.** To illustrate how the orientation of stable matching is determined when there are more than two dimensions, we make some additional assumptions. Let $R(\zeta)$ be the rotation matrix in a $(d-1)$-dimensional space through an angle ζ. For example, if $d = 3$, then $R(\zeta)$ has a form

$$R(\zeta) = \begin{bmatrix} \cos(\zeta), & \sin(\zeta), \\ \sin(\zeta), & -\cos(\zeta). \end{bmatrix}$$

Let $cR(\zeta)$ be the rotation matrix multiplied by a scalar. We assume

$$\mathcal{M}^{\hat{}P}_{mw} = I_{d-1} \text{ and } \mathcal{W}^{\hat{}P}_{mw} = R(\zeta),$$

where I_{d-1} is the $(d-1)$-dimensional identity matrix. In order to interpret this assumption, recall from section 4 that $\left(\mathcal{M}^{\hat{}P}_{mw}\right)' v = v$ is the favorite direction of rematch for man $\hat{m} + v$, where v belongs to the hyperplane tangent to woman $\hat{w}$'s indifference curve. In other words, if constrained to make choices from the hyperplane tangent to $\hat{w}$ indifference curve, man $\hat{m}+v$ would like to rematch towards women that are represented by the same vector as his own, v. Similarly, woman $\hat{w}+u$ would like to rematch towards men that are represented by vectors rotated through an angle ζ from u. Angle ζ measures the local discrepancy between men's and women's preferences. The preferences coincide only if $\zeta = 0$.

Because the weighted average of identity and a rotation matrix is (generically) a scalar-multiplied rotation matrix, there exists $c_{\hat{\theta},\zeta}$ and $\beta_{\hat{\theta},\zeta} \in (0,1)$ such that

$$\hat{W}^P = \mathcal{M}^{\hat{}P}_{mw} + \hat{\theta}\mathcal{W}^{\hat{}P}_{mw} = I + \hat{g}R(\zeta_W) = c_{\hat{\theta},\zeta} R\left(\beta_{\hat{\theta},\zeta}\zeta\right).$$

The weight on rotation angle $\beta_{\hat{\theta},\zeta}$, increases with the local bargaining power of women.

Given the assumptions (6.16), equation (6.15) implies that

$$\hat{g}^{2/d} I_{d-1} = \hat{S}\left(\hat{W}^{P\prime}\right)^{-1} I_{d-1}\left(\hat{W}^P\right)^{-1}\hat{S} = c^{-2}_{\hat{\theta},\zeta}\hat{S}\hat{S}.$$

The only positive-definite and symmetric solution to the above equation is equal to $S = \left(c_{\hat{\theta},\zeta}\hat{g}^{1/d}\right) I_{d-1}$. By (6.12),

$$\left(\hat{D}\mu\right)^P = \left(\hat{W}^P\right)^{-1}\hat{S} = \left(\hat{g}^{1/d}\right) R\left(-\beta_{\hat{\theta},\zeta}\zeta\right).$$

Thus, if the local bargaining power of women is very low and $\beta_{\hat{\theta},\zeta} \approx 0$, men around $\hat{m}$ are matched in their favorite directions. If the local bargaining power of women is very high and $\beta\left(\hat{\theta}\right) \approx 1$, men are matched in the favorite directions of women, rotated through an angle $-\zeta$. In the intermediate case, stable matching balances the favorite directions of men and women.

7. Comments

In this section, we discuss some generalizations of Theorem 1 as well as some open questions.

7.1. **Domain assumptions.** We can mildly relax the assumptions on the domains from section 5 so that the domains look like cones only locally, at some neighborhood of $\mathbf{0}$. Assume that $E_M = \phi(E^*)$ and $E_W = \psi(E^*)$ for some analytical mappings $\phi, \psi : R^d \to R^d$ such that $\phi(\mathbf{0}) = \psi(\mathbf{0}) = \mathbf{0}$ and such that their derivatives at $\mathbf{0}$, $\hat{D}\phi, \hat{D}\psi : R^d \to R^d$, are a proper linear mapping. Define proper cone matrices $\Phi = \left(\hat{D}\phi\right)' I^* \left(\hat{D}\phi\right)$ and $\Psi = \left(\hat{D}\psi\right)' I^* \left(\hat{D}\psi\right)$. Then, Theorem 1 holds without any modification of its proof.

What about non-cone shape domains? For example, one can imagine that E_M and E_W are compact subsets with smooth boundaries. If the preferences are monotonic, one can still show that there exists man $\hat{m} \in E_M$ and $\hat{w} \in E_W$ such that $\hat{w}$ is the most preferred match of man $\hat{m}$ and vice versa. One can approach the uniqueness question with such domains along the same lines as in the proof of Theorem 1. Specifically, it is enough to show that all of the higher-order $\hat{m}$-derivatives of solution μ to the system of partial differential equations (3.2) and (4.2) with boundary conditions $\mu(\operatorname{bd} E_M) = \operatorname{bd} E_W$ are uniquely determined by the fundamentals of the model. Because of the different nature of the domain around the top match $\hat{m}$, the higher-order equations are different than the equations derived in the proof of Theorem 1. Nevertheless, one can still show that the number of equations is equal to the number of unknowns and we expect that an analog of Theorem 1 holds.

7.2. **Existence.** Theorem 1 is concerned with uniqueness and does not comment on existence. In many cases, we know that smooth stable matching exists. For example, one can show that for any strictly monotonic smooth function $\mu : E_M \to E_W$ that preserves masses, any men's utility function $\mathcal{M}$, one can find women's utilty function $\mathcal{W}$ such that μ is stable matching. Whenever it exists, smooth stable matching is unique.

Of course, the question is whether smooth stable matching exists for a a given utility functions $\mathcal{M}, \mathcal{W}$, domains E_M and E_W, and densities g_M and g_W. In order to answer this question, our results suggest to look for the existence of a solution to a particular type of partial differential equations. In general, the issue of the existence of a solution to a system of differential equations is a difficult one. In the Cauchy problem, one looks for solution $f : R^d \to R$ to the system of equations of form $A(f, Df, ..., D^n f) = 0$ with boundary condition $f(y_1, ..., y_{d-1}, 0) = f_0(y_1, ..., y_{d-1})$ for some functions A and f_0. Under some regularity conditions on A and f_0, the standard off-the-shelf result, the Cauchy-Kovalevskaya Theorem proves the existence and uniqueness of a smooth solution in some neighborhood of $(0, 0, ..., 0)$.

The Cauchy-Kovalevskaya Theorem cannot be applied to the existence of stable matching because the boundary conditions (5.1) are different than in the Cauchy problem. We explain the difficulty in the case $d = 2$. The domains E_M and E_W are generated by proper cone matrices. In the coordinates used in section 6, it means that for some $\phi^0 > \phi^1$ and $\psi^0 > \psi^1$,

$$E_M = \left\{(m_1, m_2) : m_2 \leq 0 \text{ and } \phi^0 m_2 \leq m_1 \leq \phi^1 m_2\right\},$$
$$E_M = \left\{(w_1, w_2) : w_2 \leq 0 \text{ and } \psi^0 w_2 \leq w_1 \leq \psi^1 w_2\right\}.$$

Let $B^i_M = \{(\phi^i m_2, m_2) : m_2 \leq 0\}$ for $i = 0, 1$ be the boundary of men's domain E_M and similarly define B^i_W. Suppose that the values of the parameters at the top match are such that stable matching preserves orientation (see Section 6.5). Let $\mu_0 : B^0_M \to B^0_W$ be a mapping that maps the left boundary of E_M onto the left boundary of E_W. Then, under some regularity conditions on μ_0, the Cauchy-Kovalevskaya Theorem can be used to show that, in some neighborhood of $\mathbf{0}$, there exists mapping $\mu^{\mu_0} : E_M \to R^d$ such that μ^{μ_0} satisfies partial differential equations (3.2) and (4.2) and such that $\mu^{\mu_0}|_{B^0_M} = \mu_0$. However, for general choices of μ_0 it is unlikely that the solution μ^{μ_0} maps the right boundary of men's domain on the right boundary of women's domain. In fact, by Theorem 1, there exists at most one choice of μ_0 such that μ^{μ_0} maps both boundaries on each other. The (local) existence of smooth stable matching hinges on whether there exists regular μ_0 such that $\mu^{\mu_0}(B^1_M) = B^1_W$.

7.3. **Convergence.** Arguably, smooth stable matching and its uniqueness is interesting mainly if it is a limit of matching in finite population models. Specifically, fix N and consider a random matching market with N men and women drawn from distributions, respectively, F_M and F_W. For each realization of the random market, there exists at least one stable matching (that can be found, for example, by Gale-Shapley algorithm). The stable matching generally is not unique. Let $\mu_M^{(N)}$ and $\mu_W^{(N)}$ denote, respectively, the men- and the women-optimal matching given the realization of preferences of N men and women. Then, all stable matchings are contained between $\mu_M^{(N)}$ and $\mu_W^{(N)}$ (see Roth and Sotomayor (1992)). We can consider sequences of random variables $\mu_M^{(N)}$ and $\mu_W^{(N)}$ as N converges to infinite. The question is: do these sequences converge to some well-defined object? Is the limit a smooth matching? We leave these questions for future research.

Appendix A. Mathematical preliminaries

A.1. **Multi-indices.** We are going to use a multi-index notation. A multi-index $\gamma = (\gamma_1, ..., \gamma_d - 1)$ is a $(d-1)$-tuple of positive integers, $\gamma_l \geq 0$. For any two multi-indices γ and γ' we write $\gamma \geq \gamma'$ if $\gamma_l \geq \gamma'_l$ for each l. Moreover, define $\gamma + \gamma' = \left(\gamma_1 + \gamma'_1, ..., \gamma_{d-1} + \gamma'_{d-1}\right)$ for each γ, γ' and $\gamma - \gamma'$ for each $\gamma \geq \gamma'$. Finally, let $|\gamma| = \gamma_1 + ... + \gamma_{d-1}$ and $\gamma! = \gamma_1!...\gamma_d!$.

Let Γ be the space of all multi-indices and let Γ_n be the space of all multi-indices γ such that $|\gamma| = n$. Let $\varnothing \in \Gamma_0$ denote the multi-index with 0 at all positions. For each $l = 1, ..., d$, let $l^\Gamma \in \Gamma_1$ denote the multi-index that has 1 at the lth position and 0 at all other positions.

We use the multi-indices in two ways. First, they denote the powers of vectors: For each $x \in R^{d-1}$, and $\gamma \in \Gamma$, let

$$x^\gamma = x_1^{\gamma_1}...x_{d-1}^{\gamma_{d-1}}.$$

Second, they denote the derivatives of functions. For each $f : R^d \to R$, each $k \geq 0$ and $\gamma \in \Gamma$, we write

$$f_{\gamma,k} = \frac{d^{|\gamma|+k}}{dx^\gamma dy^k} f = \frac{d^{k+|\gamma|}}{dx_1^{\gamma_1}...dx_{d-1}^{\gamma_{d-1}} dy^k} f.$$

A.2. **Matrix notation and terminology.** Let A be an n-dimensional square matrix for $n = d$ or $d-1$. We write $A = \left[a_l^j\right]$, where j corresponds to rows and l corresponds to columns. Let a_{-l}^{-j} be the $jl-$cofactor of matrix A, i.e., $a_{-l}^{-j} = (-1)^{j+l} m_l^j$, where m_l^j

is the determinant of a matrix that is obtained from A by crossing out jth row and lth column. Let $C = \left[a_{-l}^{-j}\right]$ be the matrix of cofactors. Then, if A is invertible,

$$A^{-1} = \frac{1}{\det A} C'.$$

Moreover, for each row j,

$$\det A = \sum\nolimits_l a_l^j a_{-l}^{-j}.$$

It is sometimes convenient to divide matrix A into four parts that correspond to the first $n-1$ and the last coordinates: We write

$$A = \begin{bmatrix} A^P & A^{(y)} \\ A^{(yy)\prime} & A^{(0)} \end{bmatrix}$$

where $A^{(0)}$ is a number, $A^{(y)}$ and $A^{(yy)}$ are $(n-1)$-vectors, and A^P is a $(n-1)$-dimensional matrix. If A is symmetric, then $A^{(y)} = A^{(yy)}$. If $A^{(P)}$ is invertible, we use the following formula for the determinant of matrix A :

$$\det A = \det A^P \det \left(A^{(0)} - A^{(y)\prime} \left(A^P\right)^{-1} A^{(y)} \right). \tag{A.1}$$

A.3. **Quadratic matrix equation.**

Lemma 4. *Suppose that d-dimensional matrices A and B are symmetric and positively definite. Then there exists a unique symmetric and positively definite matrix S that solves*

$$SAS = B.$$

Moreover, the unique solution is an analytic function of A and B.

First, notice that for each d-dimensional symmetric and positive-definite matrix A, there exists a unique symmetric and positive-definite matrix $A^{1/2}$ such that

$$A^{1/2}A^{1/2} = A. \tag{A.2}$$

Indeed, the spectral theorem and equation (A.2) imply that $A^{1/2}$ and A each have d independent eigenvectors, that $A^{1/2}$ must have the same eigenvectors as A, and that for each eigenvector v, if $Av = \lambda v$, then $A^{1/2}v = \lambda^{1/2}v$. The eigenvalues and eigenvectors determine symmetric $A^{1/2}$.

Notice also that $A^{1/2}$ is an analytic function of A for all (symmetric and positively definite) A.

Observe that

$$A^{1/2}SASA^{1/2} = \left(A^{1/2}SA^{1/2}\right)\left(A^{1/2}SA^{1/2}\right) = A^{1/2}BA^{1/2},$$

which implies that

$$A^{1/2}SA^{1/2} = \left(A^{1/2}BA^{1/2}\right)^{1/2}, \text{ and}$$
$$S = A^{-1/2}\left(A^{1/2}BA^{1/2}\right)^{1/2}A^{-1/2}.$$

A.4. **Cone masses.** Consider a cone E generated by a proper cone matrix Φ,

$$E = \{m : m'\Phi m \leq 0, m_d \leq 0\}.$$

Because Φ is a proper cone matrix, for each vector m such that $m_d = 0$, we have $m_d'\Phi m_d > 0$, and matrix Φ^P is positively definite. Let E_1 be a part of this cone that lies between plane $y = -1$ and apex $\mathbf{0}$. The next lemma finds the volume of E_1.

Lemma 5. *There exists constant c_d that depends only on d such that*

$$\mathrm{vol}_d(E_1) = c_d(-\det\Phi)^{-\frac{1}{2}(d-1)}\left(\det\Phi^P\right)^{\frac{1}{2}(d-2)}.$$

We need a preliminary observation.

Lemma 6. *There exists constant c_{d-1} that depends only on d such that for each positively definite and symmetric $(d-1)$-dimensional matrix A, for each $b \in R^{d-1}$,*

$$\mathrm{vol}_{d-1}\left\{y \in R^{d-1} : (y-b)'A(y-b) \leq 1\right\} = c_{d-1}(\det A)^{-1/2}$$

Proof. Let c_{d-1} be the volume of $(d-1)$-dimensional ball $B_{d-1} = \left\{y \in R^{d-1} : y'y \leq 1\right\}$. For each square and positive Let $A = A^{1/2}A^{1/2}$ for some matrix $A^{1/2}$ (such a matrix exists due to assumptions about A). Then,

$$\begin{aligned}
&\left\{y \in R^{d-1} : y'Ay \leq 1\right\}\\
&= \left\{y \in R^{d-1} : \left(A^{1/2}y\right)'A^{1/2}y \leq 1\right\}\\
&= \left\{\left(A^{1/2}\right)^{-1}y \in R^{d-1} : y'y \leq 1\right\} = \left(A^{1/2}\right)^{-1}B_{d-1}.
\end{aligned}$$

Thus,

$$\mathrm{vol}_{d-1}\left\{y \in R^{d-1} : y' A y \leq 1\right\} = c_{d-1}\left(\det A\right)^{-1/2}.$$

The result follows. □

We can move to the proof of Lemma 5. First, compute the $(d-1)$-dimensional Lesbegue measure of the ellipse $\{m : m_d = -1, m'\Phi m \leq 0\}$. Notice that for each $y \in R^{d-1}$

$$\begin{aligned}
&\left(y', -1\right)\Phi\left(y, -1\right) \\
&= y'\Phi^P y - y'\Phi^{(d)} - \Phi^{(d)\prime} y + \Phi^{(0)} \\
&= \left(y' - \Phi^{(d)\prime}\left(\Phi^P\right)^{-1}\right)\Phi^P\left(y - \left(\Phi^P\right)^{-1}\Phi^{(d)}\right) + \Phi^{(0)} - \Phi^{(d)\prime}\left(\Phi^P\right)^{-1}\Phi^{(d)} \\
&= \left(y' - \Phi^{(d)\prime}\left(\Phi^P\right)^{-1}\right)\Phi^P\left(y - \left(\Phi^P\right)^{-1}\Phi^{(d)}\right) - \frac{\left(-\det\Phi\right)}{\det\Phi^P} \\
&= \frac{\left(-\det\Phi\right)}{\det\Phi^P}\left[\left(y' - \Phi^{(d)\prime}\left(\Phi^P\right)^{-1}\right)\left(\frac{\det\Phi^P}{\left(-\det\Phi\right)}\Phi^P\right)\left(y - \left(\Phi^P\right)^{-1}\Phi^{(d)}\right) - 1\right].
\end{aligned}$$

where we used formula (A.1). It follows from the definition of the proper cone matrix that $\det\Phi < 0$. By Lemma 6,

$$\begin{aligned}
&\mathrm{vol}_{d-1}\left\{m : m_d = -1, m'\Phi m \leq 0\right\} \\
&= \mathrm{vol}_{d-1}\left\{y \in R^{d-1} : \left(y' - \Phi^{(d)\prime}\left(\Phi^P\right)^{-1}\right)\left(-\frac{\det\Phi}{\det\Phi^P}\Phi^P\right)\left(y - \left(\Phi^P\right)^{-1}\Phi^{(d)}\right) \leq 1\right\} \\
&= c_{d-1}\left(-\det\Phi\right)^{-\frac{1}{2}(d-1)}\left(\det\Phi^P\right)^{\frac{1}{2}(d-2)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{vol}_d E_1 &= \frac{1}{d+1}\mathrm{vol}_{d-1}\left\{m : m_d = -1, m'\Phi m \leq 0\right\} \\
&= c_d\left(-\det\Phi\right)^{-\frac{1}{2}(d-1)}\left(\det\Phi^P\right)^{\frac{1}{2}(d-2)}
\end{aligned}$$

for some constant c_d.

A.5. **Generic solvability of a system of equations.** Take any open and convex $U \subseteq R^n$. Let λ_U be the Lesbegue measure on W_U. Then, $\lambda_U(U) > 0$.

Square matrix-valued function $A : R^n \to R^{m\times m}$ is analytic if $A_{i,j} : R^n \to R$ is analytic for each $i, j = 1, ..., m$. Let $Z_A \subseteq R^n$ denote the set of elements of its domain $z \in R^N$ such that the system of linear equations

$$A(z)\, y = 0, \tag{A.3}$$

has a non-zero solution. In other words,

$$Z_A = \{z : \det A(z) = 0\}.$$

Lemma 7. *For any open and convex $U \subseteq R^n$, any analytic function $A : R^n \to R^{(d-1)\times(d-1)}$, either $U \subseteq Z_A$ or $\lambda(Z_A \cap U) = 0$, where λ is a Lesbegue measure on R^n.*

Proof. Note that $\det A(z)$ is an analytic function on open and convex domain. Moreover, Z_A is a closed set. If $Z_A \cap U$ has a non-empty interior, then there exists an open set $W \subseteq Z_A \cap U$ such that $\det A(z) = 0$ for each $z \in W$. The properties of analytic functions imply that $\det A(z) = 0$ for each $z \in R^n$. □

By the above lemma, if function $A(.)$ is analytic, and if there exists at least one $z \in U$ such that the system of equations (A.3) has a unique solution, then A.3 has a unique solution for generic parameters $z \in U$.

Appendix B. Proof of Theorem 1

As we explain in Section 5, smooth (i.e., analytical) matching over an open connected domain is determined by its first- and higher-order derivatives computed at the top match $\hat{m}$. We are going to show that, for the generic values of the parameters $\hat{\mathcal{M}_w}, \hat{\mathcal{W}_m}, \hat{\mathcal{M}_{mw}}, \hat{\mathcal{W}_{mw}}, \phi, \psi, \hat{g}$, all higher-order derivatives of smooth stable matching at $\hat{m}$ can be uniquely determined as functions of the parameters. In section 6, we establish our claim for the first-order derivatives. Here, we show that the claim holds for all higher orders.

We describe the plan of the proof. Parts B.1 and B.2 develop the notation. The smooth stable matching must satisfy partial differential equations that are restated in part B.3.

We are going to interpret the higher-order derivatives as unknown variables in a system of linear equations. In part B.4, we classify all of the variables by their order

$n = 2, 3, ...$ and rank $k = -1, 0, ..., n-1$. Let $Var_{n,k}$ be the vector of all variables of order n and rank k. In part B.5, we show that for each n and $k \leq n-1$, variables $Var_{n,k}$ must satisfy a system of linear equations:

$$C_{n,k}\left(Parameters\right) Var_{n,k} = \text{Terms}_{n,k}\left(V_{n',k'} \text{ st. } n' < n \text{ or } n' = n \text{ and } k' < k\right),$$

where the matrix of coefficients $C_{n,k}\left(Parameters\right)$ is an analytic function of the parameters of the model, and $\text{Terms}_{n,k}$ is a function of variables of lower order or rank. The number of equations is equal to the number of variables $Var_{n,k}$ (or, in other words, $C_{n,k}$ is a square matrix). In parts B.6 and B.7, we show that for each order n and rank k, there exist values of $Parameters$ for which matrix $C_{n,k}\left(Parameters\right)$ is invertible (the former part deals with equations of rank $k < n-1$ and the latter with equations of rank $n-1$). Part B.8 concludes the argument by showing that matrices $C_{n,k}\left(Parameters\right)$ are invertible for all n and k and generic values of parameters.

B.1. **Notations and normalizations.** In order to distinguish the first $d-1$ and the last coordinates of the characteristics, for each man m and woman w, we write $m = (x_1, ..., x_{d-1}, y) \in R^d$ and $w = (a_1, ..., a_{d-1}, b) \in R^d$.

Throughout the proof of Theorem 1, we make a generic assumption that matrix $\hat{W}^P = \left(\mathcal{M}_{mw}^{\hat{}} + \hat{\theta}\mathcal{W}_{mw}^{\hat{}}\right)^P$, where $\hat{\theta}$ is given by (6.10), is invertible. We make all the normalization assumptions described in section 6.2. In particular, we assume that the coordinates are chosen so that vectors $\mathcal{W}_m{}^{\hat{}}$ and $\mathcal{M}_w{}^{\hat{}}$ have the last coordinate equal to 0 and all the other coordinates equal to 0,

$$\mathcal{W}_m^{\hat{}} = \mathcal{M}_w^{\hat{}} = (0, ..., 0, 1).$$

Let $\hat{S}$ be the $(d-1)$-dimensional matrix that is the unique symmetric and positive solution to the matrix equation (6.15). Often, we work with the inverse image of $\hat{S}$, $\hat{S}^{-1}$. We denote the elements of the inverse as $\hat{S}^{-1} = \left[s_l^{*j}\right]$, where j corresponds to rows and l corresponds to columns.

For each m, w, and each $j = 1, ..., d-1$, let $y^{*j}(m, w)$ be defined as in (4.1). Then, vector $y^{*j}(m, w)\, e^d + e^j$ is orthogonal to vector $\mathcal{W}_m(m, w)$, $\left(y^{*j}(m, w)\, e^d + e^j\right)\cdot$

$\mathcal{W}_m(m,w)=0$. Additionally, for each m,w, and each $l=1,...,d-1$, let

$$a_l^*(m,w)=\frac{(\mathcal{M}_w(m,w))_l}{(\mathcal{M}_w(m,w))_d}.$$

Then, vector $(a^*(m,w),1)$ lies in the same direction as vector $\mathcal{M}_w(m,w)$: there exists scalar $c>0$ such that

$$\left(a_1^*(m,w),...,a_{d-1}^*(m,w),1\right)=c\mathcal{M}_w(m,w).$$

Observe that for each $j,l=1,...,d$,

$$\hat{y}^{*j}=\hat{a}_l^*=0. \tag{B.1}$$

For each $j,l=1,...,d-1$, denote derivatives[5]:

$$y^{*j,y}=\frac{\partial y^{*j}}{\partial y},y^{*j,l}=\frac{\partial y^{*j}}{\partial x_l},y_l^{*j}=\frac{\partial y^{*j}}{\partial a_l},y_b^{*j}=\frac{\partial y^{*j}}{\partial b},$$
$$a_l^{*y}=\frac{\partial a_l^*}{\partial y},a_l^{*j}=\frac{\partial a_l^*}{\partial x_j},a_{l,j}^*=\frac{\partial a_l^*}{\partial a_j},a_{l,b}^*=\frac{\partial a_l^*}{\partial b}.$$

Lemma 8. *For each $l,j=1,...,d$, $\hat{y}^{*j,l}=\hat{y}^{*l,j}$ and $\hat{a}_{l,j}^*=a_{j,l}^*$. Moreover,*

$$\left[\hat{y}_l^{*j}\right]=-\mathcal{W}_{mw}^{P\hat{}},\left[\hat{a}_l^{*j}\right]=\mathcal{M}_{mw}^{P\hat{}},$$

which implies that $\hat{W}^P=\left[\hat{a}_l^{*j}\right]-\hat{\theta}\left[\hat{y}_l^{*j}\right]$. (Recall that matrix $\hat{W}$ is defined in (6.2).)

Proof. Notice that $(\mathcal{W}_m(m,w))_j=\frac{d}{dx_j}\mathcal{W}(m,w)$ and, by normalization, $\left(\hat{\mathcal{W}_m}\right)_j=(\mathcal{W}_m(\hat{m},\hat{w}))_j=0$ and $(\mathcal{W}_m(\hat{m},\hat{w}))_d=1$. Hence,

$$\begin{aligned}\frac{\partial y^{*j}}{\partial x_l}(\hat{m},\hat{w})&=-\frac{\frac{\partial}{\partial x_l}(\mathcal{W}_m(\hat{m},\hat{w}))_j}{(\mathcal{W}_m(\hat{m},\hat{w}))_d}+\frac{\left(\hat{\mathcal{W}_m}\right)_j\left(\frac{\partial}{\partial x_l}(\mathcal{W}_m(\hat{m},\hat{w}))_d\right)}{((\mathcal{W}_m(\hat{m},\hat{w}))_d)^2}\\&=-\frac{\partial^2}{\partial x_l\partial x_j}\mathcal{W}(\hat{m},\hat{w})=\frac{\partial y^{*l}}{\partial x_j}(\hat{m},\hat{w}),\end{aligned}$$

where the last equality comes from the fact that the roles of l and j can be exchanged. The argument for $\hat{a}_{l,j}^*=a_{j,l}^*$ is analogous.

[5] For each function $f:E_M\times E_W\to R$, we write f instead of $f(m,w)$ when the variables are clear from the context.

For the second part, observe that

$$\hat{y}_l^{*j} = \frac{\partial y^{*j}}{\partial a_l}(\hat{m},\hat{w}) = -\frac{\frac{\partial}{\partial a_l}\left(\mathcal{W}_m(\hat{m},\hat{w})\right)_j}{\|\hat{\mathcal{W}_m}\|} + \frac{\left(\hat{\mathcal{W}_m}\right)_j \frac{\partial}{\partial a_l}\|\mathcal{W}_m(\hat{m},\hat{w})\|}{\|\hat{\mathcal{W}_m}\|^2} = -\frac{\partial^2}{\partial x_j \partial a_l}\mathcal{W}(\hat{m},\hat{w}).$$

Similarly,

$$\hat{a}_l^{*j} = \frac{\partial a_l^*}{\partial x_j}(\hat{m},\hat{w}) = \frac{\frac{\partial}{\partial x_j}\left(\mathcal{M}_w(\hat{m},\hat{w})\right)_l}{\|\hat{\mathcal{M}_w}\|} + \frac{\left(\hat{\mathcal{M}_w}\right)_l \frac{\partial}{\partial x_j}\|\hat{\mathcal{M}_w}\|}{\|\hat{\mathcal{M}_w}\|^2} = \frac{\partial^2}{\partial x_j \partial a_l}\mathcal{M}(\hat{m},\hat{w}),$$

□

B.2. **Matching.** Let $\mu : E_M \to E_W$ be a smooth stable matching. In order to distinguish the first $d-1$ and the last other coordinates of the matching function, we write

$$\mu(x,y) = \left(\alpha^1(x,y), ..., \alpha^{d-1}(x,y), \beta(x,y)\right) = (\alpha(x,y), \beta(x,y)),$$

where $y \in R$ is a number, $x \in R^d$ is a vector, $\beta, \alpha^1, ..., \alpha^{d-1} : R^d \to R$ and $\alpha = \left(\alpha^1, ..., \alpha^{d-1}\right) : R^d \to R^{d-1}$ are functions.

Section 6 derives the first-order derivatives of μ at $\hat{m}$ as functions of the parameters of the model. In fact, we have the following result:

Lemma 9. *The first-order derivatives are analytic functions of parameters* $\left(\hat{\mathcal{M}}_{mw}, \hat{\mathcal{W}}_{mw}, \Phi, \Psi, \hat{g}\right)$:

$$\hat{\beta}_y = \hat{\theta}, \hat{\beta}_j = 0 \text{ for each } j = 1,..,d-1, \tag{B.2}$$

$$\hat{\alpha}_x = \left(\hat{W}^P\right)^{-1}\hat{S},$$

$$\hat{\alpha}_y' = \left[(\hat{\alpha}_x)^{-1}\left(\hat{g}^2\frac{\det\Psi}{\det\Phi}\right)^{\frac{1}{d}}\Phi^{(y)\prime} - \hat{\beta}_y\Psi^{(y)\prime}\right]\left(\Psi^{(x)}\right)^{-1}.$$

where we write $\hat{\alpha}_x = \left[\alpha_l^j\right]$ *for the* $(d-1)$*-dimensional matrix of derivatives of* α*.*

Proof. The formula for $\hat{\alpha}_y$ follows from equations (6.12), (6.11), and (6.14). The result follows from Lemma 4 and equation (6.10). □

B.3. **Partial differential equations.** Smooth stable matching μ satisfies the following functional equations: For each man m, we have

- *matching equation*:

$$\left|\det\begin{bmatrix}\alpha_x & \alpha_y\\ \beta_x & \beta_y\end{bmatrix}\right| - g = 0. \tag{M}$$

  Section 6 determines matrix $\begin{bmatrix}\hat{\alpha}_x & \hat{\alpha}_y\\ \hat{\beta}_x & \hat{\beta}_y\end{bmatrix}$ of derivatives at the top match, and it also determines the (non-zero) sign determinant of this matrix. In particular, in some neighborhood of the top match, the determinant is always negative or always positive. In order to focus attention, we assume that the determinant is positive and we drop the absolute value from the above equation (the alternative assumption would not affect our analysis),

- *stability equation*: Equation 4.2 implies that for each $j = 1, .., d$, vectors $\begin{bmatrix}\alpha_x & \alpha_y\\ \beta_x & \beta_y\end{bmatrix}\left(y^{*j}e^d + e^j\right)$ are orthogonal to vector $(1, a^*)$. In other words,

$$\begin{aligned}&\begin{bmatrix}\alpha_x & \alpha_y\\ \beta_x & \beta_y\end{bmatrix}\left(y^{*j}e^d + e^j\right)\cdot(1, a^*)\\ &= y^{*j}\beta_y + \beta_j + \sum\nolimits_i a_i^*\alpha_j^i + y^{*j}\sum\nolimits_i a_i^*\alpha_y^i = 0,\end{aligned} \tag{S^j}$$

- *boundary conditions*: For each $q > 0$, and each x such that

$$0 = [x', 1]\,\Phi\begin{bmatrix}x\\ 1\end{bmatrix} = \Phi^{(0)} + 2x'\Phi^{(y)} + x'\Phi^{(P)}x \tag{B^0}$$

  it must be that

$$\begin{aligned}0 &= \left[\alpha(qx, q), \beta(qx, q)\right]\Psi\begin{bmatrix}\alpha(qx, q)\\ \beta(qx, q)\end{bmatrix}\\ &= \left(\beta(qx, q)\right)^2\Psi^{(0)} + 2\beta(qx, q)\,\alpha'(qx, q)\,\Psi^{(y)} + \alpha'(qx, q)\,\Psi^{(P)}\alpha(qx, q).\end{aligned} \tag{$B(qx, x)$}$$

B.4. **Order and rank of variables.** For each $n \geq 2$, $k \geq 0$, and multi-index $\gamma \in \Gamma_{n-k}$, let $\beta_{\gamma,k}$ and $\alpha_{\gamma,k}$ be the derivatives of functions β and α (more precisely,

$\alpha_{\gamma,k}$ is a d-vector of derivatives $\alpha^j_{\gamma,k}$ for $j = 1, ..., d$). Let $\hat{\beta}_{\gamma,k}$ and $\hat{\alpha}_{\gamma,k}$ be the derivatives computed at $\hat{m}$. We say that derivatives $\hat{\beta}_{k,\gamma}$ or $\hat{\alpha}_{k,\gamma}$ have order $n = k + |\gamma|$. Additionally, we use auxiliary variables:

$$\{C_{\gamma,k} : \gamma \in \Gamma_{n+1-k} \text{ for } k \leq n\} \text{ and } \{D_{\gamma,k} : \tau \in \Gamma_{n-1-k} \text{ for } k \leq n-1\}.$$

All the above variables have order n.

Apart from the order, each variable is equipped with a rank:

- variables $\hat{\beta}_{\gamma,k}$ for $k < n$ and $\gamma \in \Gamma_{n-k}$ have rank -1,
- variable $\hat{\beta}_{\varnothing,n}$ has rank $n-1$,
- variables $\hat{\alpha}_{\gamma,k}$ for $k \leq n$ and $\gamma \in \Gamma_{n-k}$ have rank $\min(k, n-1)$,
- variables $C_{\gamma,k}$ for $k \leq n$ and $\gamma \in \Gamma_{n+1-k}$ have rank $\min(k, n-1)$, and
- variables $D_{\gamma,k}$ for $k \leq n-1$ and $\tau \in \Gamma_{n-1-k}$ have rank k.

B.5. **Equations of order** n**.** From now on, we assume that $n \geq 2$ is fixed. For each k, we write "Terms(k)" to denote the terms that depend on variables that have either order $n' < n$, or order n and rank $k' < k$. For example, equation (B.3) below means that variable $\hat{\beta}_{\gamma,k}$ (i.e., a variable of order n and rank -1) can be presented as a function of terms that include only variables of order $n-1$ or lower.

Lemma 10. *For each $k < n$ and $\gamma \in \Gamma_{n-k}$,*

$$\hat{\beta}_{\gamma,k} = \textit{Terms}(-1). \tag{B.3}$$

Proof. Take any $j = 1, ..., d$ such that $j^\Gamma \leq \gamma$ and consider the $(k, \gamma - j)$th derivative of equation S^j evaluated at the point of the top match. Because $\hat{y}^{*j} = 0$ and $\hat{a}^* = \mathbf{0}$, we get

$$0 = \frac{d^n}{dx^\gamma dy^k} S^j|_{\hat{m}} = \hat{\beta}_{k,\gamma} + \text{Terms}(-1).$$

□

Lemma 11. *There exists $\{C_{\gamma,k} : \gamma \in \Gamma_{n+1-k}$ for $k \leq n\}$ such that for each $k \leq n$ and $\gamma \in \Gamma_{n-k}$,*

$$\hat{W}^P \hat{\alpha}_{\gamma,k} = C_{\gamma+\cdot,k} + \hat{\beta}_{\varnothing,n} \left(\sum_{j,l} \mathbf{1}_{\gamma = l^\Gamma} \hat{\alpha}^j_l y^*_{j\cdot} \right) + \textit{Terms}(\min(k, n-1)),$$

where $C_{\gamma+.}$ is a vector

$$C_{\gamma+.,k}=\left[\begin{array}{c}C_{\gamma+1^{\Gamma},k}\\ \dots\\ C_{\gamma+(d-1)^{\Gamma},k}\end{array}\right]\text{ and }y_s^{*\cdot}=\left[\begin{array}{c}y_s^{*1}\\ \dots\\ y_s^{*d-1}\end{array}\right].$$

Proof. Fix $k=0,...,n$. We work with the stability equations S^j for $j=1,...,d-1$. The equation has four terms. For each $\gamma\in\Gamma_{n-k}$, and each j, we evaluate the (γ,k)-derivatives of each of the terms of equation S^j at $\hat{m}$. First, notice that

$$\begin{aligned}&\frac{d^n}{dx^\gamma dy^k}\left(y^{*j}\beta_y\right)|_{\hat{m}}\\ &=\hat{\beta}_y\left(\frac{d^n}{dx^\gamma dy^k}y^{*j}|_{\hat{m}}\right)+y^{*j}\hat{\beta}_{\gamma,k+1}\\ &+k\left(\frac{dy^{*j}}{dy}|_{\hat{m}}\right)\hat{\beta}_{\gamma,k}+\sum_l\gamma_l\left(\frac{dy^{*j}}{dx_l}|_{\hat{m}}\right)\hat{\beta}_{\gamma-l^{\Gamma},k+1}+\text{Terms}\left(-1\right),\end{aligned}$$

where all the remaining terms contain products of variables of order smaller than n. Because of (B.1), the second term of the left-hand-side is equal to 0. Due to Lemma 10, the third term has rank -1 unless $k=n$ and the fourth term has rank -1 unless $k=n-1$ (the fourth term is equal to 0 when $k=n$). Using Lemma 10 and the fact $\hat{\beta}_l=0$, we obtain

$$\begin{aligned}&\frac{d^n}{dx^\gamma dy^k}\left(y^{*j}\beta_y\right)|_{\hat{m}}\\ &=\hat{\beta}_y\left(\hat{y}_b^{*j}\hat{\beta}_{\gamma,k}+\sum\nolimits_i\hat{y}_i^{*j}\hat{\alpha}^i_{\gamma,k}\right)+\mathbf{1}_{k=n}n\left(\hat{y}^{*,j,y}+\hat{y}_b^{*,j}\hat{\beta}_y+\sum\nolimits_l\hat{y}_l^{*,j}\hat{\alpha}^l_y\right)\hat{\beta}_{\varnothing,n}\\ &+\sum\nolimits_l\mathbf{1}_{\gamma=l^{\Gamma}}\left(\hat{y}^{*j,l}+\hat{y}_b^{*,j}\hat{\beta}_l+\sum\nolimits_i\hat{y}_i^{*j}\hat{\alpha}^i_l\right)\hat{\beta}_{\varnothing,n}+\text{Terms}\left(0\right)\\ &=\hat{\beta}_y\sum\nolimits_i\hat{y}_i^{*j}\hat{\alpha}^i_{\gamma,k}+\mathbf{1}_{k=n}\left(n\hat{y}^{*,j,y}+(n+1)\,\hat{y}_b^{*,j}\hat{\beta}_y+n\sum\nolimits_l\hat{y}_l^{*,j}\hat{\alpha}^l_y\right)\hat{\beta}_{\varnothing,n}\\ &+\sum\nolimits_l\mathbf{1}_{\gamma=l^{\Gamma}}\left(\hat{y}^{*j,l}+\sum\nolimits_i\hat{y}_i^{*j}\hat{\alpha}^i_l\right)\hat{\beta}_{\varnothing,n}+\text{Terms}\left(0\right)\end{aligned}\tag{B.4}$$

Notice that $\hat{\beta}_y\hat{y}_b^{*j}\hat{\beta}_{\gamma,k}$ is a term with rank -1 unless$k=n$, in which case it has rank $n-1$.

Similarly,

$$\begin{aligned}
&\frac{d^n}{dx^\gamma dy^k}\left(\sum\nolimits_i a_i^* \alpha_j^i\right)|_{\hat{m}} \\
&= \sum\nolimits_i \left(\frac{d^n}{dx^\gamma dy^k} a_i^*|_{\hat{m}}\right)\hat{\alpha}_j^i + \sum\nolimits_i a_i^* \hat{\alpha}^i_{\gamma+j^\Gamma,k} \\
&+ k\sum\nolimits_i \left(\frac{da_i^*}{dy}|_{\hat{m}}\right)\hat{\alpha}^i_{\gamma+j^\Gamma,k-1} + \sum_l \gamma_l \sum\nolimits_i \left(\frac{da_i^*}{dx_l}|_{\hat{m}}\right)\hat{\alpha}^i_{\gamma+j^\Gamma-l^\Gamma,k} + \text{Terms}\,(-1)
\end{aligned}$$

The second term disappears due to (B.1). Because $\hat{\alpha}^i_{\gamma+j^\Gamma,k-1}$ has rank $k-1$ and $\hat{\beta}_{\gamma,k}$ has rank -1 unless $k=n$, and because $\hat{\beta}_l = 0$, we get

$$\begin{aligned}
&\frac{d^n}{dx^\gamma dy^k}\left(\sum\nolimits_i a_i^* \alpha_j^i\right)|_{\hat{m}} \\
&= \sum\nolimits_{i,l} \hat{a}^*_{i,l}\hat{\alpha}^i_j \hat{\alpha}^l_{\gamma,k} + \mathbf{1}_{k=n}\sum\nolimits_i \hat{a}^*_{i,b}\hat{\alpha}^i_j\hat{\beta}_{\varnothing,n} \\
&+ \mathbf{1}_{k=n} n \sum\nolimits_i \left(\hat{a}_i^{*y} + \hat{a}^*_{i,b}\hat{\beta}_y + \sum\nolimits_l \hat{a}^*_{i,l}\hat{\alpha}^l_y\right)\hat{\alpha}^i_{j^\Gamma,n-1} \\
&+ \sum\nolimits_{i,l}\gamma_l\left(\hat{a}_i^{*l} + \sum\nolimits_t \hat{a}^*_{i,t}\hat{\alpha}^t_l\right)\hat{\alpha}^i_{\gamma+j^\Gamma-l^\Gamma,k} + \text{Terms}\,(\min(k,n-1))\,.
\end{aligned} \tag{B.5}$$

Finally,

$$\frac{d^n}{dx^\gamma dy^k}\left(y^{*j}\sum\nolimits_i a_i^*\alpha_y^i\right)|_{\hat{m}} = \text{Terms}\,(-1)\,, \tag{B.6}$$

because any term with a variable of nth order disappears since it also contains either $\hat{y}^{*j} = 0$ or $\hat{a}_i^* = 0$.

Adding (B.4), (B.5), and (B.6) to $\hat{\beta}_{\gamma+j^\Gamma,k}$, we obtain the (γ,k)-derivative of equation S^j:

$$\begin{aligned}
0 &= \frac{d^n}{dx^\gamma dy^k} S^j|_{\hat{m}} \\
&= \hat{\beta}_{\gamma+j^\Gamma,k} + \hat{\beta}_y \sum\nolimits_i \hat{y}_i^{*j}\hat{\alpha}^i_{\gamma,k} + \sum\nolimits_{i,l}\hat{a}^*_{i,l}\hat{\alpha}^i_j\hat{\alpha}^l_{\gamma,k} + \sum\nolimits_{i,l}\gamma_l\left(\hat{a}_i^{*l} + \sum\nolimits_t \hat{a}^*_{i,t}\hat{\alpha}^t_l\right)\hat{\alpha}^i_{\gamma+j^\Gamma-l^\Gamma,k} \\
&+ \sum\nolimits_l \mathbf{1}_{\gamma=l^\Gamma}\left(\hat{y}^{*j,l} + \sum\nolimits_i \hat{y}_i^{*j}\hat{\alpha}^i_l\right)\hat{\beta}_{\varnothing,n} \\
&+ \mathbf{1}_{k=n}\left[\begin{array}{c} \sum_i \hat{a}^*_{i,b}\hat{\alpha}^i_j\hat{\beta}_{\varnothing,n} + n\sum_i\left(\hat{a}_i^{*y} + \hat{a}^*_{i,b}\hat{\beta}_y + \sum_l \hat{a}^*_{i,l}\hat{\alpha}^l_y\right)\hat{\alpha}^i_{j^\Gamma,n-1} \\ + \left(n\hat{y}^{*,j,y} + (n+1)\,\hat{y}_b^{*,j}\hat{\beta}_y + n\sum_l \hat{y}_l^{*,j}\hat{\alpha}^l_y\right)\hat{\beta}_{\varnothing,n} \end{array}\right] \\
&+ \text{Terms}\,(\min(k,n-1))\,.
\end{aligned}$$

After some rearranging of terms, we get

$$\begin{aligned}
&\sum\nolimits_i \left(\hat{a}_i^{*j} - \hat{\beta}_y \hat{y}_i^{*j}\right) \hat{\alpha}_{\gamma,k}^i - \sum\nolimits_l \mathbf{1}_{\gamma=l^\Gamma} \sum\nolimits_i \hat{y}_i^{*j} \hat{\alpha}_l^i \hat{\beta}_{\varnothing,n} \\
&= \hat{\beta}_{k,\gamma+j^\Gamma} + \sum\nolimits_{i,l} \left(\gamma + j^\Gamma\right)_l \left(\hat{a}_i^{*l} + \sum\nolimits_t \hat{a}_{i,t}^* \hat{\alpha}_l^t\right) \hat{\alpha}_{\gamma+j^\Gamma - l^\Gamma,k}^i \\
&+ \sum\nolimits_l \mathbf{1}_{\gamma=l^\Gamma} \hat{y}^{*j,l} \hat{\beta}_{\varnothing,n} \\
&+ \mathbf{1}_{k=n} \left[\begin{array}{c} \sum_i \hat{a}_{i,b}^* \hat{\alpha}_j^i \hat{\beta}_{\varnothing,n} + n \sum_i \left(\hat{a}_i^{*y} + \hat{a}_{i,b}^* \hat{\beta}_y + \sum_l \hat{a}_{i,l}^* \hat{\alpha}_y^l\right) \hat{\alpha}_{j^\Gamma,n-1}^i \\ + \left(n\hat{y}^{*,j,y} + (n+1)\, \hat{y}_b^{*,j} \hat{\beta}_y + n \sum_l \hat{y}_l^{*,j} \hat{\alpha}_y^l\right) \hat{\beta}_{\varnothing,n} \end{array} \right] \\
&+ \text{Terms}\left(\min\left(k, n-1\right)\right).
\end{aligned}$$

Because of Lemma 8, $\hat{y}^{*j,l} = \hat{y}^{*l,j}$. It follows that the right-hand side of the above equality depends only on multi-index $\gamma + j^\Gamma$ (and not, like the left-hand side, separately on γ and j). We define the right-hand side as $C_{\gamma+j^\Gamma,k}$. From another part of Lemma 8, $\hat{w}^j = \left(\hat{a}_j^* - \hat{\beta}_y \hat{y}_a^{*j}\right)'$ is the jth row of matrix $\hat{W}^P$. The result follows. □

Lemma 12. *There exists* $\{D_\gamma, \gamma \in \Gamma_{n-1-k}\}$ *such that*

- *for each* $k = 0, ..., n-1$ *and* $\tau \in \Gamma_{n+1-k},$ *we have*

$$\sum_l \tau_l \hat{\alpha}_l' \Psi^P \hat{\alpha}_{\tau - l^\Gamma,k} = \sum\nolimits_{j,l} \tau_j \left(\tau - j^\Gamma\right)_l \Phi_l^{P,j} D_{\tau - j^\Gamma - l^\Gamma,k} + \textit{Terms}\,(k)\,,$$

- *for each* $l = 1, .., d-1,$ *we get*

$$\hat{\alpha}_l' \Psi^{(y)} \hat{\beta}_{\varnothing,n} + \hat{\alpha}_l' \Psi^P \hat{\alpha}_{\varnothing,n} + n\left(\hat{\beta}_y \Psi^{(y)\prime} + \alpha_y' \Psi^P\right) \hat{\alpha}_{l^\Gamma,n-1} = \Phi_l^{(y)} D_{\varnothing,n-1} + \textit{Terms}\,(n-1)\,,$$

- *finally,*

$$\hat{\beta}_y \Psi^{(0)} \hat{\beta}_{\varnothing,n} + \hat{\alpha}_y' \Psi^{(y)} \hat{\beta}_{\varnothing,n} + \hat{\beta}_y \Psi^{(y)\prime} \hat{\alpha}_{\varnothing,n} + \hat{\alpha}_y' \Psi^P \hat{\alpha}_{\varnothing,n} = \Phi^{(0)} D_{\varnothing,n-1} + \textit{Terms}\,(n-1)$$

Proof. Let

$$B\left(qx,q\right) = \left(\beta\left(qx,q\right)\right)^2 \Psi^{(0)} + 2\beta\left(qx,q\right)\alpha\left(qx,q\right)\Psi^{(y)} + \alpha'\left(qx,q\right)\Psi^{(P)}\alpha\left(qx,q\right).$$

For each $x \in R^d$, we compute the $(n+1)$th derivative of $B(qx, x)$ with respect to q at $q = 0$:

$$\begin{aligned}
&\frac{d^{n+1}}{dq^{n+1}} B(qx, x)|_{q=0} \\
&= 2(n+1)\left(\hat{\beta}_y + \sum_l \hat{\beta}_l x_l\right) \Psi^{(0)} \left(\sum_{k=0}^{n} \sum_{\gamma \in \Gamma_{n-k}} \frac{n!}{k!\gamma!} \hat{\beta}_{\gamma,k} x^{\gamma}\right) \\
&+ 2(n+1)\left(\sum_l \hat{\alpha}_l x_l + \hat{\alpha}_y\right)' \Psi^{(y)} \left(\sum_{k=0}^{n} \sum_{\gamma \in \Gamma_{n-k}} \frac{n!}{k!\gamma!} \hat{\beta}_{\gamma,k} x^{\gamma}\right) \\
&+ 2(n+1)\left(\hat{\beta}_y + \sum_l \hat{\beta}_l x_l\right) \Psi^{(y)\prime} \sum_{k=0}^{n} \sum_{\gamma \in \Gamma_{n-k}} \frac{n!}{k!\gamma!} \hat{\alpha}_{\gamma,k} x^{\gamma} \\
&+ 2(n+1)\left(\alpha_y' + \sum_l \hat{\alpha}_l' x_l\right) \Psi^{P} \sum_{k=0}^{n} \sum_{\gamma \in \Gamma_{n-k}} \frac{n!}{k!\gamma!} \hat{\alpha}_{\gamma,k} x^{\gamma} \\
&+ \text{Terms}(-1)
\end{aligned}$$

In particular, all other terms contain only variables of an order lower than n. Due to Lemma 10 and the fact that $\hat{\beta}_l = 0$ for each $l = 1, ..., d-1$, after some rearranging of terms, we obtain:

$$\begin{aligned}
&\frac{d^{n+1}}{dq^{n+1}} B(qx, x)|_{q=0} \\
&= 2(n+1)\hat{\beta}_y \Psi^{(0)} \hat{\beta}_{\varnothing,n} \\
&+ 2(n+1)\left(\sum_l \hat{\alpha}_l x_l + \hat{\alpha}_y\right)' \Psi^{(y)} \hat{\beta}_{\varnothing,n} \\
&+ 2(n+1)\hat{\beta}_y \Psi^{(y)\prime} \sum_{k=0}^{n} \sum_{\gamma \in \Gamma_{n-k}} \frac{n!}{k!\gamma!} \hat{\alpha}_{\gamma,k} x^{\gamma} \\
&+ 2(n+1)\left(\alpha_y' + \sum_l \hat{\alpha}_l' x_l\right) \Psi^{P} \sum_{k=0}^{n} \sum_{\gamma \in \Gamma_{n-k}} \frac{n!}{k!\gamma!} \hat{\alpha}_{\gamma,k} x^{\gamma} \\
&+ \text{Terms}(0).
\end{aligned}$$

Further, due to equations (B.1), we obtain:

$$\frac{d^{n+1}}{dq^{n+1}}B\left(qx,x\right)|_{q=0}$$
$$=2\left(n+1\right)\left[\hat{\beta}_y\Psi^{(0)}\hat{\beta}_{\varnothing,n}+\hat{\alpha}_y'\Psi^{(y)}\hat{\beta}_{\varnothing,n}+\hat{\beta}_y\Psi^{(y)\prime}\hat{\alpha}_{\varnothing,n}+\hat{\alpha}_y'\Psi^{P}\hat{\alpha}_{\varnothing,n}\right]$$
$$+2\left(n+1\right)\sum_l\left[\hat{\alpha}_l'\Psi^{(y)}\hat{\beta}_{\varnothing,n}+\hat{\alpha}_l'\Psi^{P}\hat{\alpha}_{\varnothing,n}+n\left(\hat{\beta}_y\Psi^{(y)\prime}+\alpha_y'\Psi^{P}\right)\hat{\alpha}_{l^\Gamma,n-1}\right]x_l$$
$$+2\left(n+1\right)\sum_{k=0}^{n-1}\sum_{\tau\in\Gamma_{n+1-k}}\frac{n!}{k!\tau!}\left[\sum_l\tau_l\hat{\alpha}_l'\Psi^{P}\hat{\alpha}_{\tau-l^\Gamma,k}+\mathbf{1}_{k>0}\left(k-1\right)\left(\hat{\beta}_y\Psi^{(y)\prime}+\alpha_y'\Psi^{P}\right)\hat{\alpha}_{\tau,k-1}\right]x^\tau$$
$$+\text{Terms}\left(0\right). \tag{B.7}$$

The right-hand side of (B.7) is a polynomial of order $n+1$ in x. Due to the boundary conditions, (B.7) is equal to 0 for for each x such that (B^0) holds. Because the second-order polynomial defining (B^0), $\Phi^{(0)}+2x'\Phi^{(y)}+x'\Phi^{(P)}x$, is irreducible, it must be that it divides polynomial (B.7). In other words, for each x (and not only those x that satisfy (B^0)), (B.7) is equal to

$$\left(\Phi^{(0)}+2x'\Phi^{(y)}+x'\Phi^{P}x\right)\sum_{k=0}^{n-1}\sum_{\gamma\in\Gamma_{n+1-k}}\frac{1}{\gamma!}d_\gamma x^\gamma$$
$$=\sum_{k=0}^{n-1}\sum_{\tau\in\Gamma_{n+1-k}}\frac{1}{\tau!}\left(\sum\nolimits_{j,l}\tau_j\left(\tau-j^\Gamma\right)_l\Phi_l^{P,j}d_{\tau-j^\Gamma-l^\Gamma}+\sum\nolimits_j\tau_j\Phi_j^{(y)}d_{\tau-j^\Gamma}+\Phi^{(0)}d_\tau\right)x^\tau$$
$$+\sum_l\left(\Phi_l^{(y)}d_\varnothing+\Phi^{(0)}d_l\right)x_l$$
$$+\Phi^{(0)}d_\varnothing.$$

for some real coefficients $\{d_\gamma,\gamma\in\Gamma_0\cup...\cup\Gamma_{n-1}\}$.

Two polynomials are equal if and only if the coefficients associated with each of the monomials x^τ are equal. The result follows from substitution $D_{\varnothing,n-1}=\frac{1}{2(n+1)}d_\varnothing$ and for each $k=1,...,,n-1$ and $\gamma\in\Gamma_{n-1-k}$,

$$D_{\gamma,k}=\frac{k!}{2\left(n+1\right)!}d_\gamma.$$

□

Lemma 13. *For each $k \leq n-1$ and $\gamma \in \Gamma_{n-1-k}$,*

$$\hat{\beta}_y \sum_l \left(\hat{S}^{-1}\right)_{lth\ row} \hat{W}^P \alpha_{\gamma+l^\Gamma,k}$$
$$= -\mathbf{1}_{k=n-1}\left(\sum\nolimits_l (-1)^{d+l} \frac{1}{(\det \hat{\alpha}_x)} \hat{\beta}_{l^\Gamma,n-1}\left(\hat{D}\mu_{-l}^{-d}\right) + \beta_{\varnothing,n}\right) + \mathit{Terms}\,(k)\,,$$

Proof. We compute the (k,γ)-derivative of the matching equation M at the top match for $\gamma \in \Gamma_{n-1-k}$. Notice first that

$$\det \begin{bmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{bmatrix} = \beta_y \det \alpha_x + \sum\nolimits_l (-1)^{d+l} \beta_l \left(\hat{D}\mu_{-l}^{-d}\right),$$

where $\hat{D}\mu_{-l}^{-d}$ is the cofactor of matrix $\hat{D}\mu$.

Using Jacobi's formula for the derivative of a determinant, we obtain

$$\frac{d^{n-1}}{dx^\gamma dy^k}\left(\beta_y \det \alpha_x\right)$$
$$= (\det \hat{\alpha}_x)\,\beta_{\gamma,k+1} + (\det \hat{\alpha}_x)\,\hat{\beta}_y \operatorname{tr}\left(\hat{\alpha}_x^{-1} \frac{d^{n-1}}{dy^k dx^\gamma}\alpha_x\right)$$
$$= \mathbf{1}_{k=n-1}\,(\det \hat{\alpha}_x)\,\beta_{\varnothing,n} + (\det \hat{\alpha}_x)\,\hat{\beta}_y \operatorname{tr}\left(\hat{\alpha}_x^{-1} \frac{d^{n-1}}{dy^k dx^\gamma}\alpha_x\right) + \text{Terms}\,(0)\,.$$

Using (6.12), we obtain

$$\operatorname{tr}\left(\hat{\alpha}_x^{-1} \frac{d^{n-1}}{dy^k dx^\gamma}\alpha_x\right) = \sum\nolimits_l \left[\hat{\alpha}_x^{-1}\right]_{l\text{th row}} \alpha_{\gamma+l^\Gamma,k}$$
$$= \sum\nolimits_l \left(\hat{S}^{-1}\right)_{l\text{th row}} \hat{W}^P \alpha_{\gamma+l^\Gamma,k}.$$

Because $\hat{\beta}_l = 0$, we get

$$\frac{d}{dx^\gamma y^k} \sum\nolimits_l (-1)^{d+l} \beta_l \left(\hat{D}\mu_{-l}^{-d}\right)$$
$$= \mathbf{1}_{k=n-1} \sum\nolimits_l (-1)^{d+l} \hat{\beta}_{l^\Gamma,n-1}\left(\hat{D}\mu_{-l}^{-d}\right) + \text{Terms}\,(-1)\,,$$

where $\left(\hat{D}\mu_{-l}^{-d}\right)$ is the dl-cofactor of matrix $\hat{D}\mu$ obtained from $\begin{bmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{bmatrix}$ by crossing out dth row and lth column.

Finally,

$$\begin{aligned}0 &= \frac{d^{n-1}}{dx^{\gamma}dy^{k}}\left(\det\begin{bmatrix}\alpha_x & \alpha_y\\ \beta_x & \beta_y\end{bmatrix} - g\right)\\ &= (\det\hat{\alpha}_x)\,\hat{\beta}_y\sum\nolimits_l\left(\hat{S}^{-1}\right)_{l\text{th row}}\hat{W}^P\alpha_{\gamma+l^\Gamma,k}+\\ &\mathbf{1}_{k=n-1}\left[(\det\hat{\alpha}_x)\,\beta_{\varnothing,n}+\sum\nolimits_l(-1)^{d+l}\,\hat{\beta}_{l^\Gamma,n-1}\left(\hat{D}\mu_{-l}^{-d}\right)\right].\end{aligned}$$

□

B.6. **Equations of rank $k < n-1$.** Fix $k < n-1$ and consider the following system of equations:

$$\hat{W}^P\hat{\alpha}_{\gamma,k} = C_{\gamma+.,k} \text{ for each } \gamma \in \Gamma_{n-k}, \tag{B.8}$$

$$\sum_l \tau_l \hat{\alpha}_l' \Psi^P \hat{\alpha}_{\tau-l^\Gamma,k} = \sum\nolimits_{j,l}\tau_j\left(\tau - j^\Gamma\right)_l \Phi_l^{P,j} D_{\tau-j^\Gamma-l^\Gamma,k} \text{ for each } \tau \in \Gamma_{n+1-k}, \tag{B.9}$$

$$\hat{\beta}_y\sum_l\left(\hat{S}^{-1}\right)_{l\text{th row}}\hat{W}^P\alpha_{\gamma+l^\Gamma} = 0 \text{ for each } \gamma \in \Gamma_{n-1-k}. \tag{B.10}$$

The variables of rank k are the unknowns. By Lemma 9, the linear coefficients of the equations are analytic functions of the parameters.

In this part of the Appendix, we show that there exist parameters such that the above system of equations has a unique solution.

Lemma 14. *Suppose that $\mathcal{M}_{mw}^{\hat{}} = \mathcal{W}_{mw}^{\hat{}} = \frac{1}{2}I_{d-1}$, $\hat{g} = 1$, and*

$$\Psi = \Phi = \begin{bmatrix}\varepsilon & 0 & 0 & \ldots & 0 & 0\\ 0 & \varepsilon^2 & 0 & \ldots & 0 & 0\\ 0 & 0 & \varepsilon^3 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & \varepsilon^{d-1} & 0\\ 0 & 0 & 0 & \ldots & 0 & -\varepsilon\end{bmatrix}.$$

Then, for sufficiently small $\varepsilon > 0$, the system of equations (B.8)-(B.10) has a unique solution with all the variables of rank k equal to 0.

Proof. First, we compute the values of the first order derivatives. By equations (6.10), and (6.15),

$$\left(\hat{g}^2 \frac{\det \Psi}{\det \Phi}\right)^{-\frac{1}{d}} = \hat{\theta} = 1, \hat{W}^P = \left(\hat{S}^*\right)^{-1} = I_{d-1}. \tag{B.11}$$

Moreover, by Lemma 9

$$\hat{\beta}_y = 1, \hat{\alpha}_x = I_{d-1}, \text{ and } \hat{\beta}_x = \hat{\alpha}'_y = \mathbf{0}. \tag{B.12}$$

Second, using the above computations, we can restate equations (B.8)-(B.10) as

$$\hat{\alpha}^j_{\gamma,k} = C_{\gamma+j^\Gamma,k} \text{ for each } \gamma \in \Gamma_{n-k} \text{ and } j,$$

$$C_{\tau,k} = \left(\sum\nolimits_j \tau_j \varepsilon^j\right)^{-1} \left(\sum_j \tau_j (\tau_j - 1) \varepsilon^j D_{\tau-2\cdot j^\Gamma}\right) \text{ for each } \tau \in \Gamma_{n+1-k},$$

$$\sum_l C_{\gamma+2l^\Gamma,k} = 0 \text{ for each } \gamma \in \Gamma_{n-1-k}.$$

The last two equations imply that, for each $\gamma \in \Gamma_{n-1-k}$,

$$\sum_l \frac{(\gamma_l + 2)(\gamma_l + 1)\varepsilon^l D_\gamma + \sum_{j \neq l} \gamma_j (\gamma_j - 1)\varepsilon^j D_{\gamma+2l^\Gamma-2\cdot j^\Gamma}}{(\gamma_l + 2)\varepsilon^l + \sum_{j \neq l} \gamma_j \varepsilon^j} = 0. \tag{B.13}$$

For each $\gamma \in \Gamma_{n-1-k}$, let $l^*(\gamma) = \min\{i : \gamma_i > 0\}$. The left-hand side of equation (B.13) is equal to

$$= \sum_{l \leq l^*(\gamma)} \left(\frac{(\gamma_l + 2)(\gamma_l + 1)}{(\gamma_l + 2) + \sum_{j>l} \gamma_j \varepsilon^{j-l}} D_\gamma + \frac{\sum_{j>l} \gamma_j (\gamma_j - 1)\varepsilon^{j-l} D_{\gamma+2l^\Gamma-2\cdot j^\Gamma}}{(\gamma_l + 2) + \sum_{j>l} \gamma_j \varepsilon^{j-l}} \right)$$
$$+ \sum_{l > l^*(\gamma)} \left(\begin{array}{c} \frac{\gamma_{l^*(\gamma)}\left(\gamma_{l^*(\gamma)}-1\right)}{\gamma_{l^*(\gamma)}+\sum_{j>l^*(\gamma)} \gamma_j \varepsilon^{j-l^*(\gamma)} + 2\varepsilon^{l-l^*(\gamma)}} D_{\gamma+2l^\Gamma-2\cdot l^*(\gamma)^\Gamma} \\ + \frac{\gamma \sum_{j>l^*(\gamma)} \gamma_j(\gamma_j-1)\varepsilon^{j-l^*(\gamma)} D_{\gamma+2l^\Gamma-2\cdot j^\Gamma}}{\gamma_{l^*(\gamma)}+\sum_{j>l^*(\gamma)} \gamma_j \varepsilon^{j-l^*(\gamma)} + 2\varepsilon^{l-l^*(\gamma)}} \end{array} \right)$$
$$\to \left(l^*(\gamma) + \gamma_{l^*(\gamma)}\right) D(\gamma) + \left(\gamma_{l^*(\gamma)} - 1\right) \sum_{l > l^*(\gamma)} D_{\gamma+2l^\Gamma-2\cdot l^*(\gamma)^\Gamma},$$

where the convergence holds for $\varepsilon \to 0$.

Third, we claim that the system of equations

$$\left(l^*(\gamma)+\gamma_{l^*(\gamma)}\right)D(\gamma)+\left(\gamma_{l^*(\gamma)}-1\right)\sum_{l>l^*(\gamma)}D_{\gamma+2l^\Gamma-2\cdot l^*(\gamma)^\Gamma}=0 \tag{B.14}$$

for each $\gamma\in\Gamma_{n-1-k}$, has a unique solution. Indeed, define an ordering on Γ_{n-1-k} : $\gamma<\gamma'$ if and only if

$$\lim_{\varepsilon\to 0}\frac{\sum\gamma_i\varepsilon^i}{\sum\gamma_i'\varepsilon^i}\to 0.$$

Then, (B.14) implies that each $D(\gamma)$ can be represented as a function of $D(\gamma')$ for $\gamma'<\gamma$. The claim follows.

By continuity, the system of equations (B.13) for each $\gamma\in\Gamma_{n-1-k}$ has a unique solution for sufficiently small $\varepsilon>0$. The result follows. □

B.7. **Equations of order** n **and rank** $n-1$. Consider the following system of equations:

$$\hat{W}^P\hat{\alpha}_{l,n-1}=C_{l^\Gamma+\cdot,n-1}+\hat{\beta}_{\varnothing,n}\left(\sum_j\hat{\alpha}_l^j y_j^{*\cdot}\right)\text{ for each } l, \tag{B.15a}$$

$$\sum_l\tau_l\hat{\alpha}_l'\Psi^P\hat{\alpha}_{\tau-l^\Gamma,k}=\sum\nolimits_{j,l}\tau_j\left(\tau-j^\Gamma\right)_l\Phi_l^{P,j}D_{\tau-j^\Gamma-l^\Gamma,k}\text{ for each }\tau\in\Gamma_2, \tag{B.15b}$$

$$\hat{\alpha}_l'\Psi^{(y)}\hat{\beta}_{\varnothing,n}+\hat{\alpha}_l'\Psi^P\hat{\alpha}_{\varnothing,n}+n\left(\hat{\beta}_y\Psi^{(y)\prime}+\alpha_y'\Psi^P\right)\hat{\alpha}_{l^\Gamma,n-1}=\Phi_l^{(y)}D_{\varnothing,n-1}\text{ for each } l, \tag{B.15c}$$

$$\hat{\beta}_y\Psi^{(0)}\hat{\beta}_{\varnothing,n}+\hat{\alpha}_y'\Psi^{(y)}\hat{\beta}_{\varnothing,n}+\hat{\beta}_y\Psi^{(y)\prime}\hat{\alpha}_{\varnothing,n}+\hat{\alpha}_y'\Psi^P\hat{\alpha}_{\varnothing,n}=\Phi^{(0)}D_{\varnothing,n-1}, \tag{B.15d}$$

$$\hat{\beta}_y\sum_l\left(\hat{S}^{-1}\right)_{l\text{th row}}\hat{W}^P\alpha_{l^\Gamma,n-1}=-\left(\sum\nolimits_l(-1)^{d+l}\frac{1}{(\det\hat{\alpha}_x)}\hat{\beta}_{l^\Gamma,n-1}\left(\hat{D}\mu_{-l}^{-d}\right)+\beta_{\varnothing,n}\right)+\text{Terms}(k). \tag{B.15e}$$

The variables of rank $n-1$ are the unknowns. The linear coefficients are analytic functions of parameters $\left(\mathcal{M}_{mw}^{\hat{}},\mathcal{W}_{mw}^{\hat{}},\Phi,\Psi,\hat{g}\right)$.

Lemma 15. *Suppose that* $\mathcal{M}_{mw}^{\hat{}}=\mathcal{W}_{mw}^{\hat{}}=\frac{1}{2}I_{d-1},\Psi=\Phi=I_{d-1}$, *and* $\hat{g}=1$. *Then, the system of equations (B.15a)-(B.15e) has a unique solution with all the variables of rank* $n-1$ *equal to 0.*

Proof. Because of equations (6.10), and (6.15),

$$\left(\hat{g}^2 \frac{\det \Psi}{\det \Phi}\right)^{-\frac{1}{d}} = \hat{\theta} = 1, \hat{W}^P = \left(\hat{S}^*\right)^{-1} = I_{d-1}.$$

From Lemma 9,

$$\hat{\beta}_y = 1, \hat{\alpha}_x = I_{d-1}, \text{ and } \hat{\beta}_x = \hat{\alpha}'_y = \mathbf{0}.$$

Because $\hat{\beta}_x = 0$, $\hat{D}\mu_{-l}^{-d} = 0$ for each $l = 1, .., d-1$. Finally, by Lemma 8, $\left[y_j^{*l}\right] = -\frac{1}{2} I_{d-1}$.

Thus, the system of equations (B.15a)-(B.15e) can be rewritten as

$$\hat{\alpha}_{l,n-1}^{j} = \begin{cases} C_{2l^{\Gamma},n-1} - \frac{1}{2}\hat{\beta}_{\varnothing,n} & \text{if } l = j \\ C_{l^{\Gamma}+j^{\Gamma},n-1} & \text{if } l \neq j \end{cases} \quad \text{for each } l,$$

$$\hat{\alpha}_{j^{\Gamma},n-1}^{l} + \hat{\alpha}_{l^{\Gamma},n-1}^{j} = \begin{cases} 2D_{\varnothing,n-1}, & \text{if } l = j \\ 0, & \text{if } l \neq j \end{cases} \quad \text{for each } l, j,$$

$$\hat{\alpha}_{\varnothing,n}^{l} = 0 \text{ for each } l,$$

$$\hat{\beta}_{\varnothing,n} = D_{\varnothing,n-1},$$

$$\sum_l \alpha_{l^{\Gamma},n-1}^{l} + \beta_{\varnothing,n} = 0, \text{ for each } l.$$

Substituting the first and the fourth equation to the second, we get

$$C_{2l^{\Gamma},n-1} = \frac{3}{2} D_{\varnothing,n-1} \text{ for each } l, \text{ and } C_{l^{\Gamma}+j^{\Gamma},n-1} = 0 \text{ for each } l \neq j.$$

Together with the last equation, this implies that

$$0 = \sum_l \left(C_{2l^{\Gamma},n-1} - \frac{1}{2}\hat{\beta}_{\varnothing,n}\right) + \beta_{\varnothing,n} = (d-1)\, D_{\varnothing,n-1} + D_{\varnothing,n-1} = dD_{\varnothing,n-1},$$

Therefore, $D_{\varnothing,n-1} = 0$ and all other variables can be uniquely determined from the above equations. This ends the proof. □

B.8. **Proof of Theorem 1.** Let $P_d^{\hat{}}$ be the space of linear mappings $\phi : R^d \to R^d$ such that $\left(\phi^{-1}(m)\right)_d < \mathbf{0}$ for each non-apex element of the standard cone $m \in E^* \backslash \{\mathbf{0}\}$. Define $\mathcal{P}^{\hat{}} = M_d^2 \times P_d^2 \times R_+$. Then, $\mathcal{P}^{\hat{}}$ is a convex subset of Euclidean space and it can be equipped with a Lesbegue measure $\lambda^{\hat{}} \in \Delta\mathcal{P}^{\hat{}}$. We show that there exists subset $\mathcal{P}_0^{\hat{}} \subseteq \mathcal{P}^{\hat{}}$ such that $\lambda^{\hat{}}\left(\mathcal{P}^{\hat{}} \backslash \mathcal{P}_0^{\hat{}}\right) = 0$, and if (a) $\mathcal{M}_{mw}^{\hat{}} = \mathcal{W}_{mw}^{\hat{}} = (0, ...0, 1)$, (b) $\left(\mathcal{M}_{mw}^{\hat{}}, \mathcal{W}_{mw}^{\hat{}}, \phi, \psi, \hat{g}\right) \in \mathcal{P}_0^{\hat{}}$, and (c) E_M and E_W are cones generated by proper linear operators ϕ and ψ, then there is at most one smooth stable matching $\mu : E_M \to E_W$. Theorem 1 follows as a straightforward corollary.

Let $\mathcal{P}^{\hat{}}_{0,1}$ be a set of parameters such that the weighted average matrix of cross-derivatives $\hat{W}^P$ is invertible.

For each $n \geq 2$, for each $k < n-1$, let $\mathcal{P}^{\hat{}}_{n-k,n} \subseteq \mathcal{P}^{\hat{}}$ denote the set of parameters such that the system of equations (B.8)-(B.10) has a unique solution. By Lemma 14, $\mathcal{P}^{\hat{}}_{n-k,n}$ is non-empty. By Lemma 7, $\lambda^{\hat{}}\left(\mathcal{P}^{\hat{}} \backslash \mathcal{P}^{\hat{}}_{n-k,n}\right) = 0$. Similarly, let $\mathcal{P}^{\hat{}}_{n-1,n} \subseteq \mathcal{P}^{\hat{}}$ denote the set of parameters such that the system of equations (B.15a)-(B.15e) has a unique solution. By Lemma 15, $\mathcal{P}^{\hat{}}_{n-1,n}$ is non-empty. By Lemma 7, $\lambda^{\hat{}}\left(\mathcal{P}^{\hat{}} \backslash \mathcal{P}^{\hat{}}_{n-k,n}\right) = 0$.

Let

$$\mathcal{P}^{\hat{}}_0 = \bigcap_{n \geq 1} \bigcap_{k=0}^{n-1} \hat{\mathcal{P}}_{n-k,n}.$$

Then, $\lambda^{\hat{}}\left(\mathcal{P}^{\hat{}} \backslash \mathcal{P}^{\hat{}}_0\right) = 0$. Moreover, for any vector of parameters in $\mathcal{P}^{\hat{}}_0$, there exists a unique collection of variables that solves equations (B.8)-(B.10) and (B.15a)-(B.15e). By Lemmas 9, 10, 11, and 12, there exists at most one analytic function μ that satisfies equations M, S^j for each $j = 1, ..., d-1$, and $B(qx, q) = 0$ for each q and each x such that (B^0) holds. It follows that there exists at most one smooth stable matching $\mu : E_M \to E_W$.

Appendix C. Examples - calculations

C.1. **Two-dimensional examples.** In this section, we assume that $d = 2$. We compute the Taylor expansion of the unique stable matching up to its second-order derivatives.

C.1.1. *Environment.* We assume that the utility functions are equal to

$$\mathcal{M}\left(\left(m_s, m_b\right), \left(w_s, w_b\right)\right) = Cw_s + w_b + D\left(m_s w_s + m_b w_b\right),$$

$$\mathcal{W}\left(\left(m_s, m_b\right), \left(w_s, w_b\right)\right) = m_s + m_b + m_s w_s + m_b w_b$$

for some $C > 0$ and D.

We assume that the domains are equal to

$$E_M = \{(m_s, m_b) : m_s, m_b \leq 0\},$$
$$E_W = \{(w_s, w_b) : w_s, w_b \leq 0\}.$$

Finally, we assume that the density ratio is constant $g(m, w) \equiv 1$.

C.1.2. *Change of coordinates.* As in section 6.2, we multiply the utility functions by a positive scalar so that the normal vectors have unit length. Additionally, it is convenient to choose the coordinates so that the normal vectors to the indifference curves computed at the top man $\hat{m}$ and woman $\hat{w}$ are equal to $(0, 1)$. For this purpose, we define rotation matrices

$$O_M = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } O_W = \frac{1}{\sqrt{C^2+1}} \begin{bmatrix} 1 & C \\ -C & 1 \end{bmatrix},$$

and consider the following change of coordinates

$$O_M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m_s \\ m_b \end{bmatrix} \text{ and } O_W \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} w_s \\ w_b \end{bmatrix}.$$

The old and normalized preferences expressed in the new coordinates are equal to

$$\mathcal{M}^n((x, y), (a, b)) = \frac{1}{\sqrt{C^2+1}} \mathcal{M}\left(O_M \begin{bmatrix} x \\ y \end{bmatrix}, O_W \begin{bmatrix} a \\ b \end{bmatrix}\right)$$
$$= \frac{1}{\sqrt{C^2+1}} \begin{bmatrix} C & 1 \end{bmatrix} O_W \begin{bmatrix} a \\ b \end{bmatrix} + [x, y] \frac{1}{\sqrt{C^2+1}} O_M' \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} O_W \begin{bmatrix} a \\ b \end{bmatrix}$$
$$= b + [x, y] \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix},$$
$$\mathcal{W}^n((x, y), (a, b)) = \frac{\sqrt{2}}{2} \mathcal{W}((x, y), (a, b))$$
$$= y + [x, y] \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix},$$

where

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = D\frac{\sqrt{2}}{2\left(C^2+1\right)}\begin{bmatrix} C+1 & C-1 \\ 1-C & 1+C \end{bmatrix},$$
$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \frac{1}{2\sqrt{C^2+1}}\begin{bmatrix} C+1 & C-1 \\ 1-C & 1+C \end{bmatrix}.$$

Thus, consistent with the normalization, $\mathcal{M}_w^{O\hat{}} = \mathcal{W}_m^{O\hat{}} = (0,1)$. From now on, we drop the superscript "n" when we refer to the preferences expressed in the new coordinates.

We can express the domains in new coordinates. Let

$$\begin{aligned} \phi^0 &= 1 > \phi^1 = -1, \\ \psi^0 &= \frac{1}{C} > \psi^1 = -C. \end{aligned} \tag{C.1}$$

Then,

$$\begin{aligned} E_M &= \left\{m_s(1,0) + m_b(0,1) : m_s, m_b \leq 0\right\} \\ &= \left\{m_s O_M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + m_b O_M^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} : m_s, m_b \leq 0\right\} \\ &= \left\{m_s\begin{bmatrix} 1 \\ 1 \end{bmatrix} + m_b\begin{bmatrix} -1 \\ 1 \end{bmatrix} : m_s, m_b \leq 0\right\} \\ &= \left\{(x,y) : y \leq 0 \text{ and } \phi^0 y \leq x \leq \phi^1 y\right\}, \\ E_W &= \left\{(a,b) : b \leq 0 \text{ and } \psi^0 b \leq a \leq \psi^1 b\right\}. \end{aligned}$$

The densities of the distribution of men and women is not affected by the orthonormal change of coordinates O_M and O_W.

In order to distinguish the first and the second coordinates of the matching function, we write $\mu(y,x) = (\alpha(x,y), \beta(x,y))$.

Define functions y^* and a^* as

$$\begin{aligned} y^*((x,y),(a,b)) &= -\frac{\left(\mathcal{W}_m((x,y),(a,b))\right)_1}{\left(\mathcal{W}_m((x,y),(a,b))\right)_2} = -\frac{w_{11}a + w_{12}b}{1 + w_{21}a + w_{22}b}, \\ a^*((x,y),(a,b)) &= \frac{\left(\mathcal{M}_w((x,y),(a,b))\right)_1}{\left(\mathcal{M}_w((x,y),(a,b))\right)_2} = \frac{m_{11}x + m_{21}y}{1 + m_{12}x + m_{22}y}. \end{aligned}$$

Then, vector $(1, y^*)$ is tangent to woman w's indifference curves at the point of the match, and vector $(a^*, 1)$ is a scalar multiple of the normal vector $\mathcal{M}_w$. Notice that the values and the derivatives of functions y^* and a^* computed that the top match are equal to

$$\begin{aligned}
&\hat{y}^* = 0,\ \hat{a}^* = 0,\\
&\hat{y}^*_x = \hat{y}^*_y = \hat{a}^*_a = \hat{a}^*_b = 0,\\
&\hat{y}^*_a = -w_{11}, \hat{y}^*_b = -w_{12}, \hat{a}^*_x = m_{11}, \hat{a}^*_y = m_{21},
\end{aligned} \tag{C.2}$$

C.1.3. *Differential equations.* We denote the first-order derivative of α with respect to the second coordinate by α_x and the value of this derivative computed at the point of top match $\hat{m} = (0, 0)$ by $\hat{\alpha}_x$. A similar notation is used for other derivatives. The stable matching satisfies the following equations:

- the matching equation (3.2):

$$|\alpha_x \beta_y - \alpha_y \beta_x| = 1, \tag{C.3}$$

- the stability equation (4.2):

$$\begin{aligned}
&\begin{bmatrix} a^* & 1 \end{bmatrix} \begin{bmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{bmatrix} \begin{bmatrix} 1 \\ y^* \end{bmatrix}\\
&= \beta_x + y^* \beta_y + a^* \alpha_x + y^* x^* \alpha_y = 0,
\end{aligned} \tag{C.4}$$

- the boundary conditions: By the discussion in section 6.5, the mapping of boundaries depends on the orientation of the matching in the neighborhood of the top match, which in turns depends on the sign of the weighted average $\hat{W}^P$ of the cross-derivatives $\hat{\mathcal{M}}^P_{mw}$ and $\hat{\mathcal{W}}^P_{mw}$. Here, $\hat{W}^P = m_{11} + \hat{\theta} w_{11}$, and we assume that $\hat{W}^P \neq 0$. Let $o \in \{0, 1\}$ denote the orientation. There two cases: (a) if $\hat{W}^P > 0$, then the orientation is preserved and $o = 0$ (b) if $\hat{W}^P < 0$, then the orientation is reversed, and $o = 1$. Also, let

  $$\tilde{\psi}^z = \psi^{z + o \bmod 2} \text{ for each } z = 0, 1$$

  Then, the boundary conditions can be written as

  $$\tilde{\psi}^z \beta\left(\phi^z y, y\right) = \alpha\left(\phi^z y, y\right) \text{ for each } z = 0, 1. \tag{C.5}$$

C.1.4. *First-order derivatives.* We compute the values of the first-order derivatives of stable matching μ. Because of (C.2), the stability equation (C.4) evaluated at the top match implies that

$$\hat{\beta}_x = 0.$$

Differentiating the boundary conditions (C.5) and using the fact that $\hat{\beta}_x = 0$, we obtain

$$\begin{aligned}\tilde{\psi}^0 \hat{\beta}_y &= \phi^0 \hat{\alpha}_x + \hat{\alpha}_y,\\ \tilde{\psi}^1 \hat{\beta}_y &= \phi^1 \hat{\alpha}_x + \hat{\alpha}_y.\end{aligned}$$

Together with (C.1), these equations imply that

$$\begin{aligned}\hat{\alpha}_x &= \frac{1}{2}\left(\tilde{\psi}^0 - \tilde{\psi}^1\right)\hat{\beta}_y = (-1)^o \frac{C + C^{-1}}{2}\hat{\beta}_y,\\ \hat{\alpha}_y &= \frac{1}{2}\left(\tilde{\psi}^0 + \tilde{\psi}^1\right)\hat{\beta}_y = \frac{C^{-1} - C}{2}\hat{\beta}_y,\end{aligned}$$

Finally, the matching equation (C.3) implies that

$$\hat{\beta}_y = \sqrt{\frac{2}{C + C^{-1}}}.$$

(Note that $\hat{\beta}_y > 0$ by the first part of Lemma 1.) By equation (6.11), $\hat{\beta}_y = \hat{\theta}$.

Using the definition of matrix $\hat{W}^P$ as well as discussion in section 6.5, we can determine the orientation of the matching:

$$\begin{aligned}o = 0 \text{ if } D &> -\sqrt{\frac{2}{C + C^{-1}}},\\ o = 1 \text{ if } D &< -\sqrt{\frac{2}{C + C^{-1}}}.\end{aligned}$$

C.1.5. *Second-order derivatives.*

$$\begin{aligned}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} &= \frac{D\sqrt{2}}{2\left(C^2+1\right)}\begin{bmatrix} C+1 & C-1 \\ 1-C & 1+C \end{bmatrix},\\ \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} &= \frac{1}{2\sqrt{C^2+1}}\begin{bmatrix} C+1 & C-1 \\ 1-C & 1+C \end{bmatrix}.\end{aligned}$$

Take two derivatives of the stability equation (with respect to x and y) and evaluate them at the top match. Because of (C.2) and $\hat{\beta}_x = 0$, we have

$$x : \hat{\beta}_{xx} - w_{11}\hat{\alpha}_x\hat{\beta}_y + m_{11}\alpha_x = 0$$
$$y : \hat{\beta}_{xy} - w_{11}\hat{\alpha}_y\hat{\beta}_y - w_{12}\left(\hat{\beta}_y\right)^2 + m_{21}\hat{\alpha}_x = 0.$$

Using the derivation of the first-order derivatives, we obtain

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = D\frac{\sqrt{2}}{2\left(C^2+1\right)}\begin{bmatrix} C+1 & C-1 \\ 1-C & 1+C \end{bmatrix},$$
$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \frac{1}{2\sqrt{C^2+1}}\begin{bmatrix} C+1 & C-1 \\ 1-C & 1+C \end{bmatrix}.$$

$$\begin{aligned}
\hat{\beta}_{xx} &= w_{11}\hat{\alpha}_x\hat{\beta}_y - m_{11}\hat{\alpha}_x \\
&= \frac{C+1}{2\sqrt{C^2+1}}\left(1 - (-1)^o D\frac{\sqrt{2}}{\sqrt{C^2+1}}\sqrt{\frac{1}{2}\left(C+C^{-1}\right)}\right) \\
\hat{\beta}_{xy} &= w_{11}\hat{\alpha}_y\hat{\beta}_y + w_{12}\left(\hat{\beta}_y\right)^2 - m_{21}\hat{\alpha}_x \\
&= \frac{1-C}{2\sqrt{C^2+1}}\left(1 - (-1)^o D\frac{\sqrt{2}}{\sqrt{C^2+1}}\sqrt{\frac{1}{2}\left(C+C^{-1}\right)}\right)
\end{aligned}$$

Next, take the derivatives of the matching equation (notice that we can drop the absolute value sign):

$$x : \hat{\alpha}_{xx}\hat{\beta}_y + \hat{\alpha}_x\hat{\beta}_{xy} - \hat{\alpha}_y\hat{\beta}_{xx} = 0,$$
$$y : \hat{\alpha}_{xy}\hat{\beta}_y + \hat{\alpha}_x\hat{\beta}_{yy} - \hat{\alpha}_y\hat{\beta}_{xy} = 0,$$

which implies that

$$\begin{aligned}
\hat{\alpha}_{xx} &= -(-1)^o\frac{C+C^{-1}}{2}\hat{\beta}_{xy} + \frac{C^{-1}-C}{2}\hat{\beta}_{xx}, \\
\hat{\alpha}_{xy} &= -(-1)^o\frac{C+C^{-1}}{2}\hat{\beta}_{yy} + \frac{C^{-1}-C}{2}\hat{\beta}_{xy}.
\end{aligned} \tag{C.6}$$

Finally, take the second derivative of the two boundary conditions:

$$\begin{aligned}\tilde{\psi}^0\hat{\beta}_{xx}+2\tilde{\psi}^0\hat{\beta}_{xy}+\tilde{\psi}^0\hat{\beta}_{yy}&=\hat{\alpha}_{xx}+2\hat{\alpha}_{xy}+\hat{\alpha}_{yy},\\ \tilde{\psi}^1\hat{\beta}_{xx}-2\tilde{\psi}^1\hat{\beta}_{xy}+\tilde{\psi}^1\hat{\beta}_{yy}&=\hat{\alpha}_{xx}-2\hat{\alpha}_{xy}+\hat{\alpha}_{yy},\end{aligned}$$

Substituting (C.6) and rearranging some terms, we obtain

$$\begin{aligned}\hat{\beta}_{yy}&=-\frac{1}{3}\hat{\beta}_{xx},\\ \hat{\alpha}_{yy}&=\tilde{\psi}^0\hat{\beta}_{xx}+2\tilde{\psi}^0\hat{\beta}_{xy}+\tilde{\psi}^0\hat{\beta}_{yy}-2\hat{\alpha}_{xy}-\hat{\alpha}_{xx}\\ &=\frac{3}{2}\left(\tilde{\psi}^0-\tilde{\psi}^1\right)\beta_{xy}-\frac{1}{6}\left(\tilde{\psi}^0+\tilde{\psi}^1\right)\beta_{xx}\\ &=(-1)^o\,\frac{3\left(C+C^{-1}\right)}{2}\beta_{xy}-\frac{1}{3}\frac{C^{-1}-C}{2}\beta_{xx}\end{aligned}$$

C.1.6. *Plots.* Let

$$\mu^{2,n}(x,y)=\left[\begin{array}{c}\hat{\alpha}_x x+\hat{\alpha}_y y+\frac{1}{2}\hat{\alpha}_{xx}x^2+\hat{\alpha}_{xy}xy+\frac{1}{2}\hat{\alpha}_{yy}y^2\\ \hat{\beta}_x x+\hat{\beta}_y y+\frac{1}{2}\hat{\beta}_{xx}x^2+\hat{\beta}_{xy}xy+\frac{1}{2}\hat{\beta}_{yy}y^2\end{array}\right]$$

be the second-order approximation of stable matching expressed in the new coordinates. In the original coordinates, the matching function takes the form

$$\mu^2(m_s,m_b)=O_W\mu^{2,n}\left(O_M^{-1}\left[\begin{array}{c}m_s\\ m_b\end{array}\right]\right)$$

Figures 1 and 2 plot the difference between the second order-approximation of the matching function and the identity matching

$$\mu^2(m_s,m_b)-\left[\begin{array}{c}m_s\\ m_b\end{array}\right].$$

Figure 3 plots the difference between the second order-approximation of the matching function and the reverse identity matching

$$\mu^2(m_b,m_s)-\left[\begin{array}{c}m_s\\ m_b\end{array}\right].$$

C.2. **Three-dimensional example.** Next, we consider the case $d=3$.

C.2.1. *Environment.* Assume that preferences are given by

$$\mathcal{M}\left(\left(m_s, m_b, m_w\right), \left(w_s, w_b, w_w\right)\right) = w_s + w_b + w_w + m_s w_b + m_b w_w + m_w w_s,$$
$$\mathcal{W}\left(\left(m_s, m_b, m_w\right), \left(w_s, w_b, w_w\right)\right) = m_s + m_b + m_w + m_s w_s + m_b w_b + m_b w_b$$

and that the domains of men and women are equal to

$$E_M = E_W = \left\{(s, b, w) : 1\left(s^2 + b^2 + w^2\right) - 2\left(sb + sw + bw\right) \leq 0\right\}.$$

C.2.2. *Normalization and change of coordinates.* We normalize the utility and rotate the coordinate system so that the normal vectors to the indifference curve at the top match are equal to $(0, 0, 1)$. Define the rotation matrix

$$O = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix}$$

Because O is a rotation matrix, $O' = O^{-1}$. Consider new coordinates for men and women, so that, respectively,

$$O\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m_s \\ m_b \\ m_w \end{bmatrix} \text{ and } O\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_s \\ w_b \\ w_w \end{bmatrix}.$$

We can express preferences in the new coordinates:

$$\begin{aligned}
&\mathcal{M}^n\left(\left(m_s, m_b, m_w\right), \left(w_s, w_b, w_w\right)\right) \\
&= \mathcal{M}\left(O\begin{bmatrix} x \\ y \\ z \end{bmatrix}, O\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) \\
&= \begin{bmatrix} x \\ y \\ z \end{bmatrix}' O^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}' O'\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} O\begin{bmatrix} a \\ b \\ b \end{bmatrix} \\
&= \sqrt{3}z + \begin{bmatrix} x \\ y \\ z \end{bmatrix}'\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ b \end{bmatrix},
\end{aligned}$$

and

$$\mathcal{W}^n\left(\left(m_s, m_b, m_w\right), \left(w_s, w_b, w_w\right)\right) = \sqrt{3}z + \begin{bmatrix} x \\ y \\ z \end{bmatrix}'\begin{bmatrix} a \\ b \\ b \end{bmatrix}$$

From now on, we will use only the preferences expressed in the new coordinates and we drop the superscript "n".

Next, we can find the proper cone matrices expressed in the new coordinates:

$$\Phi = \Psi = O^T\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} O = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

C.2.3. *First-order derivatives.* We use the results from Section 6 to compute the first-order derivative of the matching function μ. By equation (6.10),

$$\hat{\theta} = 1.$$

We compute the average cross-derivative matrix W^P at the top match:

$$\hat{W}^P = \mathcal{M}_{mw}^{\hat{}P} + \hat{\theta}\mathcal{W}_{mw}^{\hat{}P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}.$$

Let $\hat{S} = \hat{W}^P \left(\hat{D}\mu\right)^P$. Then, $\hat{S}$ is a symmetric and positive-definite solution to the matrix equation (6.15),

$$\hat{S} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \hat{S} = \hat{S} \left(\hat{W}^{P\prime}\right)^{-1} \Psi \left(\hat{W}^P\right)^{-1} \hat{S} = \left(\hat{g}^2 \frac{\det \Psi}{\det \Phi}\right)^{\frac{1}{d}} \Phi^P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The unique solution to the above equation is the identity matrix. Therefore,

$$\left(\hat{D}\mu\right)^P = \left(\hat{W}^P\right)^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}.$$

Finally, we use equation (B.2) to deduce that $\left(\hat{D}\mu\right)^y$ is a vector of 0s and

$$\hat{D}\mu = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

C.2.4. *Plot.* Figure 4 plots the difference between the first order approximation of the unique stable matching (expressed in the original coordinates) and the identity matching:

$$O\left(\hat{D}\mu\right)O^T \begin{bmatrix} m_s \\ m_b \\ m_w \end{bmatrix} - \begin{bmatrix} m_s \\ m_b \\ m_w \end{bmatrix}.$$

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