“Reverse Bayesianism”: A Choice-Based Theory of Growing Awareness*

Edi Karni
Johns Hopkins University

Marie-Louise Vierø
Queen’s University

January 19, 2011

Abstract

This paper invokes the axiomatic approach to explore the notion of growing awareness in the context of decision making under uncertainty. It introduces a new approach to modeling the expanding universe of a decision maker in the wake of becoming aware of new consequences, new acts, and new links between acts and consequences. The expanding universe, or state space, is accompanied by extension of the set of acts. The preference relations over the expanded sets of acts are linked by unchanging preferences over the satisfaction of basic needs. The main results are representation theorems and corresponding rules for updating beliefs over expanding state spaces that have the flavor of “reverse Bayesianism.”

Keywords: Awareness, unawareness, reverse Bayesianism,

JEL classification: D8, D81, D83

*We thank Brian Hill, David Schmeidler, and Peter Wakker for comments and suggestions.

1 Introduction

According to the Bayesian paradigm, as new discoveries are made and new information becomes available, the universe shrinks: With the arrival of new information, events replace the prior universal state space to become the posterior state space, or universe of discourse. This process of "destruction" reflects the impossibility, in the Bayesian framework, of updating the probabilities of null events, coupled with the fact that conditioning on new information renders null events that, a-priori, were nonnull. Yet, experience and intuition alike contradict this view of the world. Becoming accustomed to things that were once inconceivable is part of history and our own life experience. There is a sense, therefore, in which our universe expands as we become aware of new opportunities.

This paper explores the idea of growing awareness and the behavioral implications of learning about unforeseen contingencies that expand a decision maker's perception of the universe. Invoking the revealed preference methodology, we axiomatize agents' choice behavior in a universe that expands in the wake of discoveries of new consequences, acts, and links between them. Because the Bayesian paradigm presumes that the state space is fixed, it cannot accommodate the expansion of the universe as a result of growing awareness in the sense described above.[1] In our approach, the state space expands as a decision-maker's awareness grows.

In this paper, a decision maker's initial perception of the universe is determined by a primitive set of what he considers to be feasible acts and, corresponding to each feasible act, a potential set of consequences. Matching feasible acts with their potential consequences, taking into account what the decision maker considers possible links between feasible acts and consequences, defines a *feasible state space*. The *conceivable state space* consists of all the mappings from the set of feasible acts to that of consequences.[2] The discovery of new consequences and/or new feasible acts expands both the conceivable and feasible state spaces. The discovery of new links between feasible acts and consequences expands the feasible state space but not the conceivable state space. In either case, the expansions represent the decision maker's growing level of awareness. We assume that, within a given

[1] See also Dekel, Lipman, and Rustichini (1998), who show that standard state spaces preclude unawareness. A choice theoretic approach therefore needs a more general point of departure than Savage (1954) and Anscombe and Aumann (1963).

[2] Here we follow the approach to defining a state space described in Schmeidler and Wakker (1987) and Karni and Schmeidler (1991).

universe, decision makers' choice behavior is governed by the axioms of subjective expected utility theory.

Preferences under different levels of awareness are defined over different domains. To link the preference relations representing growing awareness, we assume that decision makers have needs whose satisfaction determines their well-being. Choice behavior is motivated by the desire to satisfy these needs. The "material" consequences of acts are means by which these needs are satisfied. To model this concept of individual behavior, we invoke an approach to consumer theory, due to Lancaster (1966) and Becker (1965), according to which, the material consequences are inputs in a "household production function" generating characteristics that determine the decision maker's well-being. In our framework, these characteristics correspond to levels of satisfaction of diverse needs. We assume that decision makers are fully conscious of their needs and that growing awareness does not alter these needs.

In our model, awareness grows as a result of the discovery of new consequences or new feasible acts, or scientific discoveries and technical innovations that establish new links between feasible acts and consequences. Such discoveries expand the state space, the decision maker's perception of the universe in which he lives. Within this framework, we axiomatize the evolution of beliefs in a way that can best be described as "reverse Bayesianism": as the state space expands, probability-mass is shifted away, proportionally, from the old to the new states. This systematic evolution of beliefs makes it possible to predict, at least partially, the decision maker's behavior when something unforeseen occurs.

When a decision maker discovers a contingency he was previously unaware of, his prior conception – or "model" – of the universe is falsified. When this happens, the decision maker's prior model need not be discarded; it can still provide some guidance for behavior in the "new" expanded universe. In other words, decision makers can use their experiences and understanding of the prior state space to guide their choices when their growing awareness enables them to construct an expanded state space.

The exploration of the issue of unawareness in the literature has invoked at least three different approaches. (a) the epistemic approach (see Fagin and Halpern [1988], Modica and Rustichini [1999], Halpern [2001], Li [2009], and Hill [2010]); (b) the game-theoretic, or interactive decision making, approach (see Heifetz, Meier, and Schipper [2006], Halpern and Rego [2008], Grant and Quiggin [2009]); and (c) the choice-theoretic approach (see Kochov [2010], Schipper [2010], Li [2008]).

Our approach falls within the third category. However, unlike other studies that take this approach, we do not take the state space as given. Instead, we construct the relevant state space from the sets of feasible acts and consequences and the perceived links among them. In so doing, we abstract from concrete interpretations of the states and treat them as abstract resolutions of uncertainty. Consequently, decision makers' unawareness concerns feasible acts, feasible consequences, and/or their links.

Kochov (2010) considers a decision maker who knows that his perception of the universe may be incomplete. He characterizes the collection of foreseen events and shows that the result of the decision maker being aware of his incomplete perception of the environment is that his beliefs are represented by a non-singleton set of priors, which he updates as his perception of the environment becomes more precise.

Schipper (2010) focuses on detecting unawareness. Taking as primitive a lattice of disjoint state spaces in the Anscombe and Aumann (1963) model and defining acts as mappings from the union of these state spaces to the set of consequences, Schipper provides conditions under which unawareness can be modeled as probability zero events in the union of the disjoint state spaces in the lattice. He does not address the issue of updating preferences in the wake of growing awareness.

Li (2008) takes as primitives a fixed set of consequences and two, exogenously given, state spaces that correspond to a decision maker being less than fully aware and fully aware. Decision makers are characterized by preference relations, conditional on the level of awareness, over Anscombe-Aumann acts on the corresponding state spaces. Li considers two types of unawareness: "pure unawareness," depicting situations in which the decision maker's perception of the environment is coarse, and "partial unawareness," depicting situations in which the decision maker's perception of the universe is a subset of the full state space. Partial unawareness has a flavor of unawareness of consequences or links between acts and consequences. However, since the set of consequences and states are given, Li's model cannot accommodate the discovery of new consequences or new scientific links.

2 The Meanings of Growing Awareness: Examples and Formalization

The examples below illustrate the sense in which a decision maker's universe expands in the wake of his growing awareness.

2.1 Examples

A. Discovery of new consequences

The discovery of the New World. Columbus set out to discover a new sea route to India, presumably taking into account consequences such as ending the trip at the bottom of the ocean, having to turn back, losing some ships and crew members, reaching India, etc. He could not have included, among the set of consequences, the discovery of a new continent. This discovery expanded the universe for mankind.

The discovery of syphilis. The discovery of the New World ushered in its wake a new consequence of sexual intercourse. The risk of contracting venereal diseases was well known in the Old World. Syphilis, however, was new. Its discovery expanded the universe of the Europeans.

Discovery of a "new" consequence expands the state space and may affect the decision maker's ordinal preferences over acts. In other words, two acts that agree on the "old" state space may become distinct when associated with new consequences; as a result, one of the newly defined acts may be strictly preferred over the other.

B. Discovery of new scientific links

Yellow fever. To prevent ants from crawling into hospitals' beds, French doctors working in Panama during the French attempt to build the Panama Canal, placed the legs of the beds in bowls of water. These pools of water provided breeding grounds for the mosquitoes carrying yellow fever. Not being aware of the way the yellow fever was transmitted, the French did not conceive that their actions contributed to the propagation of the disease. Later, when the connection between stagnant water, mosquitoes, and yellow fever was understood, the Americans were able to eradicate yellow fever, eliminating a major stumbling point to the construction of the Panama Canal.

Smoking and lung cancer. The connection between smoking and lung cancer was established by the accumulated evidence of many studies. Lung cancer was not a new consequence, and smoking was not a new activity. However, the discovery of the connection between smoking and lung cancer established a new link with implications for the desirability of smoking.

DDT. During World War II, soldiers sprayed themselves and their beds with DDT to kill bugs. The connection between DDT and genetic mutations in one's offspring was not discovered until later. The possibility of genetic mutations was known at the time, so it was not the consequence itself that was new but rather the discovery of the link between DDT and genetic mutation, which had implications for the use of DDT.

C. Discovery of new feasible acts

Artificial self-sustaining nuclear chain reaction. After the discovery of nuclear fission, Szilárd and Fermi discovered neutron multiplication in uranium, proving that a nuclear chain reaction by this mechanism was possible. On December 2, 1942, Fermi created the first artificial self-sustaining nuclear chain reaction, thus making it feasible to use nuclear energy, for peaceful and military purposes.

The invention of sound recordings. By making it possible to preserve sounds, the invention of sound recording devices expanded the state space to include future replays of currently produced sounds.

The invention of new financial instruments. The invention of option trading opened up new possibilities of creating portfolios and diversifying risks.

2.2 Growing awareness formalized

We introduce a unifying framework within which the different sources of growing awareness may be described and analyzed. We also illustrate how the different notions of growing awareness can be formalized in this framework.

States of nature, or *states* for short, are abstract representations of resolutions of uncertainty. To define the state space, we invoke the approach of Schmeidler and Wakker (1987) and Karni and Schmeidler (1991).[3] According to this approach, there is a (finite,

[3] See aslo Gilboa (2009) Chpater 11, for a detailed discussion and the ingenious use of this approach to formulating the state space as means of resolving Newcomb's paradox.

nonempty) set, F, of *feasible acts*, a finite, nonempty set, C, of *feasible consequences* and a correspondence, $\varphi : F \twoheadrightarrow C$, representing the decision maker's beliefs about the possible links among feasible acts and consequences. In other words, to each $f \in F$, $\varphi(f) \subseteq C$ is the subset of consequences that, the decision maker believes, are possible if he chooses the act f.

A decision maker's perception of the universe is bounded by his awareness of the sets of feasible acts and consequences, and all the conceivable links among feasible acts and consequences. Formally, the decision maker's universe is depicted by the *conceivable state space*, C^F, depicting the resolutions of uncertainty under the presumption that any of the feasible acts could potentially result in any of the consequences.

A decision maker's beliefs about the links among feasible acts and consequences define a *feasible state space* whose elements indicate, for each feasible act, the resulting consequence that the decision maker believes could obtain. Thus, a feasible state completely resolves the uncertainty present in the decision maker's perception of the feasible universe. Formally, the feasible state space is given by $S(F, C, \varphi) := \{s : F \to C | s(f) \in \varphi(f)\}$. If, for some act f, $\varphi(f)$ is a proper subset of C then the conceivable but unfeasible event $C^F - S(F, C, \varphi)$ (that is, the set of states in which, for some $f \in F$, $c' \in C - \varphi(f)$ is assigned to f) is presumably null.

Once the set of conceivable states is fixed, the set of feasible acts is expanded to include *conceivable acts*. The notion of conceivable acts captures the idea of acts that are imaginable given the feasible acts and consequences. The expansion of the set of acts includes two steps. First, conceivable new acts are formed by the association of feasible consequences to the existing states. By itself this allows the expansion of the set of acts from F to include all the functions from the set of conceivable states C^F to the set C of consequences, that is,

$$\widetilde{F} := \{f : C^F \to C\}. \tag{1}$$

Second, the decision maker may imagine acts whose outcomes are lotteries with consequences in C as prizes. Let the set of all such lotteries be denoted by $\Delta(C)$.[4] Then the set of acts may be enlarged to include the functions in the set

$$\hat{F} := \{f : C^F \to \Delta(C)\}, \tag{2}$$

[4] To be clear, $\Delta(C) := \{p \in [0,1]^{|C|} \mid \Sigma_{c \in C} p_c = 1\}$.

which we refer to as the set of conceivable acts. We identify $c \in C$ with the degenerate lottery $\delta_c \in \Delta(C)$ that assigns c the unit probability mass. Hence, $F \subset \widetilde{F} \subset \hat{F}$.

"In practice, the distinction between feasible and conceivable acts is not always crucial, and in many applications the sets of states and consequences are taken as primitives." (Karni and Schmeidler (1991) p. 1766). In the present context the distinction between feasible and conceivable acts is crucial. It is the set of feasible acts, together with the feasible consequences and the links among feasible acts and consequences, that constitute the decision maker's level of awareness and shape his vision of the universe.

Using this framework, we discuss the various types of unawareness with which we are concerned. We use the following notational convention throughout. We denote by F, C and φ, respectively, the initial sets of feasible acts, consequences, and the correspondence representing the links between them. When new elements are introduced into each of these sets we denote the corresponding new sets by F' and $C'$and when new links are established we denote the resulting new correspondence by φ'. When new consequences are discovered, the acts and the correspondence must be redefined. We denote the new set of acts by F^* and the new correspondence by φ^*. When new links are discovered, the set of acts needs to be redefined. We denote the new set of acts by F^*.

2.2.1 Discovery of new scientific links

Imagine that a new scientific link between feasible acts and consequences is established. Formally, let φ denote the correspondence depicting the "old" links, and denote by φ' the correspondence depicting the new links. Specifically, suppose that as a result of becoming aware of a new link, the set of consequences the decision maker now thinks possible under f is $\varphi'(f) := \varphi(f) \cup \{c'\}$, where $c' \in C - \varphi(f)$.

To see how this discovery expands the state space, consider the case in which there are two feasible acts, $F = \{f_1, f_2\}$ and two consequences, $C = \{c_1, c_2\}$. Suppose that $\varphi(f_1) = C$ and $\varphi(f_2) = \{c_1\}$, then the state space is $S(F, C, \varphi) = \{s_1, s_2\}$, as described below

$F \setminus S(F, C, \varphi)$	s_1	s_2
f_1	c_1	c_2
f_2	c_1	c_1

Suppose now that it is discovered that f_2 may also result in c_2, (that is, after the new discovery, $\varphi'(f_i) = C$, $i = 1, 2$). To indicate the fact that the range of consequences associated with (some) feasible acts is now larger, we denote the set of feasible acts by F^*. Then the state space, $S(F^*, C, \varphi')$, consists of four states, $s_1, ..., s_4$, described in the following matrix:

$F^* \setminus S(F^*, C, \varphi')$	s_1	s_2	s_3	s_4
f_1	c_1	c_2	c_1	c_2
f_2	c_1	c_1	c_2	c_2

Note that following the discovery of the new (and final link) the feasible and conceivable state spaces coincide (that is, $S(F^*, C, \varphi') = C^F$).

Before the discovery of the new link, the event $\{s_3, s_4\} = C^F - S(F, C, \varphi)$ was null in the larger conceivable state space C^F. Upon the discovery of the link, the decision maker realizes that he presupposed that it was impossible to obtain a particular consequence by implementing a particular feasible act, and that this presupposition has now been falsified. In other words, before the discovery of the new link, $C^F - S(F, C, \varphi)$ was a conceivable but unfeasible and, hence, null event. Following the discovery of the new link, $C^F = S(F^*, C, \varphi')$. The event $C^F - S(F, C, \varphi)$ was regarded as impossible before the discovery of the new link and became possible following the discovery of the new link.[5]

What is a reasonable updating rule for probabilities of events that were considered impossible (null) and, as a result of scientific progress and growing understanding of the structure of the universe, become possible (nonnull)? Clearly, the Bayesian approach is useless for this purpose. Here we explore an alternative approach.

2.2.2 Discovery of new consequences

Let C denote the initial set of consequences and suppose that a new consequence, $\bar{c} \notin C$, is discovered. The set of consequences of which the decision maker is aware then expands to $C' = C \cup \{\bar{c}\}$, requiring a reformulation of the initial model, incorporating the new consequence into the range of the feasible acts. Because ranges of the feasible acts rather than the acts themselves changed, we denote the set of feasible acts with extended range by F^* and the subset of consequences the decision maker now believes possible if he chooses f by $\varphi^*(f)$. Then the corresponding extended conceivable state space is $(C')^{F^*}$ and the feasible state space is given by

[5] By the same logic, the discovery that a link that the decsion maker thought possible is, in fact, impossible, results in nullifying an event that was initially nonnull.

$$S\left(F^{*},C',\varphi^{*}\right):=\{s:F^{*}\rightarrow C'|s(f)\in\varphi^{*}\left(f\right)\}. \tag{3}$$

Define the corresponding expanded set of conceivable acts,

$$\hat{F}^{*}:=\{f:\left(C'\right)^{F^{*}}\rightarrow\Delta\left(C'\right)\}. \tag{4}$$

The event $(C')^{F^{*}}-C^{F}$ represents the expansion of the decision maker's conceivable state space, while $S\left(F^{*},C',\varphi^{*}\right)-S\left(F,C,\varphi\right)$ represents the expansion of the decision maker's feasible state space, as a result of his growing awareness of consequences.

As an illustration, suppose we start with two feasible acts, $F=\{f_1,f_2\}$, two consequences, $C=\{c_1,c_2\}$, and the links $\varphi\left(f_1\right)=\varphi\left(f_2\right)=\{c_1,c_2\}$. The resulting state space is $S(F,C,\varphi)=\{s_1,s_2,s_3,s_4\}=C^{F}$:

$F\setminus S(F,C,\varphi)$	s_1	s_2	s_3	s_4
f_1	c_1	c_2	c_1	c_2
f_2	c_1	c_1	c_2	c_2

Suppose now that a new consequence, c_3, is discovered and that it is established that the feasible act f_1 may result in c_3. The range of the act f_1 is $\varphi^{*}\left(f_1\right)=\{c_1,c_2,c_3\}$ after the discovery. The new feasible state space is $S\left(F^{*},C',\varphi^{*}\right)=\{s_1,s_2,\ldots,s_6\}$:[6]

$F^{*}\setminus S\left(F^{*},C',\varphi^{*}\right)$	s_1	s_2	s_3	s_4	s_5	s_6
f_1	c_1	c_2	c_1	c_2	c_3	c_3
f_2	c_1	c_1	c_2	c_2	c_1	c_2

2.2.3 Discovery of new feasible acts

Suppose that a new act, say f_3, becomes feasible. Instead of F, the set of feasible acts is now $F'=\{f_1,f_2,f_3\}$, and the redefined correspondence depicting the links among feasible acts and consequences is $\varphi^{*}\left(f_i\right)=C=\{c_1,c_2\}$, $i=1,2,3$. The conceivable state space is expanded to $C^{F'}$ and the feasible state space, $S\left(F',C,\varphi^{*}\right)$, now consists of eight states:

$F'\setminus S\left(F',C,\varphi^{*}\right)$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
f_1	c_1	c_2	c_1	c_2	c_1	c_2	c_1	c_2
f_2	c_1	c_1	c_2	c_2	c_1	c_1	c_2	c_2
f_3	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2

[6] The new conceivable state space consisits of 9 states.

In general, the elements of the state space $S(F', C, \varphi^*)$ constitute a finer partition of the state space $S(F, C, \varphi)$. In other words, each state in $S(F, C, \varphi)$ is a non-degenerate event in the expanded state space $S(F', C, \varphi^*)$. For example, the state $s_1 := (c_1, c_1) \in S(F, C, \varphi)$ is the event $E = \{s_1, s_5\}$ in the state space $S(F', C, \varphi^*)$. However, if $\varphi^*(f_3) = \{c_1\}$, then the number of states in the feasible state space remains the same (that is, it consists of the four states in $S(F, C, \varphi)$). In either case, the decision maker's original conception of the state space is determined by the initial sets of acts and consequences he considers feasible and the links between acts and consequences he considers possible. The act f_3 was neither conceivable nor feasible before its discovery. Now that it has become feasible, it changes the decision maker's conception of the state space.[7]

Note that, unlike in the cases of discoveries of new consequences or scientific links, in the case of discovery of new acts, the length of the vector of consequences defining each state increases. As we show later, this aspect of the evolving state space requires special treatment.

Note also that the discovery of new scientific links expands the set of feasible states but leaves the set of conceivable states intact, while the discovery of new feasible consequences and/or new feasible acts expands the sets of conceivable and feasible states at the same time. In the model below we treat the former form as updating zero probability events, and the latter as genuine expansion of the decision maker's universe.

3 The Analytical Framework

A decision maker's growing awareness of the feasibility of acts, consequences, and of the links between them expands his perception of the universe and its structure. How does growing awareness manifest itself in his choice behavior? In this section we introduce the analytical framework as well as some preliminary results used in the subsequent analysis.

[7] Ahn and Ergin (2010) model decision makers whose choice behavior depends on their perception of contingencies, represented by alternative partitions of a given state space. Unlike our work, in which the state space expands and is partitioned more finely as a result of the discovery of new acts, in Ahn and Ergin's work new acts are defined as a consequence of finer partition of the state space. These acts represent growing alertness to possibilities that were always present and were simply ignored.

3.1 Preferences, needs, and technology

Decision makers in our model are supposed to be able to express preferences among conceivable acts. Formally, let $\mathcal{F}$ be a family of sets of conceivable acts corresponding to increasing levels of awareness from all sources (that is, from the discovery of new feasible acts, consequences, and links among them). Preferences are binary relations on $\hat{F} \in \mathcal{F}$. Because the set of conceivable acts is a variable in our model, we denote the preference relation on $\hat{F}$ by $\succcurlyeq_{\hat{F}}$, and use the notation $\succ_{\hat{F}}$ and $\sim_{\hat{F}}$ to denote the asymmetric and symmetric parts of $\succcurlyeq_{\hat{F}}$, respectively. When the state space expands in the wake of discoveries of new feasible consequences and/or new links among acts and consequences, the set of conceivable acts must be expanded and the preference relations must be redefined on the extended domain. For instance, if $\hat{F}^*$ is the expanded set of conceivable acts, then the corresponding preference relation is denoted by $\succcurlyeq_{\hat{F}^*}$. If the state space is expanded in the wake of the discovery of new feasible acts, then the new set of conceivable acts is denoted by by $\hat{F}'$ and the expanded preference relation by $\succcurlyeq_{\hat{F}'}$.

Our main concern is how does the preference relation change when the decision maker's universe expands as his awareness grows? To model the change of preferences resulting from increasing awareness, we employ a variation of the model proposed by Lancaster (1966) and Becker (1965). In particular, we assume that decision makers have needs, which they seek to satisfy by means of consumption of goods and services. Let $N = \{1, \ldots, n\}$ be a list of needs (e.g., food, shelter, clothing, entertainment, social status, etc.). The trade-offs among the satisfaction of different needs are assumed to be a matter of personal taste. Let $Z \subset \mathbb{R}^n$ be a set whose elements are levels of satisfaction of these needs. In other words, $z \in Z$ is a vector whose j-th coordinate, z_j, $j \in N$, indicates the degree to which the need j is satisfied. Let $\Delta(Z)$ denote the set of simple probability measures on Z, which we refer to as *need-satisfaction lotteries*.[8] A decision maker's well-being is determined by the satisfaction of his needs. Thus, at the basic level, a decision maker is characterized by a preference relation, $\succsim$, on $\Delta(Z)$.

Let $X \subset \mathbb{R}^m$ be a finite, nonempty set of *feasible material outcomes,* or outcomes, for short. For example, $x \in X$ could be a lobster dinner, a two-bedroom apartment in an upscale neighborhood, and a James Bond movie. Let F be a finite set of feasible acts. For each $f \in F$, denote by $\varphi(f)$ the set of material outcomes that in the mind of the decision

[8] A measure is simple if it has a finite support.

maker are possible if he chooses the act f. Let $S(F, X, \varphi) := \{s : F \to X \mid s(f) \in \varphi(f)\}$ be the set of feasible states and X^F the set of conceivable states.

Denote by $\Delta(X)$ the set of lotteries on X. Then, following Anscombe and Aumann (1963), the set of *conceivable acts,* $\hat{F}$, consists of all the mappings from the set of states to the set of lotteries on outcomes. Formally, $\hat{F} := \{f : X^F \to \Delta(X)\}$ is the set of conceivable acts. Henceforth, we indicate by $\hat{F}$ the set of conceivable acts corresponding to the universe depicted by X^F.

Let $t : X \to Z$ be a mapping representing the *technology* that generates needs satisfaction from material outcomes. Put differently, t is a "production function" that transforms material outcomes into need-satisfaction levels.[9] In our example, the dinner, the apartment, and the movie allow, with the appropriate input of time, the attainment of some levels of satisfaction of the needs for nutrition, shelter, social status and entertainment. Given a technology t, $p \in \Delta(X)$ induces a lottery l_p in $\Delta(Z)$ as follows: $l_p(z) = p\left(t^{-1}(z)\right)$, for all $z \in Z$.[10]

Decision makers are characterized by a primitive preference relation $\succsim$ on need-satisfaction lotteries and preference relations $\succcurlyeq_{\hat{F}}$ on the sets of conceivable acts, for all $\hat{F} \in \mathcal{F}$. The connections between the preference relation on need-satisfaction lotteries, and the preference relations on sets of conceivable acts are at the core of our theory. They are defined and discussed in Section 4 below.

Growing awareness expands the sets of acts and states and thus alters the domain over which the corresponding sets of induced preference relations are defined. We postulate that *the preference relations corresponding to different levels of awareness are linked by a primitive, unchanging, preference relation over need-satisfaction levels.*

3.2 Expected utility theory

Let K be a convex set in a linear space and $\trianglerighteq$ a binary relation on K. The von Neumann-Morgenstern axioms applied to $\trianglerighteq$ are:

(A.1) **(Weak order)** The preference relation $\trianglerighteq$ is transitive and complete.

[9] In Lancaster (1966) the technology transforms material goods into "characteristics" and is linear. We do not insist on linearity and identify characteristics with needs satisfaction.

[10] Note that $t^{-1}(z)$ is the preimage of z under the technology, representing an isoquant of the "household production function." Formally, $t^{-1}(z) := \{x \in X \mid t(x) = z\}$.

(A.2) (**Archimedean**) For all $p, q, r \in K$, if $p \trianglerighteq q$ and $q \trianglerighteq r$ then $\alpha p + (1-\alpha)\, r \trianglerighteq q$ and $q \trianglerighteq \beta p + (1-\beta)\, r$, for some $\alpha, \beta \in (0,1)$.

(A.3) (**Independence**) For all $p, q, r \in K$ and $\alpha \in (0,1]$, $p \trianglerighteq q$ if and only if $\alpha p + (1-\alpha)\, r \trianglerighteq \alpha q + (1-\alpha)\, r$.

The von Neumann-Morgenstern theorem states that $\trianglerighteq$ on K satisfies (A.1) - (A.3) if and only if there exist a real-valued, affine function U on K that represents $\trianglerighteq$, and is unique up to positive linear transformations.

Since $\Delta(Z)$ is a convex set in a linear space, application of the von Neumann-Morgenstern theorem yields the expected utility theorem below:

Theorem 1 (**von Neumann-Morgenstern**) Let $\succsim$ be a binary relation on $\Delta(Z)$, then the following two conditions are equivalent:

(i) $\succsim$ satisfies (A.1), (A.2) and (A.3).
(ii) There exists a real-valued function, u, on Z, such that for all $l, l' \in \Delta(Z)$,

$$l \succsim l' \Leftrightarrow \sum_{z \in Supp(l)} u(z)\, l(z) \geq \sum_{z \in Supp(l')} u(z)\, l'(z)\,. \tag{5}$$

Moreover, u is unique up to positive linear transformations.

Consider the preference relation $\succcurlyeq_{\hat{F}}$ on $\hat{F}$. Note that X^F is the domain of the acts in $\hat{F}$. For any $f \in \hat{F}$, $p \in \Delta(X)$, and $s \in X^F$, let $f_{-s}p$ be the act in $\hat{F}$ obtained from f by replacing its $s-th$ coordinate with p. A state $s \in X^F$ is said to be null if $f_{-s}p \sim_{\hat{F}} f_{-s}q$ for all $p, q \in \Delta(X)$. A state is said to be nonnull if it is not null. Similarly, we denote by $f_{-E}p$ the act in $\hat{F}$ obtained from f by replacing its $s-th$ coordinate with p, for all $s \in E \subset X^F$. We suppose that the event $K := X^F - S(F, X, \varphi)$, that consists of states that the decision maker regards as conceivable but infeasible is null. Formally, henceforth we assume that $f_{-K}p \sim_{\hat{F}} f_{-K}q$, for all $p, q \in \Delta(X)$.

The following axioms are due to Anscombe and Aumann (1963).

(A.4) (**State independence**) For all $p, q \in \Delta(X)$ and nonnull $s, s' \in X^F$, $f_{-s}p \succ_{\hat{F}} f_{-s}q$ if and only if $f_{-s'}p \succ_{\hat{F}} f_{-s'}q$.

(A.5) (**Nontriviality**) $\succ_{\hat{F}} \neq \varnothing$.

For every given $\hat{F} \in \mathcal{F}$, for all $f, f' \in \hat{F}$ and $\alpha \in [0,1]$, define the convex combination $\alpha f + (1-\alpha) f' \in \hat{F}$ by

$$\left(\alpha f + (1-\alpha) f'\right)(s) = \alpha f(s) + (1-\alpha) f'(s), \ \forall s \in X^F.$$

Then $\hat{F}$ is a convex set in a linear space.[11] For future reference, we state below a version of the Anscombe-Aumann (1963) theorem.

Theorem 2 (**Anscombe-Aumann**) Let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$, then the following two conditions are equivalent:

(*i*) $\succcurlyeq_{\hat{F}}$ satisfies (A.1)-(A.5).

(*ii*) There exists a real-valued, non-constant, affine function, $U_{\hat{F}}$ on $\Delta(X)$, and a probability measure π on X^F, such that for all $f, f' \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} f' \Leftrightarrow \sum_{s \in X^F} U_{\hat{F}}(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in X^F} U_{\hat{F}}\left(f'(s)\right) \pi_{\hat{F}}(s), \tag{6}$$

Moreover, $U_{\hat{F}}$ is unique up to positive linear transformations,[12] $\pi_{\hat{F}}$ is unique, and $\pi_{\hat{F}}(s) = 0$ if and only if s is a null state.

Remark: Since $X^F - S(F, X, \varphi)$ is a null event, $\pi_{\hat{F}}\left(X^F - S(F, X, \varphi)\right) = 0$, $\pi_{\hat{F}}(S(F, X, \varphi)) = 1$, and, for all $f \in \hat{F}$,

$$\sum_{s \in X^F} U_{\hat{F}}(f(s)) \pi_{\hat{F}}(s) = \sum_{s \in S(F,X,\varphi)} U_{\hat{F}}(f(s)) \pi_{\hat{F}}(s).$$

To simplify the exposition, henceforth we disregard the null event $X^F - S(F, X, \varphi)$, and focus our attention on the feasible state space $S(F, X, \varphi)$. Notice that all the events in $S(F, X, \varphi)$ are nonnull, since the feasible state space is defined exactly by those acts-consequences links the decision maker considers possible.

[11] Throughout this paper we use Fishburn's (1970) formulation of Anscombe and Aumann (1963). According to this formulation, mixed acts, (that is, $\alpha f + (1-\alpha) f'$) are, by definition, conceivable acts.

[12] Hence, $U_{\hat{F}}(p) = \sum_{x \in Supp(p)} u_{\hat{F}}(x) p(x)$, where $u_{\hat{F}}(x) = U_{\hat{F}}(\delta_x)$, for all $x \in X$.

4 The Main Results

To study the connections among the preference relations on expanding sets of conceivable acts, we assume that these preference relations are linked together by the properties of the unchanging preference relation, $\succsim$, on the set of need-satisfaction lotteries. Using these connections we explore the impact of growing awareness on a decision maker's choice behavior.

The analysis of the effects of growing awareness on choice behavior and the evolution of decision makers' beliefs is divided into two parts. In the first part, we explore the implications of the discovery of new consequences and/or new links between acts and consequences. The discovery of new acts-consequences links expands the set of feasible states but not that of conceivable states. The discovery of new consequences increases the number of both conceivable and feasible states. In either case, nonnull events are added but the "dimension" of each state is unchanged. In the second part we explore the implications of the discovery of new feasible acts. In general, the discovery of new feasible acts increases the number of both conceivable and feasible states and, at the same time, changes the characterization of each state in such a way that what used to be a state before the discovery of the new act, is an event in the reconstructed state space following the discovery.

4.1 The discovery of new consequences and/or new acts-consequences links

Assume that a decision maker's preference relation over the set of need-satisfaction lotteries, representing his basic tastes, does not change as his awareness grows. For every given $\succcurlyeq_{\hat{F}}$ on $\hat{F}$ and $s \in S(F, X, \varphi)$, define a conditional preference relation, $\succsim^s_{\hat{F}}$, on $\Delta(Z)$ induced by $\succcurlyeq_{\hat{F}}$, as follows:

$$\text{For all } p, q \in \Delta(X), l_p \succsim^s_{\hat{F}} l_q \text{ if } f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q, \text{ for all } f \in \hat{F}.$$

The next axiom requires that the conditional preference relations, $\{\succsim^s_{\hat{F}}\}_{\hat{F} \in \mathcal{F}}$, and unconditional preference relation, $\succsim$, on need-satisfaction lotteries agree. Put differently, it asserts that a decision maker's preferences regarding his basic needs and his risk attitudes toward these needs are independent of the particular process (that is, acts) by which the need-satisfaction lotteries are obtained and by the specificity of the manner by which the uncertainty is resolved. Formally,

(A.6) (**Taste consistency**) For all $\hat{F} \in \mathcal{F}$ and nonnull $s \in S(F, X, \varphi)$, $\succsim^s_{\hat{F}} = \succsim$.

The following axiom requires that, as the decision maker's awareness of consequences and/or links among feasible acts and consequences grows and his universe expands, his preferences conditional on the prior perception of the universe remain intact. In other words, the discovery of new consequences and/or new links between feasible acts and consequences does not alter the preference relation conditional on the original set of feasible states. To formalize this idea, let $S(F^*, X', \widetilde{\varphi}) := \{s : F^* \to X' \mid s(f) \in \widetilde{\varphi}(f)\}$, where $X \subset X'$ and $\widetilde{\varphi} = \varphi^*$ (if a new consequence is discovered), or $X = X'$ and $\widetilde{\varphi} = \varphi'$ with $\varphi(f) \subset \varphi'(f)$ for some f (if a new link is discovered).

(A.7) (**Awareness consistency**) For all $\hat{F}, \hat{F}^* \in \mathcal{F}$, if $S(F^*, X', \widetilde{\varphi}) \supset S(F, X, \varphi)$ and $f', g' \in \hat{F}^*$, $f' = f$ and $g' = g$ on $S(F, X, \varphi)$ and $f' = g'$ on $S(F^*, X', \widetilde{\varphi}) - S(F, X, \varphi)$ then $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}^*} g'$.

4.2 Representation of preferences when growing awareness reflects the discovery of new consequences and/or new links

Our first result describes the evolution of a decision maker's beliefs in the wake of discoveries of new consequences and/or new links among feasible acts and consequences. Specifically, a decision maker whose preferences have the structure depicted by the axioms above is a subjective expected utility maximizer. Moreover, when he becomes aware of new consequences and/or new links among feasible acts and consequences, the decision maker updates his beliefs in such a way that likelihood ratios of events in the original state space remain intact. That is to say, probability mass is shifted away from states in the prior state space to the posterior state space, proportionally. We refer to this property as "reverse Bayesianism."

Theorem 3. For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$ then, for all $\hat{F}, \hat{F}^* \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.6) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ satisfy (A.7).

(ii) There exists a real-valued, non-constant, affine function, U on $\Delta(X)$ and, for each $\hat{F} \in \mathcal{F}$, there is a probability measure $\pi_{\hat{F}}$ on $S(F, X, \varphi)$, such that for all $f, f' \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} f' \Leftrightarrow \sum_{s \in S(F,X,\varphi)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,X,\varphi)} U\left(f'(s)\right) \pi_{\hat{F}}(s). \tag{7}$$

Moreover, U is unique up to positive linear transformations, $\pi_{\hat{F}}$ is unique and

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(s')}, \tag{8}$$

for all $\hat{F}, \hat{F}^* \in \mathcal{F}$ and $s, s' \in S(F, X, \varphi) \cap S(F^*, X', \widetilde{\varphi})$.

4.3 The discovery of new feasible acts

The introduction of new feasible acts may or may not increase the number of states. In either case, however, it increases the number of coordinates defining a state. If it also increases the number of states, the newly defined states constitute a finer partition of the original state space. Thus, if $F \subset F'$ then $S(F, X, \varphi) \cap S(F', X, \varphi^*) = \varnothing$, and for each $s \in S(F, X, \varphi)$ there corresponds an event $E(s) \subset S(F', X, \varphi^*)$ defined by $E(s) = \{s' \in S(F', X, \varphi^*) \mid \boldsymbol{P}_{S(F,X,\varphi)}(s') = s\}$, where $\boldsymbol{P}_{S(F,X,\varphi)}(\cdot)$ is the projection of $S(F', X, \varphi^*)$ on $S(F, X, \varphi)$.[13] For $s \in S(F, X, \varphi)$, we refer to the set $E(s)$ as the projection of s on $S(F', X, \varphi^*)$. Using these notations we state the next axiom, which is analogous to axiom (A.7).

(A.8) (**Projection consistency**) For all $\hat{F}, \hat{F}' \in \mathcal{F}$ such that $F \subset F'$, $p, q, \bar{p}, \bar{q} \in \Delta(X)$, $h \in \hat{F}$, $s, s' \in S(F, X, \varphi)$ and $E(s), E(s') \subset S(F', X, \varphi^*)$, if $f', g' \in \hat{F}'$ agree on $S(F', X, \varphi^*) - E(s) \cup E(s')$, $f'(t) = p, g'(t) = q$ for all $t \in E(s)$, and $f'(t) = \bar{p}, g'(t) = \bar{q}$, for all $t \in E(s')$, then $\left((h_{-s}p)_{-s'}\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\bar{q}\right)$ if and only if $f' \succcurlyeq_{F'} g'$.

[13]Suppose that $\mid F \mid = r$ and $\mid F' \mid = k > r$. Let $s = (c_1, ..., c_k) \in S(F', X, \varphi^*)$, then $\boldsymbol{P}_{S(F,X,\varphi)}(s) = (c_1, ..., c_r) \in S(F, X, \varphi)$.

4.3.1 Representation of preferences when growing awareness is due to the discovery of new feasible acts

The representation theorem below describes how a decision maker's beliefs evolve as he becomes aware of new feasible acts. As before, the decision maker is a subjective expected utility maximizer. When he becomes aware of new feasible acts, the decision maker updates his beliefs in a way that the likelihood ratios of events in the original state space remain intact. Because of the difference in the evolution of the state space, probability mass is shifted from states in the prior state space to the corresponding events the posterior state space, in such a way as to preserve the likelihood ratios of the events in the posterior state space and their corresponding projected states in the prior state space.[14]

Theorem 4 For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$. Then for all $\hat{F}, \hat{F}' \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.6) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}'}$ satisfy (A.8).

(ii) There exists a real-valued, non-constant, affine function, U on $\Delta(X)$ and, for each $\hat{F} \in \mathcal{F}$, there is a probability measure $\pi_{\hat{F}}$ on $S(F, X, \varphi)$, such that for all $f, f' \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} f' \Leftrightarrow \sum_{s \in S(F,X,\varphi)} U(f(s))\, \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,X,\varphi)} U\left(f'(s)\right) \pi_{\hat{F}}(s). \tag{9}$$

Moreover, U is unique up to positive linear transformations, $\pi_{\hat{F}}$ is unique, and if $F \subset F'$ then

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}'}(E(s))}{\pi_{\hat{F}'}(E(s'))}, \tag{10}$$

for all $s, s' \in S(F, X, \varphi)$ and $E(s), E(s') \subset S(F', X, \varphi^*)$, where $E(s)$ and $E(s')$, are the projections of s and s' on $S(F', X, \varphi^*)$.

[14] This is "reverse Bayesianism" applied to the present context. Li (2008) conjectures an axiomatization of the link between preferences under full awareness and those under pure unawareness and states a proposition linking the evolution of beliefs. This is in the spirit of Theorem 4. Li's axiom neither implies, nor is it implied by, our projection consistency axiom.

5 Concluding Remarks

In this paper we take a step toward modeling the process of growing awareness and expansion of the universe in its wake. To model the evolution of awareness, we invoke the theory of choice in the face of uncertainty. We borrow its language and structure while modifying it to fit our purpose. In particular, we allow for new consequences and feasible acts to be introduced and for the discovery of new links among acts and consequences. This enables us to expand the state space which bounds the decision maker's conception of the universe.

To link and lend structure to the likelihoods of events belonging to different state spaces, we invoke a theory of needs. We assume that there is a fundamental preference relation depicting the trade-offs among need satisfactions of the decision maker.

Within this framework, we axiomatize the evolution of beliefs in a way that can be described as "reverse Bayesianism." As the state space expands, probability mass is shifted proportionally away from events in the prior state space to events created as a result of the expansion of the state space. We note that the same process may be applied in the inverse direction. For example, the discovery that certain hypotheses about the connections among feasible acts and consequences are invalid can shrink the relevant state space, by rendering some events null. This corresponds to Bayesian updating in Savage's (1954) framework.

The interpretation of the updating is somewhat different for the discovery of new feasible acts and consequences on the one hand and the discovery of new scientific links between feasible acts and consequences on the other. The discovery of new feasible acts and consequences is a genuine increase in the level of awareness, while the discovery of new scientific links between feasible acts and consequences results in rendering events that, although conceivable, were considered null before the discovery of the new links into nonnull events. This updating of zero probability events is part of the reverse Bayesianism nature of our model.

6 Proofs

6.1 Proof of theorem 3.

(Sufficiency) By Theorems 1 and 2, for all $\hat{F} \in \mathcal{F}$, $f \in \hat{F}$ and $p, q \in \Delta(X)$,

$$f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q \Leftrightarrow \sum_{x \in Supp(p)} u_{\hat{F}}(x)\, p(x) \geq \sum_{x \in Supp(q)} u_{\hat{F}}(x)\, q(x)\,. \tag{11}$$

By definition of $\succsim^s_{\hat{F}}$ and axiom (A.6), for every $\hat{F} \in \mathcal{F}$, $f \in \hat{F}$ and $p, q \in \Delta(X)$,

$$f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q \Leftrightarrow l_p \succsim l_q. \tag{12}$$

By Theorem 1,

$$l_p \succsim l_q \Leftrightarrow \sum_{z \in Supp(l_p)} u(z)\, l_p(z) \geq \sum_{z \in Supp(l_q)} u(z)\, l_q(z). \tag{13}$$

But $l_p(z) := p\left(t^{-1}(z)\right)$ and $l_q(z) := q\left(t^{-1}(z)\right)$, for all $z \in Z$. In particular, if $p = \delta_x$ then $l_p(z) = \delta_z$, where $z = t(x)$.

Fix $z \in Z$ and let $x, x' \in t^{-1}(z)$. Then, $\delta_{t(x)} = \delta_{t(x')}$ and, by Theorem 1, $u(t(x)) = u(t(x')) = u(z)$. By (12), this implies that $f_{-s}\delta_x \sim_{\hat{F}} f_{-s}\delta_{x'}$, for all $\hat{F} \in \mathcal{F}$, $f \in \hat{F}$ and $x, x' \in t^{-1}(z)$. Thus, by the representation (11), $u_{\hat{F}}(x) = u_{\hat{F}}(x')$, for all $x, x' \in t^{-1}(z)$, and $\hat{F} \in \mathcal{F}$. We denote this fact by defining $u_{\hat{F}}\left(t^{-1}(z)\right) := u_{\hat{F}}(x)$, for $x \in t^{-1}(z)$.

Using these notations, the representation (11) may be written as follows:

$$f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q \Leftrightarrow \sum_{z \in Supp(l_p)} u_{\hat{F}}\left(t^{-1}(z)\right) p\left(t^{-1}(z)\right) \geq \sum_{z \in Supp(l_q)} u_{\hat{F}}\left(t^{-1}(z)\right) q\left(t^{-1}(z)\right). \tag{14}$$

But (12), (13), and (14) imply that

$$\sum_{z \in Supp(l_p)} u_{\hat{F}}\left(t^{-1}(z)\right) p\left(t^{-1}(z)\right) \geq \sum_{z \in Supp(l_q)} u_{\hat{F}}\left(t^{-1}(z)\right) q\left(t^{-1}(z)\right) \tag{15}$$

if and only if

$$\sum_{z \in Supp(l_p)} u(z)\, l_p(z) \geq \sum_{z \in Supp(l_q)} u(z)\, l_q(z). \tag{16}$$

Since, $l_p(z) := p\left(t^{-1}(z)\right)$, for all $z \in Z$, the equivalence of (15) and (16), and the uniqueness of the von Neumann-Morgenstern utility function imply that $u_{\hat{F}}\left(t^{-1}(z)\right) = bu(z) + a$, $b > 0$, for all $z \in Z$ and $\hat{F} \in \mathcal{F}$. Let $U(f(s)) := \sum_{z \in Supp\left(l_{f(s)}\right)} u(z)\, l_{f(s)}(z)$, for all $f \in \hat{F}$ and $s \in S(F, X, \varphi)$. Hence, we can drop the subscript $\hat{F}$ from $U_{\hat{F}}$. Therefore, by Theorem 2 and since $X^F - S(F, X, \varphi)$ is a null event, for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,X,\varphi)} U(f(s))\, \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,X,\varphi)} U(g(s))\, \pi_{\hat{F}}(s). \tag{17}$$

Let $X' \supset X$ and let $\hat{F}, \hat{F}^* \in \mathcal{F}$. Then $S(F, X, \varphi) \subset S(F^*, X', \varphi^*)$. By the previous result and Theorem 2, for all $f', g' \in \hat{F}^*$,

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s \in S(F^*, X', \varphi^*)} U\left(f'(s)\right) \pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F^*, X', \varphi^*)} U\left(g'(s)\right) \pi_{\hat{F}^*}(s). \tag{18}$$

Let $f', g' \in \hat{F}^*$ be as in Axiom (A.7) (that is, $f' = f$ and $g' = g$ on $S(F, X, \varphi)$ and $f' = g'$ on $S(F^*, X', \varphi^*) - S(F, X, \varphi)$ then, using (18) and that common terms cancel out, $f' \succcurlyeq_{\hat{F}^*} g'$ if and only if

$$\sum_{s \in S(F, X, \varphi)} U\left(f'(s)\right) \pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F, X, \varphi)} U\left(g'(s)\right) \pi_{\hat{F}^*}(s), \tag{19}$$

which, by the definition of f' and g' is equivalent to

$$\sum_{s \in S(F, X, \varphi)} U(f(s)) \pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F, X, \varphi)} U(g(s)) \pi_{\hat{F}^*}(s). \tag{20}$$

But Axiom (A.7) implies

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow f' \succcurlyeq_{\hat{F}^*} g'. \tag{21}$$

By Theorem 2 and the representation (18) ,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F, X, \varphi)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F, X, \varphi)} U(g(s)) \pi_{\hat{F}}(s). \tag{22}$$

We thus have that the expressions in (20) and (22) are equivalent. Now, by the uniqueness of the probabilities in Theorem 2,

$$\frac{\pi_{\hat{F}^*}(s)}{\sum_{s \in S(F, X, \varphi)} \pi_{\hat{F}^*}(s)} = \pi_{\hat{F}}(s), \text{ for all } s \in S(F, X, \varphi). \tag{23}$$

Consider next the case in which $X = X'$ and $\varphi(f) \subset \varphi'(f)$, for some $f \in F$. Then, replace $S(F^*, X', \varphi^*)$ in the argument above with $S(F^*, X, \varphi')$. Then the conclusion follows by the same reasoning.

(Necessity) The necessity of (A.1)-(A.5) is an implication of Theorem 2. The necessity of (A.6) and (A.7) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Theorem 2. ■

6.2 Proof of theorem 4.

(Sufficiency) By Theorem 2 and since $X^F - S(F,X,\varphi)$ is a null event, for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,X,\varphi)} U_{\hat{F}}(f(s))\,\pi_{\hat{F}}(s) \geq \sum_{s \in S(F,X,\varphi)} U_{\hat{F}}(g(s))\,\pi_{\hat{F}}(s), \tag{24}$$

where $U_{\hat{F}}$ is affine.

Let u be the von Neumann-Morgenstern utility function representing $\succsim$ on $\Delta(Z)$. Then, by the same argument as in the proof of Theorem 3, and invoking axiom (A.6), $u_{\hat{F}}\left(t^{-1}(z)\right) = bu(z) + a$, $b > 0$, for all $z \in Z$ and $\hat{F} \in \mathcal{F}$. Let $U(f(s)) := \sum_{z \in Supp\left(l_{f(s)}\right)} u(z)\, l_{f(s)}(z)$, for all $f \in \hat{F}$ and $s \in S(F,X,\varphi)$. Then, for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,X,\varphi)} U(f(s))\,\pi_{\hat{F}}(s) \geq \sum_{s \in S(F,X,\varphi)} U(g(s))\,\pi_{\hat{F}}(s). \tag{25}$$

Let $\hat{F}, \hat{F}' \in \mathcal{F}$ and, without loss of generality, suppose that $S(F', X, \varphi^*)$ is a refinement of the partition $S(F,X,\varphi)$.[15] Let $f', g' \in \hat{F}'$ be as in Axiom (A.8) (that is, $f', g' \in \hat{F}'$ agree on $S(F', X, \varphi^*) - E(s) \cup E(s')$, $f'(t) = p, g'(t) = q$ for all $t \in E(s)$, and $f'(t) = \bar{p}, g'(t) = \bar{q}$ for all $t \in E(s')$). For the specific choice of f' and g', (25) is equivalent to

$$f' \succcurlyeq_{\hat{F}'} g' \Leftrightarrow U(p)\,\pi_{\hat{F}'}(E(s)) + U(\bar{p})\,\pi_{\hat{F}'}\left(E\left(s'\right)\right) \geq U(q)\,\pi_{\hat{F}'}(E(s)) + U(\bar{q})\,\pi_{\hat{F}'}\left(E\left(s'\right)\right). \tag{26}$$

By axiom (A.8),

$$f' \succcurlyeq_{\hat{F}'} g' \Leftrightarrow \left((h_{-s}p)_{-s'}\,\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\,\bar{q}\right). \tag{27}$$

By (25),

$$\left((h_{-s}p)_{-s'}\,\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\,\bar{q}\right) \tag{28}$$

if and only if

$$\sum_{s \in S(F,X,\varphi)} U\left(\left((h_{-s}p)_{-s'}\,\bar{p}\right)(s)\right)\pi_{\hat{F}}(s) \geq \sum_{s \in S(F,X,\varphi)} U\left(\left((h_{-s}q)_{-s'}\,\bar{q}\right)(s)\right)\pi_{\hat{F}}(s), \tag{29}$$

which, since common terms cancel out, is equivalent to

$$U(p)\,\pi_{\hat{F}}(s) + U(\bar{p})\,\pi_{\hat{F}}\left(s'\right) \geq U(q)\,\pi_{\hat{F}}(s) + U(\bar{q})\,\pi_{\hat{F}}\left(s'\right). \tag{30}$$

[15] Hence, $F \subset F'$.

By (27), the expressions (26) and (30) are equivalent, which holds for all $p, \bar{p}, q, \bar{q} \in \Delta(X)$, if and only if

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}'}(E(s))}{\pi_{\hat{F}'}(E(s'))}, \tag{31}$$

for all $s, s' \in S(F, X, \varphi)$ and $E(s), E(s') \subset S(F', X, \varphi^*)$, where $E(s)$ and $E(s')$, are the projections of s and s' on $S(F', X, \varphi^*)$.

(Necessity) The necessity of (A.1)-(A.6) is an implication of Theorem 2. The necessity of (A.8) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Theorem 2. ■

References

[1] Ahn, David S. and Haluk Ergin (2010) "Framing Contingencies," *Econometrica* 78, 655-695.

[2] Anscombe, Francis J. and Aumann, Robert J. (1963) "A Definition of Subjective Probability," *Annals of Mathematical Statistics* 43, 199–205.

[3] Becker, Gary S. (1965) "A Theory of the Allocation of Time," *The Economic Journal* 75, 493-517.

[4] Dekel, Eddie, Bart Lipman, and Aldo Rustichini (1998) "Standard state-space models preclude unawareness," *Econometrica* 66, 159-173.

[5] Fagin, Ronald and Joseph Y. Halpern (1988) "Belief, Awareness and Limited Reasoning, *Artificial Intelligence* 34, 39-76.

[6] Fishburn, Peter (1970) *Utility Theory for Decision Making,* Wiley, New York

[7] Gilboa, Itzhak (2009) *Theory of Decision Making under Uncertianty.* Cambridge University Press, Cambridge.

[8] Grant, Simon and John Quiggin (2009) "Interactive Reasoning about Awareness," unpublished manuscript

[9] Halpern, Joseph, Y. (2001) "Alternative Semantics for Unawareness," *Games and Economic Behavior* 37, 321-339.

[10] Halpern, Joseph, Y. and Leandro C. Rego. (2008) "Interactive Awareness Revisited," *Games and Economic Behavior* 62, 232-262.

[11] Heifetz, Aviad, Martin Meier, and Burkhard C. Schipper (2006) "Interactive Unawareness," *Journal of Economic Theory* 130, 78-94.

[12] Hill, Brian (2010) "Awareness Dynamics," *Journal of Philosophical Logic* 39, 113-137.

[13] Karni, Edi and David Schmeidler (1991) "Utility Theory with Uncertainty," in Werner Hildenbrand and Hugo Sonnenschein, eds. *Handbook of Mathematical Economics* vol. IV. Elsevier Science Publishers B.V.

[14] Kochov, Asen (2010) "A Model of Limited Foresight," working paper, University of Rochester.

[15] Lancaster, Kelvin J. (1966) "A New Approach to Consumer Theory," *Journal of Political Economy* 74, 132-157.

[16] Li, Jing (2008) "A Note on Unawareness and Zero Probability," working paper, University of Pennsylvania.

[17] Li, Jing (2009) "Information Structures with Unawareness," *Journal of Economic Theory* 144, 977-993.

[18] Modica, Salvatore and Aldo Rustichini (1999) "Unawareness and Partitional Information Structures," *Games and Economic Behavior* 27, 256-298.

[19] Savage, Leonard J. (1954) "*The Foundations of Statistics*," Wiley, New York.

[20] Schipper, Burkhard C. (2010) "Revealed Unawareness," working paper, UC Davis.

[21] Schmeidler, David and Peter Wakker (1987) "Expected Utility and Mathematical Expectation," in John Eatwell, Murray Milgate, and Peter Newman, eds. *The New Palgrave: A Dictionary of Economics*, Macmillan Press.