# A Lemmas and Theorems

**Proof of Lemma 1.** Substituting (9) into (12):

$$\begin{aligned} \alpha k + (1-\alpha) a &= (1 - (1 - \upsilon (1-\alpha)) s_I) c + (1 - \upsilon (1-\alpha)) s_I i \\ &= \left(1 - \phi_I^{PC}\right) c + \phi_I^{PC} i \end{aligned} \quad (41)$$

Here $\phi_I^{PC} = (1 - \upsilon (1-\alpha)) s_I$.

$k(0) = 0$ and $a(0) = 0$, therefore (41) can be written as:

$$0 = \left(1 - \phi_I^{PC}\right) c(0) + \phi_I^{PC} i(0) \quad (42)$$

First, if $\upsilon (1-\alpha) > 1$, then $\phi_I^{PC} < 0$ & $\left(1 - \phi_I^{PC}\right) > 0$ . Therefore, if $\upsilon (1-\alpha) > 1$ and $c(0)$ increases then for (42) to hold $i(0)$ must also increase.

Further, if $\upsilon (1-\alpha) > 1$, then $\left(1 - \phi_I^{PC}\right) = \left(-\phi_I^{PC} + 1\right) > -\phi_I$. Therefore, if $c(0)$ increases, then for (42) to hold $i(0)$ must increase by a larger magnitude than $c(0)$, this implies $(i(0) - c(0))$ increases when $c(0)$ increases, which in turn due to (9) implies that $n(0)$ must increase.

Therefore, if $\upsilon (1-\alpha) > 1$, then consumption, investment, and hours will comove at time zero.

Second, if $\upsilon (1-\alpha) < 1$, then $\phi_I^{PC} > 0$ and $\left(1 - \phi_I^{PC}\right) > 0$. Therefore, if $c(0)$ increases then for (42) to hold $i(0)$ must decrease. Therefore, if $\upsilon (1-\alpha) < 1$, then consumption, investment and hours will not comove at time zero. □

**Proof of Lemma 2.** Substituting (9) into (10):

$$\gamma_I^{PC} i - \left(\sigma + \gamma_I^{PC}\right) c = \lambda \quad (43)$$

Here $\gamma_I^{PC} = (\upsilon - 1) - (\upsilon (1-\alpha)(1-\sigma) s_I) / (1 - s_I)$.

Further, substituting (41) into 43 and solving for $c$ at time 0 leads to:

$$c(0) = \frac{-\phi_I^{PC}}{\gamma_I^{PC} + \phi_I^{PC}\sigma}\lambda(0) \tag{44}$$

First, from the proof of lemma 1 we know that $-\phi_I^{PC} > 0$ if $\upsilon(1-\alpha) > 1$. Also, $\gamma_I^{PC} + \phi_I^{PC}\sigma > 0$ if $\upsilon(1-\alpha) > 1$.[31] From equation (44) if $\lambda(0) > 0$ then $c(0)$ will increase. If $c(0) > 0$ then from the proof of lemma 1 we know that both $i(0)$ and $n(0)$ will also increase.

As a result, if $\upsilon(1-\alpha) > 1$ and $\lambda(0) > 0$, then consumption, investment, and labor hours will comove procyclically at time zero in response to a news shock about technology in time $T > 0$.

Second, by Lemma 1 we also know that if $\upsilon(1-\alpha) < 1$ and $\lambda(0) > 0$, then consumption and investment will not comove at time zero. □

**Proof of Lemma 3.** Solving (41) and (43) simultaneously for the values of $c$ and $i$:

$$c = \tau_{c,k}^{PC} k + \tau_{c,\lambda}^{PC}\lambda + \tau_{c,a}^{PC} a \tag{45}$$

$$i = \tau_{i,k}^{PC} k + \tau_{i,\lambda}^{PC}\lambda + \tau_{i,a}^{PC} a \tag{46}$$

$$n = \tau_{n,k}^{PC} k + \tau_{n,\lambda}^{PC}\lambda + \tau_{n,a}^{PC} a \tag{47}$$

Here $\tau_{c,k}^{PC}, \tau_{c,\lambda}^{PC}\tau_{i,k}^{PC}, \tau_{i,\lambda}^{PC}, \tau_{n,k}^{PC}$, and $\tau_{n,\lambda}^{PC}$ are all positive.[32]

It follows directly that if $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$ $\forall t < T$ then $\dot{c} \geq 0$, $\dot{i} \geq 0$, and $\dot{n} \geq 0$ for all $t < T$. Again, remember for $\forall t < T$, $a(t) = 0$. □

**Proof of Lemma 4.** Recall $k(0) = 0$. As a result, the time derivatives of the $k(t)$ and $\lambda(t)$

[31] For the proof see Lemma B.3 in Appendix B (Supplementary Appendix).
[32] For the proof see Lemma B.4 in Appendix B (Supplementary Appendix).

paths for all $t < T$:

$$\dot{k}(t) = \frac{\Gamma^{PC}_{k,\lambda}\left(\mu_2^{PC}e^{\mu_2^{PC}t} - \mu_1^{PC}e^{\mu_1^{PC}t}\right)}{\mu_2^{PC} - \mu_1^{PC}}\lambda(0)$$

$$\dot{\lambda}(t) = \left[\frac{\left(\mu_2^{PC} - \Gamma^{PC}_{k,k}\right)}{\mu_2^{PC} - \mu_1^{PC}}\mu_2^{PC}e^{\mu_2^{PC}t} - \frac{\left(\mu_1^{PC} - \Gamma^{PC}_{k,k}\right)}{\mu_2^{PC} - \mu_1^{PC}}\mu_1^{PC}e^{\mu_1^{PC}t}\right]\lambda(0)$$

First, for $0 \leq t < T$: $\left(\Gamma^{PC}_{k,\lambda}\left(\mu_2^{PC}e^{\mu_2^{PC}t} - \mu_1^{PC}e^{\mu_1^{PC}t}\right)\right)/\left(\mu_2^{PC} - \mu_1^{PC}\right)$ is positive as $\Gamma^{PC}_{k,\lambda} > 0$, [33] and we know that $\mu_2^{PC} > 0$ and $\mu_1^{PC} < 0$. Therefore the $sign(\dot{k}(t)) = sign(\lambda_0)$.

Second, for the $\dot{\lambda}$ equation: $\left(\mu_2^{PC} - \Gamma^{PC}_{k,k}\right)\mu_2^{PC}e^{\mu_2^{PC}t}/\left(\mu_2^{PC} - \mu_1^{PC}\right)$ is positive because $\mu_2^{PC} - \Gamma^{PC}_{k,k} = \Gamma^{PC}_{\lambda,\lambda} - \mu_1^{PC}$,[34] and we know $\mu_1^{PC} < 0$ and $\Gamma^{PC}_{\lambda,\lambda} > 0$.[35]

$\left(\mu_1^{PC} - \Gamma^{PC}_{k,k}\right)\mu_1^{PC}e^{\mu_1^{PC}t}/\left(\mu_2^{PC} - \mu_1^{PC}\right)$ may be either positive or negative. If $\mu_1^{PC} - \Gamma^{PC}_{k,k} > 0$, then the second term on the right-hand side is positive. In this case, $\dot{\lambda}(t) > 0$. However, if $\mu_1^{PC} - \Gamma^{PC}_{k,k} < 0$, then $\left(\mu_1^{PC} - \Gamma^{PC}_{k,k}\right)\mu_1^{PC}e^{\mu_1^{PC}t}/\left(\mu_2^{PC} - \mu_1^{PC}\right)$ is negative. In this case, we must show that $\left(\mu_2^{PC} - \Gamma^{PC}_{k,k}\right)\mu_2^{PC}e^{\mu_2^{PC}t}/\left(\mu_2^{PC} - \mu_1^{PC}\right)$ is larger than $\left(\mu_1^{PC} - \Gamma^{PC}_{k,k}\right)\mu_1^{PC}e^{\mu_1^{PC}t}/\left(\mu_2^{PC} - \mu_1^{PC}\right)$ in order that $\dot{\lambda}(t) > 0$. Because $\mu_2^{PC} > 0 > \mu_1^{PC}$, in this second case, the smallest value for $\dot{\lambda}(t)$ occurs at $t = 0$.

$$\begin{aligned}
\dot{\lambda}(0) &= \frac{\lambda(0)}{\mu_2^{PC} - \mu_1^{PC}}\left[\mu_2^{PC}\left(\mu_2^{PC} - \Gamma^{PC}_{k,k}\right) - \mu_1^{PC}\left(\mu_1^{PC} - \Gamma^{PC}_{k,k}\right)\right] \\
&= \lambda_0\left[\mu_2^{PC} + \mu_1^{PC} - \Gamma^{PC}_{k,k}\right] \\
&= \lambda_0\left[\Gamma^{PC}_{k,k} + \Gamma^{PC}_{\lambda,\lambda} - \Gamma^{PC}_{k,k}\right] \\
&= \lambda_0\Gamma^{PC}_{\lambda,\lambda}
\end{aligned}$$

As $\Gamma^{PC}_{\lambda,\lambda} > 0$, this establishes that $sign(\dot{\lambda}(t)) = sign(\lambda_0)$. □

[33] For the proof see Lemma B.5 in Appendix B (Supplementary Appendix).

[34] This follows because $tr(\Gamma^{PC}) = \mu_1^{PC} + \mu_2^{PC}$.

[35] For the proof see Lemma B.5 in Appendix B (Supplementary Appendix).

**Proof of Lemma 5.** Recall $\mu_2^{PC} > 0$ and

$$k(t) = \begin{cases} \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)+\left(\mu_1^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)k(0)}{\mu_1^{PC}-\mu_2^{PC}}e^{\mu_1^{PC}t} + \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)+\left(\mu_2^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)k(0)}{\mu_2^{PC}-\mu_1^{PC}}e^{\mu_2^{PC}t} & \text{for } t \in [0,T) \\ \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)+\left(\mu_1^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)k(0)}{\mu_1^{PC}-\mu_2^{PC}}e^{\mu_1^{PC}t} + \frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC}-\Gamma_{\lambda,\lambda}^{PC}b_{k,a}^{PC}}{\mu_1^{PC}\mu_2^{PC}} + \frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC}+\left(\mu_1^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_1^{PC}\left(\mu_1^{PC}-\mu_2^{PC}\right)}e^{\mu_1^{PC}(t-T)} & t \geq T \end{cases}$$

Then as $k(0) = 0$ a non-explosive path for $[\lambda\ k]'$ requires that we choose $\lambda(0)$ such that the terms involving the explosive root $\mu_2^{PC}$ in the exponential are 'zeroed out' for all $t > T$. Otherwise the path for $k(t)$ will be explosive. This imposes the following restriction on $\lambda(0)$:

$$\left(\frac{\Gamma_{k,\lambda}^{PC}}{\mu_2^{PC} - \mu_1^{PC}}\right)\lambda_0 = -\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2^{PC} - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_2^{PC}\left(\mu_2^{PC} - \mu_1^{PC}\right)}e^{-\mu_2^{PC}T}$$

This can be re-written as:

$$\lambda_0 = -\left[\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2^{PC} - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\Gamma_{k,\lambda}^{PC}\mu_2^{PC}}\right]e^{-\mu_2^{PC}T} \tag{48}$$

Because $\Gamma_{k,\lambda}^{PC} > 0$, $\lambda(0) > 0$ if and only if $\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2 - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC} < 0$. Also, $\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2 - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC} < 0$ algebraically simplifies to $\mu_2^{PC} < (\rho + (1-\alpha)\delta)\upsilon/(\gamma_I + \sigma)$ . □

**Proof of Theorem 1.** $\Leftarrow$. If $\upsilon(1-\alpha) > 1$ and $\mu_2^{PC} < (\rho + (1-\alpha)\delta)\upsilon/\left(\gamma_I^{PC} + \sigma\right)$, then a technology news shock is procyclical. Lemmas 2 and 5 prove the procyclical comovement at $t = 0$, while Lemmas 3, 4 and 5 establish the procyclical comovement for $0 < t < T$.

$\Rightarrow$. If $\upsilon(1-\alpha) < 1$ or $\mu_2^{PC} < (\rho + (1-\alpha)\delta)\upsilon/\left(\gamma_I^{PC} + \sigma\right)$, then a technology news shock is not procyclical. This follow trivially from Lemma 2, as the procyclical comovement will not occur at time $t = 0$ if either of the above conditions are not met. □

**Proof of Lemma 6.** The condition $\mu_2^{PC} < (\rho + (1-\alpha)\delta)\upsilon/\left(\gamma_I^{PC} + \sigma\right)$ can be rewritten implicitly as $\sigma < \sigma^*$. As $\delta \to 0$ we have $\sigma^* \to 1$. The above lemma thus follows directly from Theorem 1. □

**Proof of Lemma 7.** Substituting (22) into (12):

$$\begin{aligned}\alpha k + (1-\alpha)\,a &= \left(1-\left(1-\frac{1-\alpha}{1-\gamma_N}\right)s_I\right)c + \left(1-\frac{1-\alpha}{1-\gamma_N}\right)s_I i \\ &= \left(1-\phi_I^{LE}\right)c + \phi_I^{LE} i \end{aligned} \tag{49}$$

Here $\phi_I^{LE} = \left(1-\frac{1-\alpha}{1-\gamma_{\bar{N}}}\right)s_I$.

$k(0)=0$ and $a(0)=0$, therefore (49) can be written as:

$$0 = \left(1-\phi_I^{LE}\right)c(0) + \phi_I^{LE} i(0) \tag{50}$$

First, if $\gamma_N > \alpha$, then $\phi_I^{LE} < 0$ & $\left(1-\phi_I^{LE}\right) > 0$ . Therefore, if $\gamma_N > \alpha$ and $c(0)$ increases then for (50) to hold $i(0)$ must also increase.

Further, if $\gamma_N > \alpha$, then $\left(1-\phi_I^{LE}\right) = \left(-\phi_I^{LE}+1\right) > -\phi_I^{LE}$. Therefore, if $c(0)$ increases, then for (50) to hold $i(0)$ must increase by a larger magnitude than $c(0)$, this implies $(i(0)-c(0))$ increases when $c(0)$ increases, which in turn due to (22) implies that $n(0)$ must increase.

Therefore, if $\gamma_N > \alpha$, then consumption, investment, and hours will comove at time zero.

Second, if $\gamma_N < \alpha$, then $\phi_I^{LE} > 0$ and $\left(1-\phi_I^{LE}\right) > 0$. Therefore, if $c(0)$ increases then for (50) to hold $i(0)$ must decrease. Therefore, if $\gamma_N < \alpha$, then consumption, investment and hours will not comove at time zero. □

**Proof of Lemma 8.** For a stable solution to exist one eigenvalue of $\Gamma^{LE}$ should be positive and the other negative. The product of the eigenvalues is given by the determinant of the $\Gamma^{LE}$ matrix.

$$\det\left(\Gamma^{LE}\right) = \frac{-\delta\,(\rho+\delta)}{\left(\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)\right)}\left[(1-s_I)\,(1-\alpha)\right]$$

First, if $\psi_I > \psi_I^{+} = -\frac{\gamma_I^{LE}+\phi_I^{LE}\sigma}{1-\phi_I^{LE}}$ then the product of the eigenvalues is negative and it follows that the eigenvalues have opposite signs. Further, it can be shown that tr $\left(\Gamma^{LE}\right) = \rho$ which gives the sum of the two eigenvalues.

Second, if $\psi_I < \psi_I^+ = -\frac{\gamma_I^{LE}+\phi_I^{LE}\sigma}{1-\phi_I^{LE}}$ then the product of the eigenvalues is positive and with $\text{tr}\left(\Gamma^{LE}\right) = \rho$, which gives the sum of the two eigenvalues, it follows that the eigenvalues are both positive. □

**Proof of Lemma 9.** Recall $\mu_2^{LE} > 0$. Also, $\Gamma_{k,\lambda}^{LE} > 0$.[36] For a stable solution we need:

$$\lambda_0 = -\left[\frac{\Gamma_{k,\lambda}^{LE} b_{\lambda,a}^{LE} + \left(\mu_2^{LE} - \Gamma_{\lambda,\lambda}^{LE}\right) b_{k,a}^{LE}}{\Gamma_{k,\lambda}^{LE}\mu_2^{LE}}\right] e^{-\mu_2^{LE}T} \tag{51}$$

As a result $\lambda(0) > 0$ if and only if $\Gamma_{k,\lambda}^{LE} b_{\lambda,a}^{LE} + \left(\mu_2^{LE} - \Gamma_{\lambda,\lambda}^{LE}\right) b_{k,a}^{LE} < 0$. □

**Proof of Theorem 2.** Given the proofs and results of lemmas 7 through 9, to prove this theorem we must establish that when $\gamma_N > \gamma_N^*$ and $\psi_I > \psi_I^+$ three results hold: (1) $c(0) > 0$ if and only if $\lambda(0) > 0$. (2) Consumption, investment and hours will comove procyclically for all time $t < T$ if $\forall t < T$, $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$. (3) if $\lambda(0) > 0$ then $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$.

(1): $c(0) > 0$ if and only if $\lambda(0) > 0$ and $\gamma_N > \gamma_N^*$ follows from the observation that we can substitute 22 into 23, and the result into 50 to get an equation of the form $c(0) = \zeta^{LE}\lambda(0)$ where $\zeta^{LE} = \frac{-\phi_{I,LE}}{\gamma_{I,LE}+\phi_{I,LE}\sigma+\psi_I\left(1-\phi_{I,LE}\right)}$. $\zeta^{LE} > 0$ follows trivially from $\psi_I > \psi_I^+$ and $\gamma_N > \gamma_N^* \Rightarrow \phi^{LE} < 0$.

(2): We can solve for and define $x = \tau_{x,k}^{LE}k + \tau_{x,\lambda}^{LE}\lambda + \tau_{x,a}^{LE}a$ for $x = c, i, n$. Here $\tau_{c,k}^{LE}$, $\tau_{c,\lambda}^{LE}$, $\tau_{i,k}^{LE}$, $\tau_{i,\lambda}^{LE}$, $\tau_{n,k}^{LE}$, and $\tau_{n,\lambda}^{LE}$ are all positive[37], as result it trivially follows that if $\forall t < T$, $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$ then consumption, investment and hours will comove procyclically for all time $t < T$.

(3): The dynamic system given by (26) takes the same form as the dynamic system given by (17). As a result showing that $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$ if $\lambda(0) > 0$ amounts, exactly as in lemma 4, to proving that $\Gamma_{k,\lambda}^{LE} > 0$ and $\Gamma_{\lambda,\lambda}^{LE} > 0$. [38] $\Gamma_{k,\lambda}^{LE} > 0$ and $\Gamma_{\lambda,\lambda}^{LE} > 0$ follow from $\psi_I > \psi_I^+$ and $\gamma_N > \gamma_N^* \Rightarrow \phi^{LE} < 0$.

[36]For the proof see Lemma B.9 in Appendix B (Supplementary Appendix).
[37]For the proof see Lemma B.8 in Appendix B (Supplementary Appendix).
[38]For the proof see Lemma B.9 in Appendix B (Supplementary Appendix).

Results (1) - (3) together establish that if $\gamma_N > \gamma_N^*$, $\psi_I > \psi_I^+$, and $\lambda(0) > 0$ then the labor externality model exhibits procyclical technology news shocks. From lemma 9 we further know that $\lambda(0) > 0$ if and only if $\psi_I > \psi_I^*$. □

**Proof of Theorem 3.** $\Gamma^{FC} = \Gamma^{PC}$ and $\tau_{x,y}^{FC} = \tau_{x,y}^{PC}$ for $x = i, c, n$ and $y = k, \lambda$. Hence, for a model with forward compatible investment lemmas 1 through 4 still hold as before.

Now, recall $\mu_2^{FC} = \mu_2^{PC} > 0$. Also, $\Gamma_{k,\lambda}^{FC} = \Gamma_{k,\lambda}^{PC} > 0$. For a stable solution we need:

$$\lambda(0) = -\left[\frac{\Gamma_{k,\lambda}^{FC}\left(b_{\lambda,q}^{FC} + \tau\mu_2^{FC} b_{\lambda,p}^{FC}\right) + \left(\mu_2^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right)\left(b_{k,q}^{FC} + \tau\mu_2^{FC} b_{k,p}^{FC}\right)}{\Gamma_{k,\lambda}^{FC}\mu_2^{FC}}\right] e^{-\mu_2^{FC}T} \tag{52}$$

As a result $\lambda(0) > 0$ if and only if $\tau > \frac{\Gamma_{k,\lambda}^{FC} b_{\lambda,q}^{FC} + \left(\mu_2^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right) b_{k,q}^{FC}}{\Gamma_{k,\lambda}^{FC}\mu_2^{FC} b_{\lambda,p_\epsilon}^{FC} + \left(\mu_2^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right)\mu_2^{FC} b_{k,p_\epsilon}^{FC}}$ .

$\Leftarrow$. If $\upsilon(1-\alpha) > 1$ and $\tau > \frac{\Gamma_{k,\lambda}^{FC} b_{\lambda,q}^{FC} + \left(\mu_2^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right) b_{k,q}^{FC}}{\Gamma_{k,\lambda}^{FC}\mu_2^{FC} b_{\lambda,p_\epsilon}^{FC} + \left(\mu_2^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right)\mu_2^{FC} b_{k,p_\epsilon}^{FC}}$, then a investment technology news shock is procyclical. Lemmas 2 and the result above prove the procyclical comovement at $t = 0$, while Lemmas 3, 4 and the result above establish the procyclical comovement for $0 < t < T$.

$\Rightarrow$. If $\upsilon(1-\alpha) < 1$ or $\tau > \frac{\Gamma_{k,\lambda}^{FC} b_{\lambda,q}^{FC} + \left(\mu_2^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right) b_{k,q}^{FC}}{\Gamma_{k,\lambda}^{FC}\mu_2^{FC} b_{\lambda,p_\epsilon}^{FC} + \left(\mu_2^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right)\mu_2^{FC} b_{k,p_\epsilon}^{FC}}$, then a technology news shock is not procyclical. This follow trivially from Lemma 2, as the procyclical comovement will not occur at time $t = 0$ if either of the above conditions are not met. □ □

# B The Model Economies (For Online Publication)

## B.1 A Model with Production Complementarities

### B.1.1 The Model Economy

A social planner has the following preferences

$$U = (1-\sigma)^{-1}\int_0^\infty e^{-\rho t}\left[C(t)\exp(-N(t))\right]^{1-\sigma} dt$$

over time paths for consumption $C$ and hours worked $N$. We assume this functional form for the utility to preserve balanced growth. Also, $\rho = 1/\beta - 1 > 0$ and $\sigma \geq 0$, where $\beta$ is the stochastic discount factor and $\sigma$ is the inverse of the intertemporal elasticity of substitution.

The planner is subject to the following constraints:

$$F\left[C\left(t\right),I\left(t\right)\right]=K\left(t\right)^{\alpha}\left(A\left(t\right)N\left(t\right)\right)^{1-\alpha} \tag{53}$$

$$\dot{K}\left(t\right)=I\left(t\right)-\delta K\left(t\right) \tag{54}$$

Here $K$, $I$ and $A$ represent capital, investment and the level of technology. The path of technology and the initial capital stock are exogenous. The depreciation rate, $\delta$, and the elasticity of output with respect to capital, $\alpha$, both lie between zero and one.

Further, we assume:

$$F\left(C,I\right)\equiv\left[\theta C^{\upsilon}+\left(1-\theta\right)I^{\upsilon}\right]^{1/\upsilon}$$

where $\theta\in(0,1)$ and $\upsilon\geq 1$. When $\upsilon=1$, the equation collapses to the standard neo-classical case, which has infinite substitutability between the two goods. As $\upsilon$ increases, the complementarity between the production of the two goods increases. If $\upsilon=\infty$, the production frontier takes a Leontief form.

Next, let us define the exogenous processes - the technology news shock. The planner again has perfect foresight, with

$$A\left(t\right)=\begin{cases}\bar{A} & \text{for } t\in[0,T)\\ \tilde{A}=1.01\times\bar{A} & t\geq T\end{cases}$$

For the contemporaneous improvements case $T=0$ in the above specification.

### B.1.2 The Model Economy's First Order Conditions

The social planner chooses $C$, $I$, $K$, and $N$ to maximize $U$ subject to (53) and (54) taking as given the initial condition $K\left(0\right)$ and time path of technology. We can express the problem as a current value Hamiltonian:

$$H=C^{1-\sigma}\exp\left[-\left(1-\sigma\right)N\right]+\Lambda\left(I-\delta K\right)+\Phi\left(K^{\alpha}\left(AN\right)^{1-\alpha}-F\left(C,I\right)\right)$$

The first-order necessary conditions at an interior solution satisfy :

$$-\frac{U_N}{U_C} = (1-\alpha)\,\frac{F}{N}\,(F_C)^{-1} \tag{55}$$

$$\frac{U_C}{\Lambda} = \frac{F_C}{F_I} \tag{56}$$

$$\frac{\dot{\Lambda}}{\Lambda} - \rho = \delta - \alpha\frac{F}{K}\,(F_I)^{-1} \tag{57}$$

along with our initial condition on capital and a transversality condition on $\Lambda$.

Equation (55) is the intratemporal Euler equation between consumption and labor hours, equation (56) is the intratemporal Euler equation between consumption and investment, and equation (57) is the optimal capital accumulation equation.

### B.1.3 The Model Economy Log Linearized and Simplified

Given the first order conditions in the previous section our model economy can be described by the following five log linearized equations:

$$(1-s_I)\,c + s_I i = \alpha k + (1-\alpha)\,(a+n) \tag{58}$$

$$\upsilon s_I\,(i-c) = n \tag{59}$$

$$\lambda = (1-\upsilon)\,(c-i) - \sigma c - \frac{(1-\sigma)(1-\alpha)}{(1-s_I)}n \tag{60}$$

$$\dot{k} = \delta\,(i-k) \tag{61}$$

$$\dot{\lambda} = -\,(\rho+\delta)\,[\upsilon\,(1-s_I)\,(c-i) + i - k] \tag{62}$$

Here, $s_I = \frac{\alpha\delta}{\rho+\delta}$.

We can substitute (59) into (58) to get the consumption-investment production frontier ($L_1$ line):

$$\left(1-\phi_I^{PC}\right)c + \phi_I^{PC} i = \alpha k + (1-\alpha)\,a \tag{63}$$

Here, $\phi_I^{PC} = (1-(1-\alpha)\,\upsilon)\,s_I$

We can also substitute (59) into (60) to get the consumption-investment euler equation ($L_2$ line):

$$\gamma_I^{PC} i - \left(\sigma + \gamma_I^{PC}\right)c = \lambda \tag{64}$$

Here, $\gamma_I^{PC} = (\upsilon - 1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$.

Equations (63) and (64) now give us a system of equations in $i$ and $c$ (treating $\lambda$, $k$, and $a$ as exogenous).

We also solve the system of equations above for $c$, $i$, $n$, $\dot{k}$, and $\dot{\lambda}$, assuming as given the state variable $\lambda$ and $k$, and the exogenous variable $a$:

$$c = \tau_{c,k}^{PC} k + \tau_{c,\lambda}^{PC}\lambda + \tau_{c,a}^{PC} a \tag{65}$$

$$i = \tau_{i,k}^{PC} k + \tau_{i,\lambda}^{PC}\lambda + \tau_{i,a}^{PC} a \tag{66}$$

$$n = \tau_{n,k}^{PC} k + \tau_{n,\lambda}^{PC}\lambda + \tau_{n,a}^{PC} a \tag{67}$$

$$\dot{k} = \Gamma_{k,k}^{PC} k + \Gamma_{k,\lambda}^{PC}\lambda + b_{k,a}^{PC} a$$

$$\dot{\lambda} = \Gamma^{PC}_{\lambda,k} k + \Gamma^{PC}_{\lambda,\lambda} \lambda + b^{PC}_{\lambda,a} a$$

where,

$\tau^{PC}_{c,k} = \frac{\partial c}{\partial k} = \frac{\gamma_I^{PC}\alpha}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{c,\lambda} = \frac{\partial c}{\partial \lambda} = \frac{-\phi_I^{PC}}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{c,a} = \frac{\partial c}{\partial a} = \frac{\gamma_I^{PC}(1-\alpha)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{i,k} = \frac{\partial i}{\partial k} = \frac{\alpha\left(\gamma_I^{PC}+\sigma\right)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{i,\lambda} = \frac{\partial i}{\partial \lambda} = \frac{1-\phi_I^{PC}}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{i,a} = \frac{\partial i}{\partial a} = \frac{(1-\alpha)\left(\gamma_I^{PC}+\sigma\right)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{n,k} = \frac{\partial n}{\partial k} = \frac{\upsilon s_I \alpha\sigma}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{n,\lambda} = \frac{\partial n}{\partial \lambda} = \frac{\upsilon s_I}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\tau^{PC}_{n,a} = \frac{\partial n}{\partial a} = \frac{\upsilon s_I(1-\alpha)\sigma}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\Gamma^{PC}_{k,k} = \frac{\partial \dot{k}}{\partial k} = \frac{-\delta\left((1-\alpha)\gamma_I^{PC}+\phi_I^{PC}\sigma-\alpha\sigma\right)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\Gamma^{PC}_{k,\lambda} = \frac{\partial \dot{k}}{\partial \lambda} = \frac{\delta\left(1-\phi_I^{PC}\right)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\Gamma^{PC}_{\lambda,k} = \frac{\partial \dot{\lambda}}{\partial k} = \frac{(\rho+\delta)\left((1-\alpha)\gamma_I^{PC}+\phi_I^{PC}\sigma-\alpha\sigma+\alpha\sigma\upsilon(1-s_I)\right)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$\Gamma^{PC}_{\lambda,\lambda} = \frac{\partial \dot{\lambda}}{\partial \lambda} = \frac{(\rho+\delta)\left(\phi_I^{PC}-(1-\upsilon(1-s_I))\right)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$b^{PC}_{k,a} = \frac{\partial \dot{k}}{\partial a} = \frac{\delta\left(\gamma_I^{PC}+\sigma\right)(1-\alpha)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

$b^{PC}_{\lambda,a} = \frac{\partial \dot{\lambda}}{\partial a} = \frac{(\rho+\delta)(1-\alpha)\left(\sigma(\upsilon(1-s_I)-1)-\gamma_I^{PC}\right)}{\phi_I^{PC}\sigma+\gamma_I^{PC}}$

Recall: $s_I = \frac{\alpha\delta}{\rho+\delta}$, $\phi_I^{PC} = (1-(1-\alpha)\,\upsilon)\,s_I$, and $\gamma_I^{PC} = (\upsilon-1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$

### B.1.4 The Dynamic System

Let us now solve the dynamic system:

$$\begin{bmatrix} \dot{\lambda}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} \Gamma^{PC}_{\lambda,\lambda} & \Gamma^{PC}_{\lambda,k} \\ \Gamma^{PC}_{k,\lambda} & \Gamma^{PC}_{k,k} \end{bmatrix} \begin{bmatrix} \lambda(t) \\ k(t) \end{bmatrix} + \begin{bmatrix} b^{PC}_{\lambda,a} \\ b^{PC}_{k,a} \end{bmatrix} a(t) \tag{68}$$

In order to solve this system we must first determine the eigenvalues of the Γ matrix. For now we assume that a stable solution exists and that one of the eigenvalues is positive and the other negative. We will later prove this to be true. Let us label the eigenvalues $\mu_1^{PC}$ and $\mu_2^{PC}$ and without loss of generality, we will assume henceforth that $\mu_1^{PC} < 0$ and $\mu_2^{PC} > 0$.

We now introduce the technology news shock – a permanent increase in technology in period $T$. Specifically,

$$a(t) = w(t) = \begin{cases} 0 & \text{for } t \in [0, T) \\ 1 & t \geq T \end{cases} \tag{69}$$

To analyze the resulting system, it will be useful to introduce the Laplace transform operator.

The Laplace transform of a function $p(t)$ is:

$$\mathcal{L}[p(t)] = \bar{P}(s) = \int_0^\infty e^{-st} p(t)\, dt$$

We will use $\bar{P}$ rather than $P$ to distinguish the Laplace transform of the log deviation of a variables from the level of said variable.

Moreover, we know from Theorem 6.3 from Boyce and Diprima (1969), that

$$\mathcal{L}[p'(t)] = s\mathcal{L}(p(t)) - p(0)$$

Taking the Laplace transform of the differential equations in $\begin{bmatrix} \lambda & k \end{bmatrix}'$ and applying this theorem, we get:

$$\begin{bmatrix} \bar{\Lambda}(s) \\ \bar{K}(s) \end{bmatrix} = (sI - \Gamma)^{-1} \left\{ \begin{bmatrix} \lambda(0) \\ k(0) \end{bmatrix} + \begin{bmatrix} b_{\lambda,a}^{PC} \\ b_{k,a}^{PC} \end{bmatrix} W(s) \right\} \tag{70}$$

Given (69), it can be shown that

$$\bar{W}(s) = \mathcal{L}[w(t)] = \frac{1}{s} e^{-sT}$$

Rewriting equation (70), we get:

$$\begin{bmatrix} \bar{\Lambda}(s) \\ \bar{K}(s) \end{bmatrix} = \frac{1}{\left(s-\mu_1^{PC}\right)\left(s-\mu_2^{PC}\right)} \begin{bmatrix} s-\Gamma_{k,k}^{PC} & \Gamma_{\lambda,k}^{PC} \\ \Gamma_{k,\lambda}^{PC} & s-\Gamma_{\lambda,\lambda}^{PC} \end{bmatrix} \left\{ \begin{bmatrix} \lambda(0) \\ k(0) \end{bmatrix} + \begin{bmatrix} b_{\lambda,a}^{PC} \\ b_{k,a}^{PC} \end{bmatrix} W(s) \right\} \tag{71}$$

Remember, $\mu_1^{PC}$ and $\mu_2^{PC}$ are the eigenvalues of $\Gamma^{PC}$, and $\mu_1^{PC} < 0$ and $\mu_2^{PC} > 0$.

The lower row of (71) gives us:

$$\bar{K}(s) = \frac{\Gamma_{k,\lambda}^{PC}\lambda(0) + \left(s-\Gamma_{\lambda,\lambda}^{PC}\right)k(0)}{\left(s-\mu_1^{PC}\right)\left(s-\mu_2^{PC}\right)} + \left[\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(s-\Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{s\left(s-\mu_1^{PC}\right)\left(s-\mu_2^{PC}\right)}\right]e^{-sT}$$

Next, we take the inverse Laplace transform of $K(s)$ to recover $k$ as a function of time. After some algebra,

$$\begin{aligned} k(t) \;=\; & \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)}{\mu_{1-}^{PC}\mu_2^{PC}}e^{\mu_1^{PC}t} + \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)}{\mu_{2-}^{PC}\mu_1^{PC}}e^{\mu_2^{PC}t} + \frac{\left(\mu_1^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)k(0)}{\mu_1^{PC}-\mu_2^{PC}}e^{\mu_1^{PC}t} + \frac{\left(\mu_2^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)k(0)}{\mu_2^{PC}-\mu_1^{PC}}e^{\mu_2^{PC}t} \\ & + u_T(t)\left(\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} - \Gamma_{\lambda,\lambda}^{PC}b_{k,a}^{PC}}{\mu_1^{PC}\mu_2^{PC}}\right) \\ & + u_T(t)\left(\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_1^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_1^{PC}\left(\mu_1^{PC}-\mu_2^{PC}\right)}e^{\mu_1^{PC}(t-T)}\right) \\ & + u_T(t)\left(\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_2^{PC}\left(\mu_2^{PC}-\mu_1^{PC}\right)}e^{\mu_2^{PC}(t-T)}\right) \end{aligned}$$

where $u_T(t)$ is a step function that takes on a value of one for all $t \geq T$, and zero otherwise.

Recall that we assume the initial capital stock is at the steady-state level associated with the pre-shock technology level. As such, $k(0) = 0$:

$$
\begin{aligned}
k\left(t\right) &= \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)}{\mu_{1-}^{PC}\mu_2^{PC}}e^{\mu_1^{PC}t} + \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)}{\mu_{2-}^{PC}\mu_1^{PC}}e^{\mu_2^{PC}t} \\
&+u_T\left(t\right)\left(\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} - \Gamma_{\lambda,\lambda}^{PC}b_{k,a}^{PC}}{\mu_1^{PC}\mu_2^{PC}}\right) \\
&+u_T\left(t\right)\left(\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_1^{PC} - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_1^{PC}\left(\mu_1^{PC} - \mu_2^{PC}\right)}e^{\mu_1^{PC}(t-T)}\right) \\
&+u_T\left(t\right)\left(\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2^{PC} - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_2^{PC}\left(\mu_2^{PC} - \mu_1^{PC}\right)}e^{\mu_2^{PC}(t-T)}\right)
\end{aligned}
$$

This gives us the solution to a differential equation with one undetermined variable $\lambda(0)$. We now seek a path for $\begin{bmatrix} \lambda & k \end{bmatrix}'$ that is not explosive. In order to achieve this, we choose $\lambda(0)$ such that the explosive root $\mu_2^{PC}$ is 'zeroed out' for all $t > T$. Otherwise, the path for $k\left(t\right)$ will be explosive. This restriction on $\lambda(0)$ is:

$$
\left(\frac{\Gamma_{k,\lambda}^{PC}}{\mu_2^{PC} - \mu_1^{PC}}\right)\lambda(0) = -\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2^{PC} - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_2^{PC}\left(\mu_2^{PC} - \mu_1^{PC}\right)}e^{-\mu_2^{PC}T}
$$

This can be re-written as:

$$
\lambda(0) = -\left[\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC} + \left(\mu_2^{PC} - \Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\Gamma_{k,\lambda}^{PC}\mu_2^{PC}}\right]e^{-\mu_2^{PC}T} \tag{72}
$$

Let us also solve the second half of our laplace transform. This will allow us to study the path of $\lambda(t)$ over time. The first row of (71) gives us:

$$
\bar{\Lambda}\left(s\right) = \frac{\left(s - \Gamma_{k,k}^{PC}\right)\lambda(0) + \Gamma_{\lambda,k}^{PC}k(0)}{\left(s - \mu_1^{PC}\right)\left(s - \mu_2^{PC}\right)} + \left[\frac{\left(s - \Gamma_{k,k}^{PC}\right)b_{\lambda,a}^{PC} + \Gamma_{\lambda,k}^{PC}b_{k,a}^{PC}}{s\left(s - \mu_1^{PC}\right)\left(s - \mu_2^{PC}\right)}\right]e^{-sT}
$$

Now we can take the inverse Laplace transform of $\Lambda\left(s\right)$ to recover $\lambda$ as a function of time. After some algebra and setting $k(0) = 0$ we get:

$$
\begin{aligned}
\lambda\left(t\right) &= \frac{\left(\mu_1^{PC}-\Gamma_{k,k}^{PC}\right)\lambda(0)}{\mu_{1-}^{PC}\mu_2^{PC}}e^{\mu_1^{PC}t}+\frac{\left(\mu_2^{PC}-\Gamma_{k,k}^{PC}\right)\lambda(0)}{\mu_{2-}^{PC}\mu_1^{PC}}e^{\mu_2^{PC}t} \\
&+u_T\left(t\right)\left(\frac{\Gamma_{\lambda,k}^{PC}b_{k,a}^{PC}-\Gamma_{k,k}^{PC}b_{\lambda,a}^{PC}}{\mu_1^{PC}\mu_2^{PC}}\right) \\
&+u_T\left(t\right)\left(\frac{\left(\mu_1^{PC}-\Gamma_{k,k}^{PC}\right)b_{\lambda,a}^{PC}+\Gamma_{\lambda,k}^{PC}b_{k,a}^{PC}}{\mu_1^{PC}\left(\mu_1^{PC}-\mu_2^{PC}\right)}e^{\mu_1^{PC}(t-T)}\right) \\
&+u_T\left(t\right)\left(\frac{\left(\mu_2^{PC}-\Gamma_{k,k}^{PC}\right)b_{\lambda,a}^{PC}+\Gamma_{\lambda,k}^{PC}b_{k,a}^{PC}}{\mu_2^{PC}\left(\mu_2^{PC}-\mu_1^{PC}\right)}e^{\mu_2^{PC}(t-T)}\right)
\end{aligned}
$$

Given that we choose a $\lambda(0)$ such that the explosive root $\mu_2^{PC}$ is 'zeroed out' for all $t > T$, we can simplify our equations for the time paths of $k(t)$ and $\lambda(t)$ to the following:

$$
k\left(t\right)=\begin{cases}\frac{\Gamma_{k,\lambda}^{PC}\lambda(0)}{\mu_1^{PC}-\mu_2^{PC}}e^{\mu_1^{PC}t}+\frac{\Gamma_{k,\lambda}^{PC}\lambda(0)}{\mu_2^{PC}-\mu_1^{PC}}e^{\mu_2^{PC}t} & \text{for } t\in[0,T) \\ \frac{\Gamma_{k,\lambda}^{PC}\lambda(0)}{\mu_{1-}^{PC}\mu_2^{PC}}e^{\mu_1^{PC}t}+\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC}-\Gamma_{\lambda,\lambda}^{PC}b_{k,a}^{PC}}{\mu_1^{PC}\mu_2^{PC}}+\frac{\Gamma_{k,\lambda}^{PC}b_{\lambda,a}^{PC}+\left(\mu_1^{PC}-\Gamma_{\lambda,\lambda}^{PC}\right)b_{k,a}^{PC}}{\mu_1^{PC}\left(\mu_1^{PC}-\mu_2^{PC}\right)}e^{\mu_1^{PC}(t-T)} & t\geq T\end{cases} \tag{73}
$$

$$
\lambda\left(t\right)=\begin{cases}\frac{\left(\mu_1^{PC}-\Gamma_{k,k}^{PC}\right)\lambda(0)}{\mu_1^{PC}-\mu_2^{PC}}e^{\mu_1^{PC}t}+\frac{\left(\mu_2^{PC}-\Gamma_{k,k}^{PC}\right)\lambda(0)}{\mu_2^{PC}-\mu_1^{PC}}e^{\mu_2^{PC}t} & \text{for } t\in[0,T) \\ \frac{\left(\mu_1^{PC}-\Gamma_{k,k}^{PC}\right)\lambda(0)}{\mu_1^{PC}-\mu_2^{PC}}e^{\mu_1^{PC}t}+\frac{\Gamma_{\lambda,k}^{PC}b_{k,a}^{PC}-\Gamma_{k,k}^{PC}b_{\lambda,a}^{PC}}{\mu_1^{PC}\mu_2^{PC}}+\frac{\Gamma_{\lambda,k}^{PC}b_{k,a}^{PC}+\left(\mu_1^{PC}-\Gamma_{k,k}^{PC}\right)b_{\lambda,a}^{PC}}{\mu_1^{PC}\left(\mu_1^{PC}-\mu_2^{PC}\right)}e^{\mu_1^{PC}(t-T)} & t\geq T\end{cases} \tag{74}
$$

Equations (65), (66), (67), (73), and (74), along with equation (72) give us a stable solution to our model economy for a 1% technology shock that occurs in period $T$.

### B.1.5 Proofs & Expressions

In this section we will sign the various expressions needed for Lemma 1-5 and Theorem 1.

First, let us recall the proof for Lemma 1. For consumption, investment and hours to comove at time zero (on impact of the news) we required $\upsilon(1-\alpha) > 1$ which resulted in $\phi_I^{PC} < 0$ and thus a positively sloped $L_1$ line. For this section we will assume that $\upsilon > (1-\alpha)^{-1}$

**Assumption:** $\upsilon > \upsilon^* = (1-\alpha)^{-1}$

**Lemma B.1:** $\gamma_I^{PC} > 0$

*Proof.*

$$\begin{aligned}
\gamma_I^{PC} &= (\upsilon - 1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)} \\
&= (\upsilon - 1) - \frac{(1-\alpha)\delta}{(\rho + (1-\alpha)\delta)}(1-\sigma)\upsilon\alpha \\
&> (\upsilon - 1) - \upsilon\alpha \\
&= \upsilon(1-\alpha) - 1 > 0
\end{aligned}$$

□

**Lemma B.2:** The slope of the $L_2$ line in the consumption-investment space is positive.

*Proof.* The $L_2$ line is given by:

$$i = \frac{\left(\sigma + \gamma_I^{PC}\right)}{\gamma_I^{PC}} c + \frac{1}{\gamma_I^{PC}} \lambda \tag{75}$$

If $\gamma_I^{PC} > 0$ then the slope, $\frac{\left(\sigma + \gamma_I^{PC}\right)}{\gamma_I^{PC}}$, must be positive. □

**Lemma B.3:** $\phi_I^{PC}\sigma + \gamma_I^{PC} > 0$

*Proof.*

$$\begin{aligned}
\phi_I^{PC}\sigma + \gamma_I^{PC} &= (1-(1-\alpha)\upsilon)\, s_I\sigma + (\upsilon - 1) - \frac{\upsilon(1-\alpha)(1-\sigma)\, s_I}{(1-s_I)} \\
&= s_I\sigma - (1-\alpha)\,\upsilon s_I\sigma + (\upsilon-1) - \frac{((1-\alpha)\,\upsilon s_I - (1-\alpha)\,\upsilon s_I\sigma)}{1-s_I} \\
&= s_I\sigma + (\upsilon-1) - \frac{(1-\alpha)\,\upsilon s_I}{1-s_I} + \frac{(1-\alpha)\,\upsilon s_I^2\sigma}{1-s_I} \\
&= \upsilon - (1-\sigma s_I)\left[1 + (1-\alpha)\,\upsilon\frac{s_I}{1-s_I}\right] \\
&= \upsilon\left\{1-(1-\sigma s_I)\left[\frac{1}{\upsilon} + (1-\alpha)\frac{s_I}{1-s_I}\right]\right\} > 0
\end{aligned}$$

The last inequality follows from the following observations:

If $(1-\sigma s_I) < 0$, then we are done. If $(1-\sigma s_I) > 0$, then we can define:

$$\chi(\upsilon) = \left\{1-(1-\sigma s_I)\left[\frac{1}{\upsilon} + (1-\alpha)\frac{s_I}{1-s_I}\right]\right\}$$

Now,

$$\begin{aligned}
\chi(\infty) &= \left\{1-(1-\sigma s_I)(1-\alpha)\frac{s_I}{1-s_I}\right\} \\
&= \left\{1-(1-\sigma s_I)\,\alpha\frac{(1-\alpha)\,\delta}{\rho + (1-\alpha)\,\delta}\right\} > 0
\end{aligned}$$

$$\begin{aligned}
\chi(\upsilon_c) &= \left\{1-(1-\sigma s_I)\left[(1-\alpha) + (1-\alpha)\frac{s_I}{1-s_I}\right]\right\} \\
&= \left[1-(1-\sigma s_I)\frac{(1-\alpha)}{1-s_I}\right] \\
&= \left[1-(1-\sigma s_I)\frac{(1-\alpha)\,\rho + (1-\alpha)\,\delta}{\rho + (1-\alpha)\,\delta}\right] > 0
\end{aligned}$$

$$\chi'(\upsilon) \;=\; \frac{(1-\sigma s_I)}{\upsilon^2} > 0$$

Therefore, for $\upsilon \in (\upsilon_c, \infty), \chi(\upsilon) > 0$. Given Assumption 1 this translates to $\chi(\upsilon) > 0$ $\square$

**Lemma B.4:** $\tau^{PC}_{c,k}$, $\tau^{PC}_{c,\lambda}$, $\tau^{PC}_{c,a}$, $\tau^{PC}_{i,k}$, $\tau^{PC}_{i,\lambda}$, $\tau^{PC}_{i,a}$, $\tau^{PC}_{n,k}$, $\tau^{PC}_{n,\lambda}$, and $\tau^{PC}_{n,a}$ are all positive

*Proof.* This result follows trivially lemma's B.2 and B.3 and our assumption, $\upsilon > (1-\alpha)^{-1}$, which ensures $\phi^{PC}_I < 0$. $\square$

**Lemma B.5:** $\Gamma^{PC}_{k,\lambda}$ and $\Gamma^{PC}_{\lambda,\lambda}$ are both positive.

*Proof.* $\Gamma^{PC}_{k,\lambda} > 0$ follows trivially from lemma B.3 and our assumption, $\upsilon > (1-\alpha)^{-1} \Rightarrow \phi^{PC}_I < 0$.

To prove $\Gamma^{PC}_{\lambda,\lambda} = \frac{(\rho+\delta)\left(\phi^{PC}_I - (1-\upsilon(1-s_I))\right)}{\phi^{PC}_I \sigma + \gamma^{PC}_I} > 0$ it suffices to show $\left(\phi^{PC}_I - (1-\upsilon(1-s_I))\right) > 0$. By lemma B.3 we know $\phi^{PC}_I \sigma + \gamma^{PC}_I > 0$.

$$\begin{aligned}
\phi^{PC}_I - (1-\upsilon(1-s_I)) &= \frac{(1-\upsilon(1-\alpha))\,\alpha\delta}{\rho+\delta} + \frac{(\upsilon-1)\,\rho + (\upsilon(1-\alpha)-1)\,\delta}{\rho+\delta} \\
&= \frac{(\upsilon-1)\,\rho + (1-\alpha)\,(\upsilon(1-\alpha)-1)\,\delta}{\rho+\delta} > 0
\end{aligned}$$

$\square$

Lemma B.4 above proves that $\Gamma^{PC}_{k,\lambda}$ and $\Gamma^{PC}_{\lambda,\lambda}$ are both positive. For our analysis we do not need to sign $\Gamma^{PC}_{k,k}$ and $\Gamma^{PC}_{\lambda,k}$ [39].

**Lemma B.6:** One of the eigenvalues of the $\Gamma^{PC}$ matrix is positive and other negative.

[39] It can be shown that both these variables are positive for $\sigma \in [0,1]$

*Proof.* The product of the eigenvalues is given by the determinant of $\Gamma^{PC}$:

$$\begin{aligned}
\det\left(\Gamma^{PC}\right) &= \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)^2}\left[\left(\alpha\left(\gamma_I^{PC}+\sigma\right)-\phi_I^{PC}\sigma-\gamma_I^{PC}\right)\left(\left(1-\upsilon\left(1-s_I\right)\right)-\phi_I^{PC}\right)\right] \\
&\quad+\frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)^2}\left[\left(1-\phi_I^{PC}\right)\left(\left(1-\alpha\right)\gamma_I^{PC}-\left(\alpha-\phi_I^{PC}\right)\sigma+\alpha\upsilon\left(1-s_I\right)\sigma\right)\right] \\
&= \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)^2}\left[\left(\left(\alpha-1\right)\gamma_I^{PC}-\left(\phi_I^{PC}-\alpha\right)\sigma\right)\left(\left(1-\upsilon\left(1-s_I\right)\right)-\phi_I^{PC}\right)\right] \\
&\quad+\frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)^2}\left[\left(1-\phi_I^{PC}\right)\left(\left(1-\alpha\right)\gamma_I^{PC}-\left(\alpha-\phi_I^{PC}\right)\sigma+\alpha\upsilon\left(1-s_I\right)\sigma\right)\right] \\
&= \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)^2}\left[\left(\left(1-\alpha\right)\gamma_I^{PC}-\left(\alpha-\phi_I^{PC}\right)\sigma\right)\upsilon\left(1-s_I\right)-\left(1-\phi_I^{PC}\right)\alpha\sigma\upsilon\left(1-s_I\right)\right] \\
&= \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)^2}\left[\left(\left(1-\alpha\right)\gamma_I^{PC}-\left(\alpha-\phi_I^{PC}\right)\sigma\right)\upsilon\left(1-s_I\right)-\left(1-\phi_I^{PC}\right)\alpha\sigma\upsilon\left(1-s_I\right)\right] \\
&= \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)^2}\left[\upsilon\left(1-s_I\right)\left(1-\alpha\right)\left(\gamma_I^{PC}+\sigma\phi_I^{PC}\right)\right]
\end{aligned}$$

$$= \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{PC}\sigma+\gamma_I^{PC}\right)}\left[\upsilon\left(1-s_I\right)\left(1-\alpha\right)\right]<0$$

As the product of the eigenvalues is negative it follows that the eigenvalues have opposite signs. □

**Lemma B.7:** The sum of the eigenvalues of the $\Gamma^{PC}$ matrix is $\rho$.

*Proof.* The sum of the eigenvalues is given by the trace of $\Gamma^{PC}$:

$$
\begin{aligned}
\mathrm{tr}\left(\Gamma^{PC}\right) &= \Gamma^{PC}_{\lambda,\lambda} + \Gamma^{PC}_{k,k} \\
&= \frac{\delta\left(\left(\alpha-1\right)\gamma_I^{PC} + \alpha\sigma - \phi_I^{PC}\sigma\right) - \delta\left(1-\upsilon\left(1-s_I\right)-\phi_I^{PC}\right) - \rho\left(1-\upsilon\left(1-s_I\right)-\phi_I^{PC}\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I^{PC} + \sigma\delta\left(\alpha-\phi_I^{PC}\right) + \left(\rho+\delta\right)\phi_I^{PC} - \left(\rho+\delta\right)\left(1-\upsilon\left(1-s_I\right)\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I^{PC} + \sigma\delta\left(\alpha-\phi_I^{PC}\right) + \left(1-\left(1-\alpha\right)\upsilon\right)\alpha\delta + \left(\upsilon-1\right)\rho + \left(\upsilon\left(1-\alpha\right)-1\right)\delta}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I^{PC} + \sigma\delta\left(\alpha-\phi_I^{PC}\right) + \left(\upsilon-1\right)\rho + \left(1-\alpha\right)\left(\upsilon\left(1-\alpha\right)-1\right)\delta}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I^{PC} + \sigma\delta\left(\alpha-\phi_I^{PC}\right) + \gamma_I^{PC}\left(\rho+\left(1-\alpha\right)\delta\right) - \sigma\left(\upsilon\left(1-\alpha\right)\alpha\delta\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{\rho\gamma_I^{PC} + \sigma\delta\left(\alpha-\phi_I^{PC}-\upsilon\alpha\left(1-\alpha\right)\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{\rho\gamma_I^{PC} + \sigma\delta\left(\alpha\left(1-\upsilon\left(1-\alpha\right)\right)-\phi_I^{PC}\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{\rho\gamma_I^{PC} + \sigma\delta\left(\left(\alpha-s_I\right)\left(1-\upsilon\left(1-\alpha\right)\right)\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{\rho\left(\gamma_I^{PC} + \sigma s_I\left(1-\upsilon\left(1-\alpha\right)\right)\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \frac{\rho\left(\phi_I^{PC}\sigma + \gamma_I^{PC}\right)}{\phi_I^{PC}\sigma + \gamma_I^{PC}} \\
&= \rho > 0
\end{aligned}
$$

□

## B.2 A Model with Labor Externalities

### B.2.1 The Model Economy

A private agent in the economy has has the following preferences

$$U = \left(1-\sigma\right)^{-1}\int_0^{\infty} e^{-\rho t}\left[C\left(t\right)\exp\left(-N\left(t\right)\bar{N}\left(t\right)^{-\gamma_N}\right)\right]^{1-\sigma} dt$$

over time paths for individual consumption $C$, hours worked $N$, and aggregate hours $\bar{N}$. We assume this functional form for the utility to preserve balanced growth. Also, $\rho = 1/\beta - 1 > 0$ and $\sigma \geq 0$, where $\beta$ is the stochastic discount factor and $\sigma$ is the inverse of the intertemporal elasticity of substitution. $\gamma_{\bar{N}} \in (-\infty, 1]$ measures the degree of the labor externality

The private agent is subject to the following constraints:

$$C(t) + I(t) = K(t)^{\alpha} (A(t) N(t))^{1-\alpha} \tag{76}$$

$$\dot{K}(t) = I(t) - \delta K(t) - \frac{\psi_I}{2} \left(1 - \frac{I(t)}{\delta K(t)}\right)^2 I(t) \tag{77}$$

Here $K$, $I$ and $A$ represent capital, investment and the level of technology. The path of technology and the initial capital stock are exogenous. The depreciation rate, $\delta$, and the elasticity of output with respect to capital, $\alpha$, both lie between zero and one. $\psi_I \in [0, \infty)$ gives a measure of the magnitude of the convex investment adjustment costs.

We include convex adjustment costs as a way to generate an increase in the shadow value of investment in response to a news technology shock. In the basic model we achieved this by a low IES. Unfortunately, in a model with labor externalities a low IES leads to a non-stable solution.

Next, let us define the exogenous processes - the technology news shock. The private agents have perfect foresight, with

$$A(t) = \begin{cases} \bar{A} & \text{for } t \in [0, T) \\ \tilde{A} = 1.01 \times \bar{A} & t \geq T \end{cases}$$

For the contemporaneous improvements case $T = 0$ in the above specification.

### B.2.2 The Model Economy's First Order Conditions

The private agents choose $C$, $I$, $K$, and $N$ to maximize $U$ subject to (76) and (77) taking as given the initial condition $K(0)$ and time path of technology. We can express the problem as a current value Hamiltonian:

$$H = C^{1-\sigma} \exp\left[-(1-\sigma) N \bar{N}^{-\gamma_{\bar{N}}^{LE}}\right] + \Lambda\left(I - \delta K - \frac{\psi_I}{2}\left(1 - \frac{I}{\delta K}\right)^2 I\right) + \Phi\left(K^\alpha (AN)^{1-\alpha} - F(C, I)\right)$$

The first-order necessary conditions at an interior solution in a symmetric equilibrium satisfy :

$$-\frac{U_N}{U_C} = (1-\alpha)\frac{F}{N} \tag{78}$$

$$U_C = \Lambda\left[1 - \frac{\psi_I}{2}\left(1 - \frac{I}{\delta K}\right)^2 + \psi_I\left(1 - \frac{I}{\delta K}\right)\frac{I}{\delta K}\right] \tag{79}$$

$$\frac{\dot{\Lambda}}{\Lambda} - \rho = \delta + \psi_I\left(1 - \frac{I}{\delta K}\right)\left(\frac{I}{\delta K}\right)^2 - \alpha\frac{F}{K}\frac{U_C}{\Lambda} \tag{80}$$

along with our initial condition on capital and a transversality condition on $\Lambda$.

Equation (78) is the intratemporal Euler equation between consumption and labor hours, equation (79) is the intratemporal Euler equation between consumption and investment, and equation (80) is the optimal capital accumulation equation.

### B.2.3 The Model Economy Log Linearized and Simplified

Given the first order conditions in the previous section our model economy can be described by the following five log linearized equations:

$$(1-s_I)c + s_I i = \alpha k + (1-\alpha)(a+n) \tag{81}$$

$$\frac{s_I}{1-\gamma_N}(i-c) = n \tag{82}$$

$$\lambda = -\sigma c - \frac{(1-\sigma)(1-\alpha)}{(1-s_I)}n + \psi_I(i-k) \tag{83}$$

$$\dot{k} = \delta \left( i - k \right) \tag{84}$$

$$\dot{\lambda} = - \left( \rho + \delta \right) \left[ \left( 1 - s_I \right) \left( c - i \right) + i - k \right] + \rho \psi_I \left( i - k \right) \tag{85}$$

Here, $s_I = \frac{\alpha\delta}{\rho+\delta}$. Notice that the term $\gamma_N$ only enters into equation (82).

We can substitute (82) into (81) to get the consumption-investment production frontier ($L_1$ line):

$$\left(1 - \phi_I^{LE}\right) c + \phi_I^{LE} i = \alpha k + \left(1 - \alpha\right) a \tag{86}$$

Here, $\phi_I^{LE} = \left(1 - \frac{1-\alpha}{1-\gamma_N}\right) s_I$

We can also substitute (82) into (83) to get the consumption-investment euler equation ($L_2$ line):

$$\left(\gamma_I^{LE} + \psi_I\right) i - \left(\sigma + \gamma_I^{LE}\right) c = \lambda \tag{87}$$

Here, $\gamma_I^{LE} = -\frac{(1-\alpha)(1-\sigma)s_I}{(1-s_I)(1-\gamma_N)}$.

Equations (86) and (87) now give us a system of equations in $i$ and $c$ (treating $\lambda$, $k$, and $a$ as exogenous).

We also solve the system of equations above for $c$, $i$, $n$, $\dot{k}$, and $\dot{\lambda}$, assuming as given the state variable $\lambda$ and $k$, and the exogenous variable $a$:

$$c = \tau_{c,k}^{LE} k + \tau_{c,\lambda}^{LE} \lambda + \tau_{c,a}^{LE} a \tag{88}$$

$$i = \tau_{i,k}^{LE} k + \tau_{i,\lambda}^{LE} \lambda + \tau_{i,a}^{LE} a \tag{89}$$

$$n = \tau_{n,k}^{LE} k + \tau_{n,\lambda}^{LE} \lambda + \tau_{n,a}^{LE} a \tag{90}$$

$$\dot{k} = \Gamma^{LE}_{k,k} k + \Gamma^{LE}_{k,\lambda} \lambda + b_{k,a} a$$

$$\dot{\lambda} = \Gamma^{LE}_{\lambda,k} k + \Gamma^{LE}_{\lambda,\lambda} \lambda + b^{LE}_{\lambda,a} a$$

where,

$$\tau_{c,k}^{LE} = \frac{\partial c}{\partial k} = \frac{\gamma_{I,LE}^{LE}\alpha + \psi_I\left(\alpha - \phi_{I,LE}^{LE}\right)}{\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right)} \qquad \Gamma_{k,k}^{LE} = \frac{\partial \dot{k}}{\partial k} = \frac{-\delta\left((1-\alpha)\gamma_{I,LE}^{LE} + \phi_{I,LE}^{LE}\sigma - \alpha\sigma\right)}{\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right)}$$

$$\tau_{c,\lambda}^{LE} = \frac{\partial c}{\partial \lambda} = \frac{-\phi_{I,LE}^{LE}}{\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right)} \qquad \Gamma_{k,\lambda}^{LE} = \frac{\partial \dot{k}}{\partial \lambda} = \frac{\delta\left(1-\phi_{I,LE}^{LE}\right)}{\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right)}$$

$$\tau_{c,a}^{LE} = \frac{\partial c}{\partial a} = \frac{\left(\gamma_{I,LE}^{LE} + \psi_I\right)(1-\alpha)}{\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right)} \qquad \Gamma_{\lambda,k}^{LE} = \frac{\partial \dot{\lambda}}{\partial k} = \frac{(\rho+\delta)\left((1-\alpha)\gamma_{I,LE}^{LE} + \phi_{I,LE}^{LE}\sigma - \alpha\sigma + \alpha\sigma(1-s_I)\right)}{\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right)}$$

$$\tau_{i,k}^{LE} = \frac{\partial i}{\partial k} = \frac{\alpha\left(\gamma_{I,LE}^{LE} + \sigma\right) + \psi_I\left(1-\phi_{I,LE}^{LE}\right)}{\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right)} \qquad + \frac{\psi_I\left[(\rho+\delta)(1-S_I)(1-\alpha) - \rho\left((1-\alpha)\gamma_I^{LE} + \left(\phi_I^{LE} - \alpha\right)\sigma\right)\right]}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)}$$

$$\tau_{i,\lambda}^{LE} = \frac{\partial i}{\partial \lambda} = \frac{1-\phi_I^{LE}}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)} \qquad \Gamma_{\lambda,\lambda}^{LE} = \frac{\partial \dot{\lambda}}{\partial \lambda} = \frac{(\rho+\delta)\left(\phi_I^{LE} - s_I\right) + \rho\psi_I\left(1-\phi_I^{LE}\right)}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)}$$

$$\tau_{i,a}^{LE} = \frac{\partial i}{\partial a} = \frac{(1-\alpha)\left(\gamma_I^{LE} + \sigma\right)}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)} \qquad b_{k,a}^{LE} = \frac{\partial \dot{k}}{\partial a} = \frac{\delta\left(\gamma_I^{LE} + \sigma\right)(1-\alpha)}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)}$$

$$\tau_{n,k}^{LE} = \frac{\partial n}{\partial k} = \frac{1}{1-\gamma_N}\frac{s_I(\alpha\sigma + \psi_I(1-\alpha))}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)} \qquad b_{\lambda,a}^{LE} = \frac{\partial \dot{\lambda}}{\partial a} = \frac{-(\rho+\delta)(1-\alpha)\left(\sigma s_I + \gamma_I^{LE}\right)}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)}$$

$$\tau_{n,\lambda}^{LE} = \frac{\partial n}{\partial \lambda} = \frac{1}{1-\gamma_N}\frac{s_I}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)} \qquad + \frac{\psi_I(1-\alpha)\left[\rho\left(\sigma + \gamma_I^{LE}\right) - (\rho+\delta)(1-S_I)\right]}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)}$$

$$\tau_{n,a}^{LE} = \frac{\partial n}{\partial a} = \frac{1}{1-\gamma_N}\frac{s_I(1-\alpha)(\sigma - \psi_I)}{\phi_I^{LE}\sigma + \gamma_I^{LE} + \psi_I\left(1-\phi_I^{LE}\right)}$$

Recall: $s_I = \frac{\alpha\delta}{\rho+\delta}$, $\phi_I^{LE} = \left(1 - \frac{1-\alpha}{1-\gamma_N}\right)s_I$, and $\gamma_I^{LE} = -\frac{(1-\alpha)(1-\sigma)s_I}{(1-s_I)(1-\gamma_N)}$

### B.2.4 The Dynamic System

The general solution to the dynamic system remains the same as before, but now with different coefficient values for $\tau_{x,x}^{LE}$'s, $\Gamma_{x,x}^{LE}$'s and $b_{x,x}$'s. The new values for $\tau_{x,x}^{LE}$'s, $\Gamma_{x,x}^{LE}$'s and $b_{x,x}$'s are given above. The solution to the dynamic system is:

$$\lambda(0) = -\left[\frac{\Gamma_{k,\lambda}^{LE} b_{\lambda,a}^{LE} + \left(\mu_2^{LE} - \Gamma_{\lambda,\lambda}^{LE}\right) b_{k,a}^{LE}}{\Gamma_{k,\lambda}^{LE}\mu_2^{LE}}\right] e^{-\mu_2^{LE}T} \tag{91}$$

$$k\left(t\right)=\begin{cases} \frac{\Gamma_{k,\lambda}^{LE}\lambda(0)}{\mu_{1}^{LE}-\mu_{2}^{LE}}e^{\mu_{1}^{LE}t}+\frac{\Gamma_{k,\lambda}^{LE}\lambda(0)}{\mu_{2}^{LE}-\mu_{1}^{LE}}e^{\mu_{2}^{LE}t} & \text{for } t\in[0,T) \\ \frac{\Gamma_{k,\lambda}^{LE}\lambda(0)}{\mu_{1-}^{LE}\mu_{2}^{LE}}e^{\mu_{1}^{LE}t}+\frac{\Gamma_{k,\lambda}^{LE}b_{\lambda,a}^{LE}-\Gamma_{\lambda,\lambda}^{LE}b_{k,a}^{LE}}{\mu_{1}^{LE}\mu_{2}^{LE}}+\frac{\Gamma_{k,\lambda}^{LE}b_{\lambda,a}^{LE}+\left(\mu_{1}^{LE}-\Gamma_{\lambda,\lambda}^{LE}\right)b_{k,a}^{LE}}{\mu_{1}^{LE}\left(\mu_{1}^{LE}-\mu_{2}^{LE}\right)}e^{\mu_{1}^{LE}(t-T)} & t\geq T \end{cases} \tag{92}$$

$$\lambda\left(t\right)=\begin{cases} \frac{\left(\mu_{1}^{LE}-\Gamma_{k,k}^{LE}\right)\lambda(0)}{\mu_{1}^{LE}-\mu_{2}^{LE}}e^{\mu_{1}^{LE}t}+\frac{\left(\mu_{2}^{LE}-\Gamma_{k,k}^{LE}\right)\lambda(0)}{\mu_{2}^{LE}-\mu_{1}^{LE}}e^{\mu_{2}^{LE}t} & \text{for } t\in[0,T) \\ \frac{\left(\mu_{1}^{LE}-\Gamma_{k,k}^{LE}\right)\lambda(0)}{\mu_{1}^{LE}-\mu_{2}^{LE}}e^{\mu_{1}^{LE}t}+\frac{\Gamma_{\lambda,k}^{LE}b_{k,a}^{LE}-\Gamma_{k,k}^{LE}b_{\lambda,a}^{LE}}{\mu_{1}^{LE}\mu_{2}^{LE}}+\frac{\Gamma_{\lambda,k}^{LE}b_{k,a}^{LE}+\left(\mu_{1}^{LE}-\Gamma_{k,k}^{LE}\right)b_{\lambda,a}^{LE}}{\mu_{1}^{LE}\left(\mu_{1}^{LE}-\mu_{2}^{LE}\right)}e^{\mu_{1}^{LE}(t-T)} & t\geq T \end{cases} \tag{93}$$

Equations (88), (89), (90), (92), and (93), along with equation (91) give us a stable solution to our model economy for a 1% technology shock that occurs in period $T$.

### B.2.5 Proofs & Expressions

In this section we will sign the various expressions needed for Lemma 7-9 and Theorem 2.

First, let us recall the proof for Lemma 6. For consumption, investment and hours to comove at time zero (on impact of the news) we required $\gamma_N > \alpha$ which resulted in $\phi_I^{LE} < 0$ and thus a positively sloped $L_1$ line. Also, by lemma 8 we know that for a stable solution we need $\psi_I > -\frac{\gamma_I^{LE}+\phi_I^{LE}\sigma}{1-\phi_I^{LE}}$. For this section we will assume both that $\gamma_N > \alpha$ and $\psi_I > -\frac{\gamma_I^{LE}+\phi_I^{LE}\sigma}{1-\phi_I^{LE}}$

**Assumption:** $\gamma_N >= \gamma_N^* = \alpha$ and $\psi_I > \psi_I^+ = -\frac{\gamma_I^{LE}+\phi_I^{LE}\sigma}{1-\phi_I^{LE}}$

**Lemma B.8:** $\phi_{I,LE}^{LE}\sigma + \gamma_{I,LE}^{LE} + \psi_I\left(1-\phi_{I,LE}^{LE}\right) > 0$

*Proof.* Follows trivially from our assumption that $\psi_I > -\frac{\gamma_I^{LE}+\phi_I^{LE}\sigma}{1-\phi_I^{LE}}$ □

**Lemma B.9:** $\tau_{c,k}^{LE}$, $\tau_{c,\lambda}^{LE}$, $\tau_{i,k}^{LE}$, $\tau_{i,\lambda}^{LE}$, $\tau_{n,k}^{LE}$, and $\tau_{n,\lambda}^{LE}$ are all positive

*Proof.* This result follows trivially lemma B.8 and our assumption, $\gamma_N > \alpha$, which ensures $\phi_I^{LE} < 0$. □

**Lemma B.10:** $\Gamma_{k,\lambda}^{LE}$ and $\Gamma_{\lambda,\lambda}^{LE}$ are both positive.

*Proof.* $\Gamma_{k,\lambda}^{LE} > 0$ follows trivially from lemma B.8 and our assumption, $\gamma_N > \alpha \Rightarrow \phi_I^{LE} < 0$.

On the other hand $\Gamma_{\lambda,\lambda}^{LE} > 0$ follows trivially from lemma B.8, and our assumptions, $\gamma_N > \alpha \Rightarrow \phi_I^{LE} < 0$ and $\psi_I > -\frac{\gamma_I^{LE} + \phi_I^{LE}\sigma}{1 - \phi_I^{LE}}$. □

## B.3 A Model with Forward Compatible Investment

### B.3.1 The Model Economy

A social planner has the following preferences

$$U = (1-\sigma)^{-1} \int_0^{\infty} e^{-\rho t} \left[C(t) \exp(-N(t))\right]^{1-\sigma} dt$$

over time paths for consumption $C$ and hours worked $N$. We assume this functional form for the utility to preserve balanced growth. Also, $\rho = 1/\beta - 1 > 0$ and $\sigma \geq 0$.

The planner is subject to the following constraints:

$$F(C(t), I(t)) = K(t)^{\alpha} N(t)^{1-\alpha} \tag{94}$$

$$\dot{K}(t) = Q(t) I(t) - \delta K(t) + \left(K(t) - e^{-\delta T} \bar{K}\right) P\left(\tilde{Q}, t, T, \epsilon\right) \tag{95}$$

Here $K$, $I$ and $q$ represent capital, investment and the level of technology embodied in the capital created at a point in time. The path of technology and the initial capital stock are exogenous. The depreciation rate, $\delta$, and the elasticity of output with respect to capital, $\alpha$, both lie between zero and one. For $\epsilon \to 0$, $P\left(\tilde{Q}, t, T, \epsilon\right)$ represents the level of forward compatibility. We talk about this process in detail later. In short, this represents the idea that capital might embody technology that does not become useful until a future date - forward compatibility of capital with future technology.

Further, we assume:

$$F(C, I) \equiv \left[\theta C^{\upsilon} + (1-\theta) I^{\upsilon}\right]^{1/\upsilon}$$

where $\theta \in (0,1)$ and $\upsilon \geq 1$. When $\upsilon = 1$, the equation collapses to the standard neo-classical case, which has infinite substitutability between the two goods. As $\upsilon$ increases, the complementarity between the production of the two goods increases. If $\upsilon = \infty$, the production frontier takes a Leontief form.

Next, let us define the exogenous processes - the level of capital embodied technology and the process that defines forward compatibility. We will consider two types of capital embodied technology shocks that occur at time zero: contemporaneous improvements, i.e. a *current shock*, and news of future improvements, i.e. a *news shock*. For both types of shocks, suppose the capital stock is at an initial steady state consistent with a particular fixed and unchanged level of technology $\bar{q}$. In the case of the future shock, the planner again has perfect foresight, with

$$Q(t) = \begin{cases} \bar{Q} & \text{for } t \in [0, T) \\ \tilde{Q} = 1.01 \times \bar{Q} & t \geq T \end{cases}$$

For the contemporaneous improvements case $T = 0$ in the above specification

The process that defines the forward compatibility is as follows:

$$P\left(\tilde{Q}, t, T, \varepsilon\right) = \begin{cases} 0 & \text{for } t \in [0, T) \cup (T+\epsilon, \infty) \\ \frac{\tau^{FC}\tilde{Q}}{(1-e^{-\epsilon})\bar{Q}} & t \in [T, T+\epsilon] \end{cases}$$

$\tau^{FC} \in [0,1]$ here represents the degree of forward compatibility of the capital accumulated between time 0 and the current period for $\epsilon \rightarrow 0$. When $\tau^{FC} = 0$, this capital embodies none of the technology that will be useful this period onwards. While when $\tau^{FC} = 1$, this capital embodies all of the technology that will become useful from this period onwards. At time T, all future investment becomes more productive in augmenting the capital stock. This can be equivalently thought of as a fall in the price of investment.

### B.3.2 The Model Economy's First Order Conditions

The social planner chooses $C$, $I$, $K$, and $N$ to maximize $U$ subject to (94) and (95) taking as given the initial condition $K(0)$ and time path of technology and shocks to the capital stock. We can express the problem as a current value Hamiltonian:

$$H = C^{1-\sigma} \exp\left[-(1-\sigma) N\right] + \Lambda \left(QI - \delta K + P_{\varepsilon}\mathcal{K}\right) + \Phi\left(K^{\alpha} N^{1-\alpha} - F(C, I)\right)$$

The first-order necessary conditions at an interior solution satisfy :

$$-\frac{U_N}{U_C} = (1-\alpha)\frac{F}{N}(F_C)^{-1} \tag{96}$$

$$\frac{U_C}{Q\Lambda} = \frac{F_C}{F_I} \tag{97}$$

$$\frac{\dot{\Lambda}}{\Lambda} - \rho = \delta - P_{\epsilon} - \alpha Q \frac{F}{K}(F_I)^{-1} \tag{98}$$

along with our initial condition on capital and a transversality condition on $\Lambda$.

### B.3.3 The Model Economy Log Linearized and Simplified

Given the first order conditions in the previous section our model economy can be described by the following five log linearized equations:

$$(1-s_I)\, c + s_I i = \alpha k + (1-\alpha)\, n \tag{99}$$

$$\upsilon s_I (i - c) = n \tag{100}$$

$$\lambda + q = (1-\upsilon)(c - i) - \sigma c - \frac{(1-\sigma)(1-\alpha)}{(1-s_I)} n \tag{101}$$

$$\dot{k} = \delta(q + i - k) + (1 - e^{\delta T}) p_{\epsilon} \tag{102}$$

$$\dot{\lambda} = -p_\epsilon - (\rho + \delta)\left[\upsilon\left(1 - s_I\right)(c - i) + q + i - k\right] \tag{103}$$

Here, $s_I = \frac{\alpha\delta}{\rho+\delta}$.

We can substitute (100) into (99) to get:

$$\left(1 - \phi_I^{FC}\right)c + \phi_I^{FC} i = \alpha k \tag{104}$$

Here, $\phi_I^{FC} = \left(1 - (1 - \alpha)\,\upsilon\right) s_I$

We can also substitute (100) into (101) to get:

$$\gamma_I^{FC} i - \left(\sigma + \gamma_I^{FC}\right)c = \lambda + q \tag{105}$$

Here, $\gamma_I^{FC} = (\upsilon - 1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$.

Equations (104) and (105) now give us a system of equations in $i$ and $c$ (treating $\lambda$, $k$, and $q$ as exogenous). Solving this system, we get:

$$c = \left(\frac{\phi_I^{FC}}{\phi_I^{FC}\sigma + \gamma_I^{FC}}\right)\left(\frac{\gamma_I^{FC}\alpha}{\phi_I^{FC}}k - \lambda - q\right) \tag{106}$$

$$i = \left(\frac{\gamma_I^{FC} + \sigma}{\phi_I^{FC}\sigma + \gamma_I^{FC}}\right)\alpha k + \left(\frac{1 - \phi_I^{FC}}{\phi_I^{FC}\sigma + \gamma_I^{FC}}\right)(\lambda + q) \tag{107}$$

Further, substituting these above equations into (100), we also get an expression for $n$ in terms of $\lambda$, $k$, and $q$:

$$n = \frac{\upsilon s_I}{\phi_I^{FC}}\left(\left(1 - \frac{\gamma_I^{FC}}{\phi_I^{FC}\sigma + \gamma_I^{FC}}\right)\alpha k + \frac{\phi_I^{FC}}{\phi_I^{FC}\sigma + \gamma_I^{FC}}(\lambda + q)\right) \tag{108}$$

Finally, we can also simplify our two dynamic equations (102) and (103) in terms of $\lambda$, $k$, and $q$, by using (106), (107), and (108):

$$\dot{k} = \frac{\delta}{\phi_I^{FC}\sigma + \gamma_I^{FC}} \left[ -\left( (1-\alpha)\gamma_I^{FC} + \phi_I^{FC}\sigma - \alpha\sigma \right) k + \left( 1 - \phi_I^{FC} \right) \lambda + \left( 1 - \phi_I^{FC} + \phi_I^{FC}\sigma + \gamma_I^{FC} \right) q \right] + \left( 1 - e^{-\delta T} \right) p_\epsilon \tag{109}$$

$$\dot{\lambda} = -p_\epsilon + \frac{\rho + \delta}{\phi_I^{FC}\sigma + \gamma_I^{FC}} \left[ \left( (1-\alpha)\gamma_I^{FC} + \phi_I^{FC}\sigma - \alpha\sigma + \alpha\sigma\upsilon (1 - s_I) \right) k \right] + \frac{\rho + \delta}{\phi_I^{FC}\sigma + \gamma_I^{FC}} \left[ \left( \phi_I^{FC} - (1 - \upsilon (1 - s_I)) \right) \lambda + \left( \phi_I^{FC} - (1 - \upsilon (1 - s_I)) - \phi_I^{FC}\sigma - \gamma_I^{FC} \right) q \right] \tag{110}$$

Equations (106) - (110) now give us a simplified system of equations that define a dynamic stochastic general equilibrium for our model economy. For ease of use in this appendix we take this one step further and rewrite these equations as follows:

$$c = \tau_{c,k}^{FC} k + \tau_{c,\lambda}^{FC} \lambda + \tau_{c,q}^{FC} q \tag{111}$$

$$i = \tau_{i,k}^{FC} k + \tau_{i,\lambda}^{FC} \lambda + \tau_{i,q}^{FC} q \tag{112}$$

$$n = \tau_{n,k}^{FC} k + \tau_{n,\lambda}^{FC} \lambda + \tau_{n,q}^{FC} q \tag{113}$$

$$\dot{k} = \Gamma_{k,k}^{FC} k + \Gamma_{k,\lambda}^{FC} \lambda + b_{k,q} q + b_{k,p}^{FC} p_\epsilon$$

$$\dot{\lambda} = \Gamma_{\lambda,k}^{FC} k + \Gamma_{\lambda,\lambda}^{FC} \lambda + b_{\lambda,q} q + b_{\lambda,p}^{FC} p_\epsilon$$

Here,

$$\tau_{c,k}^{FC} = \frac{\partial c}{\partial k} = \frac{\gamma_I^{FC}\alpha}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad \Gamma_{k,k}^{FC} = \frac{\partial \dot{k}}{\partial k} = \frac{-\delta\left((1-\alpha)\gamma_I^{FC}+\phi_I^{FC}\sigma-\alpha\sigma\right)}{\phi_I^{FC}\sigma+\gamma_I^{FC}}$$

$$\tau_{c,\lambda}^{FC} = \frac{\partial c}{\partial \lambda} = \frac{-\phi_I^{FC}}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad \Gamma_{k,\lambda}^{FC} = \frac{\partial \dot{k}}{\partial \lambda} = \frac{\delta\left(1-\phi_I^{FC}\right)}{\phi_I^{FC}\sigma+\gamma_I^{FC}}$$

$$\tau_{c,q}^{FC} = \frac{\partial c}{\partial q} = \frac{-\phi_I^{FC}}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad \Gamma_{\lambda,k}^{FC} = \frac{\partial \dot{\lambda}}{\partial k} = \frac{(\rho+\delta)\left((1-\alpha)\gamma_I^{FC}+\phi_I^{FC}\sigma-\alpha\sigma+\alpha\sigma\upsilon(1-s_I)\right)}{\phi_I^{FC}\sigma+\gamma_I^{FC}}$$

$$\tau_{i,k}^{FC} = \frac{\partial i}{\partial k} = \frac{\alpha\left(\gamma_I^{FC}+\sigma\right)}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad \Gamma_{\lambda,\lambda}^{FC} = \frac{\partial \dot{\lambda}}{\partial \lambda} = \frac{(\rho+\delta)\left(\phi_I^{FC}-(1-\upsilon(1-s_I))\right)}{\phi_I^{FC}\sigma+\gamma_I^{FC}}$$

$$\tau_{i,\lambda}^{FC} = \frac{\partial i}{\partial \lambda} = \frac{1-\phi_I^{FC}}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad b_{k,q} = \frac{\partial \dot{k}}{\partial q} = \delta\left(\frac{1-\phi_I^{FC}}{\phi_I^{FC}\sigma+\gamma_I^{FC}}+1\right)$$

$$\tau_{i,q}^{FC} = \frac{\partial i}{\partial q} = \frac{1-\phi_I^{FC}}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad b_{\lambda,q} = \frac{\partial \dot{\lambda}}{\partial q} = \frac{(\rho+\delta)\left(\phi_I^{FC}-(1-\upsilon(1-s_I))-\phi_I^{FC}\sigma-\gamma_I^{FC}\right)}{\phi_I^{FC}\sigma+\gamma_I^{FC}}$$

$$\tau_{n,k}^{FC} = \frac{\partial n}{\partial k} = \frac{\upsilon s_I\alpha\sigma}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad b_{k,p}^{FC} = \frac{\partial k}{\partial p_\epsilon} = \left(1-e^{-\delta T}\right)$$

$$\tau_{n,\lambda}^{FC} = \frac{\partial n}{\partial \lambda} = \frac{\upsilon s_I}{\phi_I^{FC}\sigma+\gamma_I^{FC}} \qquad b_{\lambda,p}^{FC} = \frac{\partial \lambda}{\partial p_\epsilon} = -1$$

$$\tau_{n,q}^{FC} = \frac{\partial n}{\partial q} = \frac{\upsilon s_I}{\phi_I^{FC}\sigma+\gamma_I^{FC}}$$

Recall: $s_I = \frac{\alpha\delta}{\rho+\delta}$, $\phi_I^{FC} = (1-(1-\alpha)\upsilon)s_I$, and $\gamma_I^{FC} = (\upsilon-1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$

### B.3.4 The Dynamic System

Let us now look at the dynamic system:

$$\begin{bmatrix} \dot{\lambda}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} \Gamma_{\lambda,\lambda}^{FC} & \Gamma_{\lambda,k}^{FC} \\ \Gamma_{k,\lambda}^{FC} & \Gamma_{k,k}^{FC} \end{bmatrix} \begin{bmatrix} \lambda(t) \\ k(t) \end{bmatrix} + \begin{bmatrix} b_{\lambda,q} \\ b_{k,q} \end{bmatrix} q(t) + \begin{bmatrix} b_{\lambda,p}^{FC} \\ b_{k,p}^{FC} \end{bmatrix} p_\epsilon \qquad (114)$$

Let $\mu_1^{FC}$ and $\mu_2^{FC}$ represent the two eigenvalues of $\Gamma^{FC}$. From the production complementarity model, we know that:

1.

2. $\mu_1^{FC}+\mu_2^{FC} > 0$

3. $\mu_1^{FC}\mu_2^{FC} < 0$

Therefore, one of the eigenvalues must be negative and the other positive. Without loss of generality, we will assume henceforth that $\mu_1^{FC} < 0$ and $\mu_2^{FC} > 0$.
We now introduce a permanent increase in technology in period $T$. Specifically,

$$q(t) = w(t) = \begin{cases} 0 & \text{for } t \in [0, T) \\ 1 & t \geq T \end{cases} \tag{115}$$

Further, let us define the shock to capital stock (we will take the limits later):

$$w_{p_\epsilon}(t) = \begin{cases} 0 & \text{for } t \in [0, T) \cup (T+\epsilon, \infty) \\ \frac{\tau^{FC}}{1-e^{-\epsilon}} & t \in [T, T+\epsilon] \end{cases} \tag{116}$$

To analyze the resulting system, it will be useful to introduce the Laplace transform operator.

The Laplace transform of a function $p(t)$ is:

$$\mathcal{L}[p(t)] = \bar{P}(s) = \int_0^\infty e^{-st} p(t)\, dt$$

We will use $\bar{P}$ rather than $P$ to distinguish the Laplace transform of the log deviation of a variables from the level of said variable.

Moreover, we know from Theorem 6.3 from Boyce and Diprima (1969), that

$$\mathcal{L}[p'(t)] = s\mathcal{L}(p(t)) - p(0)$$

Taking the Laplace transform of the differential equations in $\begin{bmatrix} \lambda & k \end{bmatrix}'$ and applying this theorem, we get:

$$\begin{bmatrix} \bar{\Lambda}(s) \\ \bar{K}(s) \end{bmatrix} = \left(sI - \Gamma^{FC}\right)^{-1} \left\{ \begin{bmatrix} \lambda(0) \\ k(0) \end{bmatrix} + \begin{bmatrix} b_{\lambda,q} \\ b_{k,q} \end{bmatrix} W(s) + \begin{bmatrix} b_{\lambda,p}^{FC} \\ b_{k,p}^{FC} \end{bmatrix} W_{p_\epsilon}(s) \right\} \tag{117}$$

Given (115), it can be shown that

$$\bar{W}(s) = \mathcal{L}[w(t)] = \frac{1}{s} e^{-sT}$$

Given (116), it can be shown that

$$\bar{W}_{p_\epsilon}(s) = \mathcal{L}\left[w_{\epsilon_\kappa}(t)\right] = \tau^{FC}\frac{e^{-sT} - e^{-s(T+\epsilon)}}{s(1-e^{-\epsilon})}$$

Rewriting equation (117), we get:

$$\begin{bmatrix} \bar{\Lambda}(s) \\ \bar{K}(s) \end{bmatrix} = \frac{1}{(s-\mu_1^{FC})(s-\mu_2^{FC})}\begin{bmatrix} s-\Gamma_{k,k}^{FC} & \Gamma_{\lambda,k}^{FC} \\ \Gamma_{k,\lambda}^{FC} & s-\Gamma_{\lambda,\lambda}^{FC} \end{bmatrix}\left\{\begin{bmatrix} \lambda(0) \\ k(0) \end{bmatrix} + \begin{bmatrix} b_{\lambda,q} \\ b_{k,q} \end{bmatrix} W(s) + \begin{bmatrix} b_{\lambda,p}^{FC} \\ b_{k,p}^{FC} \end{bmatrix} W_{p_\epsilon}(s)\right\} \tag{118}$$

Remember from the previous section, $\mu_1^{FC}$ and $\mu_2^{FC}$ are the eigenvalues of $\Gamma^{FC}$, and $\mu_1^{FC} < 0$ and $\mu_2^{FC} > 0$.

The lower row of (118) gives us:

$$\begin{aligned} \bar{K}(s) &= \frac{\Gamma_{k,\lambda}^{FC}\lambda(0) + \left(s-\Gamma_{\lambda,\lambda}^{FC}\right)k(0)}{\left(s-\mu_1^{FC}\right)\left(s-\mu_2^{FC}\right)} + \left[\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,q} + \left(s-\Gamma_{\lambda,\lambda}^{FC}\right)b_{k,q}}{s\left(s-\mu_1^{FC}\right)\left(s-\mu_2^{FC}\right)}\right]e^{-sT} \\ &+ \left[\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,p}^{FC} + \left(s-\Gamma_{\lambda,\lambda}^{FC}\right)b_{k,p}^{FC}}{s\left(s-\mu_1^{FC}\right)\left(s-\mu_2^{FC}\right)}\right]\frac{\tau^{FC}\left(e^{-sT} - e^{-s(T+\epsilon)}\right)}{1-e^{-\epsilon}} \end{aligned}$$

Next, we take the inverse Laplace transform of $K(s)$ to recover $k$ as a function of time. After some algebra,

$$\begin{aligned}
k(t) \;=\; & \frac{\Gamma_{k,\lambda}^{FC}\lambda(0)}{\mu_{1-}^{FC}\mu_2^{FC}}e^{\mu_1^{FC}t}+\frac{\Gamma_{k,\lambda}^{FC}\lambda(0)}{\mu_{2-}^{FC}\mu_1^{FC}}e^{\mu_2^{FC}t}+\frac{\left(\mu_1^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)k(0)}{\mu_1^{FC}-\mu_2^{FC}}e^{\mu_1^{FC}t}+\frac{\left(\mu_2^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)k(0)}{\mu_2^{FC}-\mu_1^{FC}}e^{\mu_2^{FC}t}\\
& +u_T(t)\left(\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,q}-\Gamma_{\lambda,\lambda}^{FC}b_{k,q}}{\mu_1^{FC}\mu_2^{FC}}\right)\\
& +u_T(t)\left(\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,q}+\left(\mu_1^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)b_{k,q}}{\mu_1^{FC}\left(\mu_1^{FC}-\mu_2^{FC}\right)}e^{\mu_1^{FC}(t-T)}\right)\\
& +u_T(t)\left(\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,q}+\left(\mu_2^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)b_{k,q}}{\mu_2^{FC}\left(\mu_2^{FC}-\mu_1^{FC}\right)}e^{\mu_2^{FC}(t-T)}\right)\\
& +\frac{\tau^{FC}\left(u_T(t)-u_{T+\epsilon}(t)\right)}{1-e^{\epsilon}}\left(\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,p}^{FC}-\Gamma_{\lambda,\lambda}^{FC}b_{k,p}^{FC}}{\mu_1^{FC}\mu_2^{FC}}\right)\\
& +\frac{\tau^{FC}\left(u_T(t)-u_{T+\epsilon}(t)\,e^{-\mu_1^{FC}\epsilon}\right)}{1-e^{\epsilon}}\left(\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,p}^{FC}+\left(\mu_1^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)b_{k,p}^{FC}}{\mu_1^{FC}\left(\mu_1^{FC}-\mu_2^{FC}\right)}e^{\mu_1^{FC}(t-T)}\right)\\
& +\frac{\tau^{FC}\left(u_T(t)-u_{T+\epsilon}(t)\,e^{-\mu_2^{FC}\epsilon}\right)}{1-e^{\epsilon}}\left(\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,p}^{FC}+\left(\mu_2^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)b_{k,p}^{FC}}{\mu_2^{FC}\left(\mu_2^{FC}-\mu_1^{FC}\right)}e^{\mu_2^{FC}(t-T)}\right)
\end{aligned}$$

where $u_T(t)$ is a step function that takes on a value of one for all $t \geq T$, and zero otherwise. Now taking the limit as $\epsilon \rightarrow 0$ we get:

$$\begin{aligned}
k(t) \;=\; & \frac{\Gamma_{k,\lambda}^{FC}\lambda(0)}{\mu_{1-}^{FC}\mu_2^{FC}}e^{\mu_1^{FC}t}+\frac{\Gamma_{k,\lambda}^{FC}\lambda(0)}{\mu_{2-}^{FC}\mu_1^{FC}}e^{\mu_2^{FC}t}+\frac{\left(\mu_1^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)k(0)}{\mu_1^{FC}-\mu_2^{FC}}e^{\mu_1^{FC}t}+\frac{\left(\mu_2^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)k(0)}{\mu_2^{FC}-\mu_1^{FC}}e^{\mu_2^{FC}t}\\
& +u_T(t)\left(\frac{\Gamma_{k,\lambda}^{FC}b_{\lambda,q}-\Gamma_{\lambda,\lambda}^{FC}b_{k,q}}{\mu_1^{FC}\mu_2^{FC}}\right)\\
& +u_T(t)\left(\frac{\Gamma_{k,\lambda}^{FC}\left(b_{\lambda,q}+\tau^{FC}\mu_1^{FC}b_{\lambda,p}^{FC}\right)+\left(\mu_1^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)\left(b_{k,q}+\tau^{FC}\mu_1^{FC}b_{k,p}^{FC}\right)}{\mu_1^{FC}\left(\mu_1^{FC}-\mu_2^{FC}\right)}e^{\mu_1^{FC}(t-T)}\right)\\
& +u_T(t)\left(\frac{\Gamma_{k,\lambda}^{FC}\left(b_{\lambda,q}+\tau^{FC}\mu_2^{FC}b_{\lambda,p}^{FC}\right)+\left(\mu_2^{FC}-\Gamma_{\lambda,\lambda}^{FC}\right)\left(b_{k,q}+\tau^{FC}\mu_2^{FC}b_{k,p}^{FC}\right)}{\mu_2^{FC}\left(\mu_2^{FC}-\mu_1^{FC}\right)}e^{\mu_2^{FC}(t-T)}\right)
\end{aligned}$$

Recall that we assume the initial capital stock is at the steady-state level associated with the pre-shock technology level. As such, $k(0) = 0$:

$$\begin{aligned} k\left(t\right) &= \frac{\Gamma^{FC}_{k,\lambda}\lambda(0)}{\mu^{FC}_{1-}\mu^{FC}_{2}}e^{\mu^{FC}_{1}t} + \frac{\Gamma^{FC}_{k,\lambda}\lambda(0)}{\mu^{FC}_{2-}\mu^{FC}_{1}}e^{\mu^{FC}_{2}t} + \\ &\quad + u_{T}\left(t\right)\left(\frac{\Gamma^{FC}_{k,\lambda}b_{\lambda,q} - \Gamma^{FC}_{\lambda,\lambda}b_{k,q}}{\mu^{FC}_{1}\mu^{FC}_{2}}\right) \\ &\quad + u_{T}\left(t\right)\left(\frac{\Gamma^{FC}_{k,\lambda}\left(b_{\lambda,q} + \tau^{FC}\mu^{FC}_{1}b^{FC}_{\lambda,p}\right) + \left(\mu^{FC}_{1} - \Gamma^{FC}_{\lambda,\lambda}\right)\left(b_{k,q} + \tau^{FC}\mu^{FC}_{1}b^{FC}_{k,p}\right)}{\mu^{FC}_{1}\left(\mu^{FC}_{1} - \mu^{FC}_{2}\right)}e^{\mu^{FC}_{1}(t-T)}\right) \\ &\quad + u_{T}\left(t\right)\left(\frac{\Gamma^{FC}_{k,\lambda}\left(b_{\lambda,q} + \tau^{FC}\mu^{FC}_{2}b^{FC}_{\lambda,p}\right) + \left(\mu^{FC}_{2} - \Gamma^{FC}_{\lambda,\lambda}\right)\left(b_{k,q} + \tau^{FC}\mu^{FC}_{2}b^{FC}_{k,p}\right)}{\mu^{FC}_{2}\left(\mu^{FC}_{2} - \mu^{FC}_{1}\right)}e^{\mu^{FC}_{2}(t-T)}\right) \end{aligned}$$

This gives us the solution to a differential equation with one undetermined variable $\lambda(0)$. We now seek a path for $\begin{bmatrix} \lambda & k \end{bmatrix}'$ that is not explosive. In order to achieve this, we choose $\lambda(0)$ such that the explosive root $\mu^{FC}_{2}$ is 'zeroed out' for all $t > T$. Otherwise, the path for $k\left(t\right)$ will be explosive. This restriction on $\lambda(0)$ is:

$$\left(\frac{\Gamma^{FC}_{k,\lambda}}{\mu^{FC}_{2} - \mu^{FC}_{1}}\right)\lambda(0) = -\frac{\Gamma^{FC}_{k,\lambda}\left(b_{\lambda,q} + \tau^{FC}\mu^{FC}_{2}b^{FC}_{\lambda,p}\right) + \left(\mu^{FC}_{2} - \Gamma^{FC}_{\lambda,\lambda}\right)\left(b_{k,q} + \tau^{FC}\mu^{FC}_{2}b^{FC}_{k,p}\right)}{\mu^{FC}_{2}\left(\mu^{FC}_{2} - \mu^{FC}_{1}\right)}e^{-\mu^{FC}_{2}T}$$

This can be re-written as:

$$\lambda(0) = -\left[\frac{\Gamma^{FC}_{k,\lambda}\left(b_{\lambda,q} + \tau^{FC}\mu^{FC}_{2}b^{FC}_{\lambda,p}\right) + \left(\mu^{FC}_{2} - \Gamma^{FC}_{\lambda,\lambda}\right)\left(b_{k,q} + \tau^{FC}\mu^{FC}_{2}b^{FC}_{k,p}\right)}{\Gamma^{FC}_{k,\lambda}\mu^{FC}_{2}}\right]e^{-\mu^{FC}_{2}T} \tag{119}$$

Let us also solve the second half of our laplace transform. This will allow us to study the path of $\lambda(t)$ over time. The first row of (118) gives us:

$$\begin{aligned} \bar{\Lambda}\left(s\right) &= \frac{\left(s - \Gamma^{FC}_{k,k}\right)\lambda(0) + \Gamma^{FC}_{\lambda,k}k(0)}{\left(s - \mu^{FC}_{1}\right)\left(s - \mu^{FC}_{2}\right)} + \left[\frac{\left(s - \Gamma^{FC}_{k,k}\right)b_{\lambda,q} + \Gamma^{FC}_{\lambda,k}b_{k,q}}{s\left(s - \mu^{FC}_{1}\right)\left(s - \mu^{FC}_{2}\right)}\right]e^{-sT} \\ &\quad + \left[\frac{\left(s - \Gamma^{FC}_{k,k}\right)b^{FC}_{\lambda,p} + \Gamma^{FC}_{\lambda,k}b^{FC}_{k,p}}{s\left(s - \mu^{FC}_{1}\right)\left(s - \mu^{FC}_{2}\right)}\right]\frac{\tau^{FC}\left(^{-sT} - e^{-s(T+\epsilon)}\right)}{1 - e^{\epsilon}} \end{aligned}$$

Now we can take the inverse Laplace transform of $\Lambda(s)$ to recover $\lambda$ as a function of time. After some algebra, and then similar to before, taking the limit and setting $k(0) = 0$ we get:

$$\begin{aligned}
\lambda(t) &= \frac{\left(\mu_1^{FC} - \Gamma_{k,k}^{FC}\right)\lambda(0)}{\mu_{1-}^{FC}\mu_2^{FC}} e^{\mu_1^{FC} t} + \frac{\left(\mu_2^{FC} - \Gamma_{k,k}^{FC}\right)\lambda(0)}{\mu_{2-}^{FC}\mu_1^{FC}} e^{\mu_2^{FC} t} \\
&\quad + u_T(t)\left(\frac{\Gamma_{\lambda,k}^{FC} b_{k,q} - \Gamma_{k,k}^{FC} b_{\lambda,q}}{\mu_1^{FC}\mu_2^{FC}}\right) \\
&\quad + u_T(t)\left(\frac{\left(\mu_1^{FC} - \Gamma_{k,k}^{FC}\right)\left(b_{\lambda,q} + \tau^{FC}\mu_1^{FC} b_{\lambda,p}^{FC}\right) + \Gamma_{\lambda,k}^{FC}\left(b_{k,q} + \tau^{FC}\mu_1^{FC} b_{k,p}^{FC}\right)}{\mu_1^{FC}\left(\mu_1^{FC} - \mu_2^{FC}\right)} e^{\mu_1^{FC}(t-T)}\right) \\
&\quad + u_T(t)\left(\frac{\left(\mu_2^{FC} - \Gamma_{k,k}^{FC}\right)\left(b_{\lambda,q} + \tau^{FC}\mu_2^{FC} b_{\lambda,p}^{FC}\right) + \Gamma_{\lambda,k}^{FC}\left(b_{k,q} + \tau^{FC}\mu_2^{FC} b_{k,p}^{FC}\right)}{\mu_2^{FC}\left(\mu_2^{FC} - \mu_1^{FC}\right)} e^{\mu_2^{FC}(t-T)}\right)
\end{aligned}$$

Given that we choose a $\lambda(0)$ such that the explosive root $\mu_2^{FC}$ is 'zeroed out' for all $t > T$, we can simplify our equations for the time paths of $k(t)$ and $\lambda(t)$ to the following:

$$k(t) = \begin{cases} \dfrac{\Gamma_{k,\lambda}^{FC}\lambda(0)}{\mu_1^{FC} - \mu_2^{FC}} e^{\mu_1^{FC} t} + \dfrac{\Gamma_{k,\lambda}^{FC}\lambda(0)}{\mu_2^{FC} - \mu_1^{FC}} e^{\mu_2^{FC} t} & \text{for } t \in [0, T) \\ \dfrac{\Gamma_{k,\lambda}^{FC}\lambda(0)}{\mu_{1-}^{FC}\mu_2^{FC}} e^{\mu_1^{FC} t} + \dfrac{\Gamma_{k,\lambda}^{FC} b_{\lambda,q} - \Gamma_{\lambda,\lambda}^{FC} b_{k,q}}{\mu_1^{FC}\mu_2^{FC}} + \dfrac{\Gamma_{k,\lambda}^{FC}\left(b_{\lambda,q} + \tau^{FC}\mu_1^{FC} b_{\lambda,p}^{FC}\right) + \left(\mu_1^{FC} - \Gamma_{\lambda,\lambda}^{FC}\right)\left(b_{k,q} + \tau^{FC}\mu_1^{FC} b_{k,p}^{FC}\right)}{\mu_1^{FC}\left(\mu_1^{FC} - \mu_2^{FC}\right)} e^{\mu_1^{FC}(t-T)} & t \geq T \end{cases} \tag{120}$$

$$\lambda(t) = \begin{cases} \dfrac{\left(\mu_1^{FC} - \Gamma_{k,k}^{FC}\right)\lambda(0)}{\mu_1^{FC} - \mu_2^{FC}} e^{\mu_1^{FC} t} + \dfrac{\left(\mu_2^{FC} - \Gamma_{k,k}^{FC}\right)\lambda(0)}{\mu_2^{FC} - \mu_1^{FC}} e^{\mu_2^{FC} t} & \text{for } t \in [0, T) \\ \dfrac{\left(\mu_1^{FC} - \Gamma_{k,k}^{FC}\right)\lambda(0)}{\mu_1^{FC} - \mu_2^{FC}} e^{\mu_1^{FC} t} + \dfrac{\Gamma_{\lambda,k}^{FC} b_{k,q} - \Gamma_{k,k}^{FC} b_{\lambda,q}}{\mu_1^{FC}\mu_2^{FC}} + \dfrac{\Gamma_{\lambda,k}^{FC}\left(b_{k,q} + \tau^{FC}\mu_1^{FC} b_{k,p}^{FC}\right) + \left(\mu_1^{FC} - \Gamma_{k,k}^{FC}\right)\left(b_{\lambda,q} + \tau^{FC}\mu_1^{FC} b_{\lambda,p}^{FC}\right)}{\mu_1^{FC}\left(\mu_1^{FC} - \mu_2^{FC}\right)} e^{\mu_1^{FC}(t-T)} & t \geq T \end{cases} \tag{121}$$

Equations (111), (112), (113), (120), and (121), along with equation (119) give us a stable solution to our model economy for a 1% technology shock that occurs in period $T$. Simultaneously in period $T$ the capital stock carried forward from period 0 (steady state) decreases by $\tau_\kappa^{FC}$ percent.

## B.4 A Model with Investment Adjustment Costs

### B.4.1 The Model Economy

A social planner has the following preferences

$$U = (1-\sigma)^{-1} \int_0^{\infty} e^{-\rho t} \left[ C(t) \exp\left(-N(t)\right) \right]^{1-\sigma} dt$$

over time paths for consumption $C$ and hours worked $N$. We assume this functional form for the utility to preserve balanced growth. Also, $\rho = 1/\beta - 1 > 0$ and $\sigma \geq 0$, where $\beta$ is the stochastic discount factor and $\sigma$ is the inverse of the intertemporal elasticity of substitution.

The planner is subject to the following constraints:

$$F\left[C(t), I(t)\right] = K(t)^{\alpha} \left(A(t) N(t)\right)^{1-\alpha} \tag{122}$$

$$\dot{K}(t) = I(t) - \delta K(t) - \frac{\psi_I}{2} \left(1 - \frac{I(t)}{\delta K(t)}\right)^2 I(t) \tag{123}$$

Here $K$, $I$ and $A$ represent capital, investment and the level of technology. The path of technology and the initial capital stock are exogenous. The depreciation rate, $\delta$, and the elasticity of output with respect to capital, $\alpha$, both lie between zero and one. $\psi_I \in [0, \infty)$ gives a measure of the magnitude of the convex investment adjustment costs.

Further, we assume:

$$F(C, I) \equiv \left[\theta C^{\upsilon} + (1-\theta) I^{\upsilon}\right]^{1/\upsilon}$$

where $\theta \in (0, 1)$ and $\upsilon \geq 1$. When $\upsilon = 1$, the equation collapses to the standard neo-classical case, which has infinite substitutability between the two goods. As $\upsilon$ increases, the complementarity between the production of the two goods increases. If $\upsilon = \infty$, the production frontier takes a Leontief form.

Next, let us define the exogenous processes - the technology news shock. The planner again has perfect foresight, with

$$A(t) = \begin{cases} \bar{A} & \text{for } t \in [0, T) \\ \tilde{A} = 1.01 \times \bar{A} & t \geq T \end{cases}$$

For the contemporaneous improvements case $T = 0$ in the above specification.

### B.4.2 The Model Economy's First Order Conditions

The social planner chooses $C$, $I$, $K$, and $N$ to maximize $U$ subject to (122) and (123) taking as given the initial condition $K(0)$ and time path of technology. We can express the problem as a current value Hamiltonian:

$$H = C^{1-\sigma} \exp\left[-(1-\sigma) N\right] + \Lambda \left( I - \delta K - \frac{\psi_I}{2} \left(1 - \frac{I}{\delta K}\right)^2 I \right) + \Phi \left( K^{\alpha} (AN)^{1-\alpha} - F(C, I) \right)$$

The first-order necessary conditions at an interior solution satisfy :

$$-\frac{U_N}{U_C} = (1-\alpha) \frac{F}{N} (F_C)^{-1} \tag{124}$$

$$\frac{F_I}{F_C} U_C = \Lambda \left[ 1 - \frac{\psi_I}{2} \left(1 - \frac{I}{\delta K}\right)^2 + \psi_I \left(1 - \frac{I}{\delta K}\right) \frac{I}{\delta K} \right] \tag{125}$$

$$\frac{\dot{\Lambda}}{\Lambda} - \rho = \delta + \psi_I \left(1 - \frac{I}{\delta K}\right) \left(\frac{I}{\delta K}\right)^2 - \alpha \frac{F}{K} \frac{U_C}{\Lambda F_C} \tag{126}$$

along with our initial condition on capital and a transversality condition on $\Lambda$.

Equation (124) is the intratemporal Euler equation between consumption and labor hours, equation (125) is the intratemporal Euler equation between consumption and investment, and equation (126) is the optimal capital accumulation equation.

### B.4.3 The Model Economy Log Linearized and Simplified

Given the first order conditions in the previous section our model economy can be described by the following five log linearized equations:

$$(1 - s_I) c + s_I i = \alpha k + (1-\alpha)(a + n) \tag{127}$$

$$\upsilon s_I\left(i-c\right)=n \tag{128}$$

$$\lambda=\left(1-\upsilon\right)\left(c-i\right)-\sigma c-\frac{(1-\sigma)(1-\alpha)}{(1-s_I)}n+\psi_I\left(i-k\right) \tag{129}$$

$$\dot{k}=\delta\left(i-k\right) \tag{130}$$

$$\dot{\lambda}=-\left(\rho+\delta\right)\left[\upsilon\left(1-s_I\right)\left(c-i\right)+i-k\right]+\rho\psi_I\left(i-k\right) \tag{131}$$

Here, $s_I=\frac{\alpha\delta}{\rho+\delta}$.

We can substitute (128) into (127) to get the consumption-investment production frontier ($L_1$ line):

$$\left(1-\phi_I^{INV}\right)c+\phi_I^{INV}i=\alpha k+\left(1-\alpha\right)a \tag{132}$$

Here, $\phi_I^{INV}=\left(1-\left(1-\alpha\right)\upsilon\right)s_I$

We can also substitute (128) into (129) to get the consumption-investment euler equation ($L_2$ line):

$$\left(\gamma_I^{INV}+\psi_I\right)i-\left(\sigma+\gamma_I^{INV}\right)c=\lambda+\psi_I k \tag{133}$$

Here, $\gamma_I^{INV}=\left(\upsilon-1\right)-\frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$.

Equations (132) and (133) now give us a system of equations in $i$ and $c$ (treating $\lambda$, $k$, and $a$ as exogenous).

We also solve the system of equations above for $c$, $i$, $n$, $\dot{k}$, and $\dot{\lambda}$, assuming as given the state variable $\lambda$ and $k$, and the exogenous variable $a$:

$$c = \tau_{c,k}^{INV} k + \tau_{c,\lambda}^{INV} \lambda + \tau_{c,a}^{INV} a \tag{134}$$

$$i = \tau_{i,k}^{INV} k + \tau_{i,\lambda}^{INV} \lambda + \tau_{i,a}^{INV} a \tag{135}$$

$$n = \tau_{n,k}^{INV} k + \tau_{n,\lambda}^{INV} \lambda + \tau_{n,a}^{INV} a \tag{136}$$

$$\dot{k} = \Gamma_{k,k}^{INV} k + \Gamma_{k,\lambda}^{INV} \lambda + b_{k,a}^{INV} a$$

$$\dot{\lambda} = \Gamma_{\lambda,k}^{INV} k + \Gamma_{\lambda,\lambda}^{INV} \lambda + b_{\lambda,a}^{INV} a$$

where,
Recall: $s_I = \frac{\alpha\delta}{\rho+\delta}$, $\phi_I^{INV} = (1-(1-\alpha)\upsilon) s_I$, and $\gamma_I^{INV} = (\upsilon - 1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$

#### B.4.4 The Dynamic System

The general solution to the dynamic system remains the same as before, but now with different coefficient values for $\tau_{x,x}^{INV}$'s, $\Gamma_{x,x}^{INV}$'s and $b_{x,x}$'s. The new values for $\tau_{x,x}^{INV}$'s, $\Gamma_{x,x}^{INV}$'s and $b_{x,x}$'s are given on the previous page. The solution to the dynamic system is:

$$\lambda(0) = -\left[\frac{\Gamma_{k,\lambda}^{INV} b_{\lambda,a}^{INV} + \left(\mu_2^{INV} - \Gamma_{\lambda,\lambda}^{INV}\right) b_{k,a}^{INV}}{\Gamma_{k,\lambda}^{INV} \mu_2^{INV}}\right] e^{-\mu_2^{INV} T} \tag{137}$$

$$k\left(t\right) = \begin{cases} \frac{\Gamma_{k,\lambda}^{INV}\lambda(0)}{\mu_1^{INV}-\mu_2^{INV}} e^{\mu_1^{INV} t} + \frac{\Gamma_{k,\lambda}^{INV}\lambda(0)}{\mu_2^{INV}-\mu_1^{INV}} e^{\mu_2^{INV} t} & \text{for } t \in [0,T) \\ \frac{\Gamma_{k,\lambda}^{INV}\lambda(0)}{\mu_{1-}^{INV}\mu_2^{INV}} e^{\mu_1^{INV} t} + \frac{\Gamma_{k,\lambda}^{INV} b_{\lambda,a}^{INV} - \Gamma_{\lambda,\lambda}^{INV} b_{k,a}^{INV}}{\mu_1^{INV}\mu_2^{INV}} + \frac{\Gamma_{k,\lambda}^{INV} b_{\lambda,a}^{INV} + \left(\mu_1^{INV} - \Gamma_{\lambda,\lambda}^{INV}\right) b_{k,a}^{INV}}{\mu_1^{INV}\left(\mu_1^{INV}-\mu_2^{INV}\right)} e^{\mu_1^{INV}(t-T)} & t \geq T \end{cases} \tag{138}$$

$$\tau_{c,k}^{INV} = \frac{\partial c}{\partial k} = \frac{\gamma_I^{INV}\alpha + \psi_I\left(\alpha - \phi_I^{INV}\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad \Gamma_{k,k}^{INV} = \frac{\partial \dot{k}}{\partial k} = \frac{-\delta\left((1-\alpha)\gamma_I^{INV} + \phi_I^{INV}\sigma - \alpha\sigma\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{c,\lambda}^{INV} = \frac{\partial c}{\partial \lambda} = \frac{-\phi_I^{INV}}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad \Gamma_{k,\lambda}^{INV} = \frac{\partial \dot{k}}{\partial \lambda} = \frac{\delta\left(1-\phi_I^{INV}\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{c,a}^{INV} = \frac{\partial c}{\partial a} = \frac{\left(\gamma_I^{INV} + \psi_I\right)(1-\alpha)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad \Gamma_{\lambda,k}^{INV} = \frac{\partial \dot{\lambda}}{\partial k} = \frac{(\rho+\delta)\left((1-\alpha)\gamma_I^{INV} + \phi_I^{INV}\sigma - \alpha\sigma + \alpha\sigma\upsilon(1-s_I)\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{i,k}^{INV} = \frac{\partial i}{\partial k} = \frac{\alpha\left(\gamma_I^{INV} + \sigma\right) + \psi_I\left(1-\phi_I^{INV}\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad + \frac{\psi_I\left[(\rho+\delta)\upsilon(1-S_I)(1-\alpha) - \rho\left((1-\alpha)\gamma_I^{INV} + \left(\phi_I^{INV} - \alpha\right)\sigma\right)\right]}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{i,\lambda}^{INV} = \frac{\partial i}{\partial \lambda} = \frac{1-\phi_I^{INV}}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad \Gamma_{\lambda,\lambda}^{INV} = \frac{\partial \dot{\lambda}}{\partial \lambda} = \frac{(\rho+\delta)\left(\phi_I^{INV} - (1-\upsilon(1-s_I))\right) + \rho\psi_I\left(1-\phi_I^{INV}\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{i,a}^{INV} = \frac{\partial i}{\partial a} = \frac{(1-\alpha)\left(\gamma_I^{INV} + \sigma\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad b_{k,a}^{INV} = \frac{\partial \dot{k}}{\partial a} = \frac{\delta\left(\gamma_I^{INV} + \sigma\right)(1-\alpha)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{n,k}^{INV} = \frac{\partial n}{\partial k} = \frac{\upsilon s_I(\alpha\sigma + \psi_I(1-\alpha))}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad b_{\lambda,a}^{INV} = \frac{\partial \dot{\lambda}}{\partial a} = \frac{(\rho+\delta)(1-\alpha)\left(\sigma(\upsilon(1-s_I)-1) - \gamma_I^{INV}\right)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{n,\lambda}^{INV} = \frac{\partial n}{\partial \lambda} = \frac{\upsilon s_I}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)} \qquad + \frac{\psi_I(1-\alpha)\left[\rho\left(\sigma + \gamma_I^{INV}\right) - (\rho+\delta)\upsilon(1-S_I)\right]}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\tau_{n,a}^{INV} = \frac{\partial n}{\partial a} = \frac{\upsilon s_I(1-\alpha)(\sigma - \psi_I)}{\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)}$$

$$\lambda(t) = \begin{cases} \frac{\left(\mu_1^{INV} - \Gamma_{k,k}^{INV}\right)\lambda(0)}{\mu_1^{INV} - \mu_2^{INV}} e^{\mu_1^{INV} t} + \frac{\left(\mu_2^{INV} - \Gamma_{k,k}^{INV}\right)\lambda(0)}{\mu_2^{INV} - \mu_1^{INV}} e^{\mu_2^{INV} t} & \text{for } t \in [0, T) \\ \frac{\left(\mu_1^{INV} - \Gamma_{k,k}^{INV}\right)\lambda(0)}{\mu_1^{INV} - \mu_2^{INV}} e^{\mu_1^{INV} t} + \frac{\Gamma_{\lambda,k}^{INV} b_{k,a}^{INV} - \Gamma_{k,k}^{INV} b_{\lambda,a}^{INV}}{\mu_1^{INV}\mu_2^{INV}} + \frac{\Gamma_{\lambda,k}^{INV} b_{k,a}^{INV} + \left(\mu_1^{INV} - \Gamma_{k,k}^{INV}\right) b_{\lambda,a}^{INV}}{\mu_1^{INV}\left(\mu_1^{INV} - \mu_2^{INV}\right)} e^{\mu_1^{INV}(t-T)} & t \geq T \end{cases} \tag{139}$$

Equations (134), (135), (136), (138), and (139), along with equation (137) give us a stable solution to our model economy for a 1% technology shock that occurs in period $T$.

### B.4.5 Proofs & Expressions

In this section for each lemma 1 - 5 and theorem 1 for the basic model with production complementarities we will either prove that the lemma or theorem for the basic model with production complementarities also holds for a model with investment adjustment costs, or we will present and prove an analagous lemma or theorem for a model with investment adjustment costs.

**Lemma B.11:** Lemma 1 from our analysis of the basic model also holds for a model with investment adjustment costs.

*Proof.* The consumption-investment production frontier given by equation (132) is identical to its counterpart in the basic model. The proof of lemma 1 depends purely on the consumption-investment production frontier equation. As a result Lemma 1 from the original model holds. □

**Lemma B.12:** Lemma 2 from our analysis of the basic model also holds for a model with investment adjustment costs.

*Proof.* Given that the consumption-investment production frontier is identical to the basic model, to prove that lemma 2 holds for the investment adjustment cost model we only need to show that $c(0) > 0$ if $\lambda(0) > 0$.

Substituting (132) into (133) and solving for $c$ at time 0 leads to:

$$c(0) = \frac{-\phi_I^{INV}}{\gamma_I^{INV} + \phi_I^{INV}\sigma + \psi_I\left(1 - \phi_I^{INV}\right)}\lambda(0) \tag{140}$$

If $\upsilon\left(1-\alpha\right) > 1$ then $-\phi_I^{INV} > 0$ and $\gamma_I^{INV} + \phi_I^{INV}\sigma > 0$ [40]. Also, by assumption $\psi_I > 0$. As a result from equation (140) if $\lambda(0) > 0$ then $c(0)$ will increase. □

**Lemma B.13:** Lemma 3 from our analysis of the basic model also holds for a model with investment adjustment costs.

*Proof.* To prove lemma 3 still holds we need to prove that the new $\tau_{c,k}^{INV}, \tau_{c,\lambda}^{INV}\tau_{i,k}^{INV}, \tau_{i,\lambda}^{INV}, \tau_{n,k}^{INV}$, and $\tau_{n,\lambda}^{INV}$ for the investment adjustment cost model are all positive. This follows trivially from the fact that if $\upsilon\left(1-\alpha\right) > 1$ then $-\phi_I^{INV} > 0$, $\gamma_I^{INV} > 0$, and $\gamma_I^{INV} + \phi_I^{INV}\sigma > 0$ [41], and by assumption $\psi_I > 0$. □

**Lemma B.14:** Lemma 4 from our analysis of the basic model also holds for a model with investment adjustment costs.

[40]For the proof see Appendix B (Supplementary Appendix).
[41]For the proof see Appendix B (Supplementary Appendix).

*Proof.* Identical to the basic model the dynamic system for a model with investment adjustment costs can be written as equation (17), albeit with different expressions for $\Gamma_{x,x}^{INV}$'s and $b_{x,x}$'s. As a result to prove lemma 4 still holds it suffices to show that the new $\Gamma_{k,\lambda}^{INV}$ and $\Gamma_{\lambda,\lambda}^{INV}$ are still positive. $\Gamma_{k,\lambda}^{INV} > 0$ and $\Gamma_{\lambda,\lambda}^{INV} > 0$ for a model with investment adjustment costs follows trivially from the fact that if $\upsilon(1-\alpha) > 1$ then $-\phi_I^{INV} > 0$, $\gamma_I^{INV} > 0$, $\left(\phi_I^{INV} - (1-\upsilon(1-S_I))\right) > 0$and $\gamma_I^{INV} + \phi_I^{INV}\sigma > 0$ [42], and by assumption $\psi_I > 0$. □

Lemma 5 and theorem 1 now change to reflect how movements in $\psi_I$ cause $\lambda(0)$ to change.

**Lemma B.15:** Suppose the economy experiences a positive technology news shock. Also, assume that $\upsilon > \upsilon_* = (1-\alpha)^{-1}$. $\lambda(0) > 0$ if and only if $\psi_I > \psi_I^{INV*}$ where $\psi_I^{INV*}$ is given by the equality $\Gamma_{k,\lambda}^{INV} b_{\lambda,a} + \left(\mu_2^{INV} - \Gamma_{\lambda,\lambda}^{INV}\right) b_{k,a}^{INV} = 0$.

*Proof.* Recall $\mu_2^{INV} > 0$ and $\Gamma_{k,\lambda}^{INV} > 0$, with:

$$\lambda_0 = -\left[\frac{\Gamma_{k,\lambda}^{INV} b_{\lambda,a}^{INV} + \left(\mu_2^{INV} - \Gamma_{\lambda,\lambda}^{INV}\right) b_{k,a}^{INV}}{\Gamma_{k,\lambda}^{INV}\mu_2^{INV}}\right] e^{-\mu_2^{INV}T} \tag{141}$$

As a result $\lambda(0) > 0$ if and only if $\Gamma_{k,\lambda}^{INV} b_{\lambda,a} + \left(\mu_2^{INV} - \Gamma_{\lambda,\lambda}^{INV}\right) b_{k,a}^{INV} < 0$. □

**Theorem B.1:** The investment adjusment cost model exhbits procyclical technology news shocks if and only if $\upsilon > \upsilon_*$ and $\psi_I > \psi_I^{INV*}$.

**Proof of Theorem 1.** $\Leftarrow$. If $\upsilon(1-\alpha) > 1$ and $\Gamma_{k,\lambda}^{INV} b_{\lambda,a} + \left(\mu_2^{INV} - \Gamma_{\lambda,\lambda}^{INV}\right) b_{k,a}^{INV} < 0$, then a technology news shock is procyclical. Lemmas C.1.2 and C.1.5 prove the procyclical comovement at $t = 0$, while Lemmas C.1.3, C.1.4 and C.1.5 establish the procyclical comovement for $0 < t < T$.

$\Rightarrow$. If $\upsilon(1-\alpha) < 1$ or $\Gamma_{k,\lambda}^{INV} b_{\lambda,a} + \left(\mu_2^{INV} - \Gamma_{\lambda,\lambda}^{INV}\right) b_{k,a}^{INV} > 0$, then a technology news shock is not procyclical. This follow trivially from Lemma C.1.2, as the procyclical comovement will not occur at time $t = 0$ if either of the above conditions are not met. □

**Lemma B.16:** One of the eigenvalues of the $\Gamma^{INV}$ matrix is positive and other negative.

[42] For the proof see Appendix B (Supplementary Appendix).

*Proof.* The product of the eigenvalues is given by the determinant of $\Gamma^{INV}$. The determinant of $\Gamma^{INV}$ matrix can be shown to be equal to

$$\det\left(\Gamma^{INV}\right) = \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I^{INV}\sigma + \gamma_I^{INV} + \psi_I\left(1-\phi_I^{INV}\right)\right)}\left[\upsilon\left(1-s_I\right)\left(1-\alpha\right)\right] < 0$$

As the product of the eigenvalues is negative it follows that the eigenvalues have opposite signs. Further, it can be shown that $\operatorname{tr}\left(\Gamma^{INV}\right) = \rho$. □