Information aggregation in a large two-stage market game

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Abstract

A market-game mechanism is studied in a two-good, pure-exchange setting with potential information aggregation. The mechanism has two stages. At the first stage, agents make offers, which turn out to be provisional for all but a small and randomly selected group. After first-stage offers are announced, everyone else gets to make new offers at a second stage in which payoffs are determined via a Shapley-Shubik market game. When the number of players is finite but large, there exists an essentially unique equilibrium in pure symmetric strategies, an equilibrium that is almost ex post efficient.

Key words: mechanism-design, information-aggregation, market-game, optimality.

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1 Introduction

We study strategic trade in a two-good, pure-exchange setting with potential information aggregation. There is an unobserved state-of-the-world with dispersed and incomplete information about that state in the form of private signals. The realized utility of an agent depends both on the state and on the private signal received. A long-standing theoretical challenge is to devise mechanisms for such settings that satisfy two requirements: first, the mechanism achieves good outcomes; second, the mechanism is robust in the sense that it does not rely on detailed information about the economy, such as the functional form of agents' utilities or the detailed way that private signals relate to the unobservable state-of-the-world. Robustness of this sort is part of the motivation in Hurwicz *et al.* [6].

There are two main candidate mechanisms that satisfy the robustness requirement: a double auction and a Shapley-Shubik [11] market game. In a quasi-linear setting with an

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indivisible good (agents consume either zero or one unit), Reny and Perry [10] show that a double auction achieves near ex post efficiency when the economy is sufficiently large. A similar result for divisible goods is obtained by Vives [13] in a setting with quadratic utilities and with normally distributed signals. However, in the divisible-good (or multi-unit) case, double-auction mechanisms must use demand schedules as actions for agents (see, also, Cripps and Swinkels [1]).

We use a market-game mechanism because the actions are simpler; they are quantities (see, also, Dubey *et al.* [4], page 108). Previous work on market games does not, in general, aggregate information in a way that leads to efficiency.[1] Those mechanisms fail because an agent commits to a quantity before the relevant information is revealed. (In contrast, double auctions in which actions are demand schedules permit an agent's trade to be contingent on the actions of others.) Our market-game mechanism departs from those studied previously by having two stages, the first of which is used mainly for information aggregation.

Two- or multiple-stage mechanisms used for information aggregation are not uncommon. A straw poll in a voting situation is one such mechanism. Another, which is closely related to our mechanism, is pari-mutuel betting. In pari-mutuel betting, running bet totals (and, therefore, odds) are announced before final odds are determined (via a market game). Information aggregation has also been studied in experiments in which the focus is on how to design rewards in order to elicit what agents know (see, for example, Axelrod *et al.* [9]).

Our mechanism elicits information at the first stage in the following way. At the first stage, each agent names an offer. Then, in a random fashion, the mechanism divides the agents into two groups: a small *inactive* group and a large *active* group. Those in the inactive group participate no further; their first-stage offers are executed at an exogenous price. Those in the active group participate in a second-stage market game after the histogram of their first-stage offers is announced—an announcement that resembles the announcement of running odds in pari-mutuel betting.

In such a mechanism, an agent at the first-stage faces a trade-off. Contingent on becoming inactive, the agent's first-stage offer determines his final payoff so that it is in the agent's interest to reveal his private information. Contingent on becoming active, his first-stage offer affects his final payoff only by way of its influence on the beliefs of other active agents at the second stage. Therefore, beliefs, both on and off the equilibrium path, play a crucial role in determining how the first-stage action affects the payoff contingent on becoming active. As a consequence, those beliefs determine how the first-stage trade-off between the two contingent payoffs is resolved.

We have three main results—one about existence, one about uniqueness, and one about ex post optimality. If a mild genericity condition holds and if the finite number of

[1] Palfrey [8] uses a Cournot mechanism and obtains ex post optimality, but, as Vives [12] points out, only because marginal cost is common and constant so that it does not matter how production is allocated among the firms in the model. Dubey *et al.* [4] study a dynamic market game with trades in multiple periods. They show that information may be aggregated, but only after trades and consumption at the first-period are observed. As they emphasize, this precludes ex post efficiency.

agents is sufficiently large, then there exists a symmetric (perfect bayesian) equilibrium in pure strategies in which stage-1 actions reveal the private-information held by all active agents. The main ingredient in the argument is the formulation of beliefs so that the above trade-off is resolved entirely in favor of the payoff contingent on being inactive. Our formulation of beliefs is simple and plausible: it associates each (on- and off-equilibrium) offer with an agent type and then employs Bayes' rule to derive the beliefs over the state-of-the-world. Under slightly stronger assumptions, including a restriction on off-equilibrium beliefs that is similar to the "no-signaling-what-you-don't-know" restriction in Fudenberg and Tirole [3], the trade-off is resolved in the same way in any symmetric equilibrium in pure strategies. As a consequence, any such equilibrium reveals the private-information held by all active agents. Moreover, in any such equilibrium, the stage-2 behavior converges to the competitive equilibrium under complete information.

Finally, we show that any equilibrium in pure strategies that reveals the private-information held by all active agents is *almost* ex post efficient, where the notion of efficiency is similar to that in Gul and Postlewaite [5] and in McLean and Postlewaite [7]. In closely related settings and using direct mechanisms, Gul and Postlewaite [5] and McLean and Postlewaite [7] obtain a similar efficiency result. However, their mechanisms depend on detailed features of the economy, may not have a unique equilibrium, and are not intended for actual use (see [7], page 2441). As we discuss in the concluding remarks, we view our mechanism as one that could actually be used.

2 The model and the mechanism

We describe, in turn, the environment, our mechanism, and the equilibrium concept.

2.1 Environment

Our economy is an endowment economy with two goods and N agents. (The set of agents is denoted $\mathcal{N}$.) Each agent is assigned a type, denoted x, where $x \in X$, a finite set. An agent of type $x \in X$ maximizes expected utility with ex post utility function, $u(q, r; x, z)$, where $(q, r) \in \mathbb{R}^2_+$ is the vector of quantities of the two goods consumed and $z \in Z$, a finite set, is a state-of-the-world. The function $u(\cdot, \cdot; x, z)$ is strictly increasing, strictly concave, continuously twice differentiable, and satisfies Inada conditions. For simplicity, each agent is endowed with the per capita endowment of each good, denoted $\bar{q}$ and $\bar{r}$, respectively. Finally, we assume that $u(\cdot, \cdot; x, z)$ is such that the implied complete-information competitive demands are monotone.[2] The consequences of dropping this assumption are discussed in our concluding remarks.

The sequence of events is as follows. First, nature draws a state-of-the-world $z \in Z$ with probability $\pi(z)$, a state which no one observes. Then each agent gets a type realization, $x \in X$, which is private to the agent. Conditional on the realization z, these realizations are *i.i.d.* across people. We denote the conditional probability $\mathbb{P}[x \mid z]$ by

[2] A gross-substitutes assumption about $u(\cdot, \cdot; x, z)$ is sufficient for such monotonicity.

$\mu_z(x)$, and denote the implied posterior probability $\mathbb{P}[z \mid x]$ by $\tau_x(z)$. We assume that $\pi(z) > 0$ for each $z \in Z$ and that $\mu_z(x) > 0$ for each $x \in X$ and $z \in Z$. We also assume that x is informative in the sense that $z \neq z'$ implies $\mu_z(x) \neq \mu_{z'}(x)$ for some $x \in X$. (This informativeness assumption is without loss of generality: If $\mu_z(x) = \mu_{z'}(x)$ for all $x \in X$, then we treat z and z' as a single state z'' with utility $u(q, r; x, z'') = \pi(z)u(q, r; x, z) + \pi(z')u(q, r; x, z')$.) Our interpretation is that x is an idiosyncratic taste shock and z is a common taste shock. Notice that the realized type, x, plays two roles: it serves as private information about z and it is private information about preferences. Of course, we could have formulated the types x as $x = (x^t, x^s)$, where x^t affects utility and x^s is a signal about z. However, this formulation is equivalent to ours and only complicates the notation.

2.2 The mechanism

After types are realized, each agent n chooses an offer $a^n = (a^n_q, a^n_r) \in \mathcal{O}$, where

$$\mathcal{O} = \{(o_q, o_r) \in [0, \bar{q}] \times [0, \bar{r}] : o_q o_r = 0\}. \tag{1}$$

Then agents are randomly divided into two groups in the following way. Let $\eta \in (0, 1)$ and let $\lceil (1 - \eta)N \rceil = M$ denote the smallest integer that is no less than $(1 - \eta)N$. An assignment, which assigns a number n' to each agent $n \in \mathcal{N}$ in a one-to-one fashion, is drawn from the uniform distribution over the set of all such assignments, and agent n is called *active* if $n' \leq M$ and is called *inactive* if $n' > M$. The payoff for each inactive agent is given by trade at the fixed price, $p_1 = \bar{r}/\bar{q}$. That is,

$$(q^n, r^n) = \left(\bar{q} - a^n_q + \frac{a^n_r}{p_1}, \bar{r} - a^n_r + p_1 a^n_q\right) \text{ for } n \notin \mathcal{M}, \tag{2}$$

where $\mathcal{M}$ is the set of active agents. Next, the mechanism announces the histogram of the stage-1 offers of the active agents, denoted $\nu : \mathcal{O} \to \{0, 1, 2, ..., M\}$.[3] For each $a \in \mathcal{O}$, $\nu(a)$ is the number of active agents whose stage-1 offers are a. Then, given that information, the second stage has active agents participating in a market game. Each active agent n makes an offer $b^n = (b^n_q, b^n_r) \in \mathcal{O}$ and gets payoff

$$(q^n, r^n) = \left(\bar{q} - b^n_q + \frac{b^n_r}{p_2}, \bar{r} - b^n_r + p_2 b^n_q\right) \text{ for } n \in \mathcal{M}, \tag{3}$$

where $p_2 = R/Q$ and

$$(Q, R) = \sum_{n \in \mathcal{M}} b^n + M\kappa. \tag{4}$$

Here, $\kappa = (\kappa_q, \kappa_r)$, where $\kappa_q > 0$ and $\kappa_r > 0$ are exogenous (small) quantities that avoid the need to define payoffs when there are zero-offers on one side of the market and

[3] We could let the mechanism announce two histograms, one for active agents and one for inactive agents. However, that would complicate the notation and would not change the results.

that prevent no-trade from being an equilibrium, a formulation borrowed from Dubey and Shubik [2]. Notice that p_2 functions as a "price," but a price that depends on the aggregation of offers from active agents.[4]

As it stands, this mechanism violates feasibility. The trades of inactive agents at the fixed price do not clear that market. In addition, resources are required for the positive κ_q and κ_r. We proceed as if the mechanism designer has the resources required to support this mechanism. In the concluding remarks, we suggest that a small entry fee could be used to provide those resources. In any case, the departure from feasibility (and *budget-balancedness*) can be made arbitrarily small in per capita terms by choosing η and κ to be small.

The restriction in $\mathcal{O}$ that agents can only make offers on one side of the market plays a significant role in our analysis. It is used to obtain the uniqueness of best responses. The following lemma shows that the restriction is not binding on the agent when there is complete information, which will be the case for the stage-2 game in the candidate equilibrium we construct.[5]

Lemma 1. Fix stage-2 offers of all other agents. Given those offers, for any offer $b' \in [0, \bar{q}] \times [0, \bar{r}]$, there exists $b'' \in \mathcal{O}$ that has the same payoff as b'.

Obviously, the restriction is also not binding in the same sense on payoffs for inactive agents.

2.3 Strategies, beliefs, and equilibrium

A stage-1 strategy is $s_1^n(x) \in \mathcal{O}$, while a stage-2 strategy is $s_2^n(x, a, \nu^{-a}) \in \mathcal{O}$, where the second component in the domain is the agent's stage-1 action, and the third is the announced histogram of offers of active agents *net of the agent's own action.* (That is, for any $a' \in \mathcal{O}$, $\nu^{-a}(a') = \nu(a')$ if $a \neq a'$ and $\nu^{-a}(a) = \nu(a) - 1$.) A strategy profile $\{(s_1^n, s_2^n) : n \in \mathcal{N}\}$ is a perfect bayesian equilibrium (PBE) if for each $n \in \mathcal{N}$, s_1^n is a best response to $\{(s_1^{n'}, s_2^{n'}) : n' \neq n\}$ and s_2^n is a best response to $\{s_2^{n'} : n' \neq n\}$ with respect to a belief φ^n that is consistent with Bayes' rule whenever possible.

Throughout the paper, we focus on symmetric equilibrium in pure strategies. A PBE $\{(s_1^n, s_2^n) : n \in \mathcal{N}\}$ is a *symmetric equilibrium* if for all $n \in \mathcal{N}$, $(s_1^n, s_2^n) = (s_1, s_2)$ and $\varphi^n = \varphi$. In a symmetric equilibrium, an agent's expected payoff at stage-2 depends only on his private history (x, a) and the configuration of other active agents' private histories $\theta : X \times \mathcal{O} \to \{0, 1, 2, ..., M-1\}$. Thus, we may formulate the belief $\varphi(x, a, \nu^{-a})$ as an element of $\Delta(Z \times \Theta)$, where Θ is the set of all configurations θ of type/stage-1-action of the other active agents. Then, a symmetric equilibrium is a triple (s_1, s_2, φ) such that (a) $s_1(x)$ is a best response to s_1 and s_2; (b) $s_2(x, a, \nu^{-a})$ is a best response to s_2 and $\varphi(x, a, \nu^{-a})$; (c) $\varphi(x, a, \nu^{-a})$ is derived from equilibrium behavior using Bayes' rule whenever possible.

[4] Our market game is a version of what is known as the "buy-sell" game, as opposed to the "sell-all" version which makes use of inside money and needs to be augmented by bankruptcy rules in order to be a game (see Shapley-Shubik [11]).

[5] All proofs appear in the Appendix.

It is also useful to define a *separating* equilibrium. A separating equilibrium is a symmetric equilibrium in which $x \neq y$ implies $s_1(x) \neq s_1(y)$. In a separating equilibrium, all active agents share the same belief on the equilibrium path. In particular, in such an equilibrium, ν^{-a} and the agent's own type reveal the true configuration of types of the active agents and each agent uses that true configuration and Bayes' rule to form a common posterior over Z. Thus, on the equilibrium path in a separating equilibrium, although the agent's private type matters for the agent's preferences at stage 2, all active agents share the same information at that stage.

3 Separating equilibrium: existence and characterization

We show that a separating equilibrium exists generically for sufficiently large N. We establish existence by demonstrating that it is optimal for an agent at the first stage to choose an action that is best contingent on being inactive when others do so. The main ingredient in that argument is the off-equilibrium belief formulation. The genericity qualification is very simple: it requires that the ratios of marginal utilities at the optimal consumption levels under the fixed price p_1 differ across types. We begin with existence and characterization of the stage-2 equilibrium when stage-1 is separating. That existence result is used along with explicit stage-1 strategies and beliefs to construct the candidate equilibrium.

3.1 Stage-2 equilibrium when stage-1 is separating

In a separating equilibrium (s_1, s_2), the belief φ about the type/stage-1-action configuration is degenerate on the configuration θ given by $\theta(x, s_1(x)) = \nu^{-a}(s_1(x))$ on the equilibrium path. This implies that there is common knowledge at stage 2 about the type-configuration of active agents, a configuration we denote $\sigma : X \to \{0, 1, ..., M\}$, where M is the number of active agents.[6] It also implies a common posterior over Z, denoted ϕ, which is derived from the type-configuration σ via Bayes' rule.

Therefore, the stage-2 game in a separating equilibrium only depends on the type-configuration σ. In fact, the stage-2 game along such an equilibrium path with type configuration σ can be regarded as a Bayesian game defined as follows: (a) the players are the active agents; (b) the action set for each player consists of offers $(b_q, b_r) \in \mathcal{O}$; (c) the payoffs are determined by the market game and u; (d) the number of players of type x is $\sigma(x)$, which is common knowledge among the players; (e) the common prior over Z is given by $\phi^\sigma \in \Delta(Z)$ that is derived from σ via Bayes' rule. Indeed, it can be regarded as a one-shot, complete-information game in which ϕ is a preference parameter. We denote a symmetric equilibrium for this stage-2 game by $\beta^\sigma : X \to \mathcal{O}$.

[6] In what follows, and only to simplify notation, we assume that $\sigma(x) > 0$ for all $x \in X$. Obviously, this holds with arbitrarily high probability for sufficiently large N.

Then we have

Proposition 1. For any type-configuration $\sigma : X \to \{1, ..., M\}$, β^σ exists.

The proof is a routine application of Brouwer's fixed point theorem. It does, however, depend crucially on the constraint $b_q b_r = 0$. With it, the best response, which is the mapping studied in order to get a fixed point, is a function; without that constraint, the mapping is not necessarily a convex correspondence. However, Lemma 1 implies that an equilibrium obtained in Proposition 1 is also an equilibrium in the game without the constraint $b_q b_r = 0$. Dubey and Shubik [2] give a similar existence result using this observation.

Although there is no uniqueness claim in Proposition 1, we can characterize the limit of β^σ as $N \to \infty$ for any suitable sequence of type-configurations σ^N. Our characterization result is based on the following implication of our informativeness assumption. Fix $z \in Z$ and let σ^N be the type configuration of active agents for an economy of size N. If for some $z \in Z$ the sequence $\{\sigma^N\}_{N=1}^\infty$ is such that $\lim_{N\to\infty} \sigma^N(x)/\lceil(1-\eta)N\rceil = \mu_z(x)$ for each $x \in X$ (which holds *almost surely* conditional on z), then $\lim_{N\to\infty} \phi^{\sigma^N}(z) = 1$. This follows from our informativeness assumption—namely, for any $z \neq z'$, there exists some x such that $\mu_z(x) \neq \mu_{z'}(x)$—and the full-support assumption that $\mu_z(x) > 0$ for all x, z.

We show that β^{σ^N} converges to the competitive equilibrium of the following economy. Let $\mathcal{L}^z(\kappa)$ denote an economy with z known, with fraction of type-x agents equal to $\mu_z(x)$, and with exogenous per capita trades κ.[7] A competitive equilibrium in $\mathcal{L}^z(\kappa)$ is $\{p^z, (q^z(x), r^z(x))_{x\in X}\}$ such that $(q^z(x), r^z(x))$ maximizes $u(q, r; x, z)$ subject to $p^z q + r = p^z \bar{q} + \bar{r}$ for each $x \in X$ and

$$\frac{\kappa_r}{p^z} + \sum_{x\in X} \mu_z(x) q^z(x) = \bar{q} + \kappa_q.$$

Lemma 2. The economy $\mathcal{L}^z(\kappa)$ has a unique competitive equilibrium and it is continuous in κ.

The competitive allocation for the economy $\mathcal{L}^z(\kappa)$, $(q^z(x), r^z(x))_{x\in X}$, has corresponding offers $\{\beta^z_q(x), \beta^z_r(x)\}_{x\in X}$ defined by

$$\beta^z_q(x) = \max\{\bar{q} - q^z(x), 0\} \text{ and } \beta^z_r(x) = \max\{\bar{r} - r^z(x), 0\},$$

and $\beta^z = \{\beta^z_q(x), \beta^z_r(x)\}_{x\in X}$ is our candidate limit for β^σ.

Proposition 2. Fix $z \in Z$. If the sequence $\{\sigma^N\}_{N=1}^\infty$ is such that $\lim_{N\to\infty} \sigma^N(x)/\lceil(1-\eta)N\rceil = \mu_z(x)$ for each $x \in X$, then $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^z$.

This is one of the places where positive κ plays a role. If $\kappa = 0$, then no trade could be a stage-2 equilibrium of the game even if it is far from a competitive equilibrium. The proof uses the fact that the sequence $\{\beta^{\sigma^N}\}_{N=1}^\infty$ is bounded and, therefore, has convergent subsequences. It applies the Maximum Theorem to conclude that any convergent subsequence has limit β^z. Here, again, the constraint $b_q b_r = 0$ is useful: with it, there is

[7] Of course, if $\mu_z(x)$ is not rational, then $\mathcal{L}^z(\kappa)$ has to be a continuum economy.

a unique best-response and the Maximum Theorem implies that the best-response varies continuously.[8]

3.2 Existence of separating equilibrium

Contingent on being inactive, an agent of type-x at stage-1 chooses $a \in \mathcal{O}$ to maximize

$$G_x(a) = \sum_{z \in Z} \tau_x(z) u(\bar{q} - a_q + \frac{a_r}{p_1}, \bar{r} - a_r + p_1 a_q; x, z). \tag{5}$$

By the argument in the proof of Proposition 1, a unique maximum of $G_x(a)$ exists. We denote it $\alpha^* = \{\alpha^*(x)\}_{x \in X}$. Generically, α^* is separating in the sense that $x \neq y$ implies $\alpha^*(x) \neq \alpha^*(y)$. Indeed, if α^* is not separating, then for some $x \neq y$, $\alpha^*(x) = a^* = \alpha^*(y)$ and

$$\frac{\sum_{z \in Z} \tau_x(z) u_q(q^*, r^*; x, z)}{\sum_{z \in Z} \tau_x(z) u_r(q^*, r^*; x, z)} = p_1 = \frac{\sum_{z \in Z} \tau_y(z) u_q(q^*, r^*; y, z)}{\sum_{z \in Z} \tau_y(z) u_r(q^*, r^*; y, z)},$$

where $q^* = \bar{q} - a_q^* + \frac{a_r^*}{p_1}$ and $r^* = \bar{r} - a_r^* + p_1 a_q^*$. But this restriction holds only for knife-edge cases for two distinct aspects of the environment: the probabilities, $\tau_x(\cdot)$ and $\tau_y(\cdot)$, and the utilities, $u(q^*, r^*; x, \cdot)$ and $u(q^*, r^*; y, \cdot)$.

Now we describe candidate beliefs under the assumption that α^* is separating. For any $a \in \mathcal{O}$, let

$$q_1(a) = \bar{q} - a_q + \frac{a_r}{p_1} \in [0, 2\bar{q}]. \tag{6}$$

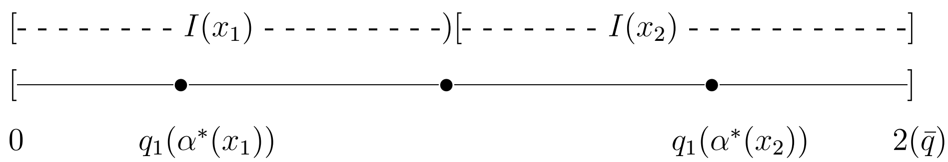
By separation of α^*, $x \neq y$ implies $q_1(\alpha^*(x)) \neq q_1(\alpha^*(y))$. Therefore, we can order the elements of X so that $q_1(\alpha^*(x_i)) < q_1(\alpha^*(x_{i+1}))$ for $i \in \{1, 2, ..., |X| - 1\}$, where $|X|$ denotes the cardinality of X. Next, partition the interval $[0, 2\bar{q}]$ into $|X|$ subintervals indexed by that ordering as follows:

$$I(x_i) = \begin{cases} \left[0, \frac{q_1(\alpha^*(x_2)) + q_1(\alpha^*(x_1))}{2}\right) & \text{for } i = 1 \\ \left[\frac{q_1(\alpha^*(x_i)) + q_1(\alpha^*(x_{i-1}))}{2}, \frac{q_1(\alpha^*(x_{i+1})) + q_1(\alpha^*(x_i))}{2}\right) & \text{for } i = 2, 3, ..., |X| - 1 \\ \left[\frac{q_1(\alpha^*(x_i)) + q_1(\alpha^*(x_{i-1}))}{2}, 2\bar{q}\right] & \text{for } i = |X| \end{cases}. \tag{7}$$

For $|X| = 2$, $I(x_1)$ and $I(x_2)$ are depicted in Figure 1.

[8] Without imposing the constraint $b_q b_r = 0$, Dubey and Shubik [2] obtain a similar result in a deterministic replica economy.

Figure 1. Intervals for beliefs: two types



An agent's belief is a joint distribution over the type/stage-1-action configuration of the other active agents and the state-of-the-world z. It is derived from the observed histogram, ν, and from knowledge of the agent's own private information and is defined for arbitrary stage-1 outcomes.

Candidate for equilibrium beliefs, φ^*: $\varphi^*(x, a, \nu^{-a})$ puts probability 1 on the configuration $\theta_{\nu^{-a}}$ defined by

$$\theta_{\nu^{-a}}(y, a') = \begin{cases} \nu^{-a}(a') \text{ if } q_1(a') \in I(y) \\ 0 \text{ otherwise} \end{cases} . \tag{8}$$

Its marginal distribution over Z is given by the posterior derived from Bayes' rule using the type-configuration of all active agents $\sigma^* : X \to \{0, 1, ..., M\}$ defined by

$$\sigma^*(y) = \begin{cases} \sum_{a' \in \mathcal{O}} \theta_{\nu^{-a}}(y, a') \text{ if } y \neq x \\ \sum_{a' \in \mathcal{O}} \theta_{\nu^{-a}}(x, a') + 1 \text{ if } y = x \end{cases} . \tag{9}$$

Condition (8) says that each agent forms a degenerate distribution over the type/stage-1-action configuration of the other active agents by treating an observed stage-1 action in $I(x_i)$ as coming from an agent of type x_i. Condition (9) says that the agent gets a type-configuration over all active agents by using the type-configuration for other active agents implied by (8) and the agent's own true type.

In order to describe the candidate for equilibrium strategies, it is helpful to distinguish between two classes of active agents according to their private histories. We call an agent of type x a *nondefector* if the agent's stage-1 action is in $I(x)$; otherwise, the agent is called a *defector*. Notice that if no one defects, then all agents' beliefs are symmetric in the sense assumed in Proposition 1: all have the same posterior on z and all active agents have the same belief about the type-configuration over all active agents, which happens to be the true configuration. If one agent defects or more than one defect, then all nondefectors have symmetric beliefs; they have the same posterior on z and any such active agent has the same belief about the type-configuration over all active agents, which, however, is not the true configuration. Each defector has a different posterior on z and a different belief about the type-configuration for the active agents.

The belief φ^* has each agent believing that other agents do not defect. Our specification for a candidate equilibrium is consistent with that belief. In particular, our candidate stage-2 strategy, which must be defined for arbitrary stage-1 actions, has each agent believing that other agents did not defect at stage-1.

Candidate for equilibrium strategies. For stage-1, our candidate is

$$s_1^*(x) = \alpha^*(x), \tag{10}$$

the offer that maximizes $G_x(a)$ (see (5)). For stage-2 strategies, consider an agent with private history (x, a, ν^{-a}) and $q_1(a) \in I(x')$. Let σ^* be the agent's belief about the type-configuration of all active agents under φ^* and let σ' be the type-configuration that he believes other agents believe (see (9)). [If $x' = x$ (nondefector), then $\sigma'(y) = \sigma^*(y)$ for all y; otherwise (defector), $\sigma'(x) = \sigma^*(x) - 1,\ \sigma'(x') = \sigma^*(x') + 1$ and $\sigma'(y) = \sigma^*(y)$ for all $y \notin \{x, x'\}$.] Then $s_2^*(x, a, \nu^{-a})$ satisfies

$$s_2^*(x, a, \nu^{-a}) \in \arg\max_{b \in \mathcal{O}} \sum_{z \in Z} \phi^{\sigma^*}(z) u\left(\bar{q} + \frac{b_r Q_- - b_q R_-}{R_- + b_r}, \bar{r} + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}; x, z\right), \tag{11}$$

where $\phi^{\sigma^*}(z)$ is derived from σ^* using Bayes' rule and where

$$(Q_-, R_-) = M\kappa + \sum_{y \neq x} \sigma^*(y)\beta^{\sigma'}(y) + (\sigma^*(x) - 1)\beta^{\sigma'}(x).$$

When the agent is a nondefector, that is, when $\sigma' = \sigma^*$, we have $s_2^*(x, a, \nu^{-a}) = \beta^{\sigma^*}(x)$. Notice that in the above construction we fix a β^σ for any σ; that is, agents coordinate on a particular proposition-1 equilibrium for any believed type-configuration.

Theorem 1. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$. There exists $\bar{N}$ such that if $N \geq \bar{N}$, then the N-agent economy has a separating equilibrium.

The proof shows that the above candidate is an equilibrium. By construction, $s_1^* = \alpha^*$ implies that φ^* is consistent with Bayes' rule. Also, by construction, $s_2^*(x, a, \nu^{-a})$ is a best response to s_2^* with respect to φ^*. That follows because, according to φ^*, the agent believes that every other active agent is a nondefector. And, if they follow s_2^*, then their actions are described by $\beta^{\sigma'}$. Therefore, what remains, and is the focus of the proof, is to show that α^* is optimal given that other agents follows the candidate equilibrium. An agent at the first stage faces a tradeoff. Conditional on being inactive, playing α^* is optimal for any N. Conditional on being active, a type-x agent could gain by playing something not in $I(x)$. By doing that, the agent influences the beliefs and, thereby, the stage-2 actions of other active agents. The proof shows that any such gain vanishes as N gets large and is, therefore, smaller than the loss implied by playing something that is not in $I(x)$—a play which, by construction, is bounded away from $\alpha^*(x)$.

The threat of being inactive plays a crucial role in the proof. Without it, there would be no penalty attached to stage-1 actions that are devoted entirely to manipulating the beliefs of others and such manipulation could be desirable for any finite N. Therefore, we strongly suspect that, in general, a separating equilibrium does not exist if $\eta = 0$. In this respect, there is a significant distinction between the model with a finite number of agents and the same model with a continuum of agents. In the continuum version as usually formulated, one agent cannot manipulate the beliefs of others and a separating equilibrium exists even if $\eta = 0$.

Finally, although our mechanism does not rely on detailed information about the structure of the model, the threshold $\bar{N}$ in this theorem and that in the uniqueness theorem below do depend on those details. In particular, both η and κ affect the convergence rate.

4 Uniqueness of equilibria

Here we show under some mild additional conditions that any equilibrium is separating. There are three such conditions. The first is a stronger assumption about the informativeness of signals; the second is a modification of the mechanism; and the third is a restriction on off-equilibrium beliefs.

A1. Let $\mathcal{Y} = \{Y_1, Y_2\}$ be any bipartition of X and let $\mu_z(Y_i) \equiv \sum_{y \in Y_i} \mu_z(y)$. For any $z \neq z'$, $\mu_z(Y_1) \neq \mu_{z'}(Y_1)$.

Assumption A1 implies that for any partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ of X with $K \geq 2$ and for any $z \neq z'$, there exists some k such that $\mu_z(Y_k) \neq \mu_{z'}(Y_k)$. Although this assumption is stronger than our original informativeness assumption, parameters for which it does not hold are nongeneric.

A2. If all agents make the same stage-1 offer, then all active agents are required to make zero offers at the second stage.

Assumption A2 says that the market shuts down after the first stage if all agents announce the same offer at the first stage. Under the assumption that $\alpha^*(x) \neq \alpha^*(y)$ for all $x \neq y$, this modification then rules out any equilibrium (s_1, s_2) such that $s_1(x) = s_1(y)$ for all $x, y \in X$, but does not change any other symmetric equilibrium if it exists. In particular, this modification does not affect the existence of a separating equilibrium. Moreover, our mechanism is still robust to the details of the environment under this modification. Finally, because all agents have the same endowments, in the rare event that every agent receives the same signal in a separating equilibrium, such shutting down is costless in terms of realized welfare because in that rare event there is no role for trade.

A3. If a single deviating offer is observed at the first stage, then it is believed to come from some set of types $A \subset X$. Moreover, that belief and the equilibrium play of other agents is used via Bayes' rule to form a belief over Z and the type configuration of other active agents.

Along the equilibrium paths of a symmetric equilibrium in pure strategies, the equilibrium belief associates each equilibrium stage-1 offer a with a set of types and then applies Bayes' rule to derive a belief about the type configuration and the state. A3 requires off-equilibrium beliefs to be derived using the same procedure, but allows there to be an arbitrary set of types, A, to be to be associated with an arbitrary deviating offer. The assumption that A is common to all nondefectors is convenient, but not crucial. The crucial part of A3 is that a set of types is assumed for the defector and that Bayes' rule is used based on that set. As a result, A3 excludes off-equilibrium beliefs that allow the deviator to signal something about other agents' types or about the state in a way that is not warranted by the deviator's private information. This requirement is essentially the requirement for "reasonable" belief systems in Fudenberg and Tirole [3]. Their requirement says that inferences drawn from a deviating action should be limited to the deviator's type (that is, no signaling about what you don't know). We need to augment their requirement with the use of Bayes' rule because of the presence in our model of a payoff-relevant state-of-the-world. Doing so is reasonable because an agent is trying to update his belief about the types and the state-of-the-world, which are exogenous.[9]

Theorem 2. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$ and that A1-A3 hold. There exists $\bar{N}$ such that if $N > \bar{N}$, then any equilibrium $s^N = (s_1^N, s_2^N)$ is separating.

Theorem 2 shows that, when the population is sufficiently large, only separating equilibrium can occur in our mechanism. Moreover, as the proof shows, for sufficiently large populations, in any equilibrium the stage-1 behavior is characterized by α^* and the stage-2 behavior is close to "price-taking" with respect to a price that is close to the unique competitive equilibrium price under a known state as described in Lemma 2.

The proof proceeds by contradiction. First, we use A2 to eliminate a complete pooling equilibrium—one in which $s_1(x) = s_1(y)$ for all $x, y \in X$. Next, we consider a semi-pooling equilibrium—one in which there is a partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ of X with $|X| > K \geq 2$ such that $s_1(y) = s_1(y')$ if $y, y' \in Y_k$ and $s_1(y) \neq s_1(y')$ if $y \in Y_k$ and $y' \in Y_{k'}$ with $k \neq k'$. Such an equilibrium is eliminated by an argument that resembles the main idea of the proof of Theorem 1: because a deviation by one agent has a vanishing effect on the beliefs of other agents, an agent is induced to defect from a semi-pooling equilibrium and play the stage-1 strategy that is best contingent on becoming inactive. However, the details differ; A1 is used to deal with the asymmetric information that exists in a semi-pooling equilibrium and A3 is used to restrict off-equilibrium beliefs.

5 Almost ex post optimality

In our setup there are three sources of exogenous uncertainty, described by three random variables: agents' types, denoted $\zeta^N = (\zeta_1, ..., \zeta_n, ...\zeta_N) \in X^N$; the assignment of each agent to one of two categories, denoted $c^N = (c_1, ..., c_N)$, where $c^N \in \mathbb{C}^N$ and $\mathbb{C}^N = \{c^N \in \{0,1\}^N : \sum_{n \in \mathcal{N}} c_n = \lceil (1-\eta)N \rceil\}$, and $c_n = 0$ means that agent n is inactive and $c_n = 1$ means that agent n is active; and the state of the world $z \in Z$. To discuss efficiency, we consider allocations as mappings from these three random variables to consumption bundles for all agents. Therefore, an allocation takes the form $\langle \omega_n : n \in \mathcal{N} \rangle$ such that for each $n \in \mathcal{N}$, ω_n is a mapping from $X^N \times \mathbb{C}^N \times Z$ to a consumption bundle (q_n, r_n) for agent n. Notice that corresponding to any strategy profile is an outcome which is an allocation of this form, but one which does not depend on z.

Now we are ready to define almost ex post efficiency. For any $(\varepsilon, \delta) \in R_+^2$, we say that an allocation $\langle \omega_n : n \in N \rangle$ is ex post (ε, δ)-efficient if two conditions hold: (a)

[9] A less restrictive extension would allow the off-equilibrium belief to associate a deviating offer with a *distribution* of types and then employ Bayes' rule to pin down the belief about the state and the type configuration of other agents. However, it is rather complicated to formulate the use of Bayes' rule under this assumption and doing so does not seem to affect our main results.

$\sum_{n\in\mathcal{N}} \omega(\zeta^N, c^N, z) \leq N(\bar{q}+\delta, \bar{r}+\delta)$ for each $(\zeta^N, c^N, z) \in X^N \times \mathbb{C}^N \times Z$; (b) there is a collection of events $E_{z,c^N} \subset X^N$ such that (i) $P[E_{z,c^N}|z, c^N] \geq 1-\varepsilon$ for each $(z, c^N) \in Z \times \mathbb{C}^N$ and (ii) there is no other allocation $\langle \omega'_n : n \in N\rangle$ satisfying

$$\sum_{n\in\mathcal{N}} \omega'_n(\zeta^N, c^N, z) \leq \max\left\{\sum_{n\in\mathcal{N}} \omega_n(\zeta^N, c^N, z), N(\bar{q}, \bar{r})\right\} \tag{12}$$

and

$$u[\omega'_n(\zeta^N, c^N, z); \zeta_n, z] > u[\omega_n(\zeta^N, c^N, z); \zeta_n, z] + \varepsilon \text{ for each } n \in \mathcal{N} \tag{13}$$

for some $(z, c^N) \in Z \times \mathbb{C}^N$ and some $\zeta^N \in E_{z,c^N}$.

When $\varepsilon = \delta = 0$, the above definition coincides with the usual definition of ex post efficiency.[10] And, except for the presence of c^N, if $\delta = 0$, then this definition coincides with the definitions in McLean and Postlewaite [7] and in Gul and Postlewaite [5]. We use $\delta > 0$ to reflect the resources that the mechanism designer may need to run the mechanism. In the concluding remarks, we discuss how to achieve almost ex post efficiency with $\delta = 0$. In any case, we do permit alternative allocations to use as much of each good as does ω_n (see (12)).

Theorem 3. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$. Let $(\varepsilon, \delta) > 0$ be given.

(i) There exists $\bar{\kappa} > 0$ and a function $N(\kappa, \eta)$ such that if $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$, $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, and $N > N(\kappa, \eta)$, then there exists a separating equilibrium whose outcome is ex post (ε, δ)-efficient.

(ii) Suppose that A1-A3 hold. Then there exists $\bar{\kappa} > 0$ and a function $N(\kappa, \eta)$ such that if $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$, $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, and $N > N(\kappa, \eta)$, then the outcome of any symmetric equilibrium in pure strategies is ex post (ε, δ)-efficient.

Theorem 3 shows that when (κ, η) is sufficiently small and when N is sufficiently large, our mechanism gives outcomes that are arbitrarily close to ex post efficiency. The two parts of Theorem 3 correspond to Theorems 1 and 2, respectively. The first part, which only depends on the genericity condition, shows existence of equilibria that are almost ex post efficient. The second part, which requires A1-A3, states that all equilibria are almost ex post efficient.

6 Concluding remarks

As noted at the outset, our mechanism violates feasibility. The payoffs of inactive agents, which are determined by the execution of their stage-1 offers at the exogenous price, $\bar{r}/\bar{q}$, and the exogenous stage-2 offers, κ, give rise to a net payout of one of the goods. Feasibility could be restored using entry fees levied on all agents before types are realized.

[10] Notice that when $\varepsilon = 0$, (13) requires the allocation $\omega'_n(\zeta^N, c^N, z)$ to be strictly better than $\omega_n(\zeta^N, c^N, z)$ for all $n \in \mathcal{N}$. This is without loss of generality: any allocation that is weakly better off for all n and strictly better for some n can be modified to be strictly better off for all n because both goods are divisible and utilities are continuous in our economy.

In particular, if the entry fee is 2κ, then η can be chosen to insure feasibility. And, provided there is sufficient motivation for trade coming from the appearance of types in the utility function, κ can be chosen to be small enough to induce participation. With such an entry fee, we could impose $\delta = 0$ in our notion of almost ex post efficiency (and in Theorem 3). That would allow us to achieve the same kind of efficiency result as those in Gul and Postlewaite [5] and in McLean and Postlewaite [7].

A mechanism that would insure feasibility except for κ and would more closely resemble pari-mutuel betting would have the stage-1 offers of the inactive agents be part of the offers that determine the "price" in the second-stage market game and would have their payoffs determined as they are for active agents. However, that would give rise to two-way interaction between the stages. In such a version, if the economy is sufficiently large, it seems as if agents at stage 1 would, as in our version, make stage-1 offers based on the presumption that they will be chosen to be inactive. Even so, they would want to predict the stage-2 price which, itself, is affected by their offers—both directly and by the information revealed by stage-1 offers. Thus, to get a fixed point, we would have to study a mapping that takes both stages into account. Moreover, the mapping would have to be defined over all feasible stage-1 actions, including stage-1 actions that give rise to asymmetric information at stage-2. Our approach decouples stage-1 payoffs from what happens at stage 2 and, therefore, is simpler. Given that it has good welfare properties, its simplicity is a virtue—both for us in analyzing the properties of the mechanism and for those who play the implied game.

We make one strong assumption about preferences; namely, that complete-information competitive demand is monotone, which assures that there is a unique competitive equilibrium (CE) in the version with no uncertainty. If, instead, there were multiple CE's, then our existence argument would fail if agents at stage 1 believed that their stage-1 actions would determine the limit to which a sequence of proposition 1 equilibria converges. If that were the case, then the influence of stage-1 actions on payoffs contingent on being active would not disappear as the size of the economy grows. One way to avoid such a belief would be to assume that there is coordination on the sequence of proposition-1 equilibria regarding the limit to which they converge. That would work if there is a sequence of proposition-1 equilibria that converges to any CE. Whether that is true seems not to be known even for complete-information versions of our market game. With a unique CE, that coordination issue does not arise.

We assume a finite number of types and a finite number of states-of-the-world. The latter plays no role. In contrast, the former is important for us. Although the realization of types is random, as the size of the economy grows, conditional independence of types gives us something that resembles replication in a deterministic version. Even more important, our existence result, via the specification of beliefs, depends on a finite number of types.[11] Another simplifying assumption is that all agents have the same endowments. For our existence result, this assumption can be replaced by any profile of endowments

[11] Reny and Perry [10] cannot use a specification with a finite number of types because they have a limit-order mechanism. With such a mechanism and a finite number of types, there can remain an indeterminacy regarding how the gains from trade are distributed. That does not happen for market-game mechanisms.

that is common knowledge. However, for our uniqueness result, if agents' endowments are heterogeneous (or there is another source of known heterogeneity), then assumption A2 would need to be modified: we would require that the second stage be shut down whenever all agents with the same endowment make the same offer at the first stage.

Regarding the information structure, two special cases of the model deserve mention. One is a specification in which the state-of-the-world does not appear in preferences. Even in that case, the state could still be a source of aggregate risk as it determines the proportions of agents' types. Therefore, stage-1 would still be useful because information aggregation would remain important for ex post efficiency. Another special case is a specification in which types do not appear as arguments of preferences. Although our results apply, this case is problematic because trade disappears at stage 2 as $\kappa \to 0$ and agents may not want to enter in the presence of an entry fee.

Like all of the previous work on strategic games for accomplishing trade with private information that we cited at the outset and all auction models, we have a two-good model. One well-known way to extend our model to $K+1$ goods is to treat good $K+1$ as *cash* and to have K simultaneous markets (trading posts) with market k having trade between cash and good k. However, then, as is well-known, we would want to have multiple rounds of trade because the proceeds of sales in one market cannot be used to make simultaneous purchases in another market.

Such an extension with multiple rounds in real time is pertinent for what we see as one important potential use of our mechanism. It could be used for spot trades in securities like the common stock of publicly traded companies. Of course, in order to use it for such trades, two time intervals have to be selected: one is the interval between stages 1 and 2 of each round; the other is the interval between market rounds.

7 Appendix: Proofs

Lemma 1. Fix stage-2 offers of all other agents. Given those offers, for any offer $b' \in [0,\bar{q}] \times [0,\bar{r}]$, there exists $b'' \in \mathcal{O}$ that has the same payoff as b'.

Proof. Let $(Q_-, R_-) \in \mathbb{R}^2_{++}$ be total offers of other agents (including the exogenous offers). For any $b \in [0,\bar{q}] \times [0,\bar{r}]$, (3) implies that the corresponding payoffs are

$$q(b_q, b_r) = \bar{q} + \frac{b_r Q_- - b_q R_-}{R_- + b_r} \text{ and } r(b_q, b_r) = \bar{r} + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}. \tag{14}$$

Case (i): $b'_r Q_- - b'_q R_- > 0$. In this case, let $b''_q = 0$ and let b''_r be the unique solution to

$$\frac{b''_r Q_-}{R_- + b''_r} = \frac{b'_r Q_- - b'_q R_-}{R_- + b'_r} \equiv \gamma, \tag{15}$$

where it follows that $\gamma \in (0, Q_-)$. The solution is $b''_r = R_-\gamma/(Q_- - \gamma)$. It follows by (15) that $q(b''_q, b''_r) = q(b'_q, b'_r)$. Also,

$$r(b''_q, b''_r) - \bar{r} = b''_r = R_-\gamma/(Q_- - \gamma) = r(b'_q, b'_r) - \bar{r},$$

where the last equality follows from the definition of γ.

Case (ii): $b'_r Q_- - b'_q R_- < 0$. This is completely analogous, but with $b''_r = 0$.

Case (iii): $b'_r Q_- - b'_q R_- = 0$. Here, of course, we let $b''_q = b''_r = 0$.■

7.1 Existence

Proposition 1. For any type-configuration $\sigma : X \to \{1, ..., M\}$, β^σ exists.

Proof. Let $S = \{[0, \bar{q}] \times [0, \bar{r}]\}^X$, which is compact and convex. We let $s = \{s^y\}_{y \in X}$ with $s^y = (s^y_q, s^y_r)$ denote a generic element of S. For $s \in S$ and $x \in X$, let $F : S \to S$ be given by

$$F_x(s) = \arg\max_{b \in \mathcal{O}} H_x(b; Q_-, R_-), \tag{16}$$

where

$$H_x(b; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q} + \frac{b_r Q_- - b_q R_-}{R_- + b_r}, \bar{r} + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}; x, z) \tag{17}$$

and

$$(Q_-, R_-) = M\kappa + \sum_{y \neq x} \sigma(y) s^y + [\sigma(x) - 1] s^x.$$

Here ϕ is the common posterior on z. We have to show that $F_x(s)$ is unique and is continuous in s. We start with uniqueness. Notice that $(Q_-, R_-) \in \mathbb{R}^2_{++}$ for any $s \in S$.

Because of the $b_q b_r = 0$ constraint in (16), it is helpful to separately consider $H_x(b_q, 0; Q_-, R_-)$ and $H_x(0, b_r; Q_-, R_-)$, where

$$H_x(b_q, 0; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q} - b_q, \bar{r} + \frac{b_q R_-}{Q_- + b_q}; x, z) \equiv g(b_q),$$

and

$$H_x(0, b_r; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q} + \frac{b_r Q_-}{R_- + b_r}, \bar{r} - b_r; x, z) \equiv h(b_r).$$

For any $(Q_-, R_-) \in \mathbb{R}^2_{++}$, the functions $f_q(b_q) = \bar{r} + \frac{b_q R_-}{Q_- + b_q}$ and $f_r(b_r) = \bar{q} + \frac{b_r Q_-}{R_- + b_r}$ are strictly concave. Then, because u is strictly concave and because a strictly increasing concave function of a concave function is strictly concave, both g and h are strictly concave. It follows that g has a unique maximum and that h has a unique maximum, denoted $\hat{b}_q$ and $\hat{b}_r$, respectively. Moreover, by the Inada conditions on u, these maxima are characterized by

$$\hat{b}_q = \begin{cases} 0 \text{ if } g'(0) \leq 0 \\ \text{satisfies } g'(\hat{b}_q) = 0 \text{ if } g'(0) > 0 \end{cases}, \tag{18}$$

and

$$\hat{b}_r = \begin{cases} 0 \text{ if } h'(0) \leq 0 \\ \text{satisfies } h'(\hat{b}_r) = 0 \text{ if } h'(0) > 0 \end{cases}. \tag{19}$$

Therefore, a sufficient condition for uniqueness is $\min\{g'(0), h'(0)\} \leq 0$. But,

$$g'(0) = \sum_{z \in Z} \phi(z) \left[-u_q\left(\bar{q}; x, z\right) + u_r\left(\bar{r}; x, z\right) \frac{R_-}{Q_-} \right],$$

and

$$h'(0) = \sum_{z \in Z} \phi(z) \left[u_q\left(\bar{q}; x, z\right) \frac{Q_-}{R_-} - u_r\left(\bar{r}; x, z\right) \right].$$

Therefore,

$$sign[h'(0)] = sign[\frac{R_-}{Q_-} h'(0)] = sign[-g'(0)] = -sign[g'(0)], \tag{20}$$

which implies $\min\{g'(0), h'(0)\} \leq 0$.

Now we turn to continuity in s, which follows if $(\hat{b}_q, \hat{b}_r)$ is continuous in (Q_-, R_-). By (20), $g'(0) = 0$ *iff* $h'(0) = 0$. That and (18) and (19) imply that $\max\{\hat{b}_q, \hat{b}_r\}$ satisfies a first-order condition with equality. Then, the implicit-function theorem applied to that first-order condition gives the required continuity.

It follows that the mapping F satisfies the hypotheses of Brouwer's fixed-point theorem. Although the domain of the mapping, S, does not satisfy $b_q b_r = 0$, the range does. Therefore, the fixed point satisfies $b_q b_r = 0$.■

Lemma 2. The economy $\mathcal{L}^z(\kappa)$ has a unique competitive equilibrium and it is continuous in κ.

Proof. Given our assumptions about $u(\cdot, \cdot; x, z)$, existence and uniqueness is entirely standard, as is continuity in κ.■

Proposition 2. Fix $z \in Z$. If the sequence $\{\sigma^N\}_{N=1}^{\infty}$ is such that $\lim_{N\to\infty} \sigma^N(x)/\lceil(1-\eta)N\rceil = \mu_z(x)$ for each $x \in X$, then $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^z$.

Proof. This proof applies the Theorem of the Maximum to a sequence of proposition 1 equilibria. We can write the best-response objective (see (17)) as

$$H_x(b; Q_-, R_-, \phi) = \sum_{z' \in Z} \phi(z') u\left(q, r; x, z'\right), \tag{21}$$

with

$$q = \bar{q} + \frac{b_r}{p(1 + \frac{b_r}{R_-})} - \frac{b_q}{1 + \frac{b_r}{R_-}}, \text{ and } r = \bar{r} - \frac{b_r}{1 + \frac{b_q}{Q_-}} + \frac{p b_q}{1 + \frac{b_q}{Q_-}},$$

and $p = R_-/Q_-$.

Now, let

$$F_x(b; p, c_1, c_2, \phi) = \sum_{z' \in Z} \phi(z') u\left(q, r; x, z'\right)$$

with

$$q = \bar{q} + \frac{b_r}{p(1 + c_2 b_r)} - \frac{b_q}{1 + c_2 b_r}, \text{ and } r = \bar{r} - \frac{b_r}{1 + c_1 b_q} + \frac{p b_q}{1 + c_1 b_q},$$

and where the domain for F_x is $A = \mathcal{O} \times \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\bar{r}+\kappa_r}{\kappa_q}\right] \times \left[0, \frac{1}{\kappa_r}\right] \times \left[0, \frac{1}{\kappa_q}\right] \times \Delta(Z)$. It follows that $F_x(b; p, 1/Q_-, 1/R_-, \phi) = H_x(b; Q_-, R_-, \phi)$. Therefore, by the argument used in the proof of proposition 1, $F_x(\cdot; p, c_1, c_2, \phi)$ has a unique maximum, $g_x(p, c_1, c_2, \phi)$. And because F_x is continuous on A, the Theorem of the Maximum implies that $g_x(p, c_1, c_2, \phi)$ is continuous.

Now consider

$$H_x(b; Q_-^N, R_-^N, \phi^N) = F_x\left(b; \frac{R_-^N}{Q_-^N}, \frac{1}{Q_-^N}, \frac{1}{R_-^N}, \phi^N\right)$$

with

$$(Q_-^N, R_-^N) = \sum_{y \in X} \sigma^N(y)(\beta^N(y) + \kappa) - \beta^N(x)$$

and with ϕ^N being the common prior derived from σ^N using Bayes' rule. Notice that $(\frac{R_-^N}{Q_-^N}, \frac{1}{Q_-^N}, \frac{1}{R_-^N}) \in \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\bar{r}+\kappa_r}{\kappa_q}\right] \times \left[0, \frac{1}{\kappa_r}\right] \times \left[0, \frac{1}{\kappa_q}\right]$. Therefore, by the definition of β^N, $\beta^N(x) = g_x(\frac{R_-^N}{Q_-^N}, \frac{1}{Q_-^N}, \frac{1}{R_-^N}, \phi)$. Because $\{\beta^N\}_{N=1}^\infty$ is bounded, it has a convergent subsequence, say $\{\beta^{N_s}\}_{s=1}^\infty$, with limit denoted $\hat{\beta}$. Notice that $\lim_{N\to\infty} \phi^N = \mathbf{1}_z$, where $\mathbf{1}_z[z'] = 1$ for $z' = z$ (and 0 otherwise). By the continuity of g_x, it follows that

$$\begin{aligned}
\hat{\beta}(x) &= \lim_{s\to\infty} g_x\left(\frac{R_-^{N_s}}{Q_-^{N_s}}, \frac{1}{Q_-^{N_s}}, \frac{1}{R_-^{N_s}}, \phi^{N_s}\right) \\
&= g_x\left(\lim_{s\to\infty} \frac{R_-^{N_s}}{Q_-^{N_s}}, \lim_{s\to\infty} \frac{1}{Q_-^{N_s}}, \lim_{s\to\infty} \frac{1}{R_-^{N_s}}, \lim_{s\to\infty} \phi^{N_s}\right) = g_x(\hat{p}, 0, 0, \mathbf{1}_z)
\end{aligned}$$

where

$$\hat{p} = \frac{\sum \mu_z(y)\hat{\beta}_r(y) + \kappa_r}{\sum \mu_z(y)\hat{\beta}_q(y) + \kappa_q}.$$

By the definition of F_x, it follows that $\hat{\beta}(x)$ maximizes $u\left(\bar{q} - b_q + \frac{b_r}{\hat{p}}, \bar{r} - b_r + \hat{p}b_q; x, z\right)$. Therefore, it is a Nash equilibrium in $\mathcal{L}^z(\kappa)$. By lemma 2, it follows that $\hat{\beta} = \beta^z$.■

Theorem 1. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$. There exists some $\bar{N}$ such that if $N \geq \bar{N}$, then the N-agent economy has a separating equilibrium.

Proof. We show that for large N's, $((s_1^*, s_2^*), \varphi^*)$ is a PBE, where $s_1^*(x) = \alpha^*(x)$ for all $x \in X$, and s_2^* and φ^* are given by (11) and (8)-(9), respectively. Notice that both s_2^* and φ^* depend on N but not s_1^*. By construction, s_2^* is a best response against s_2^* w.r.t. φ^* and φ^* is consistent with Bayes' rule. It remains to show that s_1^* is a best response to (s_1^*, s_2^*) for sufficiently large N.

Let $M^N = \lceil (1-\eta)N \rceil$ be the number of active agents and consider an agent of type x. Because the assignment into active/inactive categories is drawn independently from the types, conditional on being active, the agent's belief about other agents' types is such that those types are i.i.d. with marginal probabilities $(\mu_z(x))_{x\in X}$ conditional on each state z.

Let γ_z^N be the i.i.d. distribution over X^{M^N-1} generated by $(\mu_z(x))_{x\in X}$. Given s_2^*, the first-stage problem for the agent of type x is $\max_{a\in\mathcal{O}} G_x^N(a)$, where

$$G_x^N(a) = \eta G_x(a) + (1-\eta)F_x^N(a). \tag{22}$$

Here, G_x is the stage-1 problem contingent on being inactive; while for $a \in I(\bar{x})$,

$$F_x^N(a) = \sum_{z\in Z} \tau_x(z) \left[\sum_{\xi\in X^{M^N-1}} \gamma_z^N(\xi) \left[u(q^N(a;z,\xi), r^N(a;z,\xi); x, z)\right] \right], \tag{23}$$

where for each z and $\xi = (\xi_1, ..., \xi_{M^N-1}) \in X^{M^N-1}$, the types of the other active agents.

$$q^N(a;z,\xi) = \bar{q} + \frac{s_{2,r}^*(x,a,\nu^{\xi,-a})Q_-^N - s_{2,q}^*(x,a,\nu^{\xi,-a})R_-^N}{s_{2,r}^*(x,a,\nu^{\xi,-a}) + R_-^N},$$

and

$$r^N(a;z,\xi) = \bar{r} + \frac{s_{2,q}^*(x,a,\nu^{\xi,-a})R_-^N - s_{2,r}^*(x,a,\nu^{\xi,-a})Q_-^N}{s_{2,q}^*(x,a,\nu^{\xi,-a}) + Q_-^N}.$$

Here, $\nu^{\xi,-a}$ is the announced histogram given that other active agents' types are ξ and that other agents follow s_1^*, and Q_-^N and R_-^N are the implied stage-2 offers of other active agents according to the candidate equilibrium. That is,

$$\nu^{\xi,-a}(s_1^*(y)) = \sum_{i=1}^{M^N-1} \mathbf{1}_y(\xi_i) \text{ for each } y \in X \text{ and } \nu^{\xi,-a}(a') = 0 \text{ otherwise,} \tag{24}$$

and

$$(Q_-^N, R_-^N) = \sum_{y\in X} \sigma^\xi(y)(\beta^{\sigma^\xi}(y) + \kappa) - \beta^{\sigma^\xi}(\bar{x}) \tag{25}$$

where σ^ξ is the type-configuration believed by other active agents; namely (recall that $a \in I(\bar{x})$),

$$\sigma^\xi(y) = \sum_{i=1}^{M^N-1} \mathbf{1}_y(\xi_i) \text{ for each } y \neq \bar{x} \text{ and } \sigma^\xi(\bar{x}) = \sum_{i=1}^{M^N-1} \mathbf{1}_{\bar{x}}(\xi_i) + 1. \tag{26}$$

The theorem is proved using the following two claims. The first describes the cost of being a defector and the second describes the potential gain.

Claim 1. Let $q_1(a;x)$ be the consumption of q of a type-x agent who plays a and becomes inactive. There exists $\epsilon > 0$ such that if $q_1(a;x) \notin I(x)$, then $G_x(a) < G_x(s_1^*(x)) - \epsilon$.

Proof of claim 1. As mentioned before, $\max_{a\in\mathcal{O}} G_x(a)$ is equivalent to $\max_{q\in[0,2\bar{q}]} L_x(q)$, where

$$L_x(q) = \sum_{z\in Z} \tau_x(z) u(q, p_1\bar{q} + \bar{r} - p_1 q; x, z).$$

Let $2\delta_x = \min_{y\in X,\ y\neq x} |q_1(\alpha^*(x)) - q_1(\alpha^*(y))|$. Then, $q \notin I(x)$ implies $|q - q_1(\alpha^*(x))| \geq \delta_x$. Because $L_x(q)$ is strictly concave in q and has a maximum at $q_1(\alpha^*(x))$, it follows that $A_x = \min\{-L'_x(q_1(\alpha^*(x)) + \frac{\delta_x}{2}), L'_x(q_1(\alpha^*(x)) - \frac{\delta_x}{2})\} > 0$. Then, for any q such that $|q - q_1(\alpha^*(x))| \geq \frac{\delta_x}{2}$, $L_x(q) \leq L_x(q_1(\alpha^*(x))) - \frac{\delta_x}{2}A_x$. Take $\epsilon_x = (\delta_x/4)A_x$. Then, $q_1(a;x) \notin I(x)$ implies $G_x(a) = L_x(q_1(a;x)) \leq L_x(q_1(\alpha^*(x))) - 2\epsilon_x < G_x(s_1^*(x)) - \epsilon_x$. Finally, let $\epsilon = \min\{\epsilon_x\}_{x \in X}$.□

Claim 2. Let $\xi = (\xi_1, ..., \xi_n, ...)$ be an infinite sequence of X-valued random variables that is i.i.d. w.r.t. the marginal distribution $(\mu_z(x))_{x\in X}$ and let $\xi^{M^N-1} = (\xi_1, ..., \xi_{M^N-1})$, where ξ^{M^N-1} describes the types of the other active agents when there are M^N of them. Then

$$\lim_{N\to\infty} q^N(a; z, \xi^{M^N-1}) = \bar{q} + \frac{\beta_r^z(x)}{p^z} - \beta_q^z(x), \quad \lim_{N\to\infty} r^N(a; z, \xi^{M^N-1}) = \bar{r} + \beta_q^z(x)p^z - \beta_r^z(x),$$

in probability and

$$\lim_{N\to\infty} F_x^N(a) = \sum_{z\in Z} \tau_x(z)u(q^z(x), r^z(x); x, z), \tag{27}$$

uniformly in $a \in \mathcal{O}$, where $(q^z(x), r^z(x))$ is the CE allocation of $\mathcal{L}^z(\kappa)$ (see Lemma 2).

Proof of claim 2. By our construction of off-equilibrium beliefs, (8) and (9), $F_x^N(a)$ depends only on the interval $I(\overline{x})$ such that $a \in I(\overline{x})$. Because there are only finitely many such intervals, uniformity follows from convergence; namely, (27).

By definition, $\xi^{M^N-1} = (\xi_1, ..., \xi_{M^N-1})$ is distributed according to γ_z^N. For each N, let $\sigma^N = \sigma^{\xi^{M^N-1}}$ as defined in (26) (recall that $a \in I(\overline{x})$) and let $\nu^N = \nu^{\xi^{M^N-1},-a}$, as defined in (24). That is, σ^N is the type-configuration believed by all other agents. Then, the sequence $\{\sigma^N\}$ is such that $\sum_{y\in X} \sigma^N(y) = M^N$ and for each $y \in X$, $\lim_{N\to\infty}(\sigma^N(y)/M^N) = \mu_z(y)$ almost surely. Consider a realization of ξ for which $\lim_{N\to\infty}(\sigma^N(y)/M^N) = \mu_z(y)$. Then, by Proposition 2, we have $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^z$. This implies that

$$\lim_{N\to\infty} (\frac{Q_-^N}{M^N}, \frac{R_-^N}{M^N}) = \sum_{y\in X} \mu_z(y)(\beta^z(y) + \kappa) \text{ and } \lim_{N\to\infty} \frac{R_-^N}{Q_-^N} = p^z, \tag{28}$$

where Q_-^N and R_-^N are defined in (25) with $\xi = \xi^{M^N-1}$.

Finally, we show that $\lim_{N\to\infty} s_2^*(x, a, \nu^N) = \beta^z(x)$, where s_2^* is defined in (11). Letting $\phi^N = \text{marg}_Z \varphi^*(x, a, \nu^N)$, where φ^* is defined in (8) and (9), we have

$$\lim_{N\to\infty} \phi^N[z] = 1.$$

Notice that ϕ^N is derived from the type-configuration believed by the agent, which is different from σ^N if $x \neq \bar{x}$. For each N, $s_2^*(x, a, \nu^N)$ solves

$$\max_{b\in\mathcal{O}} H_x^N(b) = \max_{b\in\mathcal{O}} \sum_{z'\in Z} \phi^N[z']u\left(\bar{q} + \frac{b_r Q_-^N - b_q R_-^N}{R_-^N + b_r}, \bar{r} + \frac{b_q R_-^N - b_r Q_-^N}{Q_-^N + b_q}; x, z'\right). \tag{29}$$

Now, let

$$J_x(b; p, c_1, c_2, \phi) = \sum_{z'\in Z} \phi[z']u\left(q, r; x, z'\right)$$

with $q = \bar{q} + \frac{b_r}{p(1+c_2 b_r)} - \frac{b_q}{1+c_2 b_r}$ and $r = \bar{r} - \frac{b_r}{1+c_1 b_q} + \frac{p b_q}{1+c_1 b_q}$, and where the domain for J_x is $\mathcal{O} \times \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\bar{r}+\kappa_r}{\kappa_q}\right] \times \left[0, \frac{1}{\kappa_r}\right] \times \left[0, \frac{1}{\kappa_q}\right] \times \Delta(Z)$. It follows that $J_x(b; \frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N) = H_x^N(b)$. Therefore, by the argument used in the proof of Proposition 1, $J_x(\cdot; p, c_1, c_2, \phi)$ has a unique maximum, $j_x(p, c_1, c_2, \phi)$. And because J_x is continuous on its domain, the Maximum Theorem implies that $j_x(p, c_1, c_2, \phi)$ is continuous.

Now, for each N, $s_2^*(x, a, \nu^N) = j_x(\frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N)$. By (28) and the continuity of j_x, it follows that

$$b^* = \lim_{N\to\infty} s_2^*(x, a, \nu^N) = \lim_{N\to\infty} j_x\left(\frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N\right) = j_x(p^z, 0, 0, \mathbf{1}_z).$$

By the definition of J_x, it follows that b^* maximizes $u\left(\bar{q} - b_q + \frac{b_r}{p^z}, \bar{r} - b_r + p^z b_q; x, z\right)$. Therefore, it is a separating stage-2 equilibrium in the limit model. By Lemma 2, it follows that $b^* = \beta^z(x)$.□

In order to have any effect on $F_x^N(a)$, the agent must choose an offer sufficiently far from s_1^*, the offer that maximizes $G_x(a)$. Claim 1 shows that the implied loss in terms of $G_x(a)$ is bounded away from zero (and does not depend on N). By claim 2, any effect on $F_x^N(a)$ goes to zero as $N \to \infty$. Together, they imply that s_1^* is a best response to (s_1^*, s_2^*) for sufficiently large N.■

7.2 Uniqueness

Theorem 2. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$ and that A1-A3 hold. There exists $\bar{N}$ such that if $N > \bar{N}$, then any equilibrium $s^N = (s_1^N, s_2^N)$ is separating.

Proof. First we exclude complete pooling, i.e., an equilibrium s such that for some $\bar{a} \in \mathcal{O}$, $s_1(x) = \bar{a}$ for all $x \in X$.

Claim 0. For any equilibrium s, there exist $x \neq y \in X$ such that $s_1(x) \neq s_1(y)$.

Proof. Suppose that s is an equilibrium with $s_1(x) = \bar{a}$ for all $x \in X$. By A2, this implies that the realized payoff of all active agents is $(\bar{q}, \bar{r})$, no-trade. However, because α^* is separating, there exist some x such that $G_x(\bar{a}) < G_x(\alpha^*(x))$. Recall that inactive agents' payoffs are still determined by p_1 under A2. Therefore, this agent has a profitable deviation to $\alpha^*(x)$ because no-trade is feasible at stage-2 contingent on being active. □

Claim 0 implies that any candidate equilibrium that is not separating is associated with a partition $\mathcal{Y} = (Y_1, ..., Y_K)$ of X with $1 < K < |X|$. We denote such a candidate equilibrium for N agents by s^N. We prove, by way of contradiction, that s^N cannot be an equilibrium for sufficiently large N. The contradiction is that one agent, called the *target* agent, has a profitable deviation (to the stage-1 action described by α^*).

For a target agent of type-x, the stage-1 objective function is

$$G_x^N(a) = \eta G_x(a) + (1-\eta) F_x^N(a),$$

where $F_x^N(a)$ is the expected payoff implied by offer a at stage-1 conditional on being active. We show that for any $\epsilon > 0$, $F_x^N(a)$ does not vary with a by more than ϵ for sufficiently large N. By assumption A3, $F_x^N(a)$ depends on the associated set of types, $A \subseteq X$, that is believed by other active agents when a is offered at stage-1. Indeed, this holds for equilibrium beliefs as well: if a is an equilibrium offer with $a = s_1^N(y)$, then a is associated with $A = Y_k$ with $y \in Y_k$. Therefore, we may rewrite $F_x^N(a)$ as $F_x^N(A)$ with $A \subseteq X$ being the set of types associated with a. To calculate $F_x^N(A)$, we need to characterize the stage-2 offers following a public announcement ν_N such that $\nu_N(\tilde{a}) = 1$ for some $\tilde{a}$ that is associated with A and $\nu_N(a) = 0$ if $a \notin \{s_1^N(Y_k) : k = 1, ..., K\} \cup \{\tilde{a}\}$.

We divide the rest of the proof into four claims and a final argument. Each of the first three claims has two similar parts—one part for the target agent and the other for non-target agents. Claim 1 is concerned with beliefs along the equilibrium path and has nothing to do with behavior. Claim 2 provides bounds on offers that assure that consumption of each good is bounded away from zero—bounds that hold in any equilibrium. Those bounds imply bounds on the derivatives that appear in the first-order conditions that hold at all best responses. Claim 3 establishes uniform convergence of equilibrium offers, β^N, to "price-taking" offers with a known state-of-the-world and with a price that is given by the equilibrium offers of others, where the uniformity is over all possible sequences of equilibria. Claim 4 is closely related to claim 2 in the proof of Theorem 1 because it says that $F_x^N(a)$ does not vary much with a for sufficiently large N. The final argument follows the logic of the proof of Theorem 1. In what follows, let $M^N = (1-\eta)N$ be the number of active agents.

Claim 1a. Let ν_N be the public announcement which includes the offer $\tilde{a}^N$ (made by the target agent) associated with the set A, while other agents follow s_1^N. Let λ^N be the corresponding signal configuration for the non-target agents. Following the signal configuration λ^N of the non-target agents, the stage-2 beliefs of the target agent of type x are

$$\tilde{\phi}_x^N[\bar{z}] = \frac{\pi(\bar{z})\mu_{\bar{z}}(x)\prod_{k=1}^K[\mu_{\bar{z}}(Y_k)]^{\lambda^N(Y_k)}}{\sum_{z\in Z}\pi(z)\mu_z(x)\prod_{k=1}^K[\mu_z(Y_k)]^{\lambda^N(Y_k)}},\ \ \tilde{\gamma}_z^N[\xi^1, ..., \xi^K] = \prod_{k=1}^K\prod_{i=1}^{\lambda^N(Y_k)}\frac{\mu_z(\xi_i^k)}{\mu_z(Y_k)},$$

where $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those non-target agents who make the offer associated with Y_k ($\lambda^N(Y_k)$ is the number of such agents), and

$$\varphi^N(x, \tilde{a}, \nu_N^{-\tilde{a}})[z, \xi^1, ..., \xi^K] = \tilde{\phi}_x^N[z]\tilde{\gamma}_z^N[\xi^1, \xi^2, ..., \xi^K].$$

Here, $\tilde{\phi}_x^N[\bar{z}]$ is the posterior over states, $\tilde{\gamma}_z^N$ is that over the types of other active agents conditional on the state, and φ^N is the joint distribution.

Given a state $z^* \in Z$ and a type $x \in X$, for any $\epsilon > 0$, there exist $N_a^1(\epsilon)$ and $\delta_a^1(\epsilon) \leq \epsilon$ such that if $N > N_a^1(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta_a^1(\epsilon)$ for all k, then for each $y \in X$,

$$\tilde{\phi}_x^N[z^*] > 1-\epsilon \text{ and } \tilde{\gamma}_{z^*}^N\left[\left|\tilde{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \epsilon\right] > 1-\epsilon,$$

where

$$\tilde{\rho}^N(y) = \#\{\xi_i^k : \xi_i^k = y, i = 1, ..., \lambda^N(Y_k)\}/\lambda^N(Y_k),$$

the fraction of non-target agents with types in Y_k who are type y.

Claim 1b. Let ν_N and λ^N be defined as in Claim 1a. Consider a non-target agent with type x. For such an agent the relevant signal configuration is the offer $\tilde{a}^N$ and λ_-^N defined as follows: If $x \in Y_{\bar{k}}$, then $\lambda_-^N(Y_k) = \lambda^N(Y_k)$ for each $k \neq \bar{k}$ and $\lambda_-^N(Y_{\bar{k}}) = \lambda^N(Y_{\bar{k}}) - 1$. After observing the signal configuration λ_-^N and $\tilde{a}^N$, the stage-2 beliefs of such a non-target agent are

$$\hat{\phi}_x^N[\bar{z}] = \frac{\pi(\bar{z})\mu_{\bar{z}}(x)\mu_{\bar{z}}(A)\prod_{k=1}^K[\mu_{\bar{z}}(Y_k)]^{\lambda_-^N(Y_k)}}{\sum_{z\in Z}\pi(z)\mu_z(x)\mu_z(A)\prod_{k=1}^K[\mu_z(Y_k)]^{\lambda_-^N(Y_k)}},$$

and

$$\hat{\gamma}_z^N[\xi^1, ..., \xi^K, \tilde{\xi}] = \frac{\mu_z(\tilde{\xi})}{\mu_z(A)}\prod_{k=1}^K\left[\prod_{i=1}^{\lambda_-^N(Y_k)}\left(\frac{\mu_z(\xi_i^k)}{\mu_z(Y_k)}\right)\right],$$

where $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those who make the offer associated with Y_k and $\tilde{\xi}$ describes the type of the agent who offered $\tilde{a}^N$, and

$$\varphi^N(y, s_1(y), \nu_N^{-s_1(y)})[z, \xi^1, ..., \xi^K, \tilde{\xi}] = \hat{\phi}_y^N[z]\hat{\gamma}_z^N[\xi^1, ..., \xi^K, \tilde{\xi}].$$

Here, $\hat{\phi}_x^N$ is the posterior distribution over states, $\hat{\gamma}_z^N$ is that over types of the other active agents conditional on the state, and φ^N is the joint distribution.

Given a state $z^* \in Z$ and a type $x \in X$, for any $\epsilon > 0$, there exist $N_b^1(\epsilon)$ and $\delta_b^1(\epsilon) \leq \epsilon$ such that if $N > N_b^1(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta_b^1(\epsilon)$ for all k, then for each $y \in X$,

$$\hat{\phi}_x^N[z^*] > 1 - \epsilon \text{ and } \hat{\gamma}_{z^*}^N\left[\left|\hat{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \epsilon\right] > 1 - \epsilon,$$

where for each realization of *other* active non-target agents' types $(\xi^k)_{k=1,...,K}$ and for each $y \in X$,

$$\hat{\rho}^N(y) = \#\{\xi_i^k : \xi_i^k = y, i = 1, ..., \lambda_-^N(Y_k)\}/\lambda_-^N(Y_k).$$

Proof. We prove Claim 1b only. The derivation of $\hat{\phi}_x^N$ and $\hat{\gamma}_z^N$ follows directly from A3 and Bayes' rule. Therefore, consider the claim for $\hat{\rho}^N$. For each $z \in Z$, consider K infinite sequences of random variables $(\zeta^1, \zeta^2, ..., \zeta^K)$ such that ζ_i^k is Y_k-valued for all $i \in \mathbb{N}$, the K sequences are independent of each other, and ζ^k is an i.i.d. sequence with marginal distribution $(\frac{\mu_z(y)}{\mu_z(Y_k)})_{y\in Y_k}$. Let γ_z denote the joint distribution of $(\zeta^1, \zeta^2, ..., \zeta^K)$. Then, given a sequence of signal-configurations λ^N and a realization $(\zeta^1, \zeta^2, ..., \zeta^K)$, for each $k = 1, ..., K$, each $y \in Y_k$, and each N, define

$$\rho^N(y) = \#\{\zeta_i^k : \zeta_i^k = y, i = 1, ..., \lambda_-^N(Y_k)\}/\lambda_-^N(Y_k).$$

Notice that for each $y \in X$, $\rho^N(y)$ and $\hat{\rho}^N(y)$ have the same distribution. By the *law of large numbers*, for each $y \in Y_k$, $\rho^N(y)$ converges to $\mu_z(y)/\mu_z(Y_k)$ in probability under γ_z for any k as $\lambda_-^N(Y_k)$ converges to infinity. This implies the result.□

Now we turn to equilibrium behavior, where, again, we distinguish between the target agent and the other agents. We use $\tilde{\beta}^{N,A}(x)$ to denote the equilibrium offer from the target agent with type x and use $\hat{\beta}^{N,A}(x)$ to denote the equilibrium offer from a non-target agent with type x. $\tilde{\beta}^{N,A}(x)$ and $\hat{\beta}^{N,A}(x)$ then maximize $\tilde{H}_x^{\lambda^N}(b)$ and $\hat{H}_x^{\lambda^N}(b)$, respectively, subject, of course, to $b \in \mathcal{O}$ that are defined in the following.

Following a signal configuration λ^N for non-target agents, the target agent of type x has the stage-2 objective function,

$$\tilde{H}_x^{\lambda^N}(b_q, b_r) = \sum_{z \in Z} \tilde{\phi}_x^N[z] \mathbb{E}_{\tilde{\gamma}_z^N} \left[u\left(\bar{q} + \frac{b_r \tilde{Q}_-^N - b_q \tilde{R}_-^N}{\tilde{R}_-^N + b_r}, \bar{r} + \frac{b_q \tilde{R}_-^N - b_r \tilde{Q}_-^N}{\tilde{Q}_-^N + b_q}; x, z \right) \right], \tag{30}$$

where

$$(\tilde{Q}_-^N, \tilde{R}_-^N) = \sum_{k=1,..,K} \sum_{y \in Y_k} \lambda^N(Y_k) \tilde{\rho}^N(y) \hat{\beta}^{N,A}(y) + M^N \kappa.$$

Following a signal configuration λ_-^N of other non-target agents, a non-target agent of type x has the stage-2 objective function,

$$\hat{H}_x^{\lambda^N}(b_q, b_r) = \sum_{z \in Z} \hat{\phi}_x^N[z] \mathbb{E}_{\hat{\gamma}_z^N} \left[u\left(\bar{q} + \frac{b_r \hat{Q}_-^N - b_q \hat{R}_-^N}{\hat{R}_-^N + b_r}, \bar{r} + \frac{b_q \hat{R}_-^N - b_r \hat{Q}_-^N}{\hat{Q}_-^N + b_q}; x, z \right) \right], \tag{31}$$

where

$$(\hat{Q}_-^N, \hat{R}_-^N) = \sum_{k=1,..,K} \sum_{y \in Y_k} \lambda_-^N(Y_k) \hat{\rho}^N(y) \hat{\beta}^{N,A}(y) + \tilde{\beta}^{N,A}(\tilde{\xi}) + M^N \kappa.$$

Noitce that here $\tilde{\xi}$ denotes the target agent's type.

Claim 2. There exist $\bar{b} = (\bar{b}_q, \bar{b}_r) < (\bar{q}, \bar{r})$ such that $\tilde{\beta}^{N,A}(x) \leq \bar{b}$ and $\hat{\beta}^{N,A}(x) \leq \bar{b}$.

Proof. As might be expected, this follows from the Inada conditions and the bounds on prices implied by $\kappa > 0$. We prove the claim for $\tilde{\beta}^{N,A}$; the other case is exactly the same. Obviously, we are only concerned with positive offers. We spell out the details for $\tilde{\beta}_q^{N,A}(x) > 0$. To abbreviate notation denote $\tilde{\beta}_q^{N,A}(x)$ by b_q^*.

Being positive, b_q^* satisfies the first-order condition (u_q and u_r denote the first-order derivatives of u),

$$\sum_{z \in Z} \tilde{\phi}_x^N[z] \left\{ \mathbb{E}_{\tilde{\gamma}_z^N} \left[-u_q(q(b_q^*), r(b_q^*); x, z) + u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \right] \right\} = 0 \tag{32}$$

where

$$(q(b_q), r(b_q)) = \left(\bar{q} - b_q, \bar{r} + \frac{b_q \tilde{R}_-^N}{\tilde{Q}_-^N + b_q} \right). \tag{33}$$

For any $b_q \in [0, \bar{q}]$, $\tilde{Q}_-^N \in [M^N \kappa_q, (M^N - 1)\bar{q} + M^N \kappa_q)]$, and $\tilde{R}_-^N \in [M^N \kappa_r, (M^N - 1)\bar{r} + M^N \kappa_r)]$,

$$u_r(q(b_q), r(b_q); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \leq \max_{q \in [0, \bar{q}], z \in Z} u_r(q, \bar{r}; x, z) \frac{(\bar{q} + \kappa_q)(\bar{r} + \kappa_r)}{\kappa_q^2} \equiv A.$$

For each $b_q \in [0, \bar{q})$, let

$$J(b_q) = \min_{r \in [\bar{r}, \bar{r} + \frac{\bar{q}(\bar{r}+\kappa_r)}{\kappa_q}], z \in Z} u_q(\bar{q} - b_q, r; x, z).$$

Because $[\bar{r}, \bar{r} + \frac{\bar{q}(\bar{r}+\kappa_r)}{\kappa_q}]$ is compact and X and Z are both finite, J is well-defined, positive, strictly increasing, $\lim_{b_q \to \bar{q}} J(b_q) = \infty$, and, of course, $J(b_q) \leq u_q(q(b_q), r(b_q); x, z)$.

Let $\gamma > 1$ be such that there is a solution for b_q to $J(b_q) = \gamma A$. Denote the solution, which is unique, $\tilde{b}_q(x)$. We next show that $\tilde{\beta}_q^{N,A}(x) = b_q^* \leq \tilde{b}_q(x) < \bar{q}$. The second inequality follows from $\gamma A < \infty$. Suppose the first inequality does not hold. Then, by (32), for some $(z, \tilde{Q}_-^N, \tilde{R}_-^N)$, we must have

$$u_q(q(b_q^*), r(b_q^*); x, z) - u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \leq 0$$

with $(q(b_q), r(b_q))$ as in (33). Because $b_q^* > \tilde{b}_q(x)$,

$$u_q(q(b_q^*)), r(b_q^*); x, z) > \gamma A \text{ and } u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \leq \gamma A,$$

a contradiction. The argument for $\tilde{\beta}_r^{N,A}(x) > 0$ is exactly analogous.

Finally, take $\bar{b}_q = \max\{\tilde{b}_q(x), \hat{b}_q(x) : x \in X\}$ and $\bar{b}_r = \max\{\tilde{b}_r(x), \hat{b}_r(x) : x \in X\}$, where $\hat{b}_q(x)$ and $\hat{b}_r(x)$ are the analogous bounds for $\hat{\beta}^{N,A}(x)$.□

Claim 3. Fix a state $z^* \in Z$. For any $p > 0$ let $\chi(x; p) = (\chi_q(x; p), \chi_r(x; p))$ be the unique solution to

$$\max_{b \in \mathcal{O}} H_x(b; p) = \max_{b \in \mathcal{O}} u(\bar{q} - b_q + \frac{b_r}{p}, \bar{r} - b_r + b_q p; x, z^*).$$

For any $\epsilon > 0$, there exists $N^3(\epsilon)$ and $\delta^3(\epsilon) \leq \epsilon$ such that if $N > N^3(\epsilon)$ and if $\left| \frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k) \right| < \delta^3(\epsilon)$ for all k, then for each $x \in X$,

$$|\tilde{\beta}_q^{N,A}(x) - \chi_q(x; p^N)| + |\tilde{\beta}_r^{N,A}(x) - \chi_r(x; p^N)| < \epsilon$$

and

$$|\hat{\beta}_q^{N,A}(x) - \chi_q(x; p^N)| + |\hat{\beta}_r^{N,A}(x) - \chi_r(x; p^N)| < \epsilon,$$

where

$$p^N = \frac{\sum_{y \in X} \mu_{z^*}(y) \hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{y \in X} \mu_{z^*}(y) \hat{\beta}_q^{N,A}(y) + \kappa_q}.$$

Proof. We first prove the claim for $\tilde{\beta}^{N,A}$. The objective $\tilde{H}_x^{\lambda^N}(b)$, defined in (30), for which $\tilde{\beta}^{N,A}(x)$ is a best response, differs from $H_x(b; p^N)$, for which $\chi(x; p^N)$ is a best response, in two respects. In $H_x(b;p^N)$, offers of others are weighted by limit weights, while in $\tilde{H}_x^{\lambda^N}(b)$ they are weighted by the agent's posterior over the types of others. And, in $H_x(b;p^N)$, the price is unaffected by the agent's own offer, while in $\tilde{H}_x^{\lambda^N}(b)$ it responds to the agent's offer as in the market game. The proof of the claim shows that both differences disappear for sufficiently large N.

Let $d = \frac{1}{2}\min\{\bar{q} - \bar{b}_q, \bar{r} - \bar{b}_r\}$. First we show that, for any $\epsilon > 0$, there exist $N^2(\epsilon)$ and $\delta^2(\epsilon)$ such that if $N > N^2(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^2(\epsilon)$ for all k, then for all $x \in X$ and for all $b_q \in [0, \bar{q} - d]$ and all $b_r \in [0, \bar{r} - d]$,

$$\left|\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) - \frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(b_q, 0)\right| < \epsilon \text{ and } \left|\frac{\partial}{\partial b_r} H_x(0, b_r; p^N) - \frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0, b_r)\right| < \epsilon. \tag{34}$$

Because the arguments are essentially the same, we only prove the first of these.

Fix some $x \in X$ and let

$$L(b_q, p_1, p_2) = u_q(\bar{q} - b_q, \bar{r} + b_q p_1; x, z^*) - u_r(\bar{q} - b_q, \bar{r} + b_q p_1; x, z^*) p_2.$$

$L(b_q, p_1, p_2)$ is continuous over $[0, \bar{q} - d] \times \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\kappa_r + \bar{r}}{\kappa_q}\right]^2$ and, hence, is uniformly continuous. Therefore, for any $\epsilon > 0$, there exists some $\hat{\delta}(\epsilon) \le \epsilon$ such that

$$|p_1 - p_1'| < \hat{\delta}(\epsilon) \text{ and } |p_2 - p_2'| < \hat{\delta}(\epsilon) \Rightarrow |L(b_q, p_1, p_2) - L(b_q, p_1', p_2')| < \epsilon. \tag{35}$$

Notice that $\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) = L(b_q, p^N, p^N)$ and that

$$\frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(b_q, 0) = \sum_{z \in Z} \tilde{\phi}_x^N[z] \mathbb{E}_{\tilde{\gamma}_z^N}\left[L\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right)\right].$$

Hence, it is sufficient to show that p^N is close to both $\frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}$ and $\frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}$ as N becomes large and as λ^N / M^N converges uniformly.

Because

$$p^N - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N} = \frac{\sum_{y \in X} \mu_{z^*}(y) \hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{y \in X} \mu_{z^*}(y) \hat{\beta}_q^{N,A}(y) + \kappa_q} - \frac{\sum_{k=1,..,K} \sum_{y \in Y_k} \frac{\lambda_-^N(Y_k)}{M^N} \tilde{\rho}^N(y) \hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{k=1,..,K} \sum_{y \in Y_k} \frac{\lambda_-^N(Y_k)}{M^N} \tilde{\rho}^N(y) \hat{\beta}_q^{N,A}(y) + \kappa_q},$$

we have

$$\left|p^N - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N}\right| \le \frac{2(\bar{q} + \kappa_q)(\bar{r} + \kappa_r)}{\kappa_q^2} \sum_{k=1}^K \left|\mu_{z^*}(Y_k) - \frac{\lambda_-^N(Y_k)}{M^N} \tilde{\rho}^N(y)\right|.$$

Moreover,

$$\left|\frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N}\right| \le \frac{\bar{r} + \kappa_r}{\kappa_q} \frac{\bar{q}(2\kappa_q + \frac{\bar{q}}{M^N})}{\kappa_q^2 M^N} \text{ and } \left|\frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q} - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N}\right| \le \frac{\bar{r} + \kappa_r}{\kappa_q} \frac{\bar{q}}{\kappa_q M^N}.$$

Hence, for any $\epsilon > 0$ there exist $\tilde{N}(\epsilon)$ and $\tilde{\delta}(\epsilon) \leq \epsilon$ such that if $N > \tilde{N}(\epsilon)$ and if $\left|\tilde{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \tilde{\delta}(\epsilon)$ and $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \tilde{\delta}(\epsilon)$ for all k, then

$$\left|\frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} - p^N\right| < \epsilon \text{ and } \left|\frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q} - p^N\right| < \epsilon \text{ for all } b_q. \tag{36}$$

Let $B = 2\max\{1, u_q(d,r;x,z), u_r(q,d;x,z) : r \in \left[0, \bar{r} + 2\bar{q}\frac{\bar{r}+\kappa_r}{\kappa_q}\right], q \in [0,\bar{q}], z \in Z\}$. Then $|L(b_q,p_1,p_2)| \leq \frac{1}{2}B$ for all $(b_q,p_1,p_2) \in [0,\bar{q}-d] \times [\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\kappa_r+\bar{r}}{\kappa_q}]^2$. Let $\delta' = \hat{\delta}(\frac{\epsilon}{10B})$ (see (35)). Let

$$\delta^2(\epsilon) = \min\{\delta^1(\frac{\epsilon}{10B}), \delta^1(\tilde{\delta}(\delta'))\} \text{ and } N^2(\epsilon) = \max\{N^1(\frac{\epsilon}{10B}), \tilde{N}(\delta^2(\epsilon))\},$$

where $\tilde{N}$ and $\tilde{\delta}$ are given in (36) and $\delta^1(\epsilon) = \min\{\delta_a^1(\epsilon), \delta_b^1(\epsilon)\}$ and $N^1(\epsilon) = \max\{N_a^1(\epsilon), N_b^1(\epsilon)\}$ with $\delta_a^1(\epsilon), \delta_b^1(\epsilon), N_a^1(\epsilon), N_b^1(\epsilon)$ given in Claim 1.

Suppose that $N > N^2(\epsilon)$ and that $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^2(\epsilon)$ for all k. By Claim 1a, because $N > N^2(\epsilon) \geq N^1(\epsilon/10B)$ and for all k, $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^2(\epsilon) \leq \delta^1(\epsilon/10B)$, we have $\tilde{\phi}_x^N[z^*] > 1 - \epsilon/10B$.

Moreover, because $N > N^1(\tilde{\delta}(\delta'))$ and because $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^1(\tilde{\delta}(\delta'))$ for all k, it follows from Claim 1a that

$$\tilde{\gamma}_{z^*}^N\left[\left|\tilde{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\delta')\right] > 1 - \tilde{\delta}(\delta') \geq 1 - \frac{\epsilon}{10B}.$$

Now, by (36), it follows that if $\left|\tilde{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \tilde{\delta}(\delta')$, if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \tilde{\delta}(\delta')$, and if $N > \tilde{N}(\delta')$, then

$$\max\left\{\left|p^N - \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}\right|, \left|p^N - \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right|\right\} < \delta' = \hat{\delta}(\frac{\epsilon}{10B}).$$

This and (35) imply

$$\left|L\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L(b_q, p^N, p^N)\right| < \frac{\epsilon}{10B}.$$

Therefore, $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \tilde{\delta}(\delta')$ and $N > \tilde{N}(\delta')$ imply that

$$\tilde{\gamma}_{z^*}^N\left[\left|L\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L(b_q, p^N, p^N)\right| < \frac{\epsilon}{10B}\right] > 1 - \frac{\epsilon}{10B}.$$

Combining these results we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial b_q}H_x(b_q,0;p^N)-\frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(b_q,0)\right| \\
\leq\ & \mathbb{E}_{\tilde{\gamma}^N_{z^*}}\left[L\left(b_q,\frac{\tilde{R}^N_-}{\tilde{Q}^N_-+b_q},\frac{\tilde{Q}^N_-\tilde{R}^N_-}{(\tilde{Q}^N_-+b_q)^2}\right)-L(b_q,p^N,p^N)\right]+(1-\tilde{\phi}^N_x[z^*])B \\
<\ & [\frac{\epsilon}{10}+\frac{\epsilon}{10B}(2B)]+[\epsilon/10B]B<\epsilon.
\end{aligned}$$

This establishes (34).

Now we complete the proof of Claim 3 for $\tilde{\beta}^{N,A}$. Let $Q(b_q,b_r;p)=\bar{q}-b_q+\frac{b_r}{p}$. It is straightforward to check that there exists a $D_1>0$ such that

$$|b_q-b'_q|+|b_r-b'_r|<D_1|Q(b;p)-Q(b';p)| \text{ for all } (b,b',p)\in\mathcal{O}^2\times\left[\frac{\kappa_r}{\bar{q}+\kappa_q},\frac{\kappa_r+\bar{r}}{\kappa_q}\right].$$

Also, letting

$$M(q;p)=u_q(q,p\bar{q}+\bar{r}-pq;x,z^*)-u_r(q,p\bar{q}+\bar{r}-pq;x,z^*)p,$$

we have

$$\frac{\partial}{\partial b_q}H_x(b_q,0;p^N)=-M(Q(b_q,0;p^N);p^N) \text{ and } \frac{\partial}{\partial b_r}H_x(0,b_r;p^N)=M(Q(0,b_r;p^N);p^N)/p^N.$$

Now, let D_2 satisfies

$$\begin{aligned}
1/D_2\ =\ & -\max\{u_{qq}(q,r;x,z^*)-2pu_{qr}(q,r;x,z^*)+p^2u_{rr}(q,r;x,z^*): \\
& (q,r,p)\in\left[d,\bar{q}+\frac{\bar{r}(\bar{q}+\kappa_q)}{\kappa_r}\right]\times\left[d,\bar{r}+\frac{\bar{q}(\bar{r}+\kappa_r)}{\kappa_q}\right]\times\left[\frac{\kappa_r}{\bar{q}+\kappa_q},\frac{\kappa_r+\bar{r}}{\kappa_q}\right]\},
\end{aligned}$$

where u_{qq}, u_{qr}, and u_{rr} denote second-order derivatives of u. Because u is strictly concave and continuously twice differentiable, D_2 is well-defined and $D_2>0$. Moreover,

$$M'(q;p^N)=u_{qq}(q,r;x,z^*)-2p^Nu_{qr}(q,r;x,z^*)+(p^N)^2u_{rr}(q,r;x,z^*)$$

with $r=p^N\bar{q}+\bar{r}-p^Nq$. Hence, $M'(q;p^N)<-1/D_2$ for all $q=Q(b;p^N)$ with $(b,p^N)\in\mathcal{O}^2\times\left[\frac{\kappa_r}{\bar{q}+\kappa_q},\frac{\kappa_r+\bar{r}}{\kappa_q}\right]$.

Because the offer $\chi(x;p^N)$ is a "price-taking" offer, it satisfies the first-order conditions at equality, i.e., $M(Q(\chi(x;p^N);p^N);p^N)=0$. Therefore, by the Mean Value Theorem, for any $\epsilon>0$, if $|M(\bar{q}-b_q+\frac{b_r}{p^N};p^N)|<\epsilon/D_1D_2$ with $b_qb_r=0$, then

$$|b_q-\chi_q(x;p^N)|+|b_r-\chi_r(x;p^N)|<\epsilon. \tag{37}$$

Let $D=2D_1D_2\frac{\bar{r}+\kappa_r}{\kappa_q}$. Then, for any $p^N\in\left[\frac{\kappa_r}{\bar{q}+\kappa_q},\frac{\kappa_r+\bar{r}}{\kappa_q}\right]$, $p^ND_1D_2<D$.

Now, let $N^3(\epsilon) = N^2(\epsilon/D)$ and $\delta^3(\epsilon) = \delta^2(\epsilon/D)$, where N^2 and δ^2 are given by (34). Suppose that $N > N^2(\epsilon)$ and that $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^2(\epsilon)$ for all k. We consider three cases.

(a) $\tilde{\beta}_q^{N,A}(x) > 0$.

Then, $\frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(\tilde{\beta}_q^{N,A}(x), 0) = 0$. By (34), we have $\left|\frac{\partial}{\partial b_q} H_x(\tilde{\beta}_q^{N,A}(x), 0; p^N)\right| < \epsilon/D$ and hence $|M(Q(\tilde{\beta}_q^{N,A}(x), 0; p^N); p^N)| < \epsilon/D$. This, by (37), implies that

$$|\tilde{\beta}_q^{N,A}(x) - \chi_q(x; p^N)| + |\chi_r(x; p^N)| < \epsilon.$$

(b) $\tilde{\beta}_r^{N,A}(x) > 0$.

Then, $\frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0, \tilde{\beta}_r^{N,A}(x)) = 0$. By (34), we have $\left|\frac{\partial}{\partial b_r} H_x(0, \tilde{\beta}_r^{N,A}(x); p^N)\right| < \epsilon/D$ and hence $|M(Q(\tilde{\beta}_q^{N,A}(x), 0; p^N))/p^N| < \epsilon/D$. This, (37), and $D/p^N > D_1 D_2$ for all $p^N \in \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\kappa_r+\bar{r}}{\kappa_q}\right]$ imply

$$|\chi_q(x; p^N)| + |\tilde{\beta}_r^{N,A}(x) - \chi_r(x; p^N)| < \epsilon.$$

(c) $\tilde{\beta}_q^{N,A}(x) = 0 = \tilde{\beta}_r^{N,A}(x)$.

Then, $\frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(0,0) \leq 0$ and $\frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0,0) \leq 0$. By (34), we have

$$-M(Q(0,0;p^N);p^N) = \frac{\partial}{\partial b_q} H_x(0,0;p^N) < \frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(0,0) + \epsilon/D \leq \epsilon/D$$

and

$$M(Q(0,0;p^N);p^N)/p^N = \frac{\partial}{\partial b_r} H_x(0,0;p^N) < \frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0,0) + \epsilon/D \leq \epsilon/D.$$

It then follows that $|M(Q(0,0;p^N);p^N)| < \epsilon/D_1 D_2$ and hence

$$|\chi_q(x; p^N)| + |\chi_r(x; p^N)| < \epsilon.$$

This concludes the proof of Claim 3 for $\tilde{\beta}^{N,A}$.

The argument is identical for $\hat{\beta}^{N,A}$, except that we need an additional argument to show that for any $\epsilon > 0$, there exists $N^2(\epsilon)$ and $\delta^2(\epsilon)$ such that if $N > N^2(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^2(\epsilon)$ for all k, then for all $b_q \in [0, \bar{q} - d]$ and all $b_r \in [0, \bar{r} - d]$,

$$\left|\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) - \frac{\partial}{\partial b_q}\hat{H}_x^{\lambda^N}(b_q, 0)\right| < \epsilon \text{ and } \left|\frac{\partial}{\partial b_r} H_x(0, b_r; p^N) - \frac{\partial}{\partial b_r}\hat{H}_x^{\lambda^N}(0, b_r)\right| < \epsilon. \tag{38}$$

Although (38) is completely analogous to (34), an additional argument is required because $\tilde{\beta}^{N,A}$ appears in $(\hat{Q}_-^N, \hat{R}_-^N)$, while p^N only involves $\hat{\beta}^{N,A}$.

Notice that

$$p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N} = \frac{\sum_{y\in X}\mu_{z^*}(y)\hat{\beta}_r^{N,A}(y)+\kappa_r}{\sum_{y\in X}\mu_{z^*}(y)\hat{\beta}_q^{N,A}(y)+\kappa_q} - \frac{\sum_{k=1,..,K}\sum_{y\in Y_k}\frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\hat{\beta}_r^{N,A}(y)+\frac{1}{M^N}\tilde{\beta}_r^{N,A}(\tilde{\xi})+\kappa_r}{\sum_{k=1,..,K}\sum_{y\in Y_k}\frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\hat{\beta}_q^{N,A}(y)+\frac{1}{M^N}\tilde{\beta}_q^{N,A}(\tilde{\xi})+\kappa_q}.$$

Therefore,

$$\left|p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{2(\bar{q}+\kappa_q)(\bar{r}+\kappa_r)}{\kappa_q^2}\left[\sum_{k=1}^K\left|\mu_{z^*}(Y_k)-\frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\right| + \left|\frac{\bar{q}}{M^N}+\frac{\bar{r}}{M^N}\right|\right].$$

Also,

$$\left|\frac{\hat{Q}_-^N\hat{R}_-^N}{(\hat{Q}_-^N+b_q)^2} - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{\bar{r}+\kappa_r}{\kappa_q}\frac{\bar{q}(2\kappa_q+\frac{\bar{q}}{M^N})}{\kappa_q^2 M^N} \text{ and } \left|\frac{\hat{R}_-^N}{\hat{Q}_-^N+b_q} - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{\bar{r}+\kappa_r}{\kappa_q}\frac{\bar{q}}{\kappa_q M^N}.$$

Thus, for any $\epsilon > 0$ there exist $\tilde{N}(\epsilon)$ and $\tilde{\delta}(\epsilon) \leq \epsilon$ such that if $N > \tilde{N}(\epsilon)$ and if $\left|\hat{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \tilde{\delta}(\epsilon)$ and $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \tilde{\delta}(\epsilon)$, then

$$\left|p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| < \epsilon,\ \left|\frac{\hat{Q}_-^N\hat{R}_-^N}{(\hat{Q}_-^N+b_q)^2} - p^N\right| < \epsilon, \text{ and } \left|\frac{\hat{R}_-^N}{\hat{Q}_-^N+b_q} - p^N\right| < \epsilon$$

for all b_q. Given these results, the rest of the argument is exactly the same as for $\tilde{\beta}^{N,A}$.□

Claim 4. For any $\epsilon > 0$, there exists $N^4(\epsilon)$ such that if $N > N^4(\epsilon)$, then for all $x \in X$,

$$|F_x^N(A) - F_x^N(s_1^N(x))| < \epsilon \text{ for any } A \subseteq X, \tag{39}$$

where, recall, $F_x^N(A)$ is the expected payoff from offer a that is associated with the set A at stage-1 conditional on being active.

Proof. Consider a state z. First we show that for any $\epsilon > 0$, there exist $N^5(\epsilon)$ and $\delta^5(\epsilon)$ such that if $N \geq N^5(\epsilon)$ and if $\left|\lambda^N(Y_k)/M^N - \mu_z(Y_k)\right| < \delta^5(\epsilon)$, then for each $x \in X$,

$$\| \hat{\beta}^{N,A}(x) - \beta^z(x) \| < \epsilon \text{ and } \| \tilde{\beta}^{N,A}(x) - \beta^z(x) \| < \epsilon, \tag{40}$$

where $\| b - b' \| = |b_q - b'_q| + |b_r - b'_r|$ for all $b, b' \in \mathcal{O}$. We establish (40) for $\hat{\beta}^{N,A}$ and a fixed state z^*. The other case is exactly the same. For any $\delta = (\delta_q, \delta_r) \geq (0,0)$, let β^δ be the unique offers corresponding to the competitive equilibrium in the economy $\mathcal{L}^{z^*}(\kappa+\delta)$ as defined in Lemma 2. Then, $\beta^{z^*} = \beta^{\delta=0}$ and β^δ is continuous in δ by Lemma 2.

If we set

$$\tilde{\delta} = (\tilde{\delta}_q, \tilde{\delta}_r) = \sum_{x\in X}\mu_{z^*}(x)[\beta^N(x) - \chi(x;p^N)],$$

then, by construction, $(\chi(x;p^N))_{x\in X}$ satisfies

$$p^N = \frac{\sum_{x\in X}\mu_{z^*}(x)\chi_r(x;p^N)+\kappa_r+\tilde{\delta}_r}{\sum_{x\in X}\mu_{z^*}(x)\chi_q(x;p^N)+\kappa_q+\tilde{\delta}_q}.$$

That is, $(\chi(x;p^N))_{x\in X}=\beta^{\tilde{\delta}}$. Because β^{δ} is continuous in δ, for any $\epsilon>0$, there exists a $\delta^P(\epsilon)\le\epsilon$ such that if $\max\{|\tilde{\delta}_q|,|\tilde{\delta}_r|\}\le\delta^P(\epsilon)$, then

$$\|\beta^{z^*}(x)-\chi(x;p^N)\|<\epsilon \text{ for all } x\in X. \tag{41}$$

Now for any $\epsilon>0$, let $\delta'=\delta^P(\epsilon/2)$ and let $\delta^5(\epsilon)=\delta^3(\delta')$. Let $N^5(\epsilon)=N^3(\delta')$. By Claim 3, if $\left|\frac{\lambda^N(Y_k)}{M^N}-\mu_{z^*}(Y_k)\right|<\delta^3(\delta')$ for all k and if $N>N^3(\delta')$, then

$$\|\tilde{\beta}^{N,A}(x)-\chi(x;p^N)\|<\delta'\le\frac{\epsilon}{2} \text{ for all } x\in X.$$

This then implies that

$$|\tilde{\delta}_q|+|\tilde{\delta}_r|\le\sum_{x\in X}\mu_{z^*}(x)|\tilde{\beta}_q^{N,A}(x)-\chi_q(x;p^N)|+\sum_{x\in X}\mu_{z^*}(x)|\tilde{\beta}_r^{N,A}(x)-\chi_r(x;p^N)|<\delta'=\delta^P(\epsilon/2).$$

By (41), this implies that $\|\beta^{z^*}(x)-\chi(x;p)\|<\epsilon/2$. Thus, for each x,

$$\|\tilde{\beta}^{N,A}(x)-\beta^{z^*}(x)\|\le\|\tilde{\beta}^{N,A}(x)-\chi(x;p^N)\|+\|\beta^{z^*}(x)-\chi(x;p^N)\|<\epsilon,$$

which is (40).

Let $\bar{\gamma}_z^N$ be the probability distribution over other active agents' types conditional on state z, that is, $\bar{\gamma}_z^N[\xi_1,...,\xi_{M^N-1}]=\prod_{t=1}^{M^N}\mu_z(\xi_t)$. For any nonempty $A\subseteq X$,

$$F_x^N(A)=\sum_{z\in Z}\tau_x[z]\mathbb{E}_{\bar{\gamma}_z^N}\left[u\left(\bar{q}+\frac{\tilde{\beta}_r^{N,A}(x)Q_-^N-\tilde{\beta}_q^{N,A}(x)R_-^N}{\tilde{\beta}_r^{N,A}(x)+R_-^N},\bar{r}+\frac{\tilde{\beta}_q^{N,A}(x)R_-^N-\tilde{\beta}_r^{N,A}(x)Q_-^N}{\tilde{\beta}_q^{N,A}(x)+Q_-^N};x,z\right)\right],$$

where

$$(Q_-^N,R_-^N)=\sum_{y\in X}\sigma(y)\hat{\beta}^{N,A}(y)+M^N\kappa \text{ and } \sigma(y)=\#\{\xi_t:t=1,...,M^N-1,\xi_t=y\}.$$

By the *law of large numbers*, $\sigma(y)/M^N$ converges to $\mu_z(y)$ in probability under $\bar{\gamma}_z^N$. As a result, $\lambda^N(Y_k)/M^N$ converges to $\mu_z(Y_k)$ in probability under $\bar{\gamma}_z^N$. Therefore, by (40), for any $\epsilon'>0$ there exists $N^z(\epsilon')$ such that if $N>N^z(\epsilon')$, then

$$\bar{\gamma}_z^N\left[\left|\left(\frac{\tilde{\beta}_r^{N,A}(x)Q_-^N-\tilde{\beta}_q^{N,A}(x)R_-^N}{\tilde{\beta}_r^{N,A}(x)+R_-^N}\right)-\left(\frac{\beta_r^z(x)}{p^z}-\beta_q^z(x)\right)\right|<\epsilon'\right]>1-\epsilon'$$

and

$$\bar{\gamma}_z^N\left[\left|\left(\frac{\tilde{\beta}_q^{N,A}(x)R_-^N-\tilde{\beta}_r^{N,A}(x)Q_-^N}{\tilde{\beta}_q^{N,A}(x)+Q_-^N}\right)-\left(\beta_q^z(x)p^z-\beta_r^z(x)\right)\right|<\epsilon'\right]>1-\epsilon'.$$

With appropriate ϵ''s, this implies that

$$\left|F_x^N(A) - \sum_{z \in Z} \tau_x[z] u\left(\bar{q} + \frac{\beta_r^z(x)}{p^z} - \beta_q^z(x), \bar{r} + \beta_q^z(x)p^z - \beta_r^z(x); x, z\right)\right| < \epsilon.$$

Claim 4 follows from the fact that there are only finitely many nonempty $A \subseteq X$.□

Now we complete the proof. Recall that we begin with a candidate semi-pooling equilibrium associated with the partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ with $1 < K < |X|$. Because such an equilibrium does not exist when $|X| = 2$, we may assume that for some $y^1, y^2 \in X$, $y^1 \neq y^2 \in Y_1$. Recall that $G_x(a)$ is the objective function for a type-x agent at stage-1 conditional on being inactive (see (5)). Because $y^1 \neq y^2$, there exists $C > 0$ such that for any $a \in \mathcal{O}$, either $G_{y^1}(a) < G_{y^1}(\alpha^*(y^1)) - C$ or $G_{y^2}(a) < G_{y^2}(\alpha^*(y^1)) - C$. Assume without loss of generality that $G_{y^1}(s_1^N(Y_1)) < G_{y^1}(\alpha^*(y^1)) - C$ so that $s_1^N(y^1) \neq \alpha^*(y^1)$.

Let $\bar{N} = N^4(\frac{\eta C}{2(1-\eta)})$. Then, by claim 4, if $N \geq \bar{N}$,

$$|F_{y^1}^N(s_1^N(Y_1)) - F_{y^1}^N(\alpha^*(y^1))| < \frac{\eta C}{2(1-\eta)} \text{ and } |F_{y^2}^N(s_1^N(Y_1)) - F_{y^2}^N(\alpha^*(y^2))| < \frac{\eta C}{2(1-\eta)}.$$

Then, for $N \geq \bar{N}$,

$$\begin{aligned} & \eta G_{y^1}(s_1^N(Y_1)) + (1-\eta)F_{y^1}^N(s_1^N(Y_1)) \\ < \;& \eta(G_{y^1}(\alpha^*(y^1)) - C) + (1-\eta)(F_{y^1}^N(\alpha^*(y^1)) + \frac{\eta C}{2(1-\eta)}) \\ = \;& \eta G_{y^1}(\alpha^*(y^1)) + (1-\eta)F_{y^1}^N(\alpha^*(y^1)) - \frac{\eta C}{2}. \end{aligned}$$

Hence, deviating from $s_1^N(y^1)$ to $\alpha^*(y^1)$ is profitable, a contradiction.■

7.3 Optimality

Theorem 3. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$. Let $(\varepsilon, \delta) > 0$ be given.

(i) There exists $\bar{\kappa} > 0$ and a function $N(\kappa, \eta)$ such that if $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$, $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, and $N > N(\kappa, \eta)$, then there exists a separating equilibrium whose outcome is ex post (ε, δ)-efficient.

(ii) Suppose that A1-A3 hold. Then there exists $\bar{\kappa} > 0$ and a function $N(\kappa, \eta)$ such that if $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$, $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, and $N > N(\kappa, \eta)$, then the outcome of any symmetric equilibrium in pure strategies is ex post (ε, δ)-efficient.

Proof of (i). Recall that $\{(q^z(x; \kappa), r^z(x; \kappa))\}_{x \in X}$ stands for the competitive allocation in the economy $\mathcal{L}^z(\kappa)$ (see Lemma 2). Consider another economy $\mathcal{J}(\kappa, \rho)$, where $\rho : X \to [0, 1]$ stands for the proportion of agents with different types in the economy, and each agent has endowment $(\bar{q} + \kappa_q, \bar{r} + \kappa_r)$. Let $\{(\tilde{q}^z(x; \kappa, \rho), \tilde{r}^z(x; \kappa, \rho))\}_{x \in X}$ denote the competitive allocation for $\mathcal{J}(\kappa, \rho)$ under known state-of-the-world z. We omit the proof of the following claim, which only asserts continuity of competitive allocations w.r.t. endowment parameters and the proportion of different types.

Claim 1. Let $\{(q^z(x;\kappa), r^z(x;\kappa))\}_{x\in X}$ and $\{(\tilde{q}^z(x;2\kappa,\rho), \tilde{r}^z(x;2\kappa,\rho))\}_{x\in X}$ be defined as above. For any $\epsilon > 0$, there is a $\delta^1(\epsilon) > 0$ such that if $|\rho(x) - \mu_z(x)| < \delta^1(\epsilon)$ for each $x \in X$ and if $\max\{\kappa_q, \kappa_r\} < \delta^1(\epsilon)$, then for each $x \in X$,

$$|u(q^z(x;\kappa), r^z(x;\kappa); x, z) - u(\tilde{q}^z(x;2\kappa,\rho), \tilde{r}^z(x;2\kappa,\rho); x, z)| < \epsilon.$$

The next claim constructs the high probability event $E_{z,c^N}(\epsilon)$ that we need to establish (ε, δ) ex post optimality of a Theorem-1 equilibrium. The event E_{z,c^N} will be the intersection of two events, E^1_{z,c^N} and E^2_{z,c^N}: the first involves only exogenous random variables; the second depends on a selected equilibrium.

Fix some $(\kappa, \eta) > 0$. For any realization ζ^N and c^N, there is a unique corresponding type-configuration for active agents, denoted $\sigma(\zeta^N, c^N) = (\sigma(\zeta^N, c^N)(x) : x \in X)$. For each $(z, c^N) \in Z \times \mathbb{C}^N$ and for any $\epsilon > 0$, define the event $E^1_{z,c^N}(\epsilon)$ as

$$E^1_{z,c^N}(\epsilon) = \left\{\zeta^N : (\forall x) \left|\sigma(\zeta^N, c^N)(x)/M^N - \mu_z(x)\right| < \delta^1(\epsilon)\right\}, \tag{42}$$

where $\delta^1(\epsilon)$ is defined in Claim 1 above. By Theorem 1, there exists a number $\bar{N}(\kappa, \eta)$ such that if $N > \bar{N}(\kappa, \eta)$, then there exists a separating equilibrium (s_1^N, s_2^N). As above, we use β^σ to denote the stage-2 offers along the corresponding equilibrium path for a realization of type-configuration σ for active agents; we also use $(q^\sigma(x), r^\sigma(x))$ to denote the corresponding payoffs as determined in (3) from offers β^σ. The event $E^2_{z,c^N}(\epsilon)$ is then defined as

$$E^2_{z,c^N}(\epsilon) = \left\{\zeta^N : (\forall x) \left|u(q^{\sigma(\zeta^N,c^N)}(x), r^{\sigma(\zeta^N,c^N)}(x); x, z) - u(q^z(x;\kappa), r^z(x;\kappa); x, z)\right| < \epsilon\right\}, \tag{43}$$

where $(q^z(x;\kappa), r^z(x;\kappa))\}_{x\in X}$ denotes the competitive allocation in the economy $\mathcal{L}^z(\kappa)$. Finally, let $E_{z,c^N}(\epsilon) = E^1_{z,c^N}(\epsilon) \cap E^2_{z,c^N}(\epsilon)$.

Claim 2. Let (κ, η) be given. Fix a sequence of separating equilibria $\{s_1^N, s_2^N\}_{N>\bar{N}(\kappa,\eta)}$ and let E_{z,c^N} be defined w.r.t. $\{s_1^N, s_2^N\}_{N>\bar{N}(\kappa,\eta)}$. There exists a number $N^2(\kappa, \eta, \epsilon)$ such that if $N > N^2(\kappa, \zeta, \epsilon)$, then for any $(z, c^N) \in Z \times \mathbb{C}^N$, $\mathbb{P}\left[E_{z,c^N}(\epsilon) \mid c^N, z\right] > 1 - \epsilon$.

Proof. Let ξ be an infinite sequence of i.i.d. X-valued random variables with marginal distribution given by μ_z. Because c^N is independent of the realization of types and the state-of-the-world, for any N the sequence $\{\zeta_n : 1 \le n \le N,\ c_n = 1\}$ and the sequence $\{\xi_m : m = 1, ..., M^N\}$ have the same distribution conditional on z and c^N. For each N and $\xi^{M^N} = (\xi_1, ..., \xi_{M^N})$, let β^{σ^N} describe the equilibrium stage-2 offers under (s_1^N, s_2^N) along the equilibrium path with $\sigma^N(x) = \#\{1 \le m \le M^N : \xi_m = x\}$ and let $(q^{\sigma^N}, r^{\sigma^N})$ describe the corresponding equilibrium payoffs for active agents. By Proposition 2, it follows that $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^z$ in probability conditional on z. Hence, by continuity of u, for any z,

$$\lim_{N\to\infty} u\left(q^{\sigma^N}(x), r^{\sigma^N}(x); x, z\right) = u(q^z(x;\kappa), r^z(x;\kappa); x, z)$$

in probability conditional on z. Moreover, by the law of large numbers, for any z and $x \in X$,

$$\lim_{N\to\infty} \sigma^N(x)/M^N = \mu_z(x)$$

in probability conditional on z.■

Now we can complete the proof. Given (ε, δ), let $\bar{\kappa} = \min\{\delta/2, \delta^1(\frac{\varepsilon}{3})\}$ and consider our mechanism with $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$ and with $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$.

Given a separating equilibrium (s_1^N, s_2^N) for N agents, and given a type realization, $\zeta^N = (\zeta_1, ..., \zeta_N) \in X^N$, and an activeness-status realization, $c^N = (c_1, ..., c_N)$, the corresponding allocation is as follows: if $c_n = 0$, then

$$\omega_n^{s^N}(\zeta^N, c^N, z) = \left(\bar{q} - \alpha_q(\zeta_n) + \frac{\alpha_r(\zeta_n)}{p^1}, \bar{r} - \alpha_r(\zeta_n) + \alpha_q(\zeta_n)p^1\right),$$

where $s_1^N(x) = (\alpha_q(x), \alpha_r(x))$ for all $x \in X$; if $c_n = 1$, then

$$\omega_n^{s^N}(\zeta^N, c^N, z) = (q^{\sigma(\zeta^N, c^N)}(\zeta_n), r^{\sigma(\zeta^N, c^N)}(\zeta_n)).$$

Let $N(\kappa, \eta) = N^2(\kappa, \eta, \frac{\varepsilon}{3})$. Now we show that if $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$ and $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, and if $N > N(\kappa, \eta)$, then the allocation $\{\omega_n^{s^N} : n \in \mathcal{N}\}$ corresponding to the separating equilibrium (s_1^N, s_2^N) from the sequence in Claim 2 is ex post (ε, δ)-efficient. For any realization $\zeta^N = (\zeta_1, ..., \zeta_N) \in X^N$ and $c^N = (c_1, ..., c^N)$, let $\mathcal{M}(\zeta^N, c^N) = \{m \in \mathcal{N} : c_m = 1\}$.

Then,

$$\begin{aligned}
\sum_{n\in\mathcal{N}} \omega_n^{s^N}(\zeta^N, c^N, z) &= \sum_{m\in\mathcal{M}(\zeta^N,c^N)} \omega_m^{s^N}(\zeta^N, c^N, z) + \sum_{n\in\mathcal{N}-\mathcal{M}(\zeta^N,c^N)} \omega_n^{s^N}(\zeta^N, c^N, z) \\
&\leq (1-\eta)N(\bar{q} + \kappa_q, \bar{r} + \kappa_r) + \eta N(2\bar{q}, 2\bar{r}) \\
&\leq N[(\bar{q}, \bar{r}) + (\kappa_q, \kappa_r) + (\frac{\kappa_q}{2}, \frac{\kappa_r}{2})] < N[(\bar{q}, \bar{r}) + (\delta, \delta)].
\end{aligned}$$

which gives condition (a) in our definition. For each z and for each c^N, by claim 2, $N > N^2(\kappa, \eta, \frac{\varepsilon}{3})$ implies $\mathbb{P}[E_{z,c^N}(\varepsilon/3)|z, c^N] > 1 - \varepsilon$, condition b(i). As for b(ii), suppose, by way of contradiction, that ω'_n satisfies (12) and (13) for some z, c^N and for some $\zeta^N \in E_{z,c^N}(\varepsilon/3)$. Let $\omega'_n(\zeta^N, c^N, z) = (q'_n, r'_n)$ for each $n \in \mathcal{N}$.

Let $\rho(x) = \sigma(\zeta^N, c^N)(x)/M^N$. Then, $\{(\tilde{q}^z(x; 2\kappa, \rho), \tilde{r}^z(x; 2\kappa, \rho)) : x \in X\}$ is the competitive allocation for a finite economy which has $\sigma(\zeta^N, c^N)(x)$ agents of type-x for each x and in which each agent has endowment $(\bar{q} + 2\kappa_q, \bar{r} + 2\kappa_r)$. Now, consider the allocation $\{(q''_n, r''_n) : n \in \mathcal{N}\}$ given by $(q''_n, r''_n) = (\tilde{q}^z(x; 2\kappa, \rho), \tilde{r}^z(x; 2\kappa, \rho))$ if $c_n = 1$ and $\zeta_n = x$ and by $(q''_n, r''_n) = (0, 0)$ if $c_n = 0$. Because $\zeta^N \in E^1_{z,c^N}(\varepsilon/3)$ and $\max\{\kappa_q, \kappa_r\} < \delta^1(\varepsilon/3)$, it follows from Claim 1 that

$$|u(q^z(x; \kappa), r^z(x; \kappa); x, z) - u(\tilde{q}^z(x; 2\kappa, \rho), \tilde{r}^z(x; 2\kappa, \rho); x, z)| < \varepsilon/3.$$

Moreover, because $\zeta^N \in E^2_{z,c^N}(\varepsilon/3)$, we have

$$\left|u(q^{\sigma(\zeta^N,c^N)}(x), r^{\sigma(\zeta^N,c^N)}(x); x, z) - u(q^z(x;\kappa), r^z(x;\kappa); x, z)\right| < \varepsilon/3.$$

Thus,

$$u(\tilde{q}^z(x; 2\kappa, \rho), \tilde{r}^z(x; 2\kappa, \rho); x, z) < u(q^{\sigma(\zeta^N,c^N)}(x), r^{\sigma(\zeta^N,c^N)}(x); x, z) + 2\varepsilon/3.$$

Now, for each n such that $c_n = 1$ and $\zeta_n = x$,

$$u(q''_n, r''_n; x, z) < u(\omega_n^{s^N}(\zeta^N, c^N, z); x, z) + 2\varepsilon/3 < u(q'_n, r'_n; x, z) - \varepsilon/3; \tag{44}$$

while for each n such that $c_n = 0$ and $\zeta_n = x$,

$$u(q''_n, r''_n; x, z) = u(0, 0; x, z) \le u(\omega_n^{s^N}(\zeta^N, c^N, z); x, z) < u(q'_n, r'_n; x, z) - \varepsilon, \tag{45}$$

where the second inequality in each of (44) and (45) follows from (13), the contradicting assumption. Therefore, $\{(q'_n, r'_n) : n \in \mathcal{N}\}$ Pareto dominates $\{(q''_n, r''_n) : n \in \mathcal{N}\}$.

However, $\{(q''_n, r''_n) : n \in \mathcal{N}\}$ is a competitive allocation (with inactive agents having zero endowments) for an economy with total resources no less than that for the allocation $\{(q'_n, r'_n) : n \in \mathcal{N}\}$. By the first fundamental theorem of welfare economics, it follows that $\{(q''_n, r''_n) : n \in \mathcal{N}\}$ cannot be Pareto dominated by $\{(q'_n, r'_n) : n \in \mathcal{N}\}$.

Proof of (ii). Under A1-A3 and by Theorems 1 and 2, it follows that there is a number $\tilde{N}(\kappa, \eta)$ such that if $N > \tilde{N}(\kappa, \eta)$, then any symmetric equilibrium in pure strategies (s_1^N, s_2^N) is a separating equilibrium. For any $(\varepsilon, \delta) > 0$ we can then construct $\bar{\kappa}$ and $N(\kappa, \eta)$ as in the proof of (i) to show almost ex post efficiency for (ε, δ). However, we need to modify Claim 2 in that proof so that the convergence rate does not depend on the sequence of equilibria that we choose. Claim 3 is the modified version. There, as in Claim 2, δ^1 comes from Claim 1.

Claim 3. Let (κ, η) be given. Let $N > \tilde{N}(\kappa, \eta)$. Then, for any $\epsilon > 0$, there exists $N^3(\kappa, \eta, \epsilon) > \tilde{N}(\kappa, \eta)$ such that if $N > N^3(\kappa, \eta, \epsilon)$, then for any equilibrium (s_1^N, s_2^N) and for any $(z, c^N) \in Z \times \mathbb{C}^N$, the event $E_{z,c^N}(\epsilon)$ given by (42) and (43) w.r.t. the equilibrium (s_1^N, s_2^N) satisfies $\mathbb{P}\left[E_{z,c^N}(\epsilon) \mid c^N, z\right] > 1 - \epsilon$.

Proof. In the proof of Theorem 2, we establish that for any semi-pooling equilibrium (including the separating equilibrium), the convergence rate of β^{σ^N} to β^z (as defined in Proposition 2) is uniformly bounded across all equilibria. When $N > \tilde{N}(\kappa, \eta)$, all such equilibria are separating. Indeed, by equation (40) (taking $\mathcal{Y} = \{\{x\} : x \in X\}$ and assuming that the target agent follows equilibrium behavior at stage-1), for any $\epsilon > 0$ there exist $N^5(\epsilon)$ and $\delta^5(\epsilon)$ such that if $|\sigma^N(x)/M^N - \mu_z(x)| < \delta^5(\epsilon)$ for all x and if $N > N^5(\epsilon)$, then $\| \beta^{\sigma^N}(x) - \beta^z(x) \| < \epsilon$ for all $x \in X$. The claim then follows from continuity of u and the law of large numbers, which implies that for any z and $x \in X$, $\lim_{N\to\infty} \sigma^N(x)/M^N = \mu_z(x)$ in probability conditional on z. ■

Given Claim 1 and Claim 3, the rest of the proof is the same as the proof of (i), except for letting $N(\kappa, \eta) = N^3(\kappa, \eta, \frac{\varepsilon}{3})$. □

References

[1] Cripps, M. W. and J. M. Swinkels, Efficiency of large double auctions. *Econometrica*, 74 (2006) 47-92.

[2] Dubey, P. S. and M. Shubik, A theory of money and financial institutions. 28. The non-cooperative equilibria of a closed trading economy with market supply and bidding strategies. *Journal of Economic Theory*, 17 (1978) 1-20.

[3] Fudenberg, D. and J. Tirole, Perfect bayesian equilibrium and sequential equilibrium. *Journal of Economic Theory*, 53 (1991) 236-260.

[4] Dubey, P.S., J. Geanakoplos, and M. Shubik, The revelation of information in strategic market games. *Journal of Mathematical Economics,* 16 (1987) 105-137.

[5] Gul, F. and A. Postlewaite, Aymptotic efficiency in large exchange economies with asymmetric information. *Econometrica*, 60 (1992) 1273-92.

[6] Hurwicz, L., E. Maskin, and A. Postlewaite, Feasible Nash implementation of social choice rules when the designer does not know endowments or production sets. In *The Economics of Informational Decentralization: Complexity, Efficiency, and Stability*, edited by J. Ledyard, Kluwer Academic Publishers, Boston/Dordrecht/London, 1995, pages 367-433.

[7] McLean, R. and A. Postlewaite, Informational size and incentive compatibility. *Econometrica*, 70 (2002) 2421-53.

[8] Palfrey, T.R., Uncertainty resolution, private information aggregation and the Cournot competitive limit. *The Review of Economic Studies*, 52 (1985) 69-83.

[9] Axelrod, B., B. Kulich, C. Plott, and K. Roust, Design improved parimutuel-type information aggregation mechanisms: inaccuracies and the long-shot bias as disequilibrium phenomena. *Journal of Economic Behavior and Organization,* 69 (2009) 170-181.

[10] Reny, P.J. and M. Perry, Toward a strategic foundation for rational expectations equilibrium. *Econometrica*, 74 (2006) 1231–69.

[11] Shapley, L.S. and M. Shubik, Trade using one commodity as a means of payment, *Journal of Political Economy,* 85 (1977) 937-68.

[12] Xavier Vives, Aggregation of information in large cournot markets. *Econometrica*, 56 (1988) 851-76.

[13] Xavier Vives, Strategic supply function competition with private information. *Econometrica*, 79 (2011) 1119-66.