Aggregating the single crossing property

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Abstract: The single crossing property plays a crucial role in economic theory, yet there are important instances where the property cannot be directly assumed or easily derived. Difficulties often arise because the property cannot be aggregated: the sum or convex combination of two functions with the single crossing property need not have that property. We introduce a new condition characterizing when the single crossing property is stable under aggregation and also identify sufficient conditions for the preservation of the single crossing property under multidimensional aggregation. We use our results to establish properties of objective functions (convexity, logsupermodularity), the monotonicity of optimal decisions under uncertainty, and the existence of monotone equilibria in Bayesian-games.

Keywords: monotone comparative statics, single crossing property, Bayesian games, monotone strategies, logsupermodularity, signed-ratio monotonicity

1. Introduction

Consider the following problem, pervasive in comparative statics analysis: an agent chooses action $x \in X \subset \mathbb{R}$ to maximize her objective $V(x; s)$, where $s \in S \subset \mathbb{R}$ is some parameter; how does $\text{argmax}_{x\in X} V(x; s)$ vary with s? It is well-known that $\text{argmax}_{x\in X} V(x; s)$ increases with s if the family $\{V(\cdot; s)\}_{s\in S}$ obeys *single crossing differences*; this means that, for any $x'' > x'$, the function

$$\Delta(s) = V(x''; s) - V(x'; s)$$

has the single crossing property, in the sense that Δ crosses the horizontal axis just once, from negative to positive, as s increases (see Milgrom and Shannon (1994)).[1] This simple but powerful result is useful when one is interested in comparative statics for its own sake (for

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[1] Our use of the term *single crossing differences* follows Milgrom (2004).

example, when considering an agent's portfolio allocation problem) or when monotonicity is needed for establishing some other result (like equilibrium existence in supermodular games (see Milgrom and Roberts (1990) and Vives (1990)).

However, single crossing differences cannot always be directly assumed or easily derived from primitive assumptions. We give two such cases.

Case 1. Consider an agent who maximizes expected payoff $V(x; s) = \int_T v(x; s, t)\, dF(t)$, where t represents possible states of the world and F the distribution over those states. Suppose, for any given t, $\{v(\cdot; s, t)\}_{s \in S}$ obeys single crossing differences, so that the optimal action increases with parameter s if the state t is known; in general, this is *not* sufficient to guarantee that $\{V(\cdot; s)\}_{s \in S}$ obeys single crossing differences, so we cannot conclude that $\text{argmax}_{x \in X} V(x; s)$ increases with s.

Notice that this difficulty will not arise if $\{v(\cdot; s, t)\}_{s \in S}$ obeys increasing differences, i.e., if $v(x'', s, t) - v(x', s, t)$ is increasing in s at any t, since $\Delta(s)$ will then be increasing in s as well. However, increasing differences is often too strong an assumption. For example, suppose $v(x, s, t) = u(\Pi(x, s, t))$ where u is the agent's Bernoulli utility function and $\Pi(x, s, t)$ is the monetary payoff in state t and parameter s. If $\{\Pi(\cdot; s, t)\}_{s \in S}$ obeys increasing differences, then $\{v(\cdot; s, t)\}_{s \in S}$ will obey single crossing differences, but will not typically have increasing differences (unless u is linear).

Case 2. Consider an n-player Bayesian game in which each player i takes an action after observing a signal $s_i \in [0, 1]$. Signal s_i may convey direct information on player i's payoff function as well as indirect information on the actions of other players in the game (through the joint distribution on players' signals). In this case, it can be shown that the player i's objective function takes the form $V_i(x; s_i) = \int_{[0,1]^{n-1}} v_i(x; s)\, dF(s_{-i}|s_i)\, ds_{-i}$, where $F(s_{-i}|s_i)$ is the distribution of s_{-i} conditional on observing s_i. The existence of a Bayesian-Nash equilibrium in which each player plays a monotone strategy (i.e., a strategy where the action increases with the player's signal) hinges on whether a particular player has an optimal strategy that is monotone, given that other players are playing monotone strategies (see Athey (2001)). To ensure that $\text{argmax}_{x \in X} V_i(x; s_i)$ increases with s_i, it suffices to have $\{V_i(\cdot; s_i)\}_{s_i \in [0,1]}$ obey single crossing differences; however, this property may not hold, even when $\{v_i(\cdot; s)\}_{s \in [0,1]^n}$ obeys single crossing differences and the signals are affiliated.

While these problems may be solved in specific contexts using various ad hoc techniques, there has been no attempt at developing a general theory that addresses them systematically.

We think that these problems are best understood as arising from the fact that the single crossing property is not preserved under aggregation; i.e., the sum of two functions with the single crossing property does not generally satisfy this property.

In this paper, we characterize the conditions under which the single crossing property *is* preserved with aggregation. We show that any weighted sum of two functions with the single crossing property also has this property if and only if the ratio of these functions is monotone in a particular sense; we refer to this relation between functions as *signed-ratio monotonicity*. Applying our results, we find conditions under which a risk averse monopolist who faces uncertainty in demand will increase output when there is a fall in marginal cost. This is an instance of a Case 1 problem. For a Case 2 problem, we look at a Bertrand oligopoly with risk averse firms selling differentiated products, where each firm receives a signal on the state of demand for his output. We find conditions under which each firm's pricing decision is monotone in the signal he receives, given that other firms are playing monotone strategies. Finally, we illustrate how our results can be used to check that a function is quasiconcave or concave-convex; note that the former can be characterized as a function with a single crossing first derivative and the latter as a function with a single crossing second derivative.

2. Aggregating single crossing functions

Let $(S, \geq)$ be a partially ordered set; a function $f : S \to \mathbb{R}$ is said to have the *single crossing property* if it satisfies the following:

$$f(s') \geq (>)\, 0 \Longrightarrow f(s'') \geq (>)\, 0 \;\text{ whenever } s'' > s'.^{2} \tag{1}$$

When S is an interval of $\mathbb{R}$, the graph of f is a curve that crosses the horizontal axis just once, hence the term 'single crossing'. We refer to a function that obeys the single crossing property as a *single crossing function* or an $\mathcal{SC}$ function.

It is easy to see that the sum of two single crossing functions is not necessarily a single crossing function.[3] This section is devoted to identifying the condition under which the single crossing property is preserved with aggregation. Consider two single-crossing functions f and g defined on the partially ordered set $(S, \geq)$.

[2]We mean that $f(s') \geq 0 \Longrightarrow f(s'') \geq 0$ whenever $s'' > s'$ and $f(s') > 0 \Longrightarrow f(s'') > 0$ whenever $s'' > s'$.
[3]For example, take $f(s) = \sin(s) + 2$ and $g(s) = -2$.

Definition 1 We say that f and g obey *signed-ratio monotonicity* if they satisfy the following conditions: (a) at any $s' \in S$, such that $g(s') < 0$ and $f(s') > 0$, we have

$$-\frac{g(s')}{f(s')} \geq -\frac{g(s'')}{f(s'')} \text{ when } s'' > s'; \textit{ and} \tag{2}$$

(b) at any $s' \in S$, such that $f(s') < 0$ and $g(s') > 0$, we have

$$-\frac{f(s')}{g(s')} \geq -\frac{f(s'')}{g(s'')} \text{ when } s'' > s'.$$

The significance of this property is clear from the following result.

Proposition 1. *Let f and g be two $\mathcal{SC}$ functions. Then $\alpha f + \beta g$ is an $\mathcal{SC}$ function for any nonnegative scalars α and β if and only if f and g obey signed-ratio monotonicity.*

Proof: Suppose $g(s') < 0$ and $f(s') > 0$ and let $\alpha' = -g(s')/f(s')$, so that $\alpha' f(s') + g(s') = 0$. Since $\alpha' f + g$ is an $\mathcal{SC}$ function, $\alpha' f(s'') + g(s'') \geq 0$ for $s'' > s'$. Re-arranging this inequality and bearing in mind that $f(s'') > 0$ (since f is an $\mathcal{S}$ function and $f(s') > 0$), we obtain

$$\alpha' = -\frac{g(s')}{f(s')} \geq -\frac{g(s'')}{f(s'')}.$$

For the proof of the converse we may, without loss of generality, assume that $\beta = 1$ (since the single crossing property is preserved under positive scalar multiplication). Suppose

$$\alpha f(s') + g(s') \geq (>)\, 0. \tag{3}$$

If $g(s') \geq 0$ and $f(s') \geq 0$, then we have $g(s'') \geq 0$ and $f(s'') \geq 0$ since f and g are $\mathcal{SC}$ functions. It follows that $\alpha f(s'') + g(s'') \geq 0$. This inequality is strict if (3) is a strict inequality since either $f(s') > 0$ or $g(s') > 0$.

Now consider the case where (3) holds but one of the two functions is negative at s'. Suppose that $g(s') < 0$. Then $f(s') > 0$ since (3) holds. For any $s'' > s'$,

$$\alpha \geq (>) -\frac{g(s')}{f(s')} \geq -\frac{g(s'')}{f(s'')}$$

where the first inequality follows from (3) and the second from signed-ratio monotonicity. Re-arranging this inequality, we obtain $\alpha f(s'') + g(s'') \geq (>)\, 0$. (Note that $f(s'') > 0$ since f is an $\mathcal{SC}$ function and $f(s') > 0$.) **QED**

It is easy to check that two increasing functions obey signed-ratio monotonicity, as one would expect, since any positive linear combination of increasing functions is increasing (and thus a single crossing function). For another illustration of Proposition 1, suppose $S = \mathbb{R}$, $f(s) = s^2 + 1$, and $g(s) = s^3$. In this case f is not increasing, but any positive linear combination of f and g is still an $\mathcal{SC}$ function. We can confirm this by checking that f and g have signed-ratio monotonicity: for $s > 0$, we have $-g(s)/f(s) < 0$, while for $s < 0$, the ratio $-g(s)/f(s) = -s^3/(s^2+1)$ is positive and decreasing in s. Now consider a third function $h(s) = e^{-s}$; since this function is positive for all s, h and f have signed-ratio monotonicity, but it is easy to check that this is not the case for h and g. Therefore, signed-ratio monotonicity is not a transitive relation between functions (though it is clearly reflexive and symmetric).

The next result extends Proposition 1; it says that any positive linear combination of a family single crossing functions is also a single crossing function, so long as members of the family obey signed-ratio monotonicity pairwise. The proof can be found in the Appendix.

Theorem 1. *Let (T, Σ, μ) be a finite measure space and suppose that for each $s \in S$, $f(s,t)$ is a bounded and measurable function of $t \in T$. Then $F(s) = \int_T f(s,t)d\mu(t)$ is an $\mathcal{SC}$ function if, for all t, $t' \in T$, the pair of functions $f(s,t)$ and $f(s,t')$ of $s \in S$ satisfy signed-ratio monotonicity. This condition is also necessary if Σ contains all singleton sets and F is required to be an $\mathcal{SC}$ function for any finite measure μ.*

In applications, T will typically be interpreted as the set of possible states of the world and when those states could be represented by a one-dimensional random variable, the assumption of pairwise signed-ratio monotonicity in Theorem 1 is often reasonable. However, in certain applications T is necessarily multi-dimensional. For example, establishing the monotonicity of a player's action with respect to his type in an n-player Bayesian game will involve interpreting t as the *vector* of types of other players in the game. In this case, signed-ratio monotonicity for all possible pairs t and t' is a very strong condition. The next theorem gets round this difficulty by imposing more structure on T and requiring signed-ratio monotonicity only for ordered pairs of t and t'.

Theorem 2. *Let (T_i, Σ_i, μ_i) (for $i = 1, 2, ...n$) be finite measure spaces such that $T_i \subseteq \mathbb{R}$ and let $f : S \times T \to \mathbb{R}$ be a bounded and measurable function of $t \in T = \Pi_{i=1}^n T_i$ and an $\mathcal{SC}$ function of (s,t). Suppose that $\forall i$, $\forall s' \in S$, and $\forall\, t'' > t'$,*
(i) the functions $f(s,t')$ and $f(s,t'')$ of $s \in S$ satisfy signed-ratio monotonicity and
(ii) the functions $f(s', t_i, t'_{-i})$ and $f(s', t_i, t''_{-i})$ of $t_i \in T_i$ satisfy signed-ratio monotonicity.
Then $F(s) = \int_T f(s,t)d\mu(t)$ (where μ is the product measure) is an $\mathcal{SC}$ function.

Remark: It is straightforward to check that if f obeys (i) and (ii) then so does $\tilde{f}(s,t) = f(s,t)h(s,t)$ where h is a logsupermodular function of (s,t).[4] Therefore, Theorem 2 tells us that the map $F(s) = \int_T f(s,t)h(s,t)d\mu(t)$ is also an $\mathcal{SC}$ function. In particular, F is an $\mathcal{SC}$ function if f is increasing in (s,t). In the context of Bayesian games, μ is the Lebesgue measure, h is the posterior density function on the types of other players, conditional on the player receiving signal s; h will be logsupermodular if types are assumed to be affiliated.

3. Applications of the aggregation theorems

Example 1. A quasiconvex function defined on an interval I in $\mathbb{R}$ can be characterized as a function with a derivative that is a single crossing function, while a function on I is concave-convex[5] if its second derivative has the single crossing property. Theorem 1 provides a useful way of checking that a function has such properties. For example, the convex-concave property of the function $G(s) = as^b - cs^d + \alpha s - \beta$ where $s > 0$, $a > 0$, $b < 0$, $c > 0$, and $d > 1$, plays an important role in the entry-exit model of Dixit (1989).[6] To see that G is convex-concave, note that $-G''(s) = f(s) + g(s)$ where $f(s) = -ab(b-1)s^{b-2}$ and $g(s) = cd(d-1)s^{d-2}$. Since $f(s) < 0$ and $g(s) > 0$, both f and g are $\mathcal{SC}$ functions. While f is an increasing function, g may be increasing or decreasing, depending on the value of d. Nonetheless, it is easy to check that these functions satisfy signed-ratio monotonicity. Hence $-G''$ is an $\mathcal{SC}$ function and G is a convex-concave function.

For another simple example, consider the profit function $\Pi(x) = xP(x) - C(x)$, where P is the inverse demand function and C the cost function. It is known that Π is quasiconcave if P is positive, decreasing, and log-concave and C is convex. To recover this result, note that $-\Pi'(x) = f(x) + g(x)$, where $f(x) = -P(x)$ and $g(x) = -xP'(x) + C'(x)$. It is easy to check that the conditions on P and C imply that f and g satisfy signed-ratio monotonicity and so Π' has the single crossing property.

Example 2. Theorem 2 implies the well-known result that logsupermodularity is preserved under integration (see Karlin and Rinott (1980)); for economic applications of this result see Jewitt (1991), Gollier (2001) and Athey (2001)). In the statement below, $X = \Pi_{i=1}^m X_i$ and $Y = \Pi_{j=1}^n Y_j$, where X_i (for $i = 1, 2, ..., m$) are subsets of $\mathbb{R}$ and Y_j (for $j = 1, 2, ..., n$) are

[4] The precise property of h we need (which makes sense even when S is not a lattice) is the following: $\forall i$, $\forall s' \in S$, and $\forall t'' > t'$, $h(s,t'')/h(s,t')$ is increasing in s and $h(s', t_i, t''_{-i})/h(s', t_i, t'_{-i})$ is increasing in t_i.

[5] By this we mean that there is $\bar{s} \in I$ such that f is concave for $s < \bar{s}$ and convex for $s > \bar{s}$.

[6] In that model, s is the price of the product and G represents the difference in the continuation value between being in and out of the market.

nonempty compact intervals of $\mathbb{R}$.

Corollary 1. *Let ϕ be a function from $X \times Y$ to the positive real numbers such that, for any x, $\phi(x, \cdot)$ is a bounded and measurable function of $y \in Y$. If ϕ is logsupermodular in (x, y) then Φ, defined by $\Phi(x) = \int_Y \phi(x, y)dy$ is a logsupermodular function.*

Proof: Let $K \subset M = \{1, 2, ...m\}$, and suppose $a'' > a'$, for a'' and a' in $\Pi_{i \in K} X_i$. Let $b^{**} > b^*$ be two vectors in $\Pi_{i \in M \setminus K} X_i$. Suppose $\Phi(b^*, a'') = Q\, \Phi(b^*, a')$; then

$$\int_Y [\phi(b^*, a'', y) - Q\phi(b^*, a', y)]\, dy = 0. \tag{4}$$

Define the function $G : \Pi_{i \in M \setminus K} X_i \to \mathbb{R}$ by $G(b) = \int_Y [\phi(b, a'', y) - Q\phi(b, a', y)]\, dy$. Note that the integrand may be written as

$$\left[\frac{\phi(b, a'', y)}{\phi(b, a', y)} - Q \right] \phi(b, a', y).$$

The term in the square brackets is increasing in (b, y) (because ϕ is logsupermodular); so G is an $\mathcal{SC}$ function of b (see Remark following Theorem 2). Since $G(b^*) = 0$ (by (4)), we obtain $G(b^{**}) \geq 0$. Therefore,

$$\frac{\Phi(b^{**}, a'')}{\Phi(b^{**}, a')} \geq Q = \frac{\Phi(b^*, a'')}{\Phi(b^*, a')},$$

which establishes the logsupermodularity of Φ. **QED**

Example 3. We now apply Theorem 1 to solve a problem belonging to the first case discussed in the Introduction. A firm has to decide on its optimal output level $x > 0$. Its profit function is $\Pi(x; s) = xP(x) - C(x; s)$, where P is the inverse demand function and $C(\cdot; s)$ is the cost function, parameterized by s in $(S, \geq)$. It is well-known that a *decrease in marginal cost leads to a rise in the profit-maximizing output.* To model this formally, assume that the family $\{C(\cdot; s)\}_{s \in S}$ obeys decreasing differences; by this we mean that, for all $x'' > x'$, the difference $C(x''; s) - C(x'; s)$ is decreasing in s. If C is differentiable, this is equivalent to marginal cost dC/dx decreasing with s. It follows that $\{\Pi(\cdot; s)\}_{s \in S}$ obeys increasing (hence, single crossing) differences, so an application of Milgrom and Shannon's monotone comparative statics theorem guarantees that the profit-maximizing output increases with s.

Now consider a more general setting where the firm faces uncertainty over the demand for its output. We assume that the profit at state $t \in T \subset \mathbb{R}$ is given by

$$\Pi(x; s, t) = xP(x; t) - C(x; s) \tag{5}$$

and that the firm maximizes $V(x;s) = \int_T u\left(\Pi(x;s,t)\right)\lambda(t)\,dt$, where $\lambda(t)$ is the subjective probability of state t and $u:\mathbb{R}\to\mathbb{R}$ is the Bernoulli utility function representing the monopolist's attitude towards uncertainty. We would like to identify conditions under which $\{V(\cdot;s)\}_{s\in S}$ obeys single crossing differences, so that we could guarantee that the optimal output level increases with s. In other words, for any $x'' > x'$, we require

$$\Delta(s) = \int_T \left[u\left(\Pi(x'';s,t)\right) - u\left(\Pi(x';s,t)\right)\right]\lambda(t)\,dt$$

to be an $\mathcal{SC}$ function. For each t, $\delta(s,t) = u\left(\Pi(x'';s,t)\right) - u\left(\Pi(x';s,t)\right)$ is an $\mathcal{SC}$ function of s if $\{C(\cdot;s)\}_{s\in S}$ obeys decreasing differences. However, unless u is linear, δ will not in general be increasing in s. Hence we face a problem of precisely the type that Theorem 1 is meant to address. Theorem 1 says that Δ is an $\mathcal{SC}$ function if signed-ratio monotonicity holds; the next result gives conditions under which that is satisfied. The proof is in the Appendix.

Proposition 2. *Suppose that (i) C is increasing x and decreasing in s and $\{C(\cdot;s)\}_{s\in\mathbb{S}}$ obeys decreasing differences; (ii) P is decreasing in x and increasing in t and $\{\ln P(\cdot;t)\}_{t\in T}$ obeys increasing differences; and (iii) $u:\mathbb{R}\to\mathbb{R}$ is twice differentiable, with $u'>0$, and obeys DARA. Then for any t, $t'\in T$, the functions $\delta(s,t)$ and $\delta(s,t')$ of s obey signed-ratio monotonicity.*

Remark: When P is differentiable, $\{\ln P(\cdot;t)\}_{t\in T}$ obeys increasing differences if and only if

$$-\frac{1}{P(x;t'')}\frac{\partial P}{\partial x}(x;t'') \;\leq\; -\frac{1}{P(x;t')}\frac{\partial P}{\partial x}(x;t') \quad \forall x>0,\ t''>t'.$$

Therefore, Condition (ii) says that in a high state, the market clearing price is high and the elasticity (i.e., the proportional response) of the market clearing price with respect to output is low. Note also that condition (iii) does *not* require u to be concave (i.e., the firm need not be risk averse) and even if it were, the firm need not face a concave maximization problem because Π need not be concave in x.

Sandmo (1971) considers the behavior of a price-taking firm under uncertainty, with the market price experiencing additive shocks; in our notation, he assumes that $P(x;t) = \bar{P} + t$. He showed that an increase in $\bar{P}$ leads to higher output if u obeys DARA. This result is a special case of ours since there is no formal difference between a rise in the price of the good by (say) q and fall in its marginal cost by q.

Milgrom (1994) did not specifically examine the question we posed but pointed out that a large class of seemingly distinct comparative statics problems has the same solution because they all rely on the same Spence-Mirrlees condition. A special case of our problem can indeed be solved this way and checking the Spence-Mirrlees condition provides another application of Theorem 1. Indeed, suppose $P(x,t) = \bar{P}(x) + t$; by Theorem 1 in Milgrom (1994), optimal output increases with s if the ratio W_x/W_y is increasing in s, where $W(x,y,s) = \int u(y + tx - C(x,s))\lambda(t)dt$. To see when this may occur, suppose $W_x(x,y;s^*)/W_y(x,y;s^*) = \alpha$. The function

$$F(s) = \int u'(y + tx - C(x;s))[t - C_x(x;s) - \alpha]\lambda(t)dt \tag{6}$$

satisfies $F(s^*) = 0$. Clearly, $W_x(x,y;s^{**})/W_y(x,y;s^{**}) \geq \alpha$ if F is an $\mathcal{SC}$ function. By Theorem 1, it suffices that the integrand in (6) is an $\mathcal{SC}$ function of s and that any two such functions (at different values of t) obey signed-ratio monotonicity. The reader can check that the conditions of Proposition 2 do imply those properties.

Example 4. This is an application of Theorem 2 to solve a problem belonging to the second case discussed in the Introduction. Consider a Bertrand oligopoly with n firms, each selling a single differentiated product. We focus our discussion on firm 1 (the situation of the other firms being analogous). Firm 1 has a constant unit cost of production of c_1; the demand for its output if it charges price p_1 and the other firms charge p_{-1} (for their respective products) is given by $D(p_1, p_{-1}; s_1)$, where s_1 is some parameter affecting Firm 1's demand that is observed by firm 1. In general, firm j observes s_j but not s_k for $k \neq j$. At the price vector $p = (p_1, p_{-1})$ and the parameter s_1, firm 1's profit is $\Pi(p_1, p_{-1}; s_1) = (p_1 - c_1)D(p_1, p_{-1}; s_1)$. Suppose that firm $j \neq 1$ charges the price $\psi_j(s_j)$ whenever it observes s_j. If so, Firm 1 chooses p_1 to maximize its expected utility

$$V(p_1; s_1) = \int_{\mathbb{S}_{-1}} u(\Pi(p_1, [\psi_j(s_j)]_{j\neq 1}; s_1)\lambda(s_{-1}|s_1)\, ds_{-1},$$

where u is the firm's Bernoulli utility function and $\lambda(\cdot|s_1)$ is the distribution of s_{-1}, conditional on observing s_1.

We know from Athey (2001) that a Bayesian Nash equilibrium (with equilibrium decision rules that are increasing in the signal) exists if each firm has an *optimal* decision rule that is increasing, given that all other firms are playing increasing decision rules.[7] Therefore we are interested in conditions under which $\mathrm{argmax}_{p_1 > c_1} V(p_1; s_1)$ is increasing in s_1, which holds if $\Delta(s_1) = V(p_1''; s_1) - V(p_1'; s_1)$ is an $\mathcal{SC}$ function for any $p_1'' > p_1' > c_1$. Consider in the first instance the case where the agent is risk neutral, so u is the identity function. Suppose that D is a logsupermodular function of $(p_1, p_{-1}; s_1)$. This condition has a very

[7]For generalizations of Athey's work, see McAdams (2003) and Reny (2009).

simple interpretation in terms of the elasticity of demand. Define

$$\epsilon_i(p; s_1) = \frac{p_i}{D(p; s_1)} \frac{\partial D}{\partial p_i}(p; s_1)\,;$$

the logsupermodularity of D is equivalent to ϵ_i being increasing in s_1 and in p_k for $k \neq i$ (for all i). It is straightforward to check that if D is (i) increasing in p_{-1} (i.e., good 1 is a substitute for good $k \neq 1$ in the sense that an increase in the price of k raises the demand for 1), (ii) logsupermodular, and (iii) ψ_j is increasing for all $j \neq 1$, then $\Pi(p_1, [\psi_j(s_j)]_{j\neq 1}; s_1)$ is logsupermodular in $(p_1; s)$. If, in addition, $\lambda(\cdot|\cdot)$ is logsupermodular (which holds if the types are affiliated), then V is logsupermodular in $(p_1; s_1)$. This in turn guarantees that Δ is a single crossing function.

When the firm is not risk neutral, Δ is an $\mathcal{SC}$ function if

$$\delta(s_1; s_{-1}) = u(\Pi(p_1'', [\psi_j(s_j)]_{j\neq 1}; s_1) - u(\Pi(p_1', [\psi_j(s_j)]_{j\neq 1}; s_1)$$

obeys conditions in (i) and (ii) in Theorem 2 (with $s = s_1$ and $t = s_{-1}$). The following result (which we prove in the Appendix) gives conditions under which this holds.

Proposition 3. *Suppose that ψ_j is increasing for all $j \neq 1$, $\lambda(\cdot|\cdot)$ is logsupermodular, and D is increasing in p_{-1} and in s_1, with ϵ_1 increasing in s_1 and in p_k for all $k \neq 1$. Then conditions (i) and (ii) in Theorem 2 (for $s = s_1$ and $t = s_{-1}$) are satisfied if* any *of the following conditions hold:*

(a) *$u(z) = \ln z$, i.e., the coefficient of relative risk aversion is identically 1;*

(b) *The coefficient of relative risk aversion is bounded above by 1 and is decreasing and, for $i \neq 1$, ϵ_i is increasing in s_1 and in p_k for all $k \neq \{i, 1\}$;*

(c) *The coefficient of relative risk aversion is bounded below by 1 and is decreasing and, for $i \neq 1$, ϵ_i is decreasing in s_1 and in p_k for all $k \neq \{i, 1\}$.*[8]

Appendix

The proof of Theorem 1 requires the following lemma.

Lemma 1. *Let $\mathcal{F} = \{f_i\}_{1\leq i\leq M}$ be a family of $\mathcal{SC}$ functions such that any two members obey signed-ratio monotonicity. Then $\sum_{i=1}^{M} \alpha_i\, f_i$, where $\alpha_i \geq 0$ for all i, is an $\mathcal{SC}$ function.*

[8] The conditions on ϵ_1, together with the (b) conditions on ϵ_i, for $i \neq 1$, are equivalent to the logsupermodularity of D. The case where h is linear is covered under (b).

Proof: By Proposition 1, we need only show that $F = \sum_{i=1}^{M} f_i$ is an $\mathcal{SC}$ function. Suppose that $F(s') \geq 0$; we are required to show that $F(s'') \geq 0$ for any $s'' > s'$. If $f_i(s') \geq 0$ for all i, then $f_i(s'') \geq 0$ for all i, so we obtain $F(s'') \geq 0$. Consider next the case where $f_i(s') < 0$ for some i. In this case, we may partition $\mathcal{F}$ into three subsets; for $f_i \in \mathcal{F}^1$, we have $f_i(s') < 0$; for $f_i \in \mathcal{F}^2$, we have $f_i(s') > 0$; and for $f_i \in \mathcal{F}^3$, we have $f_i(s') = 0$. Clearly, $f_i(s'') \geq 0$ for $f_i \in \mathcal{F}^3$. Since $\mathcal{F}_1$ is nonempty, so is $\mathcal{F}_2$. We may write the sum $\sum_{f_i \in \mathcal{F}^1 \cup \mathcal{F}^2} f_i$ in the form $\sum_{j=1}^{L} h_j$, where each h_j is a positive linear combination of *at most two* functions in the family $\mathcal{F}^1 \cup \mathcal{F}^2$, such that $h_j(s') \geq 0$ for all j.[9] By Proposition 1, h_j is an $\mathcal{SC}$ function, so we have $h_j(s'') \geq 0$ for all j. This gives $F(s'') \geq 0$. If $F(s') > 0$, then $h_j(s') > 0$ for some j, so we obtain $F(s'') > 0$. **QED**

Proof of Theorem 1: If $f(s,t')$ and $f(s,t'')$ violate signed-ratio monotonicity, Proposition 1 says that there are α, $\beta > 0$ such that $\alpha f(s,t') + \beta f(s,t'')$ is not an $\mathcal{SC}$ function. The necessity of signed-ratio monotonicity is clear since we can choose μ to be the measure with $\mu(\{t'\}) = \alpha$ and $\mu(\{t''\}) = \beta$ and $\mu(J) = 0$ for all $J \in \Sigma$ not containing t' or t''.

To prove the sufficiency of the condition, consider $s'' > s'$ and suppose the ranges of $f(s'',\cdot)$ and $f(s',\cdot)$ are contained in some bounded interval I. Partition I into disjoint intervals I_j, $j = 1, 2, ..., K$, with a mesh of $1/m$. Denote by $\bar{T}(j,k)$ the subset of T such that for $t \in \bar{T}(j,k)$, we have $f(s',t) \in I_j$ and $f(s'',t) \in I_k$. Note that the collection of sets $\bar{T}(j,k)$, with j and k ranging between 1 and K form a partition of T. Define the simple functions[10] $f^m(s',\cdot)$ and $f^m(s'',\cdot)$ in the following way: for $t \in \bar{T}(j,k)$, choose $f^m(s',t) = f(s',\hat{t}_{jk})$ and $f^m(s'',t) = f(s'',\hat{t}_{jk})$ for some $\hat{t}_{jk} \in \bar{T}(j,k)$. This construction guarantees that for $s = s'$, s'', $f^m(s,\cdot)$ tends $f(s',\cdot)$ pointwise; it also guarantees that for $s = s'$, s'', $\int_T f^m(s,t)\,d\mu(t)$ is a finite sum of the form $\sum_{1 \leq j, k \leq K} \alpha_{jk} f(s,\hat{t}_{jk})$. Lemma 1 guarantees that, if $\int_T f^m(s',t)\,d\mu(t) > 0$ then $\int_T f^m(s'',t)\,d\mu(t) > 0$ (because both integrals are finite sums). Since f is bounded and μ is a finite measure, the dominated convergence theorem is applicable. Taking limits, we obtain the following result ($\star$): *if* $\int_T f(s',t)\,d\mu(t) > 0$ *then* $\int_T f(s'',t)\,d\mu(t) \geq 0$.

To complete the proof we need to establish the following claims: (a) if $\int_T f(s',t)\,d\mu(t) \geq 0$ then $\int_T f(s'',t)\,d\mu(t) \geq 0$, and (b) that $\int_T f(s',t)\,d\mu(t) > 0$ then $\int_T f(s'',t)\,d\mu(t) > 0$. To

[9]One procedure is the following. Choose the f_i function with the smallest absolute value at s' and add it to a fraction of a function having the opposite sign at s', so that they sum to zero. Repeat the procedure with the remaining functions; this will terminate after finitely many steps. For example, suppose $f_1(s') = 1$, $f_2(s') = -2$, $f_3(s') = 3$, then we may decompose $f_1(s') + f_2(s') + f_3(s')$ into $\left[f_1(') + \frac{1}{2} f_2(s')\right] + \left[\frac{1}{2} f_2(s') + \frac{1}{3} f_3(s')\right] + \left[\frac{2}{3} f_3(s')\right]$, where each square bracketed term gives an h_j function.

[10]By a simple function we mean a measurable function that takes finitely many distinct values.

show (a), suppose $\int_T f(s',t)\,d\mu(t) \geq 0$. Claim (a) is trivial if $f(s',t) \geq 0$ μ-a.e. Assuming this is *not* the case, there must be $\tilde{t}$ such that $f(s',\tilde{t}) > 0$. Thus, for any $\alpha > 0$, $\int_T f(s',t)\,d\mu(t) + \alpha f(s',\tilde{t}) > 0$. Result $(\star)$ guarantees that $\int_T f(s'',t)\,d\mu(t) + \alpha f(s'',\tilde{t}) \geq 0$.[11] Since this is true for any $\alpha > 0$, we conclude that $\int_T f(s'',t)\,d\mu(t) \geq 0$. To prove (b), first note that the problem is trivial if $f(s'',t) \geq 0$ for all t. Suppose instead that there is $\hat{t}$ such that $f(s'',\hat{t}) < 0$. Assuming that $\int_T f(s',t)\,d\mu(t) > 0$, choose $\beta > 0$ sufficiently small, so that $\int_T f(s',t)\,d\mu(t) + \beta f(s',\hat{t}) > 0$. By result $(\star)$, $\int_T f(s'',t)\,d\mu(t) + \beta f(s'',\hat{t}) \geq 0$. Since $f(s'',\hat{t}) < 0$, we obtain $\int_T f(s'',t)\,d\mu(t) > 0$. **QED**

Proof of Theorem 2: Denote the vectors in $\Pi_{i=1}^{n-1} T_i$ by $\tilde{t}$. Define $\tilde{F}(s,\tilde{t}) = \int_{T_n} f(s,\tilde{t},t_n) d\mu_n(t_n)$. Conditions (i) and (ii) guarantee (by Theorem 1) that $\tilde{F}$ is an $\mathcal{SC}$ function of $(s,\tilde{t})$. We claim that $\tilde{F}$ has the following properties: $\forall\, i < n$, $\forall\, s' \in S$, and $\forall\, \tilde{t}'' > \tilde{t}'$, (I) the functions $\tilde{F}(s,\tilde{t}')$ and $\tilde{F}(s,\tilde{t}'')$ of $s \in S$ satisfy signed-ratio monotonicity and (II) the functions $\tilde{F}(s',t_i,\tilde{t}'_{-i})$ and $\tilde{F}(s',t_i,\tilde{t}''_{-i})$ of $t_i \in T_i$ satisfy signed-ratio monotonicity. In other words, $\tilde{F}$ inherit the properties (i) and (ii) from F. We can then repeat the exercise and integrate $\tilde{F}$ by t_{n-1} and so on. Eventually, we obtain signed-ratio monotonicity for the functions $G(s,t''_1)$ and $G(s,t'_1)$ of s, where

$$G(s,t_1) = \int_{T_2}\int_{T_3} \ldots \int_{T_n} f(s,t_1,t_2,\ldots,t_{n-1},t_n)\,d\mu_2(t_2)\,d\mu_3(t_3)\ldots d\mu_n(t_n)\,.$$

By Theorem 1, $F(s) = \int_{T_1} G(s,t_1) d\mu_1(t_1)$ is an $\mathcal{SC}$ function.

It is clear that if (i) and (ii) implies (I), then another application of the same result guarantees that (II) follows from (ii).[12] So we need only show (I). By Proposition 1, it suffices to show that $\tilde{F}(s,\tilde{t}') + \Gamma\tilde{F}(s,\tilde{t}'')$ is an $\mathcal{SC}$ function of s for any $\Gamma > 0$. By Theorem 1, this is true if, for any t''_n and t'_n in T_n, the functions $g(s,t''_n)$ and $g(s,t'_n)$ of s obey signed-ratio monotonicity, where $g(s,t_n) = f(s,\tilde{t}',t_n) + \Gamma f(s,\tilde{t}'',t_n)$. Note that, by Proposition 1 and condition (i), g is an $\mathcal{SC}$ function of s. By Proposition 1 again, we need only show that $\phi(s) = g(s,t'_n) + Qg(s,t''_n)$ is an $\mathcal{SC}$ function. Suppose $\phi(s^*) \geq (>)\, 0$. We need to show that $\phi(s^{**}) \geq (>)\, 0$, where $s^{**} > s^*$. This holds if we can construct A^1 and A^2 such that (a) A^1 and A^2 are $\mathcal{SC}$ functions, (b) $\phi(s) = A^1(s) + A^2(s)$ and (c) $A^1(s^*) \geq 0$ and $A^2(s^*) \geq 0$. Since T_n is totally ordered, there is no loss of generality in assuming that $t''_n \geq t'_n$. Note that $g(s,t_n)$ is an $\mathcal{SC}$ function of t_n, by condition (ii), so if $g(s^*,t'_n) \geq 0$ we also have $g(s^*,t''_n) \geq 0$. In this case, we can let $A^1(s) = g(s,t''_n)$ and $A^2(s) = g(s,t'_n)$. Suppose instead that $g(s^*,t''_n) > 0$ and $g(s^*,t'_n) < 0$. Since f is an $\mathcal{SC}$ function, $f(s^*,\tilde{t}'',t''_n) > 0$ while $f(s^*,\tilde{t}',t''_n)$ may be negative

[11] Note that we applying $(\star)$ to the finite measure in which μ is supplement by an atom at $\tilde{t}$.

[12] In this application, t_i takes the place of s and $\hat{t} = (t_1,\ldots,t_{i-1},t_{i+1},\ldots t_n)$ takes the place of t.

(Case 1) or nonnegative (Case 2).

Case 1. Given that $g(s^*, t_n'') > 0$, there is $\alpha \in (0,1)$ so that $A^1(s) = Qf(s, \tilde{t}', t_n'') + \alpha Q\Gamma f(s, \tilde{t}'', t_n'')$ satisfies $A^1(s^*) = 0$. Since $(\tilde{t}'', t_n'')$ and $(\tilde{t}', t_n'')$ are ordered, condition (i) guarantees that A^1 is an $\mathcal{SC}$ function. Define $A^2(s) = \phi(s) - A^1(s)$. Since $A^1(s^*) = 0$, $A^2(s^*) \geq (>)\, 0$. Observe that $A^2(s) = (1-\alpha)Q\Gamma f(s, \tilde{t}'', t_n'') + \Gamma f(s, \tilde{t}'', t_n') + f(s, \tilde{t}', t_n')$, where $(\tilde{t}'', t_n'') \geq (\tilde{t}'', t_n') \geq (\tilde{t}', t_n')$, so condition (i) guarantees that A^2 is an $\mathcal{SC}$ function. So A^1 and A^2 have properties (a)–(c).

Case 2. Notice that $\phi(s) = \left[Qf(s, \tilde{t}', t_n'') + f(s, \tilde{t}', t_n')\right] + \Gamma\left[Qf(s, \tilde{t}'', t_n'') + f(s, \tilde{t}'', t_n')\right]$. The map from $\tilde{t}$ to $Qf(s, \tilde{t}, t_n'') + f(s, \tilde{t}, t_n')$ is an $\mathcal{SC}$ map. This follows from repeated application of condition (ii). Since $\phi(s^*) \geq 0$, we must have $Qf(s^*, \tilde{t}'', t_n'') + f(s^*, \tilde{t}'', t_n') \geq (>)\, 0$ while $Qf(s^*, \tilde{t}', t_n'') + f(s^*, \tilde{t}', t_n')$ may be of either sign. Define A^1 and A^2 by

$$\begin{aligned} A^1(s) &= Qf(s, \tilde{t}', t_n'') + \beta f(s, \tilde{t}', t_n') \text{ and} \\ A^2(s) &= \Gamma\left[Qf(s, \tilde{t}'', t_n'') + f(s, \tilde{t}'', t_n')\right] + (1-\beta) f(s, \tilde{t}', t_n'). \end{aligned}$$

If $Qf(s^*, \tilde{t}', t_n'') + f(s^*, \tilde{t}', t_n') \geq 0$, choose $\beta = 1$. Otherwise, choose $\beta \in [0,1)$ to guarantee that $A^1(s^*) = 0$; this is possible since $f(s^*, \tilde{t}', t_n'') \geq 0$ so $f(s^*, \tilde{t}', t_n') < 0$. Notice that the component functions that make up A^1 and A^2 are all ordered with respect to t, so condition (i) guarantees that A^1 and A^2 are $\mathcal{SC}$ functions. So A^1 and A^2 have properties (a)–(c).**QED**

The proofs of Propositions 2 and 3 require the following two lemmas.

Lemma 2. *Suppose that $u : \mathbb{R} \to \mathbb{R}$ is twice differentiable, with $u' > 0$ and $-u''(z)/u'(z)$ decreasing in z, i.e., obeys DARA. Then for any $a_1 < a_2$, $b_1 < b_2$, $a_1 \leq b_1$, and $a_2 \leq b_2$,*

$$\frac{u(a_2) - u(a_1)}{u(b_2) - u(b_1)} \geq \frac{u(a_2 + w) - u(a_1 + w)}{u(b_2 + w) - u(b_1 + w)} \quad \text{where } w \geq 0. \tag{7}$$

Proof: It suffices to show that $F(x) = \ln\left(u(a_2 + x) - u(a_1 + x)\right) - \ln\left(u(b_2 + x) - u(b_1 + x)\right)$ is decreasing in $x \geq 0$. Denoting $u' \circ u^{-1}$ by f, the derivative

$$\begin{aligned} \frac{dF}{dx} &= \frac{u'(a_2 + x) - u'(a_1 + x)}{u(a_2 + x) - u(a_1 + x)} - \frac{u'(b_2 + x) - u'(b_1 + x)}{u(b_2 + x) - u(b_1 + x)} \\ &= \frac{f(u(a_2 + x)) - f(u(a_1 + x))}{u(a_2 + x) - u(a_1 + x)} - \frac{f(u(b_2 + x)) - f(u(b_1 + x))}{u(b_2 + x) - u(b_1 + x)} \leq 0, \end{aligned}$$

where the final inequality holds because DARA guarantees that f is convex and, since u is strictly increasing, $u(a_1 + x) < u(a_2 + x)$, $u(b_1 + x) < u(b_2 + x)$, $u(a_1 + x) \leq u(b_1 + x)$, and $u(a_2 + x) \leq u(b_2 + x)$. **QED**

Lemma 3. *Let X and T be totally ordered sets, S a partially ordered set, and ϕ a map from $X \times S \times T$ to $\mathbb{R}$. For x'', $x' \in X$ with $x'' > x'$, suppose that*
(i) for every $t \in T$, $\phi(x''; s, t) - \phi(x'; s, t)$ is increasing in s;
(ii) for every $s \in S$, $\phi(x''; s, t) - \phi(x'; s, t)$ is an $\mathcal{SC}$ function of t;
(iii) u obeys the conditions of Lemma 2;
(iv) ϕ is increasing in (s, t); and
(v) any one *of the following conditions holds for $x = x'$, x'', and every $s'' > s'$:*

(a) *$\phi(x, s'', t) - \phi(x, s', t)$ is independent of t;*
(b) *u is concave and $\phi(x, s'', t) - \phi(x, s', t)$ decreases with t;*
(c) *u is convex and $\phi(x, s'', t) - \phi(x, s', t)$ increases with t.*

Then the functions $\delta(s, t')$ and $\delta(s, t'')$ of $s \in S$ obey signed-ratio monotonicity, where $\delta(s, t) = u(\phi(x'', s, t)) - u(\phi(x', s, t))$.

Proof: Suppose that $\delta(s^*, t'') > 0$ and $\delta(s^*, t') < 0$. This means that $\phi(x''; s^*, t'') - \phi(x'; s^*, t'') > 0$ and $\phi(x''; s^*, t') - \phi(x'; s^*, t') < 0$. Given (ii), this can only occur if $t' < t''$. Now (iv) guarantees that $\phi(x', s^*, t') \leq \phi(x'; s^*, t'')$, so we obtain

$$\phi(x''; s^*, t') < \phi(x', s^*, t') \leq \phi(x'; s^*, t'') < \phi(x''; s^*, t'').$$

If $s^{**} > s^*$ we have

$$-\frac{\delta(s^*, t')}{\delta(s^*, t'')} = -\frac{u(\phi(x''; s^*, t')) - u(\phi(x'; s^*, t'))}{u(\phi(x''; s^*, t'')) - u(\phi(x'; s^*, t''))}$$

$$\begin{aligned}
\geq\ & \frac{u\left(\phi(x'; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - u\left(\phi(x''; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right)}{u\left(\phi(x''; s^*, t'') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - u\left(\phi(x'; s^*, t'') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right)} \\
\geq\ & \frac{u\left(\phi(x'; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - u\left(\phi(x''; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right)}{u\left(\phi(x''; s^*, t'') + [\phi(x'; s^{**}, t'') - \phi(x'; s^*, t'')]\right) - u\left(\phi(x'; s^*, t'') + [\phi(x'; s^{**}, t'') - \phi(x'; s^*, t'')]\right)} \\
\geq\ & \frac{u\left(\phi(x'; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - u\left(\phi(x''; s^*, t') + [\phi(x''; s^{**}, t') - \phi(x''; s^*, t')]\right)}{u\left(\phi(x''; s^*, t'') + [\phi(x''; s^{**}, t'') - \phi(x''; s^*, t'')]\right) - u\left(\phi(x'; s^*, t'') + [\phi(x'; s^{**}, t'') - \phi(x'; s^*, t'')]\right)} \\
=\ & -\frac{u(\phi(x''; s^{**}, t')) - u(\phi(x'; s^{**}, t'))}{u(\phi(x''; s^{**}, t'')) - u(\phi(x'; s^{**}, t''))} = -\frac{\delta(s^{**}, t')}{\delta(s^{**}, t'')}
\end{aligned}$$

The first inequality follows from Lemma 2, the second inequality from condition (v) (any of (a), (b), or (c)), and the third inequality from the assumption that u is increasing and condition (i). **QED**

Proof of Proposition 2: We check that Π and u satisfy the conditions of Lemma 3. Π is clearly increasing in (s, t). For every t, $\Pi(x''; s, t) - \Pi(x'; s, t)$ is increasing in s since

$\{C(\cdot; s)\}_{s \in \mathbb{S}}$ obeys decreasing differences. Furthermore, for every x and $s'' > s'$, $\Pi(x; s'', t) - \Pi(x; s', t) = C(x; s') - C(x; s'')$, which is independent of t (so version (a) of condition (v) in Lemma 3 is satisfied). It remains to show that, for any s, $\Pi(x''; s, t) - \Pi(x'; s, t)$ is an $\mathcal{SC}$ function of t. Suppose $\Pi(x''; s, t') - \Pi(x'; s, t') \geq (>)\, 0$. Then $x''P(x''; t') - x'P(x'; t') \geq\, (>)\, 0$ since C is increasing in x. Furthermore, we have[13]

$$\begin{aligned} 0 \leq (<)\, x''P(x''; t') - x'P(x'; t') &= \left[x'' \frac{P(x''; t')}{P(x'; t')} - x'\right] P(x'; t') \\ &\leq \left[x'' \frac{P(x''; t'')}{P(x'; t'')} - x'\right] P(x'; t'') \\ &= x''P(x''; t'') - x'P(x'; t'') \end{aligned}$$

The inequality is true since $\{\ln P(\cdot; t)\}_{t \in T}$ obeys increasing differences and P is increasing in t. We conclude that $\Pi(x''; s, t'') - \Pi(x'; s, t'') \geq\, (>)\, 0$. **QED**

Proof of Proposition 3: We may write $u(\Pi(p_1, (\psi_j(s_j))_{j \in N_1}; s_1))$ as $\tilde{u}(\tilde{\pi}(p_1; s))$ where $\tilde{u}(\cdot) = u(\exp(\cdot))$ and $\tilde{\pi}(p_1; s) = \ln \Pi(p_1, (\psi_j(s_j))_{j \in N_1}; s_1)$. The conditions on ϵ_1 guarantee the property (P1): $\tilde{\pi}(p_1''; s) - \tilde{\pi}(p_1'; s)$ is increasing in s. We also have the property (P2): $\tilde{\pi}$ is increasing in s. For case (a), we have $\delta(s) = \tilde{\pi}(p_1''; s) - \tilde{\pi}(p_1'; s)$, which certainly means that (i) and (ii) holds since δ is an increasing function (by (P1)).

For cases (b) and (c), we first note that (P1) guarantees that δ is an $\mathcal{SC}$ function. To confirm that (i) and (ii) holds, it suffices to check that $\delta(s_{N \setminus K}, s_K'')$ and $\delta(s_{N \setminus K}, s_K')$ (thought of as functions of $s_{N \setminus K}$) obey signed-ratio monotonicity whenever $s_K'' > s_K'$, for $K \subset N$. This can be obtained via Lemma 3 (with $T = \{s_K'', s_K'\}$). Consider the assumptions under case (b). Those assumptions guarantee property (P3): for any p_1, $\tilde{\pi}(p_1; s_{N \setminus K}^{**}, s_K) - \tilde{\pi}(p_1; s_{N \setminus K}^{*}, s_K)$ is increasing in s_K, for any $s_{N \setminus K}^{**} > s_{N \setminus K}^{*}$; they also guarantee (P4): $\tilde{u}$ is a convex function with DARA. Properties (P1), (P2), (P3), and (P4) together ensure that conditions (i), (ii), (iii), (iv), and (v-c) in Lemma 3 are satisfied. We conclude that $\delta(s_{N \setminus K}, s_K'')$ and $\delta(s_{N \setminus K}, s_K')$ obey signed-ratio monotonicity. The same conclusion obtains under case (c). This is because those conditions imply property (P3$'$): for any p_1, $\tilde{\pi}(p_1; s_{N \setminus K}^{**}, s_K) - \tilde{\pi}(p_1; s_{N \setminus K}^{*}, s_K)$ is decreasing in s_K, for any $s_{N \setminus K}^{**} > s_{N \setminus K}^{*}$; they also guarantee property (P4$'$): $\tilde{u}$ is concave with DARA. [Note the contrast between (P3) and (P3$'$) and between (P4) and (P4$'$).] In this case, conditions (i), (ii), (iii), (iv), and (v-b) in Lemma 3 are satisfied. **QED**

[13] This argument is an adaptation of the one used by Amir (1996) to guarantee that reaction curves in the Cournot model are downward sloping.

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