Class Notes on Monotone Comparative Statics

In this lecture I will give a quick survey of the following topics:

1. The comparative statics of one-dimensional optimization problems
2. Supermodular Games
3. Comparative statics of optimization problems under uncertainty

The two important tools used for solving these problems are the *single crossing property* and the *interval dominance order*.

1. One-dimensional comparative statics

Let $X \subseteq R$ and let $f(\cdot; s) : X \to R$ be a family of functions parameterized by $s \in S$ (a poset).

We are interested in how $\arg\max_{x \in X} f(x; s)$ varies with s.

Standard approach:

> Assume X is a compact interval and $f(\cdot; s)$ are quasi-concave functions of x. Let $\bar{x}(s)$ be the unique maximizer of $f(\cdot; s)$. Then $f'(\bar{x}(s), s') = 0$. Show $f'(\bar{x}(s'), s'') \geq 0$ for $s'' > s'$. Then optimum has shifted to the right.

This approach makes various assumptions, most notably the quasi-concavity of $f(\cdot; s)$. Not the most natural assumption; example:

let x be output, P the inverse demand function, and c the marginal cost of producing good. The profit function $\Pi(x; c) = xP(x) - cx$ is not naturally concave in x.

The approach via *monotone comparative statics* avoids this assumption and others.

Assume that $f(\cdot; s)$ is continuous in $x \in X$ and X is compact. Then $\arg\max_{x \in X} f(x; s)$ is nonempty. But it need not be singleton or an interval.

First question: how do we compare sets?

Definition: Let S' and S'' be subsets of R. S'' dominates S' in the *strong set order* ($S'' \geq S'$) if for any for x'' in S'' and x' in S', we have $\max\{x'', x'\}$ in S'' and $\min\{x'', x'\}$ in S'.

Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$ but $\{3, 4, 5, 6, 7\} \geq \{1, 3, 4, 5, 6\}$.

Note: if $S'' = \{x''\}$ and $S' = \{x'\}$, then $x'' \geq x'$.

When S'' and/or S' are non-singleton,
largest element in S'' is *larger than* the largest element in S';

smallest element in S'' is *larger than* the smallest element in S'.

Definition: Let S be a poset and $\phi : S \to R$. Then ϕ has the *single crossing property* (is a single crossing function) if

$$\phi(s') \geq (>)\, 0 \Longrightarrow \phi(s'') \geq (>)\, 0 \;\text{ where } s'' > s'.$$

Definition: The family of functions $\{f(\cdot, s)\}_{s \in S}$ obeys *single crossing differences* if for all $x'' > x'$, the function

$$\delta(s) = f(x''; s) - f(x'; s) \;\text{ is a single crossing function.}$$

These are ordinal properties. In particular, if $\{f(\cdot, s)\}_{s \in S}$ obey single crossing differences, then so does $\{g(\cdot; s)\}_{s \in S}$ where there is a function $H(\cdot; s)$, strictly increasing in x, such that $g(x; s) = H(f(x; s); s)$.

Definition: The family of functions $\{f(\cdot, s)\}_{s \in S}$ obeys *increasing differences* if for all $x'' > x'$, the function

$$\delta(s) = f(x''; s) - f(x'; s) \;\text{ is an increasing function.}$$

Very often, the easiest way to show that a family obeys single crossing differences is to show that some strictly increasing transformation of this family has increasing differences.

Proposition 1: *Let S be an open subset of R^l and X an open interval. Then a sufficient (and necessary) condition for the family $\{f(\cdot, s)\}_{s \in S}$ to obey increasing differences is that*

$$\frac{\partial^2 f}{\partial x \partial s_i}(x, s) \geq 0$$

at every point (x, s) and for all i.

Theorem 1: (Milgrom-Shannon) *The family* $\{f(\cdot; s)\}_{s\in S}$ *obeys single crossing differences if and only if* $\arg\max_{x\in Y} f(x; s)$ *is increasing in* s *for all* $Y \subseteq X$.

Proof: Assume $s'' > s'$ and $x'' \in \arg\max_{x\in Y} f(x; s'')$, and $x' \in \arg\max_{x\in Y} f(x; s')$. We have to show that
$\max\{x', x''\} \in \arg\max_{x\in Y} f(x; s'')$ and $\min\{x', x''\} \in \arg\max_{x\in Y} f(x; s')$.
We need only consider the case where $x' > x''$.

Since $x' \in \arg\max_{x\in Y} f(x; s')$, we have $f(x'; s') \geq f(x''; s')$. By single crossing differences, $f(x'; s'') \geq f(x''; s'')$ so $x' \in \arg\max_{x\in Y} f(x; s'')$.

Furthermore, $f(x'; s') = f(x''; s')$ so that $x'' \in \arg\max_{x\in Y} f(x; s')$. If not, $f(x'; s') > f(x''; s')$ which implies (by single crossing differences) that $f(x'; s'') > f(x''; s'')$, contradicting the assumption that $f(\cdot; s'')$ is maximized at x''.

Necessity: follows from definition of single crossing differences! **QED**

Application: Let $\Pi(x; -c) = xP(x) - cx$. Then $\{\Pi(\cdot, -c)\}_{-c\in R_-}$ obey increasing differences, since

$$\frac{\partial^2 \Pi}{\partial x \partial c} = -1.$$

By MCS Theorem, $\arg\max_{x\in X} \Pi(x, -c)$ is increasing in $-c$.

Application: Bertrand Oligopoly with differentiated products, with

$$\begin{aligned}\Pi_a(p_a, p_{-a}) &= (p_a - c_a) D_a(p_a, p_{-a}) \\ \ln \Pi_a(p_a, p_{-a}) &= \ln(p_a - c_a) + \ln D_a(p_a, p_{-a})\end{aligned}$$

So $\{\Pi_a(\cdot, p_{-a})\}_{-a\in -A}$ has single crossing differences if $\{\ln \Pi_a(\cdot, p_{-a})\}_{-a\in -A}$ has increasing differences. We require

$$\frac{\partial^2}{\partial p_a \partial p_{-a}} [\ln \Pi_a] \geq 0; \text{ Equivalently,}$$

$$\frac{\partial}{\partial p_{-a}} \left[-\frac{p_a}{D_a} \frac{\partial D_a}{\partial p_a} \right] \leq 0;$$

i.e., firm a's own-price elasticity of demand decreases with p_{-a}. If this assumption holds, $\arg\max_{p_a\in P} \Pi_a(p_a, p_{-a})$ increases with p_{-a}.

In other words, firm a's optimal price is *increasing* in the price charged by other firms. Firms' strategies are *complements*.

2. Supermodular Games

The Bertrand game is an example of a *supermodular game.*

A supermodular game is one where the best response of each agent is increasing with the strategies of the other agents.

In the next few slides we take a brief look at the properties of these games.

We do not assume that the payoff functions of the agents in the game are quasiconcave. Therefore, the best response map need not be a convex-valued, upper hemi-continuous correspondence. For this reason, the 'standard' proof of equilibrium existence via Kakutani's fixed point theorem cannot be applied.

Instead, we appeal to the monotonicity of the best response map and use another fixed point theorem - Tarski's.

Let $X = \Pi_{i=1}^{N} X_i$, where each X_i is a compact interval of R.

Theorem 2: (Tarski) *Suppose $\phi : X \to X$ is an increasing function. Then ϕ has a fixed point. In fact,*

$$x^{**} = \sup\{x \in X : x \leq \phi(x)\}$$

is a fixed point and is the largest *fixed point, i.e., for any other fixed point x^*, we have $x^* \leq x^{**}$.*

Note: ϕ need not be continuous.

Theorem 3: *Suppose $\phi(\cdot, t) : X \to X$ is increasing in (x, t). Then the largest fixed point of $\phi(\cdot, t)$ is increasing in t.*

Bertrand Oligopoly: assume the set of firms is A; the typical firm a chooses its price from the compact interval P to maximize $\Pi_a(p_a, p_{-a}) = (p_a - c_a)D_a(p_a, p_{-a})$.

Recall: if own-price elasticity is decreasing in p_{-a} then $\{\Pi_a(\cdot, p_{-a})\}_{p_{-a} \in P_{-a}}$ obeys single crossing differences.

Consequently, firm a's best response set

$$B_a(p_{-a}) = \arg\max_{p_a \in P} \Pi_a(p_a, p_{-a}) \text{ is increasing in } p_{-a}.$$

Define $\bar{B}_a(p_{-a}) = \max\left[\arg\max_{p_a \in P} \Pi_a(p_a, p_{-a})\right]$; this is the *largest* best response to p_{-a}.

$\bar{B}_a$ is an increasing function of p_{-a}.

Define $\bar{P} = P \times P \times ... \times P$ and the map $\bar{B} : \bar{P} \to \bar{P}$ by

$$\bar{B}(p) = \left(\bar{B}_a(p_{-a})\right)_{a \in A}.$$

A fixed point of this map is a NE of the game.

Since $\bar{B}$ is an increasing function, Tarski's Fixed Point Theorem guarantees that a fixed point exists.

Specifically,

$$p^* = \sup\{p \in \bar{P} : p \leq \bar{B}(p)\}$$

is a fixed point of the map $\bar{B}$ and thus a NE. In fact, this is the *largest* NE, i.e., suppose $\hat{p}$ is another NE; then $p^* > \hat{p}$.

We can do comparative statics exercises on the largest NE...

What happens to the largest NE when firm $\tilde{a}$ experiences an increase in marginal cost from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$? Recall

$$\ln \Pi_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}, c_{\tilde{a}}) = \ln(p_{\tilde{a}} - c_{\tilde{a}}) + \ln D_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}).$$

Observe that

$$\frac{\partial}{\partial p_{\tilde{a}} \partial c_{\tilde{a}}} \left[\ln \Pi_{\tilde{a}}\right] > 0.$$

By the MCS Theorem, firm a's best response increase with $c_{\tilde{a}}$ (for fixed p_{-a}). Formally,

$$B_{\tilde{a}}(p_{-\tilde{a}}, c'_{\tilde{a}}) \geq B_{\tilde{a}}(p_{-\tilde{a}}, c_{\tilde{a}}).$$

This implies that $\bar{B}(p, c'_{\tilde{a}}) \geq \bar{B}(p, c_{\tilde{a}})$. So largest fixed point of $\bar{B}(\cdot, c'_{\tilde{a}})$ is larger than the largest fixed point of $\bar{B}(\cdot, c_{\tilde{a}})$ (by Theorem 3).

In other words, if firm $\tilde{a}$'s marginal cost increases from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$, the largest NE increases: *every firm increases its price.*

3. SCD and optimization under uncertainty

Suppose $\{v(\cdot; s)\}_{s\in S}$ obeys single crossing differences, so $\arg\max_{x\in X} v(x; s)$ is increasing in s.

Now interpret s as the state of the world, which is unknown when x is chosen. Formally, x is chosen to maximize

$$V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)\, ds$$

where $\lambda(\cdot, \theta)$ is the density function over $s \in S \subset R$ (one-dimensional!).

Since action is increasing in state if state is known, we would expect the optimal action to be higher if higher states are more likely.

Suppose "higher state are more likely" when θ is higher, then it suffices that $\{V(\cdot, \theta)\}_{\theta\in\Theta}$ obeys single crossing differences.

Definition: $\{\lambda(\cdot, \theta)\}_{\theta\in\Theta}$ obeys the *monotone likelihood ratio order* if

$$\frac{\lambda(s, \theta'')}{\lambda(s, \theta')} \text{ is increasing in } s \text{ whenever } \theta'' > \theta'.$$

Theorem 4: *Let $S \subset R$ and suppose $\delta : S \to R$ is a single crossing function and $\{\lambda(\cdot, \theta)\}_{t\in\Theta}$ obeys the MLR order. Then*

$$\Delta(\theta) = \int_S \delta(s)\lambda(s, \theta)\, ds \text{ is a single crossing function (of } \theta).$$

Corollary 1: *Suppose that $\{v(\cdot; s)\}_{s\in S}$ obeys single crossing differences and $\{\lambda(\cdot, \theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order. Then $\{V(\cdot; s)\}_{\theta\in\Theta}$ obeys single crossing differences, where*

$$V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)\, ds.$$

Consequently, $\arg\max_{x\in X} V(x; \theta)$ *is increasing in θ.*

Proof: Note that

$$\Delta(\theta) = V(x'', \theta) - V(x', \theta) = \int_S [v(x'', s) - v(x', s)]\lambda(s, \theta)\, ds = \int_S \delta(s)\lambda(s, \theta)\, ds$$

Since $\{v(\cdot; s)\}_{s\in S}$ obeys single crossing differences, δ is a single crossing function. Conclusion follows immediately from theorem above. **QED**

Proof of Theorem 4: Let $\theta'' > \theta'$. We split $\Delta(\theta'') = \int_S \delta(s)\lambda(s,\theta'')\,ds$ into two parts:

$$\Delta(\theta'') = \int_{-\infty}^{s_0} \delta(s)\lambda(s,\theta')\,\frac{\lambda(s,\theta'')}{\lambda(s,\theta')}\,ds + \int_{s_0}^{\infty} \delta(s)\lambda(s,\theta')\,\frac{\lambda(s,\theta'')}{\lambda(s,\theta')}\,ds,$$

where $\delta(s) \leq 0$ for $s < s_0$ and $\delta(s) > 0$ for $s > s_0$. The first term on the right is greater than

$$\frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_{-\infty}^{s_0} \delta(s)\lambda(s,\theta')\,ds$$

while the second term is greater than

$$\frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_{s_0}^{\infty} \delta(s)\lambda(s,t_1)\,ds.$$

Adding up the two lower bounds gives us

$$\Delta(\theta'') \geq \frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_S \delta(s)\lambda(s,\theta')\,ds = \frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\Delta(\theta').$$

So $\Delta(\theta') \geq (>)\, 0$ implies $\Delta(\theta'') \geq (>)\, 0$. **QED**

Application: Consider a firm that maximizes profit

$$\Pi(x,-c) = xP(x) - cx.$$

Since $\frac{\partial^2 \Pi}{\partial x \partial} = -1$, the family $\{\Pi(\cdot,-c)\}_{c\in R_+}$ obeys increasing (hence single crossing) differences.

Theorem 1 says that $\arg\max_{x\geq 0} \Pi(x,-c)$ is increasing in $-c$.

Now suppose that the firm has to choose x *before* c is known. Given its Bernoulli utility function u, the firm's objective function is

$$V(x;t) = \int u(\Pi(x,-c))\lambda(c,\theta)\,dc$$

where $\lambda(\cdot,\theta)$ is a density function (defined over c). Note that $v(x;-c) \equiv u(\Pi(x,-c))$ obeys single crossing differences.

Corollary 1 says that when higher c becomes more likely (in the MLR sense), then the firm will choose to produce less.

4. The interval dominance order

Single crossing differences is not a panacea...

Single crossing differences does *not* hold – in a canonical case!

Let $X \subseteq R$. The set $Y \subseteq X$ is an *interval of* X if, whenever x^* and x^{**} are in X, then any $x \in X$ such that $x^* < x < x^{**}$ is also in Y.

Notation: $[x^*, x^{**}] = \{x \in X : x^* \leq x \leq x^{**}\}$.

Definition: The family $\{f(\cdot; s)\}_{s \in S}$ obeys the *interval dominance order* if for any $x'' > x'$ and $s'' > s'$, such that

$$f(x''; s') - f(x; s') \geq 0 \text{ for all } x \in [x', x''], \text{ we have}$$

$$f(x''; s') - f(x'; s') \geq (>)\, 0 \implies f(x''; s'') - f(x'; s'') \geq (>)\, 0.$$

Definition: The family $\{f(\cdot; s)\}_{s \in S}$ obeys the *interval dominance order* if for any $x'' > x'$ and $s'' > s'$, such that $f(x''; s') - f(x; s) \geq 0$ *for all* $x \in [x', x'']$, we have

$$f(x''; s') - f(x'; s') \geq (>)\, 0 \implies f(x''; s'') - f(x'; s'') \geq (>)\, 0.$$

Theorem 5: (Quah-Strulovici) *Suppose the family* $\{f(\cdot; s)\}_{s \in S}$ *obeys the interval dominance order if and only if* $\arg\max_{x \in Y} f(x; s)$ *is increasing in* s *for all* intervals $Y \subseteq X$.

Consider the case where X is an interval of R. A simple sufficient condition for $\{f(\cdot;s)\}_{s\in S}$ to obey single crossing differences is the following:
for any $\bar{s} > s$, there is scalar $k > 0$ such that $f'(x;\bar{s}) \geq kf'(x;s)$ for all $x \in X$.

This is clear, since the condition guarantees that, for any $x^{**} > x^*$,

$$f(x^{**};\bar{s}) - f(x^*;\bar{s}) \geq k\left[f(x^{**};s) - f(x^*;s)\right].$$

Proposition 2: *Let X be an interval of R and let $\{f(\cdot;s)\}_{s\in S}$ be family of real-valued functions with the following property: for any $\bar{s} > s$, there is a nondecreasing positive function $\alpha : X \to R$ such that*

$$f'(x;\bar{s}) \geq \alpha(x)f'(x;s) \text{ for all } x \in X.$$

Application: (The optimal stopping time problem) At each moment in time, agent gains profit of $\pi(t)$, which can be positive or negative. If agent decides to stop at time x, the present value of his accumulated profit is

$$V(x;r) = \int_0^x e^{-rt}\pi(t)dt$$

where $r > 0$ is the discount rate.

How does optimal stopping time vary with discount rate?

Note that $V'(x;r) = e^{-rx}\pi(x)$. So (i) there are lots of turning points and (ii) turning points do not vary with the discount rate.

Proposition 3: *Suppose*

$$V(x;r) = \int_0^x e^{-rt}\pi(t)dt.$$

If $r > \bar{r} > 0$ then $\arg\max_{x\geq 0} V(x;\bar{r}) \geq \arg\max_{x\geq 0} V(x;r)$.

Proof: We have

$$V'(x;\bar{r}) = e^{-\bar{r}x}\pi(x) = e^{(r-\bar{r})x}V'(x;r).$$

Note that the function $\alpha(x) = e^{(r-\bar{r})x}$ is positive and increasing.

So $\{V(\cdot;r)\}_{r>0}$ obeys the interval dominance order (strictly speaking, with respect to $-r$). **QED**

Like single crossing differences, IDO is 'preserved' under uncertainty.

Theorem 6: *Suppose that* $\{v(\cdot; s)\}_{s\in S}$ *obeys the interval dominance order and* $\{\lambda(\cdot, \theta)\}_{\theta\in\Theta}$ *obeys the monotone likelihood ratio order. Then* $\{V(\cdot; s)\}_{\theta\in\Theta}$ *obeys the interval dominance order, where*

$$V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)\, ds.$$

Consequently, $\arg\max_{x\in X} V(x; \theta)$ *is increasing in* θ.

Application: A family of quasiconcave functions $\{v(\cdot; s)\}_{s\in S}$ parameterized by their peaks obeys the interval dominance order, but not necessarily single crossing differences.

Note that $V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)\, ds$ need not be a quasiconcave function of x and little is known of its shape.

Nonetheless, we know by Theorem 6 that, if $\{\lambda(\cdot, \theta)\}_{\theta\in\Theta}$ obeys the MLR order, then $\arg\max_{x\in X} V(x; \theta)$ is increasing in θ.