# *The Economics of Counterfeiting**

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**Abstract**

This paper develops a new tractable strategic theory of counterfeiting as a multi-market large game played by good and bad guys. There is free entry of bad guys, who choose whether to counterfeit, and what quality to produce. Opposing them is a continuum of good guys who select a costly verification effort. A counterfeiting equilibrium consists of a "cat and mouse" game between effort and quality, and a collateral "hot-potato" passing game among good guys. With log-concave verification costs, counterfeiters producer better quality at higher notes, but verifiers try sufficiently harder that the verification rate still rises. We prove that the passed and counterfeiting rates vanish for low and high notes. We develop and use a graphical framework for deducing comparative statics.

Our theory applies to fixed-value counterfeits, like checks, money orders, or money. Focusing on counterfeit money, we assemble a unique data set from the U.S. Secret Service. We identify some new time series and cross-sectional patterns, and explain them: $(1)$ the ratio of all counterfeit money (*seized* or *passed*) to passed money rises in the note, but less than proportionately; $(2)$ the passed-circulation ratio rises in the note, and is very small at $1 notes; $(3)$ the vast majority of counterfeit money used to be *seized* before circulation, but this is no longer true; and $(4)$ the ratio of the internal Federal Reserve Banks passed rate to the economy-wide average falls in the note until the $100 note. Our theory explains how to estimate from data the counterfeiting rate, the street price of counterfeit notes, and the incredibly small costs expended verifying each note.

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*This is a radical revision of a 2009 submission to this journal. The paper began in 2005 as "Counterfeit $$$", as a model just of the hot potato game; the cat and mouse game was developed while Lones visited the Cowles Foundation in 2006. We have profited from the insights and/or data of Charles Bruce (Director, National Check Fraud Center), Pierre Duguay (Deputy Governor, Bank of Canada), Antti Heinonen (European Central Bank, Counterfeit Deterrence Chairman), Ruth Judson (Federal Reserve), John Mackenzie (counterfeit specialist, Bank of Canada), Stephen Morris, and Lorelei Pagano (former Special Agent, Secret Service) — as well as comments at I.G.I.E.R. (Bocconi), the 2006 Bonn Matching Conference, the 2006 SED in Vancouver, the Workshop on Money at the Federal Reserve Bank of Cleveland, Tulane, Michigan, the Bank of Canada, the 2007 NBER-NSF GE conference at Northwestern, the 2008 Midwest Theory Conference in Columbus, the 2011 Yale Summer Theory Conference, Maryland, Pittsburgh, Stanford, and Georgetown.

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# Contents

# 1 Introduction

Counterfeiting is a major economic problem, called "the world's fastest growing crime wave" (Phillips, 2005). Specifically, counterfeiting of stated value financial documents like money, checks, or money orders, is both centuries-old *and* a large and growing economic problem. Domestic losses from check fraud, for instance, may have exceeded $20 billion in 2003 — due to Nigerian scams.[1] Counterfeit money is much less common but still quite costly: The counterfeiting rate of the U.S. dollar is about one per 10,000 notes, with the direct cost to the domestic public exceeding $80 million in 2011, more than doubling since 2003. The indirect counterfeiting costs for money are much greater, forcing a U.S. currency re-design every 7–10 years. As well, many costs are borne by the public in checking the authenticity of their currency.[2]

When we write *counterfeit* money (or checks), we have in mind two manifestations of it. *Seized* money is confiscated before it enters circulation. *Passed* money is found at a later stage, and leads to losses by the public. We have gathered an original data set mostly from the Secret Service on seized and passed money over time and across denominations. In the USA, all passed counterfeit currency must be handed over to the Secret Service, and so very good data is available (in principle). The stylized facts are best expressed in terms of two measures — the *counterfeit-passed ratio* (seized plus passed over passed) and the *passed rate* (passed over circulation). The key facts are:

*#1. The counterfeit-passed ratio* $(a)$ *rises in the note, but* $(b)$ *less than proportionately.*
*#2. The passed rate is small for low notes, greatly rises, and levels off or drops.*
*#3. Since the 1970s, the counterfeit-passed ratio has drammatically fallen about 90%.*
*#4. Compared to the average passed rates, Federal Reserve Banks find proportionately fewer counterfeits of higher than lower notes — until the $100 bill.*

We build a strategic model of the *cat and mouse game* between bad guys who may counterfeit and good guys who must transact anonymously. Good guys expend efforts screening out passed counterfeit money handed them; more effort yields stochastically better scrutiny. Since some fake notes change hands, a larger collateral game emerges: Good guys unwittingly pass on the counterfeit notes they acquire in an anonymous random matching exchange economy. This *hot potato game* is one of strategic complements (it is *supermodular*) — the more others verify, the more one should do likewise to protect oneself, *ceteris paribus*. In a unique stable equilibrium of this multi-market continuum player game, the counterfeiting rate emerges as a market-clearing quantity.

[1] Data here is sketchy. This estimate owes to a widely-cited Nilson Report (www.nilsonreport.com).
[2] Arguably, the $500M budget of the Bureau of Printing and Engraving, and maybe $1B of the Secret Service and Treasury budgets owe to anti-counterfeiting. Also, there is a private sector industry.

Counterfeiting is inherently a deception exercise, and so any good theory should capture the rival efforts of bad guys to successfully fool victims (*quality*), and of good guys to avoid being fooled (*effort*). Our cat and mouse game models just such a conflict. Costly effort and counterfeit quality jointly determine the chance bad notes are caught: the *verification rate*. Our verification function confers a cardinal meaning on both effort and quality, with each subject to diminishing returns. We explore how the clash between effort and quality unfolds, as the denomination stakes amplify, or other elements of the counterfeiting game change — like the legal, production, or verification costs. Some comparative statics are surprising, as payoffs changes on one side of a game often immediately require accommodating behavioral shifts by the *rival* side.

Our model admits expressions for several economically meaningful variables. The ratio of passed notes to all fake notes equals the passing fraction. The passed rate equals the unobserved counterfeiting rate times the discovery rate of notes. And fake money at Federal Reserve Banks (FRB) has passed and then escaped bank notice.

For a sample of the rich comparative statics, consider what happens when the denomination increases from a lower threshold. In the cat and mouse game, verification effort and quality both rise from zero. But which wins out is far from obvious. With a convex but log-concave verification cost function and a monotonic cost of quality elasticity — effort rises proportionately faster, and so the verification rate rises. An example argues that both cost function assumptions are needed for this conclusion. This yields a falling *passing fraction*, and thus a falling counterfeit-passed ratio (fact #1-a). Since quality rises in the note, so must counterfeiting costs — explaining why the counterfeit-passed ratio does not rise 1-for-1 with the note (fact #1-b). Finally, the rising verification rate explains why the FRB finds most counterfeit $1 notes (fact #4).

Next, in the adjoined hot potato game, the individual verification efforts rise in each of the denomination, the counterfeiting rate, and the verification efforts of others. This affords conclusions about the counterfeiting rate, and thereby the passed rate. The counterfeiting rate ultimately arises from the third choice variable we admit — namely, free entry of bad guys. Since the counterfeiting rate rises in the verification rate, and the verification rate rises in the note, the passed rate initially rises in the note (part of fact #2). Its later fall reflects the rising marginal cost of quality, and is explained later.

Among our normative predictions, we estimate the unobserved counterfeiting rate from our data, and approximate the street price of counterfeit notes — agreeing with anecdotal evidence. Most curiously, we back out marginal verification costs from the passed rate. They equal the passed rate times the denomination, peaking around 1/4 cent for the $100 bill! That such tiny verification costs explain the data attests to the explanatory power of slight incentives: microeconomics becomes "nano-economics".

RELATIONSHIP TO THE LITERATURE. Counterfeit money has not been studied much by economists. There are theoretical papers inspired by the classic money matching model of Kiyotaki and Wright (1989). For a useful point of comparison, Williamson and Wright (1994) assumes that transactors observe *fixed signals* of the authenticity of money, albeit *after* acquiring it. We instead build an entire theory on costly verification efforts that individuals expend before accepting money. Their work could not explain any counterfeiting data, since the signals in no way respond to the payoff stakes. Simply put, we argue that *exogenous attention cannot rationalize the facts of counterfeiting* — the assumption common to almost all existing work.[3,4]

The domestic price of U.S. notes is fixed, for by protocol they do not sell at a discount. The margin that does adjust is the verification rate. For a bigger picture on our model, this rate acts as an *implicit price* and the counterfeiting rate as a market-clearing quantity in a multi-market equilibrium. In contrast to earlier games with a continuum of players (Schmeidler, 1973; Green, 1984), ours involves two submarkets, the enmeshed hot-potato game and the cat and mouse game. Both games are novel. Recently, large games has seen a rebirth in macroeconomics. As in Angeletos and Pavan (2007), our payoffs depend on the average action, one's own action, and a state variable. But here, the counterfeiting rate state variable is endogenous.

Our assumption of costly verification is reminiscent of the spirit underlying a recent agenda in macroeconomics on "rational inattention" (Sims, 2003). That literature vein assumes that agents cannot observe the true state, but are constrained by bandwidth. Here, we explicitly model the cost of acquiring a more accurate signal about the state.

We lay out the model in §2. We develop our verification function and then prove equilibrium existence and uniqueness in §3. We then illustrate it in a solved example with geometric verification and counterfeit quality cost functions. By formally establishing the slopes of equilibrium equations in the appendix, we can use familiar graphical reasoning to establish in §4 almost all comparative statics in legal and productions costs of counterfeiting, verification costs, and the denomination. We hope that this graphical apparatus is a useful contribution for practitioners. Each derived result then makes sense in §5 of data or facts of seized and passed counterfeit money.

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[3] In Green and Weber (1996), only government agents can descry fake notes, whose stock is assumed exogenous, unlike here. Williamson (2002) admits counterfeits of private bank notes that are found with fixed chance; counterfeiting does not occur in most of his equilibria. Verification is also random and exogenous in Nosal and Wallace (2007), who find no counterfeiting in equilibrium with a high enough counterfeiting cost. Li and Rocheteau (2011) subsequently questioned this.

[4] An outlier in this literature is Banerjee and Maskin (1996). In our language, their verification is either perfect or worthless for each good: Agents either can or cannot distinguish good and bad qualities.

## 2 The Model

**A. Overview.** We use the language of counterfeit (or fake or hot) money. This is a dynamic discrete time story unfolding in periods 1,2,3,.... There are two types of risk neutral maximizing agents: In one sector of the economy is a continuum of *bad guys* who are potential counterfeiters. In another is a continuum of *good guys* who transact.

We'll see that endogenous quality is essential to explain the data, but that variable production is not. So motivated, each period, bad guys choose whether to counterfeit, and if so, what quality $q \geq 0$ of notes to produce. Notes have a common denomination $\Delta > 0$. The *counterfeiting rate* is the fraction $\kappa$ of transacted notes that are fake. The supply of counterfeit and genuine notes has *value* $M[\Delta] > 0$, assumed fixed. There is an infinitely elastic supply of identical bad guys who may freely enter. Each earns zero profits every period, net of legal penalty ("crime does not pay"). Counterfeiters try to pass all production. The value of *seized money,* i.e. taken by bad guys, is $S[\Delta]$.

Good guys randomly meet someone, possibly and unwittingly a bad guy, every period, at which point notes exogenously change hands for unmodeled reasons. Notes are just held for one period: half of the good guys always acquire notes in odd periods and spend them in even periods, while the rest do the opposite. A good guy possibly spends his notes at a bank; bad guys never do so. A good guy can only reject another's note if he notices that it is "hot"; the note then becomes worthless *passed money*[5] — whose total value is $P[\Delta]$. If found to be fake, the passer loses the face value $\Delta$.

These sectors interact, since counterfeit passed money circulates. Everyone is anonymous, with counterfeiters indistinguishable from good guys. So money changes hands not only from bad guy to good guy, but also from good guy to good guy.

Aware that they may be knowingly or unknowingly handed counterfeit currency, good guys expend *effort* $e \geq 0$ scrutinizing any note before accepting it. Checking notes is a stochastic endeavor that transpires note by note, and is our core novel feature. Real notes are never mistaken for counterfeit. The *verification rate* is the chance $v \in [0, 1]$ that a fake note is so noticed. This intuitively should rise in effort $e$ and fall in quality $q$. Verification efforts also help police keep bad money out of circulation.[6]

Everyone acts competitively, thinking he cannot affect the actions of others. We explore the *steady-state equilibrium* of this model, in which the verification rate is an "implicit price" on everyone, and the counterfeiting rate is a market-clearing quantity.

[5] Knowingly passing on fake currency is illegal by Title 18, Section 472 of the U.S. Criminal Code. We assume that no one engages in this crime of "uttering", seeking a "greater fool" to accept bad money.

[6] On its web page, the Secret Service also advises anyone receiving suspected counterfeit money: "Do not return it to the passer. Delay the passer if possible. Observe the passer's description."

**B. Currency Verification and Counterfeit Quality.** Good guys choose how much effort to expend checking the authenticity of money before accepting it. They notice counterfeit notes with chance $v \in [0,1]$, the verification rate; they never think a real note is fake. Better quality fakes look and feel more real, which impairs verification.

The function $e = q\chi(v)$ translates effort and quality $e, q \geq 0$ into a verification rate — to wit, doubling the quality requires twice the effort to secure the same verification rate. So verification is the derived smooth function $v = V(e,q) = \chi^{-1}(e/q)$ of effort and quality if $e < q\chi(1)$, and flat at $V(e,q) = 1$ for all $e \geq q\chi(1)$. Verification is perfect with zero quality ($V(e,0)=1$) and any $e \geq 0$, or for low quality $q>0$ if $e>0$.

Effort costs are twice smooth and increasing in verification, $\chi'(v) > 0$ for $v > 0$, but $\chi(0) = \chi'(0) = 0$. Also, $\chi'(v)/v$ is weakly increasing. Thus, $\chi$ is strictly convex, and also $v\chi''(v)/\chi'(v) \geq 1$,[7] whence the limit elasticity $\lim_{v \to 0} v\chi'(v)/\chi(v) \geq 2$ exists, by l'Hopital's rule. To rule out a rapidly rising or spiking marginal cost function, we assume that $\chi$ is strictly log-concave: $(\log \chi)'' < 0$, and so $(\chi'/\chi)' < 0$, or $\chi''/\chi' < \chi'/\chi$. All assumptions hold for any geometric cost function $\chi(v) = v^r$ with $r \geq 2$.

**C. The Verifier's Problem.** In spending periods, good guys meet random transactors with fixed chances $\beta \in (0,1)$, and otherwise go to a bank. Banks have professional staff that replicate a fixed chance $\alpha > 0$ of finding bad money.[8] Counterfeit money is thus found in transactions at the *discovery rate* $\delta(v) = \beta\alpha + (1-\beta)v$. If not signed over to another person, checks are deposited into a bank with chance $\beta' > \beta$. So good or bad guys with fake or real notes, have random meetings with transactors (or banks), who might or might not verify correctly. All events in this chain are independent.[9]

In periods that he acquires a note, a good guy first invests verification effort $e \geq 0$ examining it. His losses are the verification costs plus the expected note losses from the three independent events that $(i)$ he is handed a fake note, and *given that it is fake*, $(ii)$ his verifying efforts miss this fact, and $(iii)$ the next transaction catches it. Faced with an average verification rate $v$, in selling periods, good guys choose their effort $e$ to minimize their verification costs plus expected counterfeit losses next period:[10]

$$q\chi(V(e,q)) + \kappa(1 - V(e,q))\delta(v)\Delta \tag{1}$$

[7]Weak convexity is clear: One can secure a verification chance $v$ at cost $(\chi(v-\varepsilon)+\chi(v+\varepsilon))/2$ by flipping a coin, and verifying at rates $v-\varepsilon$ or $v+\varepsilon$. In other words, $\chi(v) \leq (\chi(v-\varepsilon)+\chi(v+\varepsilon))/2$.

[8]Bank tellers told us that they used set protocols, but were not encouraged or incentivized to treat different notes according to their value. As evidence of $\alpha < 1$, ATMs even dispensed counterfeit money (personal communication, John Mackenzie, Bank of Canada).

[9]As is the norm, we ignore technicalities of randomness and independence for a continuum of events, and assume simply that probabilities of individual events correspond to measures of aggregate events.

[10]We assume that $\chi$ absorbs any discounting between periods in this simple optimization.

The unobserved stock $\kappa M[\Delta]$ of counterfeit money is observably manifested by the passed money outflow $P[\Delta] = \delta(v)\kappa M[\Delta]$. Consequently, the *passed rate* equals:

$$p[\Delta] = P[\Delta]/M[\Delta] = \delta(v)\kappa \tag{2}$$

**D. Verification and the Counterfeit Passing Fraction.** Police may seize counterfeit money before it is passed onto the public. The *passing fraction* $f(v)$ is the share of fake notes passed. We assume that it is a smooth, falling function obeying $f(v) \leq 1 - v$ and $f(0) > 0$. So perfect verification chokes off passing ($f(1) = 0$), and some passing occurs if no one verifies. Intuitively, the first verifier catches a fraction $v$ of notes, and police seize a share $1 - v - f(v)$. We also assume that $f$ is weakly convex — in other words, we posit diminishing returns to police seizure efficacy. For instance, if police seize a fraction $\gamma \in [0, 1)$ of the fake notes missed by verifiers, then $f(v) = (1 - \gamma)(1 - v)$. We also ask for strict log-concavity in the verification rate: $(\log f)'' = (f'/f)' < 0$, and so $f'(0) > -\infty$, with limit $f'(v)/f(v) \downarrow -\infty$ as $v \uparrow 1$.

**E. The Counterfeiter's Problem.** Among the myriad of decisions counterfeiters must make, we center our theory on the entry and quality choices. Bad guys freely enter if they can make positive profits. We assume that bad guys produce a fixed finite expected quantity $x > 0$ of notes if they enter.[11] There are legal, production, and distribution costs of counterfeiting. The human and physical capital cost $c(q)$ of the counterfeit quality $q$ is smooth, with $c', c'' > 0$ for $q > 0$, $c(0) = 0$, and $c'(q) \to \infty$ as $q \uparrow \infty$. We assume a monotone cost of quality elasticity, and so a well-defined limit $\eta = \lim_{q\to 0} qc'(q)/c(q) \geq 2$:

$$\left(\frac{qc'(q)}{c(q)}\right)' \geq 0 \tag{3}$$

Next, since counterfeiters are invariably eventually caught,[12] and the stated penalty is the same across notes, we assume a fixed average present value of the punishment loss $\ell > 0$. A counterfeiter cares about his quality, and how carefully his notes are scrutinized. Counterfeiters maximize profits equal to expected revenues $f(v)x\Delta$ less costs $c(q) + \ell$:

$$\Pi(e, q, \Delta) \equiv f(V(e, q))x\Delta - c(q) - \ell \tag{4}$$

[11]Because each passing attempt risks discovery, the *marginal* distribution costs rise in output. "If a counterfeiter goes out there and, you know, prints a million dollars, he's going to get caught right away because when you flood the market with that much fake currency, the Secret Service is going to be all over you very quickly. They will find out where it's coming from." — interview with Jason Kersten, author of Kersten (2005) [*All Things Considered*, July 23, 2005].

[12]The Secret Service estimates that the conviction rate for counterfeiting arrests is close to 99%.

# 3 Equilibrium Derivation

## 3.1 The Cat and Mouse Game

We solve our large game in halves, focusing first on the struggle between the quality of bad guys and the effort choice of good guys. We need only consider how verifier effort holds counterfeiting profits to zero; effort optimization occurs in the next game.

We now describe the verification function in the competitive cat and mouse game, exploring how it embeds diminishing marginal returns to verification effort or counterfeit quality. Since $\chi$ is smooth, we conclude that $V$ is smooth: For the identity $q\chi(V(e,q)) \equiv e$ yields $q\chi' V_q + \chi \equiv 0$ and $q\chi' V_e \equiv 1$ in the range $e < q\chi(1)$. Hence:

**Lemma 1 (First Derivatives)** *Fix the verification effort $e > 0$ and counterfeit quality $q > 0$ so that $v = V(e,q) < 1$. The verification intensity rises in $e$ and falls in $q$:*
$(a)$ *Verification rises in effort, with $V_e(e,q) = 1/q\chi'(v) > 0$ and $V_e(e,q) \uparrow \infty$ as $e \downarrow 0$.*
$(b)$ *Verification falls in quality, with slope $V_q(e,q) = -\chi(v)/q\chi'(v) < 0$.*

Given our multiplicative cost structure, strictly log-concave costs delivers the intuitive result that while greater quality inhibits verification, this reduction itself obeys the law of diminishing returns, or $V_{qq} > 0 > V_q$. Summarizing all second derivative properties:

**Lemma 2 (Second Derivatives)** *Fix effort $e > 0$ and quality $q > 0$ so that $v = V(e,q) < 1$. Then each has falling marginal returns, or $V_{qq} > 0 > V_{ee}$, and the verification function is submodular in effort and quality, namely $V_{eq} < 0$.*

*Proof:* Let $e < q\chi(1)$. Differentiating $q\chi(V(e,q)) \equiv e$ yields $q\chi' V_{ee} + q\chi'' V_e^2 \equiv 0$, so that $q^2 V_{ee}(e,q) = -\chi''(v)/(\chi'(v))^3 < 0$. Derive $V_{eq}$ and $V_{qq}$ by differentiating the identity $q\chi'(V(e,q))V_e(e,q) \equiv 1$ in $q$ and $e$, similarly. Since $\chi$ is strictly log-concave:

$$q^2 V_{qq} \;=\; \frac{\chi}{\chi'} + \left(\frac{\chi}{\chi'}\right)^2 \left(\frac{\chi'}{\chi} - \frac{\chi''}{\chi'}\right) > 0 > \frac{\chi}{(\chi')^2}\left(\frac{\chi''}{\chi'} - \frac{\chi'}{\chi}\right) = q^2 V_{eq}$$

Effort and quality are substitutes for good guys but complements for bad guys: quality blunts the marginal fruits of effort, but effort raises the marginal efficacy of quality.

Given free entry by bad guys, expected profits (4) vanish. In $(q,v)$-space, this becomes

$$\Delta x f(v) - c(q) - \ell = 0 \tag{5}$$

A *cat and mouse equilibrium* is a pair $(q,e)$ yielding counterfeiters zero profits (5) and for which quality $q$ maximizes their profits (4) given verifier effort $e = q\chi(v)$.

Define the threshold $\underline{\Delta} \equiv \ell/(xf(0)) > 0$— suggestively, the *least counterfeit note*.

**Theorem 0 (Non-Existence)** *No cat and mouse equilibrium exists for notes $\Delta \leq \underline{\Delta}$.*

For if $\Delta < \underline{\Delta}$, then profits would be less than $\underline{\Delta} x f(0) - \ell = 0$, namely that with nothing verified and no quality costs. If $\Delta = \underline{\Delta}$, then zero profits requires that quality vanish. Verification is then perfect for all $e > 0$, and counterfeiters lose at least $\ell > 0$.

We henceforth restrict focus to notes $\Delta > \underline{\Delta}$. In this case, effort and quality are positive in any cat and mouse equilibrium, obeying $q < e/\chi(1)$. First, with zero effort, profits are strictly positive for small enough $q$, where $c(q) > f(0)x\Delta - \ell$. So effort $e$ is positive. Next, when quality obeys $q < e/\chi(1)$, no fake notes pass, and counterfeit losses are at least $\ell$. In this range, $V(e,q)$ is smooth. Then the next FOC holds at an optimum. It captures the tradeoff that higher quality notes pass more readily but cost more:

$$\Pi_q(e,q,\Delta) \equiv \Delta x f'(V(e,q))V_q(e,q) - c'(q) = 0 \tag{6}$$

Lemma 1-$(b)$ allows us to express the optimality condition (6) in $(q,v)$-space as:

$$-\Delta x f'(v)\frac{\chi(v)}{\chi'(v)} \;=\; qc'(q) \tag{7}$$

Taking logarithms, we define two convenient functions of the verification rate, namely, $F(v) \equiv \log[xf(v)]$ and $G(v) \equiv \log[-xf'(v)\chi(v)/\chi'(v)]$. These have ranked slopes:

$$G'(v) - F'(v) \equiv \frac{f''}{f'} - \frac{f'}{f} + \frac{\chi'}{\chi} - \frac{\chi''}{\chi'} > 0 \tag{8}$$

in light of our respective log-concavity assumptions on the fraction $f$ and costs $\chi$. Defining $T(q) \equiv \log[c(q)+\ell]$ and $U(q) \equiv \log[qc'(q)]$, we may rewrite the cat and mouse equilibrium equations (5) and (7) in the equivalent additively separable forms:

$$F(v) + \log\Delta \;=\; T(q) \tag{9}$$

$$G(v) + \log\Delta \;=\; U(q) \tag{10}$$

Because $T'(q) > 0 > F'(v)$, *the zero-profit locus $\bar{\Pi}$ solving (5) or (9) slopes down.* For since profits fall in quality and verification, these are inversely related along $\bar{\Pi}$. Given $\Delta x f(0) > \ell$, there exists a verification rate $v_\Delta > 0$ and quality $q_\Delta > 0$ with $\Delta x f(v_\Delta) = \ell$ and $\Delta x f(0) = c(q_\Delta) + \ell$. Also, $v \uparrow v_\Delta$ as $q \downarrow 0$ and $q \uparrow q_\Delta$ as $v \downarrow 0$.

Next, we analyze the *optimal quality locus* $Q^*$ solving (7) or (10). Consider first its behavior for the lowest verification rates. Since the limit of $v\chi'(v)/\chi(v)$ as $v \downarrow 0$ finitely exists, and $-\infty < f'(0) < 0$, the locus $Q^*$ starts at $q = v = 0$, and initially rises in $q$. Its initial slope vanishes: $v/q = -[v\chi'(v)/\chi(v)][c'(q)/\Delta x f'(v)] \to 0$ as

$q, v \to 0$. So quality cannot explode near perfect verification, for $1 - v \geq f(v) > 0$ and the convex passing fraction $f$ implies a bounded slope $f'(v) \geq -1$ as $v \uparrow 1$.

For the global behavior of the $Q^*$ locus, first assume constant marginal returns to police interdiction, as captured by a linear passing fraction $f(v)$. For instance, $f(v) = 1 - v$ in the extreme case without police. Log-concavity of $\chi$ then implies $G' \geq 0$. Since $U' > 0$, in this case $Q^*$ monotonely slopes upwards.

Next assume diminishing marginal returns to police interdiction — i.e., a strictly convex passing function $f(v)$. Then $G' < 0$, and so $Q^*$ still slopes upward, provided $f(v)$ is not so convex that:

$$\frac{f''}{f'} + \frac{\chi'}{\chi} - \frac{\chi''}{\chi'} > 0 \tag{11}$$

This is stronger than the joint log-concavity inequality (8), because it lacks the middle positive term $-f'/f$. Intuitively, the $Q^*$ curve bends back when the map $q \mapsto \chi(v) \equiv e/q$ falls in $q$, and so surely when the function $q \mapsto e$ is falling. Now, given Topkis (1998), the maximization of profits (4) yields an implied falling map $e \mapsto q$ for a submodular passing fraction $f(V(e, q))$. By Rockafellar (1970), the composition of an increasing and convex function $g(v) = -f(v)$ with an increasing and supermodular one $W(e, -q) = V(e, q)$ is supermodular. So when the passing fraction $f$ is convex enough to secure inequality (11), $f(V(e, q))$ is supermodular enough that $Q^*$ slopes up.

Regardless of its monotonicity, the $Q^*$ locus hits $v = 1$ at a finite quality $q_\Delta$, satisfying $q_\Delta c'(q_\Delta) = -\Delta f'(1)\chi(1)/\chi'(1) > 0$. Summarizing these insights:

**Lemma 3 (The $Q^*$ Curve)** *The optimal quality locus $Q^*$ rises from $(0, 0)$ to $(q_\Delta, 1)$, for some $q_\Delta < \infty$. Its slope is initially zero, then positive, and always so given (11). If $Q^*$ slopes down at an equilibrium, then it is steeper than the zero profit curve $\bar{\Pi}$.*

We illustrate this with a convenient parameterized class of passing fractions $f(v) = (1 - v)(1 - \gamma v)$. When $\gamma = 0$, this reduces to no police interdiction. But when $0 \leq \gamma < 1$, the passing fraction $f$ is monotone decreasing, convex and log-concave, with $f(0) > 0 = f(1)$. In the special case of geometric verification costs $\chi(v) = v^B$, inequality (11) reduces to $vf''(v)/f'(v) \geq -1$, which obtains whenever $\gamma \leq 1/3$. So $Q^*$ slopes upward for a robust class of models with diminishing police efficacy.

Figure 1 depicts the two possible $\bar{\Pi}$ and $Q^*$ cases. Equilibrium existence follows from the Intermediate Value Theorem, provided (5) and (7) admit solutions continuous in $\Delta$. Moreover, given the slopes of the $Q^*$ and $\bar{\Pi}$ curves, the equilibrium is unique.

**Theorem 1** *For any $\Delta > \underline{\Delta}$, there is a unique cat and mouse equilibrium $(e^*, q^*)$. The verification rate, effort, and quality are all positive, and differentiable in $\Delta$, and verification is imperfect: $v^* < 1$*

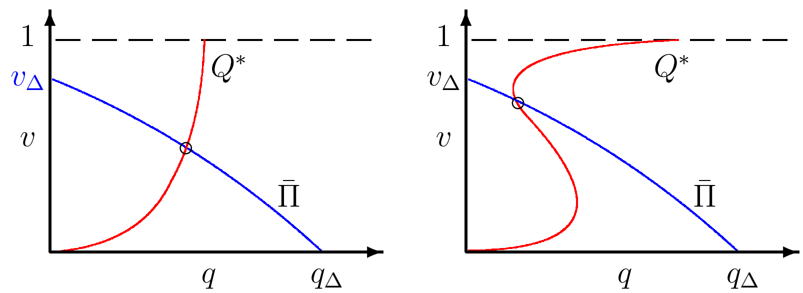


Figure 1: **Zero Profit and Optimal Quality Curves**. The zero profit curve $\bar{\Pi}$ solving (5) falls from $(0, v_\Delta)$ to $(q_\Delta, 0)$, and the optimal quality locus $Q^*$ solving (7) rises from $(0,0)$ to $(q_\Delta, 1)$. Any negatively-sloped portion of $Q^*$ is steeper than the zero profit curve $\bar{\Pi}$ at an equilibrium (right). A monotone $Q^*$ curve (left) arises given (11).

## 3.2 The Hot Potato Passing Game

While this passing game requires solving for the verification effort $e$ given a counterfeiting rate $\kappa$, we proceed in reverse, deducing the $\kappa$ that justifies a pre-determined effort $e$. In equilibrium, counterfeit quality $q$ is known, and thus an effort choice is tantamount to a selection of the verification rate $\hat{v} = V(e, q)$. We may rewrite (1) as

$$\min_{0 \leq \hat{v} \leq 1} q\chi(\hat{v}) + \kappa(1 - \hat{v})\delta(v)\Delta \tag{12}$$

Fixing the counterfeiting rate (as no good guy can affect it), one's verification rate $\hat{v}$ is a strategic complement in (1) to the average rate $v$. Intuitively, one should examine a note more closely the more intensely it will be checked. The best reply $\hat{v}$ in (12) thus rises in $v$. *Supermodular games* may have multiple equilibria (Milgrom and Roberts, 1990), as increasing best reply functions may multiply cross. But this is moot here, for by Theorem 1, the cat and mouse equilibrium pins down a unique verification rate $v$.

Since benefits in (1) are linear in verification, and costs $\chi$ are strictly convex with $\chi'(0) = 0$, any FOC solution with imperfect verification must be the global minimum, where:

$$q\chi'(\hat{v}) = \kappa\delta(v)\Delta \tag{13}$$

Facing any average verification rate $v$, the homogeneous good guys naturally choose the same best response $\hat{v}$. So there is a unique and symmetric *hot potato passing game equilibrium* with $\hat{v} = v > 0$. Since the verification rate and quality are determined in Theorem 1, we instead write it as the counterfeiting rate $\kappa$ solving (14). This admits the economic interpretation as the ratio of marginal costs and benefits of verification per note:

$$\kappa = \frac{q\chi'(v)}{\delta(v)\Delta} = \frac{\text{marginal verification cost}}{\text{discovery rate} \times \text{denomination}} \tag{14}$$

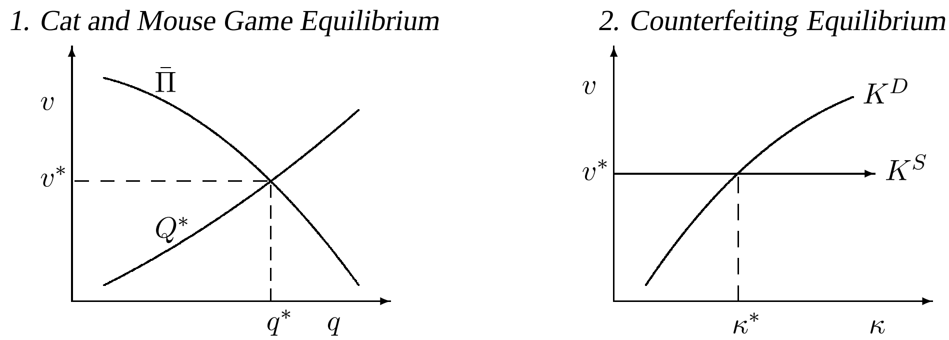

Figure 2: **Two Sector Equilibrium Logic.** One verification rate "price" clears two markets — for criminals and verifiers. The cat and mouse equilibrium in $(q, v)$-space (left) yields the infinitely elastic counterfeiting supply at $v^*$ in $(\kappa, v)$-space (right).

## 3.3 A Stable Multimarket Counterfeiting Equilibrium

A *counterfeiting equilibrium* is a triple $(e^*, q^*, \kappa^*)$ yielding equilibrium in each game:

- Verifiers' effort $e^*$ and counterfeit quality $q^*$ are a cat and mouse equilibrium.
- Given counterfeit quality $q^*$, the effort $e^*$ by good guys is an equilibrium of the hot potato passing game for the counterfeiting rate $\kappa^* \in (0, 1)$.

This equilibrium admits a useful recursive structure: The unique cat and mouse equilibrium $(q^*, e^*)$ fixes the verification rate $v^*$. Then the hot potato game determines $\kappa^*$. This yields the infinitely elastic *counterfeiting supply curve* $K^S$ in Figure 2.[13]

Next, think of the map $v \mapsto \kappa$ in (14) as the *derived counterfeiting demand curve*. For the verification rate $v^*$ is the "price" paid to deter the counterfeiting rate $\kappa$. This demand curve, $K^D$ in Figure 2, intuitively slopes upward, since fake notes are a "bad": For $\chi'(v)/\delta(v) = [\chi'(v)/v][v/\delta(v)]$ is a product of a weakly and a strictly increasing function. An equilibrium $(e^*, q^*, \kappa^*)$ is *stable* if it is robust to a "price adjustment" process. When the verification rate differs from $v^* = V(e^*, q^*)$, say $v < v^*$, bad guys seeking profits enter; this raises the counterfeiting rate above $\kappa^*$. On the other hand, lower verification requires that good guys think the counterfeiting rate lies below $\kappa^*$. All told, supply rises and derived demand falls. These two discordant realities push the verification rate back up towards $v^*$, as the verifiers infer the error of their ways.

**Theorem 2** *If $\Delta > \underline{\Delta}$, there is a unique stable counterfeiting equilibrium $(e^*, q^*, \kappa^*)$. Verification is imperfect and positive, and counterfeiting is positive but non-runaway, and bounded by:*

$$\kappa(v) \leq \frac{\sqrt{3}xf(0)\chi'(1)}{(1-\beta)c(1)^{1/\eta}\ell^{1-1/\eta}} \tag{15}$$

[13]The supply curve would slope down with heterogeneous bad guys. This would not be recursively solvable, *greatly* complicating the analysis.

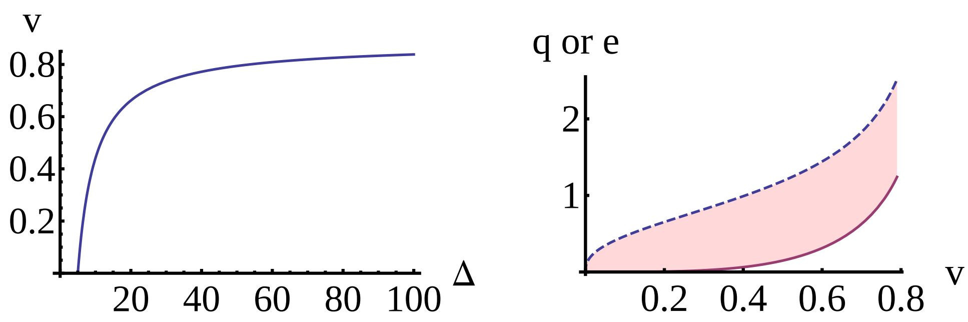


Figure 3: **Effort, Quality, and Verification in Example.** When $A = 5, B = 3, x = 2$ and $\ell = 10$, the verification rate (left), rises from the least counterfeit note $\underline{\Delta} = 5$ toward $\bar{v} = 0.8$. At right, verifier effort (solid) and counterfeit quality (dashed) both rise.

We see that counterfeiting never vanishes, but can spiral out of control. A counterfeiting rate below one (*non runaway*) is mathematically immaterial in the good guys' optimization (1), but is clearly mandated by economic sense. The bound (15) rises if counterfeiting is easier — either lower legal costs $\ell$, or unit quality counterfeit costs $c(1)$, or higher production $x$ or passing rate $f(0)$. The bound falls with more effective verification — a higher bank chance $\beta$, or lower verification marginal costs $\chi'(1)$.

## 3.4 An Illustrative Example of a Counterfeiting Equilibrium

A geometric verification cost function $\chi(v) = v^B$ is log-concave, and when $B \geq 2$, it is strictly convex with $\chi'(v)/v$ weakly increasing. A geometric counterfeiting cost function $c(q) = q^A$ is convex and obeys our elasticity condition (3) when $A \geq 2$.

Consistent with the monotonicity and curvature of the plots in the left panel of Figure 2, the zero profit equation (5) and optimal quality equation (7) reduce to:

$$\Delta x(1-v) - q^A - \ell = 0 \qquad \text{and} \qquad Aq^A - \Delta x v/B = 0 \tag{16}$$

Solving the zero profit condition in (16), verification vanishes for notes $\Delta$ approaching $\underline{\Delta} = \ell/x$. And as $\Delta \uparrow \infty$, the verification rate tends to $\bar{v} = AB/(1+AB) < 1$, since:[14]

$$q^A = (1-\bar{v})(\Delta - \underline{\Delta}) \qquad \text{and} \qquad v = \bar{v}(1 - \underline{\Delta}/\Delta) \tag{17}$$

So verification rises in the note $\Delta$, but is forever imperfect. While effort $e = qv^B$ rises in $\Delta$, quality rises much faster, and infinitely so initially as $B > 0$, as seen in Figure 3:

$$e = (1-\bar{v})^{1/A}\bar{v}^B \Delta^{-B}(\Delta - \underline{\Delta})^{B+1/A} \tag{18}$$

[14] Had we assumed our richer passing function $f(v) = (1-v)(1-\gamma v)$, a quadratic equation would have determined the verification rate $v$. In this case, positive $\gamma$ would depress $v$, and elevate quality $q$.

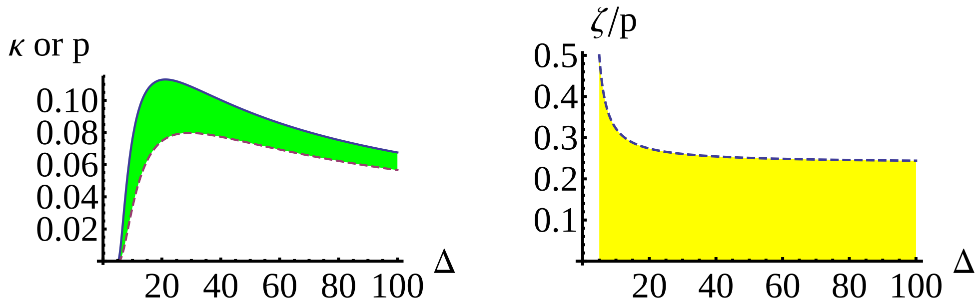

Figure 4: **Counterfeiting and Passed Rates, plus FRB Passed Ratio.** These plots obtain for $\alpha = 4/5$ and $\beta = 1/4$. The counterfeiting and passed rate (2) curves (solid/dashed) at left vanish both initially and eventually. At right, is the banking sector story: the ratio of the FRB and average passed rates $\zeta/p$ in (27) falls in $\Delta$.

So far, the economic logic turns solely on incentives in the cat and mouse game. We now consider the hot potato game to compute the counterfeiting rate. Substituting quality and verification from (17) into (14) yields the equilibrium counterfeiting rate $\kappa = Bqv^{B-1}/(\delta(v)\Delta)$, given the increasing discovery rate $\delta(v) = \beta\alpha + (1-\beta)v$. Not only does counterfeiting occur for all notes $\Delta > \underline{\Delta}$, but the counterfeiting rate $\kappa$ is a unimodal function of the note, vanishing for both $\Delta \downarrow \underline{\Delta}$ and $\Delta \uparrow \infty$, since $A > 1$:

$$\kappa = \frac{B(1-\bar{v})^{1/A}\bar{v}^{B-1}\Delta^{2-B}(\Delta - \underline{\Delta})^{B-1+1/A}}{\beta\alpha\Delta + (1-\beta)\bar{v}(\Delta - \underline{\Delta})} \tag{19}$$

Figure 4 (left) also depicts the similarly-shaped plot of the passed rate $p = \delta(v)\kappa$ from (2). The passed rate understates the counterfeiting rate, but their ratio $p/\kappa$ rises in $\Delta$, tending to $\bar{v} < 1$. Since $B > 1 + 1/A$, the passed and counterfeiting rates both vanish for notes $\Delta \downarrow \underline{\Delta}$. For notes $\Delta \uparrow \infty$, both rates vanish as fast as $\Delta^{1/A-1}$.

# 4 Equilibrium Comparative Statics

Towards a common tractable graphical framework for both the hot-potato and the cat and mouse games that will afford a common comparative statics analysis of the triple $(e, q, \kappa)$, we next superimpose the *constant counterfeiting rate locus* $\bar{K}$ in $q$-$v$ space. For since as argued in §3.3, the ratio $\chi'(v)/\delta(v)$ is increasing, anything that raises the quality or verification rate also inflates the counterfeiting rate, by (14). So the constant counterfeiting locus $\bar{K}$ slopes down in $q$-$v$ space. This locus, seen in Figure 5, is sandwiched between $\bar{\Pi}$ and $Q^*$ under a *henceforth assumed new bound, jointly limiting the convexity of the passing fraction and the verification cost elasticity:*

$$\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} \geq 1 \tag{20}$$

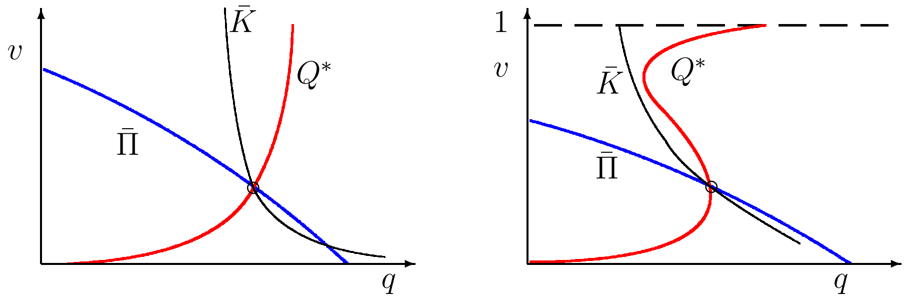


Figure 5: **Hot Potato and Cat and Mouse Equilibria, Superimposed**. The locus $\bar{K}$ with a constant counterfeiting rate is sandwiched between the optimal quality and zero profit loci $Q^*$ and $\bar{\Pi}$, given the inequality (20).

This inequality holds in the no police special case $f(v) = 1-v$. But it robustly holds in our parameterized example $f(v)=(1-v)(1-\gamma v)$, reducing to $\gamma \leq (2B-1)/(2B+1)$. Since $B>1$, this is less restrictive than the bound $\gamma \leq 1/3$ for the inequality (11).

**Lemma 4 (Slope of the $\bar{K}$ Curve)** *The slope of the constant counterfeiting locus $\bar{K}$ is negative, but greater than $\bar{\Pi}$. In addition, the slope of $\bar{K}$ is less than $Q^*$, given (20).*

The slope of $Q^*$ obviously exceeds that of $\bar{K}$ when $Q^*$ slopes up, and not surprisingly, inequality (20) is weaker than inequality (11), given assumption $v\chi''(v)/\chi'(v) \geq 1$.

## 4.1 Shifts of One Curve Only: Legal and Verification Costs

Differentiate the zero-profit identity (5) in legal costs $\ell$ to get[15] $\Pi_q \dot{q} + \Pi_e \dot{e} + \Pi_\ell = 0$. Since the firm optimizes on quality, the first term cancels, by the Envelope Theorem.[16] Given $\Pi_e = \Delta f' V_e < 0$ and $\Pi_\ell = -1 < 0$, effort falls when legal costs rise: $\dot{e} < 0$.

To deduce the impact on quality and verification, we use the graphical framework. When $\ell$ rises, the zero profit curve $\bar{\Pi}$ shifts down at each quality, because counterfeiters require less verification effort to avoid losses. Obviously, the least notes can no longer be profitably counterfeited with greater legal costs (i.e. $\underline{\Delta}$ rises in $\ell$). As the optimal quality locus $Q^*$ in (10) is unaffected by $\ell$, the shape of $Q^*$ alone governs changes in $(q, v)$. Verification unambiguously falls, for either $Q^*$ slopes up, or slopes down and is steeper than $\bar{\Pi}$. Finally, if $Q^*$ is monotone, higher legal costs depress both quality and the verification rate, thus lowering the counterfeiting rate — as seen in Figure 5.

**Proposition 1** *If legal costs rise, verification effort and rate fall. Counterfeit quality falls at low and high $\Delta$, and always if $Q^*$ is monotone. The counterfeiting rate falls.*

[15]The notation $\dot{x}$ denotes the derivative of $x$ in $\ell$. Later, it denotes derivatives in other parameters.

[16]If $q > 0$ then the first order condition $\Pi_q = 0$ holds. If $q = 0$ in an open interval, then $\dot{q} = 0 \geq \Pi_q$. By continuity of $\Pi_q$, this happens also if $q > 0$ for notes $\Delta'$ arbitrarily close to $\Delta$, and thus at $\Delta$.

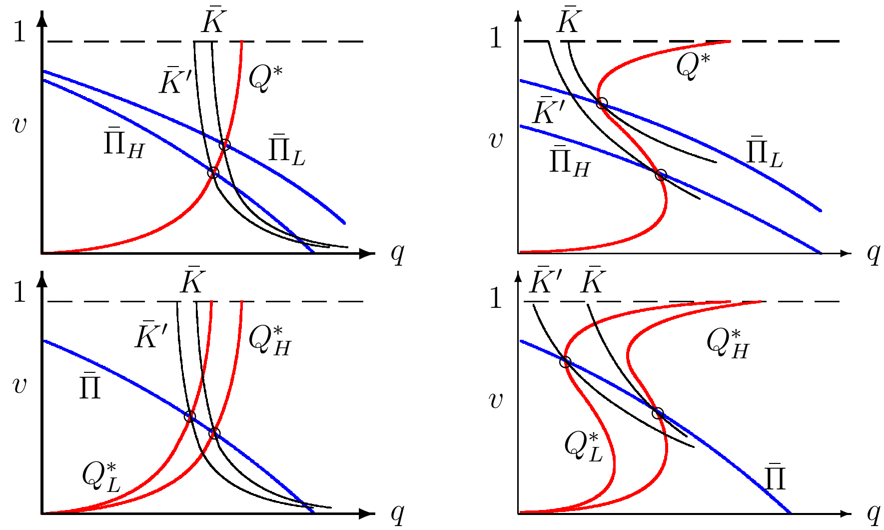


Figure 6: **Shifting Legal or Verification Costs: Propositions 1 and 2 Proved.** Top: When legal costs rise, the zero profit curve $\bar{\Pi}$ shifts down ($\bar{\Pi}_L$ to $\bar{\Pi}_H$). So the verification rate $v$ falls, while quality $q$ falls if $Q^*$ is locally rising — i.e. surely for low and high notes. The counterfeiting rate falls, as we shift to a higher $\bar{K}$ locus (thin curves). Bottom: When verification costs fall, the $Q^*$ locus shifts left ($Q^*_H$ to $Q^*_L$). Verification improves and quality falls, and so the counterfeiting locus $\bar{K}$ shifts down to $\bar{K}'$.

Altogether, with greater legal cost, counterfeiters exit, the counterfeiting rate drops, so verification effort falls, and the verification rate falls despite usually lower quality.

Next assume a new technology renders money more readily verified. Verification costs only affect the optimal quality locus $Q^*$. To capture all smooth technological improvement, let verification rate $v$ with technology $t$ cost the same as $\mathcal{V}(v,t)$, where $\mathcal{V}(v,0) \equiv v$, with $\mathcal{V}(v,t)$ falling in $t$ and rising in $v$, or $\mathcal{V}_t < 0 < \mathcal{V}_v$. The zero profit identity (5) is:

$$\Delta x f(\mathcal{V}(V(e,q),t)) - c(q) - \ell = 0$$

Differentiate in $t$. Its $q$ derivatives cancel by the Envelope Theorem. Given $V_e > 0$ (Lemma $1(a)$) and $\mathcal{V}_v V_e \dot{e} + \mathcal{V}_t = 0$, effort $e$ rises in $t$. Since $\chi''/\chi' \le \chi'/\chi$ by log-concavity of $\chi$, the ratio $\chi'(\mathcal{V}(v,t))/\chi(\mathcal{V}(v,t))$ rises in $t$. So $Q^*$ in (7) shifts left in $t$.

**Proposition 2** *If the verification technology improves, the verification effort and rate both rise, the counterfeit quality falls, and the counterfeiting rate falls.*

Figure 6 graphically proves this result, except that the counterfeiting rate also falls for an exogenous reason — because the cost function $\chi$ in (14) drops: we not only shift to $\bar{K}'$ from $\bar{K}$, but this new curve $\bar{K}'$ corresponds to a lower counterfeiting rate.

In our cat and mouse game, do counterfeiters reply with improved quality to better elude capture? No. Easier verification is met by *lower quality counterfeits*. Increased losses from a poorer passing technology force counterfeiters to spend less on quality.

## 4.2 Shifts of Both Curves: Changing Technology

The counterfeiting technology improves when production costs fall for any quality: As with verification costs, we generally capture this by a smooth function $q \mapsto \mathcal{Q}(q, \tau)$, i.e., the quality that costs $c(q)$ given counterfeiting technology $\tau$. Then $\mathcal{Q}(q, 0) \equiv q$, with $\mathcal{Q}(q, \tau) < q$ when $\tau > 0$, falling in $\tau$ and rising in $q$, or $\mathcal{Q}_\tau < 0 < \mathcal{Q}_q$. In order to sustain zero profits, verification effort must rise in the technology $\tau$. For differentiate the zero profit identity $\Pi(q, e, \Delta) \equiv 0$ in $\tau$, using $\Pi_\tau > 0 > \Pi_e$ and $\Pi_q = 0$ to get $\dot{e} > 0$.

We employ the graphical framework to determine how quality and the verification rate change. The cost function affects both the $\bar{\Pi}$ and $Q^*$ curves. Since $T', U' > 0$, by equations (9) and (10), both $\bar{\Pi}$ and $Q^*$ shift right when $\tau$ rises. As seen in Figure 7, *the verification rate $v$ falls exactly when $Q^*$ shifts right more than $\bar{\Pi}$ does.* This happens:

$$\frac{d}{d\tau}[U(\mathcal{Q}(q, \tau)) - T(\mathcal{Q}(q, \tau))]\Big|_{\tau=0} = \mathcal{Q}_\tau[U'(q) - T'(q)] < 0$$

Now, since $c'(q) > 0$ and $c''(q) > 0$, if average costs $[c(q) + \ell]/q$ fall in $q$, then

$$U'(q) - T'(q) \equiv \frac{d}{dq} \log\left(\frac{qc'(q)}{c(q) + \ell}\right) = \frac{d}{dq}\left(\log c'(q)\right) - \frac{d}{dq} \log\left(\frac{c(q) + \ell}{q}\right) > 0 \quad (21)$$

The middle term is positive when $\ell > qc'(q) - c(q)$, true for small $q$: legal costs exceed producer surplus. And since $c(q)/(c(q)+\ell)$ rises in $q$, the last term is positive given (3).

Next, when $G'(v) > 0$, the optimal quality locus $Q^*$ slopes up, and quality naturally rises when $Q^*$ and $\bar{\Pi}$ shift right. But if $G'(v) < 0$, then $Q^*$ slopes down, and the analysis is more subtle. But since $Q^*$ is steeper than $\bar{\Pi}$ at an equilibrium (Lemma 3), *quality $q$ rises in $\tau$ exactly when $Q^*$ falls more than $\bar{\Pi}$ for fixed $q$* (top right panel of Figure 7). By (9) and (10), this occurs because $|F'(v)| > |G'(v)|$ by inequality (8).

The counterfeiting rate rises, for this shifts to the higher counterfeiting locus $\bar{K}'$.

**Proposition 3** *If the cost of counterfeiting money falls, then the counterfeit quality and verification effort rises, the verification rate falls, and the counterfeiting rate rises.*

## 4.3 Shifts of Both Curves: A Rising Denomination

A rising denomination is the hardest comparative statics exercise, as it exogenously shifts both $\bar{\Pi}$ and $Q^*$ loci, *and also* exogenously depresses the counterfeiting rate (14).

First, higher notes command closer scrutiny — if not, they could be profitably counterfeited. Differentiate the zero-profit identity (5) in $\Delta$ to get $\Pi_q\dot{q} + \Pi_e\dot{e} + \Pi_\Delta = 0$:

$$\Pi_e\dot{e} + \Pi_\Delta = 0 \quad (22)$$

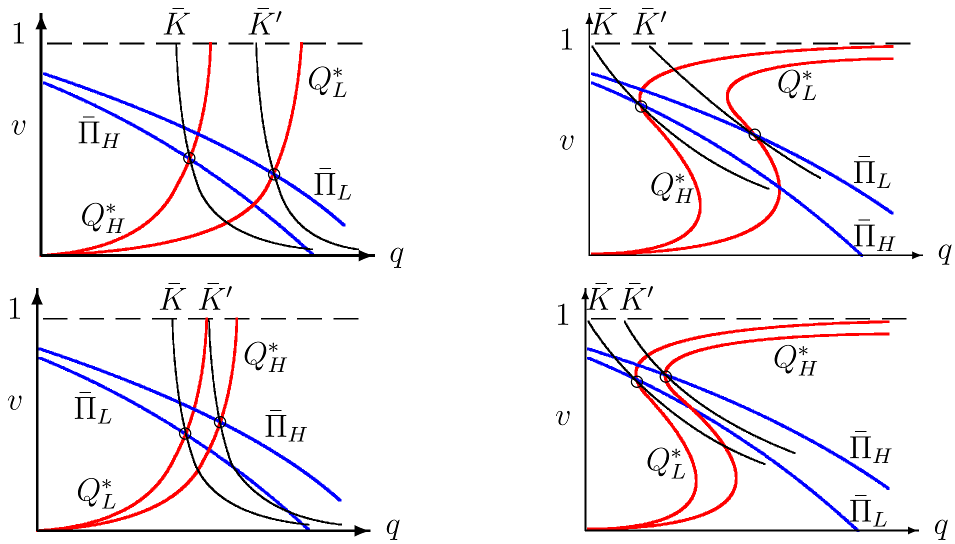


Figure 7: **Changing Technology or Denomination: Propositions 3 and 4 Depicted**. Top: The counterfeiting costs fall from $H$ to $L$, pushing the zero profit locus $\bar{\Pi}$ right more than the optimal quality locus $Q^*$, raising quality but lowering the verification rate. The counterfeiting locus shifts right to $\bar{K}'$. Bottom: When the note $\Delta$ rises from $L$ to $H$, the locus $\bar{\Pi}$ shifts right more than $Q^*$, raising both quality and verification. The counterfeiting locus shifts right to $\bar{K}'$, but the rate might not rise, since $\Delta$ is higher.

since $\Pi_q = 0$. So $\dot{e} > 0$. Next, we exploit the graphical framework. When $\Delta$ rises, $\bar{\Pi}$ and $Q^*$ shift right, and so the logic of §6 applies: If $Q^*$ slopes upward, then quality rises, and the verification rate rises at low and high notes $\Delta$, and always rises given (3). But when $Q^*$ slopes down, then as seen in Figure 7, quality rises if $\bar{\Pi}$ shifts up more than $Q^*$ when $\Delta$ rises. This holds when $|F'(v)| < |G'(v)|$, true by log-concavity (8).

**Proposition 4** *The verification effort and rate, and counterfeit quality all vanish as the note $\Delta \downarrow \underline{\Delta}$. Effort, quality, and the verification rate monotonically rise in the note if $\Delta > \underline{\Delta}$, and effort and quality explode as the note $\Delta \uparrow \infty$.*

Intuitively, a counterfeit \$100 note has higher quality than a counterfeit \$5 note, and yet passes less readily (as we shall see) because it is sufficiently more carefully inspected.

Now, consider how the counterfeiting rate changes. While the $\bar{K}'$ locus in Figure 7 (bottom) is right of $\bar{K}$, the counterfeiting rate (14) is also exogenously depressed by $\Delta$. In fact, the example in Figure 4 for geometric costs and a linear passing function suggests a counterfeiting rate that is unimodal in $\Delta$. But it is impossible to deduce this strong result from our weak inequality assumptions on the cost functions $\chi(v)$ and $c(q)$. Still, we next argue that it vanishes near the least and highest counterfeit notes.

**Proposition 5** *The counterfeiting rate vanishes for notes $\Delta \downarrow \underline{\Delta}$ or $\Delta \uparrow \infty$.*

Proposition 5 not only rationalizes the data, but it also refutes arguments against issuing yet higher denominations for fear that they would be heavily counterfeited.

The proof exploits the hot potato equilibrium equation (14). First, consider low notes. The counterfeiting rate vanishes for $\Delta$ tending to the least counterfeit note $\underline{\Delta} > 0$ since quality and the verification rate from the cat and mouse game vanish in (14) by Proposition 4, while the discovery rate obeys $\delta(v) \geq \beta\alpha > 0$. Next, assume $\Delta \uparrow \infty$. Substitute the optimal quality condition (6) and Lemma 1 into (14):

$$\kappa = \frac{q\chi'(v)}{\delta(v)\Delta} = \frac{q\chi'(v)}{\delta(v)} \frac{xf'(v)V_q(e,q)}{c'(q)} = \frac{-xf'(v)\chi(v)}{\delta(v)c'(q)} \tag{23}$$

Since quality explodes by Proposition 4, so too does marginal cost $c'(q)$ (Appendix A.4). Now, $\chi(v) \leq \chi(1) < \infty$, and $-f'(1) \leq -f'(0) < \infty$ as $f$ is convex. So $\kappa \to 0$.

## 5 Empirical Evidence via Seized and Passed Money

Our model admits expressions for the levels of seized and passed money that afford many normative insights, and positive predictions that make sense of a novel data set. We explore these below for the case of the USA denominations (except once, where we turn to the Euro). For simplicity, we proceed according to the logical topical sequence.

1. Estimating the Verification Rate. Using a steady-state approximation, the counterfeit passage into circulation balances the passed money outflow: $P[\Delta] = f(v)C[\Delta]$, and counterfeit production replenishes the outflow of seized and passed money, or $C[\Delta] \equiv S[\Delta] + P[\Delta]$. The *counterfeit-passed ratio* is therefore

$$C[\Delta]/P[\Delta] = 1/f(v[\Delta]) \tag{24}$$

Accordingly, the seized-counterfeit ratio bounds verification: $v \leq 1 - f(v[\Delta]) = 1 - P[\Delta]/C[\Delta] = S[\Delta]/C[\Delta]$. This ratio has varied from $0.19 = .23/1.23$ for the \$1 note to $0.55 = 1.2/2.2$ for the \$100 note (see Figure 8 and its caption).[17]

2. The counterfeit-passed ratio rises in the note, but far less than proportionately so. This unambiguous trend holds in the U.S. denominations \$1, \$5, ..., \$100 over the samples of millions of passed and seized notes, as well as in Canada's six paper denominations.[18] For instance, slopes (elasticities) in this log-log

[17] Barring highly varying police seizure efficacy across notes, the verification rate is nonconstant, refuting the assumption that verifiers observe fixed authenticity signals — as in Williamson (2002).

[18] For Canada, from 1980-2005, the counterfeit-passed ratios are 0.095, 0.145, 0.161, 0.184, 0.202, and 3.054 for (respectively) \$5, \$10, \$20, \$50, \$100, and \$1000. The \$1000 note was ended in 2000.

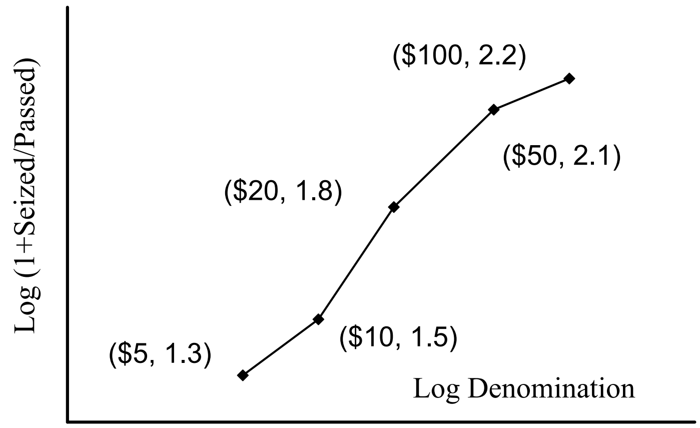


Figure 8: **Counterfeit Over Passed, Across Denominations.** These are the counterfeit-passed ratios, labeled by pairs $(\Delta, C(\Delta)/P(\Delta))$, averaged over 1995–2008, for non-Colombian counterfeits in the USA. *The sample includes almost ten million passed notes, and about half as many seized notes.* We do not have data for this time span for the \$1 note; it averages 1.23 for the years 1998 and 2005–8. For this log-log graph, slopes are elasticities — positive and below one.

diagram of Figure 8 are positive but far below 1, averaging 0.18 between \$5 and \$100.

The verification rate rises in the note $\Delta$ by Proposition 4, and thus the passing fraction $f(v[\Delta])$ falls. But $1/f(v[\Delta])$ does not rise in proportion to $\Delta$, for quality $q[\Delta]$ rises.

$$\frac{c(q[\Delta]) + \ell}{x\Delta} = f(v[\Delta]) \tag{25}$$

All told, the counterfeit-passed ratio (24) has elasticity $\mathcal{E}_\Delta(C/P) = -\mathcal{E}_\Delta(f) \in (0, 1)$.

In other words, with fixed quality, zero profits (5) would require that the passing fraction scale by half moving from \$5 to \$10 to \$20. The denomination elasticity would then be $-1$. But quality optimally rises in the note, thereby increasing costs. So the passing fraction falls less than inversely to the note, and its elasticity exceeds $-1$.

3. THE COUNTERFEIT-PASSED RATIO HAS GREATLY FALLEN OVER TIME.[19] There has been a sea change in the seized and passed money since 1980. Historically, seized vastly exceeded passed counterfeit money (Figure 9). But starting in 1986, and accelerating in 1995, the counterfeit-passed ratio began to tumble. Nowadays, most counterfeit money is passed,[20] as the passing fraction has skyrocketed roughly from

[19] We justify our comparative statics using comparison of steady-states, which is often done in many settings, like growth theory. Recently, eg., Acemoglu and Hawkins (2010) do this for a search model. The simple fact is that comparison of steady-states invariably secures the right signs of changes.

[20] The Annual Reports of the USSS supplied earlier data, and the Secret Service itself gave us more recent data. Seized is a more volatile series, as seen in Figure 9, as it owes to random, maybe large, counterfeiting discoveries, and is also contemporaneous counterfeit money. By contrast, passed money is twice averaged: It has been found by thousands of individuals, and may have long been circulating.

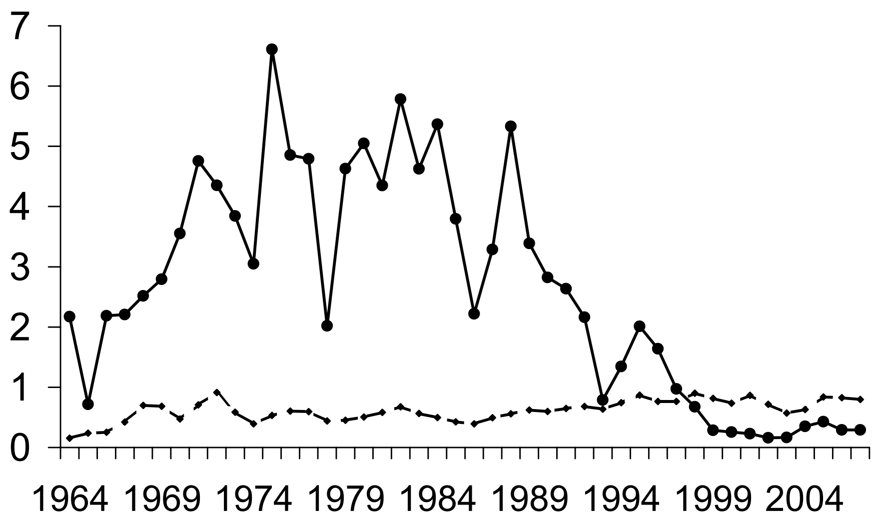


Figure 9: **USA Passed and Seized, 1964–2007.** The units here are per thousand dollars of circulation across all denominations. The solid line represents seizures, and the dashed line passed money. From 1970–85, the vast majority of counterfeit money (about 90%) was seized. The reverse holds (about 20%) for 2000–2007. Two down-spikes in 1986 and 1996 roughly correspond to the years of technological shifts.

10% to 80%. Table 1 documents a digital counterfeiting revolution that explains this massive swing using our theory: For the verification rate falls when the counterfeiting cost falls by Proposition 3, and with note value-eroding inflation, by Proposition 4.

4. COUNTERFEIT QUALITY RISES IN THE NOTE. As Table 1 depicts, the fraction of cheaper digitally-produced counterfeits falls in the note, i.e. quality rises, just as Proposition 4 predicts. In lieu of digital production, Judson and Porter (2003) find that 73.6% of passed \$100 bills were high quality *circulars*, but only 19.2% of \$50 bills, and less than 3% of all others. For instance, the "Supernote" (circular 14342) is the highest quality counterfeit on record. North Korea made this highly deceptive counterfeit \$100 note from bleached \$1 notes, with the intaglio printing process used by the Bureau of Engraving and Printing, and so is missed by commercial banks.

5. THE STREET PRICE OF COUNTERFEIT NOTES. The "street price" of counterfeit notes is at most the average costs. Expressions (24) and (25) imply that average costs equal the note times the passing fraction, and thus the counterfeit-passed ratio:[21]

$$\text{street price} \leq \text{average costs} = f(v[\Delta])\Delta = \frac{P[\Delta]}{S[\Delta] + P[\Delta]}$$

[21]We thank Pierre Duguay for this insight; he said the predicted street prices are realistic. In one recent American case, a Mexican counterfeiting ring discovered this year sold counterfeit \$100 notes at 18% of face value to distributors, who then resold the counterfeit notes for 25–40% of face value. The money was transported across the border by women couriers, carrying the money.

| Note | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | avg. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \$5 | .250 | .306 | .807 | .851 | .962 | .972 | .986 | .980 | .974 | .981 | .901 |
| \$10 | .041 | .095 | .506 | .851 | .908 | .911 | .961 | .963 | .971 | .978 | .756 |
| \$20 | .139 | .295 | .619 | .882 | .902 | .926 | .929 | .961 | .974 | .983 | .823 |
| \$50 | .276 | .335 | .546 | .768 | .777 | .854 | .911 | .828 | .822 | .857 | .755 |
| \$100 | .059 | .066 | .147 | .263 | .239 | .314 | .267 | .251 | .307 | .399 | .250 |

Table 1: **Fraction of Notes Digitally Produced, 1995–2004.** This Secret Service data encompasses all 8,541,972 passed and 5,594,062 seized counterfeit notes in the USA, 1995–2004. Observe $(a)$ the growth of inexpensive digital methods of production, and $(b)$ lower denomination notes are more often digitally produced.

The implied US street price ceilings for the \$5, \$10, \$20, \$50, and \$100 notes can be computed from Figure 8, to get \$3.37, \$5.95, \$9.30, \$19.20, \$35.70, respectively.

6. ESTIMATING THE TRUE COUNTERFEITING RATE. The counterfeiting rate $\kappa[\Delta]$ is unobserved, and its observable manifestation, the passed-rate $p[\Delta] = \delta[\Delta]\kappa[\Delta]$, is an imperfect proxy. Since the discovery rate $\delta(v[\Delta])$ increases in the note $\Delta$ by Proposition 4, so too is the ratio $p[\Delta]/\kappa[\Delta]$. *The passed rate increasingly understates the actual counterfeiting problem at lower notes,* and so the peak counterfeiting rate occurs at a lower note than the peak passed rate. For a specific estimate, we approximate the bank verification rate by the equilibrium rate $v$, then $\delta(v[\Delta]) = \beta\alpha + (1-\beta)v[\Delta] \sim v[\Delta] \leq S[\Delta]/C[\Delta]$. The implied lower bounds on the ratios of true counterfeit rates to passed rates for the notes \$5 through \$100 are $4.3, 3, 2.2, 1.9, 1.8$. Eg., using the last factor, we estimate that the true domestic counterfeiting rate for the \$100 note has been at least $1.8 \times 100.81 \approx 181$ per 100,000 notes (see Figure 10).

7. ESTIMATING THE MARGINAL VERIFICATION COSTS. Substituting the expression for the passed rate into the hot potato game equilibrium equation (14):

$$p[\Delta] = \delta[\Delta]\kappa[\Delta] = \frac{q[\Delta]\chi'(v[\Delta])}{\Delta} = \frac{\text{marginal verification cost}}{\text{denomination}} \tag{26}$$

The implied verification costs in (26) are easily measured by $\Delta p[\Delta]$. These are quite miniscule even for the highest notes. The passed rate is at most 1 per 10,000 *annually*. Suppose the \$100 note transacts at least four times per year. Then the passed rate $p[\Delta]$ is at most 1 in 40,000, and marginal verification costs are at most \$100/40,000, or *one quarter penny per note*. Yet such tiny verification costs drive our theory. Surprisingly, incentives explain behavior even when costs are very small.

8. THE PASSED RATE VANISHES FOR LOW NOTES, AND DROPS FOR LARGE NOTES. The first is strongly predictive of the U.S. dollar and Euro data, and obtains

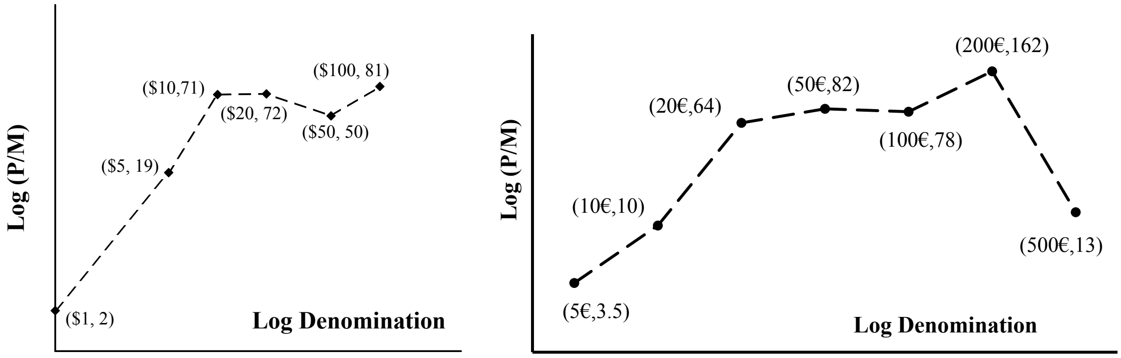


Figure 10: **Passed Over Circulation, Dollar and Euro.** At left are the average ratios of passed domestic counterfeit notes to the (June) circulation of the $1 note for 1990-96, 1998, 2005–7, and the $5, $10, $20, $50, $100 notes for 1990–2007, all scaled by $10^6$. Euro data is at right. The data points are labeled as $(\Delta, P(\Delta)/M(\Delta))$.

without any of our stronger cost assumptions. Figure 10 plots at the left the average fractions $p[\Delta]$ of passed notes by denomination over a long time horizon.[22] The possibility highlighted in Proposition 5 of a falling passed rate at sufficiently high notes is not realized in the US data. Yet the Euro offers two higher value notes, and *the passed rate of the 500 Euro note is less than one twelfth that of the 200 Euro note* in Figure 10.

Our theory assumes that notes trade hands once per "period". Unlike with the counterfeit-passed ratio, the passed rate is a flow over a stock, which skews the per period meaning. Yet the velocity is intuitively falling in the note.[23] The higher the note, fewer transaction opportunities a year represents. Interpreting annualized passed data in this light, the relevant "per transaction passed rate" rises from $50 to $100 note, and might always rise in the denomination. Yet this falling velocity surely cannot account for the more than twelve-fold drop in the passed rate at the 500 Euro note.

9. COMPARED TO PASSED RATES, THE FRB FINDS PROPORTIONATELY FEWER COUNTERFEITS OF HIGHER THAN LOWER NOTES — UNTIL THE $100 BILL. The banking sector offers a reverse test of our model, since counterfeit money hitting banks *missed* earlier detection. Commercial banks transfer damaged or unneeded notes to the Federal Reserve Banks (FRB), who find about $5–10 million of fake money yearly. The FRB computes its own internal passed money rates, and we determined that for the years 1998, 2002, and 2005 with available data, the ratio of the internal FRB and

[22]These ratios per million have averaged $1.96$, $19.46$, $71.21$, $72.03$, $49.94$, $81.43$, respectively. The common claim that the most counterfeited note domestically on an annualized basis is the $20 is false over our time span. Accounting for the higher velocity of the $20, on a per-transaction basis (the relevant measure for decision-making), the $100 note is unambiguously the most counterfeited note.

[23]Lower denomination notes wear out faster, surely due to a higher velocity. Longevity estimates by the Federal Reserve Bank of NY [www.newyorkfed.org/aboutthefed/fedpoint/fed01.html] are 1.8, 1.3, 1.5, 2, 4.6, and 7.4 months, respectively, for $1,…,$100. FRB (2003) has close longevity estimates.

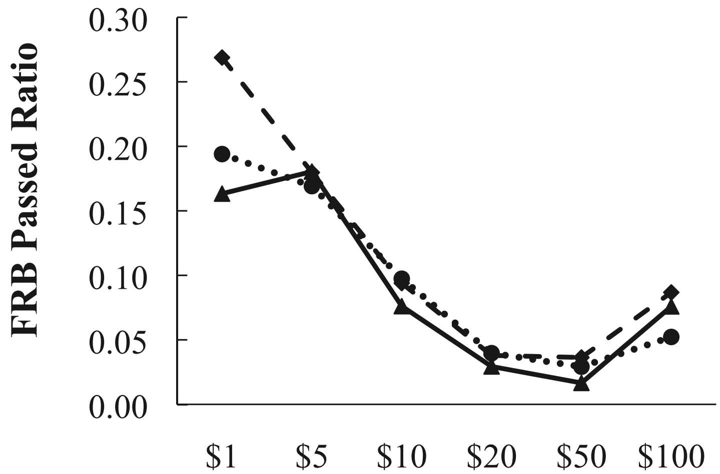


Figure 11: **Internal FRB / Average Passed Rate.** These are the ratios of internal FRB and average passed rates in 1998 (dashed), 2002 (dotted), and 2005 (solid).

passed rates monotonically falls from \$1 through \$50 (Figure 11). This general reverse monotonicity should appear surprising, as the lowest notes are the poorest quality counterfeits, and so easiest for innocent verifiers to catch before deposit into a bank.[24] As seen earlier for our example depicted in Figure 4, we make sense of this puzzle.

Assume that commercial banks transfer a fraction $\phi[\Delta]$ of $\Delta$ notes to the FRB each period. A fake note lands at an FRB if the following sequence of independent events transpires: it is fake, is deposited into a bank, it is not found, and then it is transferred to an FRB. With its perfect counterfeit detection, any counterfeit buck stops at an FRB. The *internal FRB passed rate* is the counterfeit fraction of transferred notes:

$$\zeta = \frac{\text{fake notes hitting FRB}}{\text{total notes hitting FRB}} = \frac{\kappa\beta(1-\alpha)\phi}{\beta(1-\kappa)\phi + \kappa\beta(1-\alpha)\phi} \approx \kappa(1-\alpha)$$

The approximation is accurate within $\kappa \approx 0.0001$, or 0.01%. While this depends on the unobserved counterfeiting rate, its quotient with the passed rate (2) — the *FRB ratio* — does not:

$$\frac{\zeta[\Delta]}{p[\Delta]} \approx \frac{1-\alpha}{\delta[\Delta]} \tag{27}$$

With constant $\alpha$, the discovery rate $\delta[\Delta]$ rises in $\delta$, since $v'[\Delta] > 0$, by Proposition 4. So our theory predicts a monotonically falling FRB ratio, almost matching Figure 11.

The FRB ratio turns up at the \$100 bill. Our simplifying assumption of constant $\alpha$ is most strained here, since \$100 note is renowned for high quality fakes. Its bank detection chance may be sufficiently lower, $\alpha[100] < \alpha[50]$, that the FRB ratio rises.

[24]See Table 6.1 in Treasury (2000), Table 6.3 in Treasury (2003), and Table 5 in Judson and Porter (2003). See also Treasury (2006).

# 6 Conclusion

Counterfeiting is a crime that induces two linked conflicts: first, counterfeiters against verifiers and law enforcement, and then verifiers against each other. The focus on the police-counterfeiter conflict in the small extant literature bipasses the key role of the second conflict in explaining passed counterfeit money. In fact, seized money has fallen from 90% to 10% of counterfeit money between the 1970s until the late 1990s.

We develop a behavioral strategic theory with a continuum of players, integrating new analyses of both passed and seized conflicts. It is centered on an assumption that is new in the money literature: endogenous verification. In the *cat and mouse game*, bad guys wish to profitably forge counterfeits that pass for the real thing. Higher quality fakes cost more, but better deceive good guys, and so pass more often. In the *hot potato game*, good guys try to avoid being saddled with bad money. This game is a new use of supermodular games in economics.[25] The counterfeiting rate emerges as a market-clearing chance justifying verification efforts. This is reminiscent of Knowles, Persico, and Todd (2001), where a police search chance incentivizes a decision to carry drugs. But for us, good guys are pitted against each other, and the effort choice is not binary, and only co-determines losses with the counterfeiting rate.

Economic models ideally clarify causation. Here, the verification efforts of good guys and the entry and quality choices of counterfeiters equilibrate in two interacting large games. Good guys' verification efforts affect both criminals in the cat and mouse game, and other good guys in the hot potato game, but criminals only affect good guys. Given homogeneous bad guys, and fixed counterfeit quantity, our model is recursive, and so tractable: The counterfeiting rate is solely fixed by counterfeiters' entry, and so is a free variable, computable after solving the cat and mouse game. Our graphical framework easily captures changes in notes, counterfeiting costs, or verification ease.

In our theory, the verification rate emerges in a struggle between verification effort and counterfeit quality. Our functional form crucially ensures diminishing returns to expenditures by both good and bad guys in this conflict. As attested by a novel data set we provide, our model is parsimonious: For with fixed verification effort, counterfeit optimization would ensure that quality rises in the note; thus, the verification rate would fall in the note, as would the counterfeit-passed ratio — contrary to data. If we instead tied the counterfeiters' hands and fixed the quality, then to ensure zero profits, the passing fraction would move inversely to the note, and the counterfeit-

[25]The search-matching macroeconomics model of Diamond (1982) is supermodular in production costs. Diamond studies multiple equilibria, but ours has a unique equilibrium forced by an entry game.

passed ratio would rise in proportion to the note. In fact, it rises *much* slower. And loosely, endogenous verification effort explains why counterfeiting rate rises at low notes, while endogenous counterfeit quality justifies its eventual decline.

Existing work on counterfeiting is predicated on a general equilibrium value of money. Our point of departure is thus to replace a priced asset with a new decision margin — verification effort. This can be thought of as a metaphorical currency itself, whose price clears an "implicit" market. If we *added general equilibrium effects* to our model, they would be second order and add little or nothing to our explanations of passed and seized money at the current counterfeiting rates, for they would only discount prices infinitesimally. On the other hand, adding endogenous verification to general equilibrium effects surely makes sense for counterfeit goods, where discounting of goods of dubious authenticity can be substantial (Grossman and Shapiro, 1988).

Our cat and mouse game should prove of independent interest as a model of other variable intensity deception games like warfare and tax evasion. Likewise, our hot potato game offers a tractable inroad for analyzing other passing games, and in fact, we are now applying it to model contagious diseases.

Future work can consider endogenous quantity in the cat and mouse game. We omitted it, as it immensely complicates the theory, and isn't needed to explain our data.

# A Appendix: Omitted Proofs

## A.1 Optimal Quality and Zero Profit Curves: Proof of Lemma 3

**Claim 1 (Strict SOC)** *The second order condition at an optimum is strict:* $\Pi_{qq} < 0$.

*Proof of Claim:* The SOC for maximizing $\Pi(e, q, \Delta)$ is locally necessary:

$$\Pi_{qq} = \Delta x f' V_{qq} + \Delta x f'' V_q^2 - c'' \leq 0 \tag{28}$$

The derivative of the quality first order condition (6) in the note $\Delta$ yields:

$$0 \;=\; \Pi_{qq}\dot{q} + \Pi_{qe}\dot{e} + \Pi_{q\Delta} \tag{29}$$

For a contradiction, assume $\Pi_{qq} = 0$. Then (22) and (29) must be linearly dependent. Since $\Pi_{qe} = \Delta\left(f' V_{qe} + f'' V_e V_q\right)$ and $\Pi_{q\Delta} = f' V_q$, then exploiting Lemmas 1 and 2:

$$\frac{f' V_{qe} + f'' V_e V_q}{f' V_q} = \frac{f' V_e}{f} \qquad \Rightarrow \qquad 0 < \frac{V_{qe}}{V_q} = \left(\frac{f'}{f} - \frac{f''}{f'}\right) V_e$$

This is a contradiction, for $V_e > 0$ and $f'/f < f''/f'$ by strict log-concavity of $f$. □

We now argue that the SOC reduces to $G'(v)T'(q) > F'(v)U'(q)$. Since the respective slopes of the $\bar{\Pi}$ and $Q^*$ curves are $T'(q)/F'(v)$ and $U'(q)/G'(v)$, this says that *if $Q^*$ is negatively sloped, then it is absolutely steeper than $\bar{\Pi}$ — in other words, $G'(v) < 0$ implies $T'(q)/F'(v) > U'(q)/G'(v)$*. Reformulating the SOC (28), we find:

$$\begin{aligned} 0 > \Pi_{qq}(v,q,\Delta) &= c'\left[\frac{V_{qq}}{V_q} + \frac{f''}{f'}V_q\right] - c''(q) \\ &= c'\left[\frac{-1}{q}\left(1 + \frac{\chi}{\chi'}\left(\frac{\chi'}{\chi} - \frac{\chi''}{\chi'}\right)\right) - \frac{f''}{f'}\left(\frac{\chi}{q\chi'}\right)\right] - c''(q) \end{aligned} \tag{30}$$

by (5) and Lemmas 1–2. Taking the quotient of (6) and (5), using $V_q = -\chi/(q\chi')$, we find:

$$\frac{f'}{f} = -\frac{qc'(q)}{c(q)+\ell}\frac{\chi'}{\chi} \qquad \Rightarrow \qquad \frac{q\chi'}{\chi} = -F'(v)/T'(q) \tag{31}$$

That $G'(v)T'(q) > F'(v)U'(q)$ follows from (8), (30), and (31), for they yield

$$\begin{aligned} F'(v) - G'(v) = \frac{\chi''}{\chi'} - \frac{\chi'}{\chi} + \frac{f'}{f} - \frac{f''}{f'} &< \frac{\chi'}{\chi}\frac{qc''(q)}{c'(q)} + \frac{f'}{f} + \frac{\chi'}{\chi} \\ &= \frac{q\chi'}{\chi}\left(\frac{qc''(q)+c'(q)}{qc'(q)} - \frac{c'(q)}{c(q)+\ell}\right) \\ &= -\frac{F'(v)}{T'(q)}\left(U'(q) - T'(q)\right) \end{aligned}$$

## A.2 Constant Counterfeiting Rate Curve Slope: Proof of Lemma 4

Differentiating the log of (14), the proportionate change in the counterfeiting rate is

$$\frac{d\kappa}{\kappa} = \frac{dq}{q} + \left(\frac{v\chi''(v)}{\chi'(v)} - \frac{v\rho'(v)}{\rho(v)}\right)\frac{dv}{v} - \frac{d\Delta}{\Delta}$$

Holding $\kappa$ and $\Delta$ fixed, the change in quality along the $\bar{K}$ locus obeys

$$\left.\frac{dq}{q}\right|_{\bar{K}} = \left(\frac{v\rho'(v)}{\rho} - \frac{v\chi''}{\chi'(v)}\right)\frac{dv}{v} \tag{32}$$

Along the $\bar{\Pi}$ locus, the change in quality obeys

$$\left.\frac{dq}{q}\right|_{\bar{\Pi}} = \frac{\Delta x v f'(v)}{qc'(q)}\frac{dv}{v} = -\frac{v\chi'(v)}{\chi(v)}\frac{dv}{v} \tag{33}$$

after substituting (7). By log-concavity of $\chi$, we see that (32) strictly exceeds (33). Thus, the slope of $\bar{K}$ exceeds that of $\bar{\Pi}$, but we now show that it is less than the slope of $Q^*$. This is clear when $Q^*$ has positive slope. Indeed, log-differentiating (7):

$$\left(1 + \frac{qc''(q)}{c'(q)}\right) \frac{dq}{q}\bigg|_{Q^*} = \left(\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} - \frac{v\chi''(v)}{\chi'(v)}\right) \frac{dv}{v}$$

When $Q^*$ has negative slope, it is steeper than $\bar{K}$ since $c''(q)/c'(q) \geq 0$ and by (20):

$$\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} \geq 1 > \frac{v\rho'(v)}{\rho(v)}$$

## A.3 Existence and Uniqueness: Proof of Theorem 1

The existence proof proceeds in $(q, v)$ space, and the uniqueness proof in $(e, q)$ space.

STEP 1: EXISTENCE FOR $\Delta > \underline{\Delta}$. In this case, we exhibit a solution to the zero profit and optimal quality equations (5) and (7), at the left of Figure 2. Since $f' < 0 < c'$, the zero profit equation (5) implicitly defines a continuous and decreasing function $q = Q_0(v)$. We must have $Q_0(0) > 0$, because $c(Q_0(0)) = \Delta x f(0) - \ell > 0$ when $\Delta > \underline{\Delta}$. Since $\Delta x f(0) > \ell$ and $f(1) = 0$, we may choose $\hat{v} < 1$ so that $\Delta x f(\hat{v}) = \ell$. Then $Q_0(v) \to 0$ as $v \to \hat{v}$. By the Implicit Function Theorem (IFT), because $qc'(q)$ is strictly increasing, the quality FOC (7) implicitly defines a differentiable function $q = Q_1(v)$. Since the limit $v\chi'(v)/\chi(v)$ exists and is positive as $v \to 0$, both sides of (7) vanish, and so $Q_1(0) = 0$. Easily, (7) is positive at $v = \hat{v}$, and thus $Q_1(\hat{v}) > 0$. Given $Q_1(0) = 0 < Q_0(0)$ and $Q_1(\hat{v}) > 0 = Q_0(\hat{v})$, the Intermediate Value Theorem yields $v \in (0, \hat{v})$ with $Q_0(v) = Q_1(v)$. But then $0 < v < 1$ and $0 < q = Q_1(v) = Q_0(v) < \infty$. So $\kappa > 0$ by (13). Finally, since $Q_0(v), Q_1(v)$ are differentiable in $\Delta$, so is $q[\Delta]$ and $v[\Delta]$. (This conclusion also follows by applying the IFT to the system (5) and (7).)

STEP 2: UNIQUENESS. Assume two equilibria $(e_1, q_1)$ and $(e_2, q_2)$ for a note $\Delta$. If $q_1 = q_2$ then $e_1 = e_2$, since profits fall in effort. Assume $q_1 < q_2$. Consider how profits $\Pi(e, q)$ change from $(e_1, q_1)$ to $(e_2, q_2)$ along the smooth optimal quality curve $\mathcal{Q}^* = \{(e, q) : \Pi_q(e, q) = 0, q_1 \leq q \leq q_2\}$. A line integral yields:

$$0 - 0 = \Pi(e_2, q_2) - \Pi(e_1, q_1) = \int_{\mathcal{Q}^*} (\Pi_e, \Pi_q) \cdot (de, dq) = \int_{e_1}^{e_2} \Pi_e de \qquad \square$$

Since $\Pi_e < 0$, we must have $e_1 = e_2$. Then $v_1 > v_2$, and thus profits are higher moving from $(e_1, q_1)$ to $(e_2, q_2)$, which is a contradiction. (That $0 < v_i < 1$ follows since $\bar{\Pi}$ has positive intercepts and $Q^*$ rises from the origin.)

## A.4 The Peak Counterfeiting Rate: Proof of Theorem 2

STEP 1. Modifying the counterfeiting rate formula (23) for zero profits (5), we find:

$$\kappa(v) = -\frac{xf'(v)\chi(v)}{\delta(v)c'(q)} = \frac{xf(v)\chi'(v)}{\delta(v)}\frac{q}{c(q)+\ell}$$

Since $(c(q)+\ell)/q$ is minimized when $qc'(q) - c(q) = \ell$, where it equals the marginal cost $c'(q)$, and since $\delta(v) \geq (1-\beta)v$ and $\chi'(v)/v$ is weakly increasing, we have:

$$\kappa(v) \leq \frac{xf(0)\chi'(1)}{(1-\beta)c'(q)} \tag{34}$$

STEP 2: A LOWER BOUND ON THE COST AND MARGINAL COST OF QUALITY. Since $qc'(q)/c(q)$ is weakly increasing by (3), we have $c'(q)/c(q) \geq \eta/q$ if $q > 0$. Integrating this inequality on $[1, q)$ yields $\log c(q) - \log c(1) \geq \log q^\eta$ if $q \geq 1$. So $c(q) \geq c(1)q^\eta$. Given $c'(q)/c(q) \geq \eta/q$, we have $c'(q) \geq c(1)\eta q^{\eta-1}$.

STEP 3: A FIXED UPPER BOUND FOR THE COUNTERFEITING RATE. Define producer surplus $\pi(q) \equiv qc'(q) - c(q)$. Let $Q(\ell)$ be the quality that yields producer surplus $\pi(Q(\ell)) \equiv \ell$. Then by the cost bounds in Step 2, we deduce

$$\ell = \pi(Q(\ell)) = Q(\ell)c'(Q(\ell)) - c(Q(\ell)) \geq c(1)\eta Q(\ell)^\eta - c(1)Q(\ell)^\eta$$

This implies the following lower bound that allows us to simplify (34):

$$c'(Q(\ell)) > \frac{\pi(Q(\ell))}{Q(\ell)} \geq \frac{\ell}{Q(\ell)} \geq \frac{\ell}{(\ell/c(1)(\eta+1))^{1/\eta}} \geq c(1)^{1/\eta}\ell^{1-1/\eta}/\sqrt{3}$$

since $(1+\eta)^{1/\eta}$ is monotone decreasing in $\eta > 1$, and we assumed $\eta \geq 2$.

## A.5 Note Comparative Statics: Rest of Proof of Proposition 4

A. QUALITY EXPLODES AT LARGE NOTES. Since $v$ increases in $\Delta$ by Theorem 4, $\chi(v)/\chi'(v)$ is nondecreasing by log-concavity of $\chi$, and $-f'(v) \geq -f'(0) > 0$, the right side of the FOC (7) explodes as $\Delta \uparrow \infty$. So $qc'(q) \uparrow \infty$, and quality $q \to \infty$. □

B. INITIAL QUALITY, EFFORT, AND VERIFICATION.

By continuity of (5) and (6), the limits as $\Delta \downarrow \underline{\Delta}$ of $e$ and $q$, and so $v$, exist.

$(a)$ *Quality.* If $\underline{q} = \lim_{n\to\infty} q[\Delta_n] > 0$ for some subsequence $\Delta_n \downarrow \underline{\Delta}$, then $\Pi(\underline{q}, \underline{v}, \Delta) = \Delta x f(\underline{v}) - c(\underline{q}) - \ell \leq \Delta x f(0) - \ell < 0$. But then counterfeiters earn negative profits for $\Delta_n$ near $\underline{\Delta}$, which is impossible in equilibrium. So $\underline{q} = 0$.

$(b)$ *Effort.* If any limit $\underline{e} = \lim_{n\to\infty} e[\Delta_n] > 0$ as $\Delta_n \downarrow \underline{\Delta}$, then $\chi(v[\Delta_n]) = e[\Delta_n]/q[\Delta_n]$ must explode as $n \to \infty$. This is impossible because $\log \chi$ is concave.

$(c)$ *Verification.* Let $\underline{v} = \lim_{n\to\infty} v[\Delta_n] > 0$ for a sequence $\Delta_n \downarrow \underline{\Delta}$. Then

$$\lim_{n\to\infty} \Pi_q(q[\Delta_n], e[\Delta_n], \Delta_n) = -f'(\underline{v}) \cdot \frac{\chi(\underline{v})}{\underline{v}\chi'(\underline{v})} \cdot \underline{\Delta} x \cdot \lim_{n\to\infty} \frac{v[\Delta_n]}{q[\Delta_n]} - \lim_{n\to\infty} c'(q[\Delta_n]) \quad (35)$$

Since $-f'(\underline{v}) > 0$, while $q[\Delta] \to 0$ provided $\Delta \downarrow \underline{\Delta}$ by part $(b)$, and $c'(0) < \infty$, the right side of (35) explodes if $\underline{v} > 0$, contrary to the quality FOC (7). So $\underline{v} = 0$. □

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