

# Rational Asset Pricing Bubbles and Debt Constraints\*

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**Abstract:** Rational price bubble arises when the price of an asset exceeds the asset's fundamental value, that is, the present value of future dividend payments. The important result of Santos and Woodford (1997) says that price bubbles cannot exist in equilibrium in the standard dynamic asset pricing model with rational agents facing borrowing constraints as long as assets are in strictly positive supply and the present value of total future resources is finite. This paper explores the possibility of asset price bubbles under endogenous debt constraints induced by limited enforcement of debt repayment. Equilibria with endogenous debt constraints are prone to have infinite present value of total resources. We show that asset price bubble are likely to exist in such equilibria. Further, we demonstrate that there always exist equilibria with price bubbles on assets in zero supply. We discuss alternative concepts of price bubbles including the speculative bubble.

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## 1. Introduction

In times of instability in financial markets, news of price bubbles are frequent. The Global Financial Crisis of 2007-2009 provides several examples: real-estate market in the U.S, mortgage-backed securities, etc. Price bubble arises when the price of an asset exceeds the asset's fundamental value. As the notion of fundamental value invites many different interpretations, news of price bubbles can often be ascribed to misjudgement of the fundamental value. Economists generally agree though that the "dot-com" bubble of 1998-2000 and the Japanese stock market bubble of the 1980s were genuine price bubbles. Internet stock prices reached staggering levels in the U.S. in 2000. Nasdaq composite index rose by 200 % between 1998 and 2000. Returns on some internet stocks over this period were nearly 1000 %. Then, the market crashed down to virtually no value. Asset prices in Japan rose sharply in the 1980s. The Nikkei index soared by more than 400 % over the decade. Market capitalization of Japanese stocks exceeded the value of U.S. stock market in 1990. By 1991 prices plummeted down to levels from the early 1980s.

Financial economic theory does not provide a satisfactory answer to the question of whether and how price bubble can arise in asset markets. The theory of rational asset pricing bubbles defines the fundamental value of an asset as the present value of future dividend payments. The main result of the seminal paper by Manuel Santos and Michael Woodford (1997) says that price bubbles cannot exist in equilibrium in the standard dynamic asset pricing model as long as assets are in strictly positive supply and the present value of total future resources over the infinite time is finite. As most assets are believed to be in strictly positive supply and equilibria with infinite present value of aggregate resources are usually considered exotic, the Santos-Woodford result is interpreted as saying that price bubbles are unlikely to happen.

The no-bubble theorem of Santos and Woodford (1997) has been established for a particular type of constraint on agents' portfolio holdings - the borrowing constraint which restricts the amount of wealth an agent can borrow on a portfolio at any date. The primary role of borrowing constraints in dynamic asset markets is to prevent agents from engaging in Ponzi schemes. There are many alternative portfolio constraints that could be considered: debt (or solvency) constraints, col-

lateral constraints, short sales constraints, etc. This paper explores the possibility of asset price bubbles in dynamic asset markets under debt constraints. Debt constraints restrict the amount of debt an agent can carry on a portfolio at every date. Our main focus is on endogenous (or self-enforcing) debt constraints induced by limited commitment to market transactions (see Alvarez and Jermann (2000), Hellwig and Lorenzoni (2009), Azariadis and Kaas (2008), V. F. Martins-da-Rocha and Y. Vailakis (2011), and others). If default on payoff of a portfolio is permitted and has precisely specified consequences, self-enforcing debt constraints are defined by a sequence of debt bounds such that an agent is unwilling to default even if his indebtedness is at the maximum allowed level.

We show that the no-bubble theorem extends to debt constraints including endogenous debt constraints. The peculiar feature of equilibria with endogenous debt constraints is that they often give rise to infinite present value of the aggregate endowment, or “low interest rates.” This is precisely when the sufficient condition for non-existence of equilibrium price bubbles on assets in strictly positive supply is violated. Our main result, Theorem 3, shows that if there is an equilibrium with endogenous debt constraints in which debt bounds are non-zero, then price bubbles may be “injected” on assets in strictly positive supply and present value of the aggregate endowment must be infinite. The term injecting means that a suitably chosen sequence of positive price bubbles can be added to equilibrium asset prices so that equilibrium be preserved with unchanged consumption plans. The idea of injecting bubbles is due to Kocherlakota (2008) whose results we extend and clarify. Related results can be found in Bidian and Bejan (2010). We provide two examples of equilibria in asset markets under endogenous debt constraints with price bubbles on assets in strictly positive supply. One of the examples is the classical example of price bubble on zero-dividend asset (i.e., fiat money) in strictly positive supply due to Bewley (1980) (see also Kocherlakota (1992)). We show that debt bounds in this example are self enforcing when the punishment for default is exclusion from further borrowing. The second example is a variation of the leading example in Hellwig and Lorenzoni (2009).

We show in Theorem 4 that there always exist equilibria with price bubbles on infinitely-lived assets that are in zero supply. There is a multiplicity of equilibria, and some equilibrium prices have bubbles.

An alternative notion of price bubble is the speculative bubble. It relies on a different notion of the fundamental value of dividends. Instead of the present value of future dividend payments, it is the marginal valuation of those payments by an agent whose such valuation is the highest. In presence of portfolio constraints and infinite-time horizon, the two notions may be different. Harrison and Kreps (1978) were the first to explore the possibility of speculative bubbles in equilibrium under short sales constraints and heterogeneous beliefs. The term speculative bubble has only recently been introduced by Scheinkman and Xiong (2003). Here, we extend the notion of speculative bubbles to debt constraints. We point out in Section 7 that speculative bubbles are in no way special to heterogeneous beliefs. Their existence does not require any of the restrictive conditions of the no-bubble theorem for rational asset price bubbles. Ofek and Richardson (2005) attributed the dot-com bubble to heterogeneous beliefs and short-sale constraints in a way that resembles speculative bubbles.

The literature on asset price bubbles is vast and diverse. Gilles and LeRoy (1992) define price bubbles as manifestation of the lack of countable additivity of pricing. Araujo et al (2009) explore the possibility of rational price bubbles when asset trades are collateralized by durable goods. Bottazzi et al (2011) study bubbles under repo trading.

## 2. Dynamic Asset Markets with Debt Constraints

Time is discrete with infinite horizon and indexed by  $t = 0, 1, \dots$ . Uncertainty is described by a set  $\mathcal{S}$  of states of the world and an increasing sequence of finite partitions  $\{\mathcal{F}_t\}_{t=0}^{\infty}$  of  $\mathcal{S}$ . The partition  $\mathcal{F}_t$  specifies sets of states that can be verified by the information available at date  $t$ . An element  $s_t \in \mathcal{F}_t$  is called a date- $t$  event. The subset relation  $s_\tau \subset s_t$  for  $\tau \geq t$  indicates that event  $s_\tau$  is a successor of  $s_t$ . We use  $S^t$  to denote the set of all successors of event  $s_t$  from  $t$  to infinity, and  $S^{t+}$  the set of all successors of  $s_t$  excluding  $s_t$ . The set of one-period (date- $(t+1)$ ) successors of  $s_t$  is denoted by  $s_t^+$ . The unique one-period predecessor of  $s_t$  is denoted by  $s_t^-$ .

There is a single consumption good. A consumption plan is a scalar-valued process  $c = \{c(s_t)\}_{s_t \in \mathcal{E}}$  adapted to  $\{\mathcal{F}_t\}_{t=0}^{\infty}$ . The consumption space is the space  $C$  of all adapted processes and it can be identified with  $\mathcal{R}^{\infty}$ . There are  $I$  consumers. Each consumer  $i$  has the consumption set  $C_+$  - the cone of nonnegative processes

in  $C$  - a strictly increasing utility function  $u^i$  on  $C_+$ , and an initial endowment  $w^i \in C_+$ . The aggregate endowment  $\bar{w} \equiv \sum_i w^i$  is assumed positive, i.e.,  $\bar{w} \geq 0$ .

Asset markets consist of  $J$  infinitely-lived assets traded at every date. The dividend process  $x_j$  of asset  $j$  is adapted to  $\{\mathcal{F}_t\}_{t=0}^\infty$  and positive, i.e.,  $x(s_t) \geq 0$  for every  $s_t$ . The ex-dividend price of asset  $j$  in event  $s_t$  is denoted by  $p_j(s_t)$ . A portfolio of  $J$  assets held after trade at  $s_t$  is  $h(s_t)$ . Each agent has an initial portfolio  $\alpha_0^i$  at date 0. The supply of assets is  $\bar{\alpha}_0 \equiv \sum_i \alpha_0^i$ .

Agent  $i$  faces the following budget constraints when trading in asset markets

$$c(s_0) + p(s_0)h(s_0) = w^i(s_0) + p(s_0)\alpha_0^i, \quad (1)$$

$$c(s_t) + p(s_t)h(s_t) = w^i(s_t) + [p(s_t) + x(s_t)]h(s_t^-) \quad \forall s_t \neq s_0. \quad (2)$$

In addition to these budget constraints some restriction on portfolio holdings has to be imposed for otherwise agents could engage in a Ponzi scheme, that is, borrow any amount of wealth at any date and roll-over the debt forever. In this paper we focus on debt constraints which impose limits on debt carried on a portfolio strategy at every date and in every event. Formally, the *debt constraints* on portfolio strategy  $h$  are

$$[p(s_{t+1}) + x(s_{t+1})]h(s_t) \geq -D(s_{t+1}), \quad \forall s_{t+1} \subset s_t \quad (3)$$

for every  $s_t$ . Bounds  $D$  are assumed to be positive in every event.

Alternative constraints considered in the literature are the *borrowing constraints*,

$$p(s_t)h(s_t) \geq -B(s_t), \quad (4)$$

seen in Santos and Woodford (1997), and the *short-sales constraints*

$$h_j(s_t) \geq -b_j(s_t), \quad \forall j. \quad (5)$$

The set of consumption plans  $c$  satisfying budget constraints (1 - 2) and debt constraints (3) is denoted by  $B_0^i(p, D^i, p_0\alpha_0^i)$ . We included date-0 financial wealth  $p_0\alpha_0^i$  among the determinants of the budget set for the use later when variations of financial wealth will be considered.

An *equilibrium under debt constraints* is a price process  $p$  and consumption-portfolio allocation  $\{c^i, h^i\}_{i=1}^I$  such that (i) for each  $i$ , consumption plan  $c^i$  and

portfolio strategy  $h^i$  maximize  $u^i(c)$  over  $(c, h) \in B_0^i(p, D^i, p(s_0)\alpha_0^i)$ , and (ii) markets clear, that is

$$\sum_i h^i(s_t) = \bar{\alpha}_0, \quad \sum_i c^i(s_t) = \bar{w}(s_t) + x(s_t)\bar{\alpha}_0,$$

for all  $s_t$ . We restrict our attention throughout to equilibria with positive prices.

### 3. Arbitrage, Event Prices, and Bubbles.

Portfolio  $\hat{h}(s_t)$  is a *one-period arbitrage* in event  $s_t$  if

$$[p(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) \geq 0, \quad \forall s_{t+1} \subset s_t \quad (6)$$

and

$$p(s_t)\hat{h}(s_t) \leq 0, \quad (7)$$

with at least one strict inequality.

One-period arbitrage cannot exist in an equilibrium under debt constraints in any event.<sup>1</sup> The reason is that one-period arbitrage portfolio could be added to an agent's equilibrium portfolio without violating debt constraints. This would result in higher consumption contradicting optimality of the equilibrium portfolio. It follows from Stiemke's Lemma that there is no one-period arbitrage if and only if there exist strictly positive numbers  $q(s_t)$  for all  $s_t$  such that

$$q(s_t)p_j(s_t) = \sum_{s_{t+1} \subset s_t} q(s_{t+1})[p_j(s_{t+1}) + x_j(s_{t+1})] \quad (8)$$

for every  $s_t$  and every  $j$ . Strictly positive numbers  $q(s_t)$  satisfying (8) and normalized so that  $q(s_0) = 1$  are called *event prices*.

If asset prices admit event prices  $q \gg 0$  satisfying (8), then the present value of an asset and the bubble can be defined using any of those event prices. The *present value* of asset  $j$  in  $s_t$  under event prices  $q$  is

$$\frac{1}{q(s_t)} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \in s_t} q(s_\tau)x_j(s_\tau) \quad (9)$$

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<sup>1</sup>The same result holds for borrowing constraints, but it does not hold for short-sales constraints. Equilibrium prices under short-sales constraints may permit one-period arbitrage, see an example in LeRoy and Werner (2001).

This intuitive definition of present value can be given more solid foundations from the view point of the theory of valuation of contingent claims, see Huang (2002). Price bubble is the difference between the price and the present value of an asset. *Price bubble* on asset  $j$  at  $s_t$  is

$$\sigma_j(s_t) \equiv p_j(s_t) - \frac{1}{q(s_t)} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \in s_t} q(s_\tau) x_j(s_\tau) \quad (10)$$

If there exist multiple event prices, the price bubble may depend on the choice of event prices.<sup>2</sup> Our notation does not reflect that possibility, but we shall keep this mind. Basic properties of price bubbles are stated in the following proposition, the proof of which is standard and therefore omitted.

**Proposition 1:** *Suppose that  $p$  admits strictly positive event prices  $q$ . Then*

(i) *Price bubbles are non-negative and do not exceed asset prices,*

$$0 \leq \sigma_j(s_t) \leq p_j(s_t), \quad \forall s_t \forall j. \quad (11)$$

(ii) *If asset  $j$  is of finite maturity (that is,  $x_{jt} = 0$  for  $t \geq \tau$  for some  $\tau$ , and that asset is not traded after date  $\tau$ ), then  $\sigma_j(s_t) = 0$  for all  $s_t$ .*

(iii) *It holds*

$$q(s_t) \sigma_j(s_t) = \sum_{s_{t+1} \subset s_t} q(s_{t+1}) \sigma_j(s_{t+1}) \quad (12)$$

*for every  $s_t$  and every  $j$ .*

Property (12) is referred to as the *discounted martingale property* of bubble  $\sigma_j$  with respect to event prices  $q$ . The reason for this terminology is that if a risk-free payoff lies in the one-period asset span for every event  $s_t$ , then the discount factor  $\rho(s_t)$  can be defined as the product of one-period risk-free returns along the path of events from  $s_0$  to  $s_t$  so that event prices rescaled by the discount factor  $q(s_t)/\rho(s_t)$  become probabilities.<sup>3</sup> Property (12) says that discounted bubble  $\rho(s_t)\sigma_j(s_t)$  is a martingale with respect to these probabilities.

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<sup>2</sup>See Huang (2002).

<sup>3</sup>For details, see LeRoy and Werner (2001).

The discounted martingale property of price bubbles together with their non-negativity have strong implications on the dynamics of bubbles. It follows that price bubble can be zero in an arbitrary event if and only if it is zero in every immediate successor of that event. Thus, non-zero price bubble can exist on an asset only if date-0 bubble is non-zero. Further, if price bubble is non-zero at date 0 (the issuance date of the asset), then there must exist a sequence of events throughout the event tree such that the bubble is strictly positive in every event of that sequence.

#### 4. No-Bubble Theorems

In this section we establish sufficient conditions for non-existence of price bubbles in equilibrium and discuss their necessity. The first result concerns complete markets. Asset markets are said to be *complete* at prices  $p$  if, for every event  $s_t$ , the one-period payoff matrix  $[p_j(s_t^+) + x_j(s_t^+)]_{j \in J}$  has rank equal to the number of one-period successors of  $s_t$ . Of course, if asset markets are complete at  $p$  and  $p$  admits event prices  $q$ , then  $q$  is unique.

**Theorem 1:** *Let  $p$  be an equilibrium price system under debt constraints. Suppose that asset markets are complete at  $p$ , and let  $q$  be the unique system of strictly positive event prices. If present value of the aggregate endowment is finite,*

$$\sum_{t=1}^{\infty} \sum_{s_t \in F_t} q(s_t) \bar{w}(s_t) < \infty, \quad (13)$$

*and assets are in strictly positive supply,*

$$\bar{\alpha}_0 \gg 0, \quad (14)$$

*then price bubbles are zero.*

The hypothesis of Theorem 1 remains true for incomplete markets under an additional assumption on agents' utility functions. For a consumption plan  $c$ , let  $c(-S^t)$  denote the consumption plan for all events not lying in  $S^t$ , and  $c(S^{t+})$  the consumption plan for all events in  $S^{t+}$ . The assumption is

**(A1)** For every  $i$ , there exists  $0 \leq \gamma^i < 1$  such that

$$u^i(c^i(-S^t), c^i(s_t) + \hat{w}(s_t), \gamma^i c^i(S^{t+})) > u^i(c^i), \quad (15)$$

for every  $s_t$  and every  $c^i$  such that  $c^i \leq \hat{w}$ , where  $\hat{w}(s_t) = \bar{w}(s_t) + \bar{\alpha}_0 x(s_t)$ .

Condition (15) concerns the tradeoff in terms of utility between current consumption and consumption over the infinite future. It says that adding the aggregate (cum dividend) endowment to an agent's consumption in event  $s_t$  and scaling down her future consumption by scale-factor  $\gamma^i$  leaves the agent strictly better off. The restrictiveness of assumption (A1) lies in the requirement that factor  $\gamma^i$  is uniform over all feasible consumption plans and all events. It is important to note that condition (A1) holds for discounted (time-separable) expected utility with strictly increasing, continuous period-utility function as long as the aggregate endowment  $\hat{w}$  is bounded above and bounded away from zero.<sup>4</sup>

**Theorem 2:** *Let  $p$  be an equilibrium price system under debt constraints. Suppose that A1 holds and assets are in strictly positive supply (14). For every system of strictly positive event prices  $q$  such that present value of the aggregate endowments is finite (13), price bubbles are zero.*

Theorems 1 and 2 are extensions of the main results of Santos and Woodford (1997) from borrowing to debt constraints. We provide a proof of Theorem 2 in the Appendix.

If either one of conditions (13) or (14) is violated, there may exist price bubbles. We show in Section 6 that there always exists equilibria with price bubbles on assets in zero supply. Here, we present an example of an asset price bubble in an equilibrium with infinite present value of the aggregate endowment. The condition of infinite present value of the aggregate endowment is often referred to as *low interest rates*. The example is due to Bewley (1980) (see also Kocherlakota (1992)) and it shows an equilibrium with price bubble on a zero-dividend asset, that is, fiat money with strictly positive price.

**Example 1:** There is no uncertainty. There are two agents with utility functions

$$u^i(c) = \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

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<sup>4</sup>Yet, (A1) is not an innocuous restriction on utility functions. An example of utility function that does not satisfy (A1) in the model with no uncertainty is  $\inf_t c_t + \sum_{t=0}^{\infty} \beta^t c_t$ , see Werner (1997) and Araujo et al (2011).

where  $0 < \beta < 1$ . Their endowments are  $w_t^1 = B$  and  $w_t^2 = A$  for even dates  $t \geq 2$ , and  $w_t^1 = A$  and  $w_t^2 = B$  for odd dates  $t \geq 1$ , where  $A > B$ . Date-0 endowments will be specified later.

There is one asset that pays zero dividend at every date, that is, fiat money. Initial asset holdings are  $\alpha_0^1 = 1$  and  $\alpha_0^2 = 0$  so that the total supply is 1. Debt bounds are  $D_t = p_t$  so that agents can short sell at most one share of the asset. It will be seen in Section 5 that these debt limits are self enforcing.

There exists a stationary equilibrium with consumption plans that depend only on current endowment, strictly positive prices  $p_t$ , and debt constraint binding the agent with low endowment at every date  $t$ . Such equilibrium has consumption plans  $c_t^1 = B + \eta$  and  $w_t^2 = A - \eta$  for even dates  $t \geq 0$ , and  $c_t^1 = A - \eta$  and  $w_t^2 = B + \eta$  for odd dates  $t \geq 1$ , asset holdings  $h_t^1 = -1$  and  $h_t^2 = 2$  for even dates and  $h_t^1 = 2$  and  $h_t^2 = -1$  for odd dates, and constant prices

$$p_t = \frac{1}{3}\eta. \quad (16)$$

The first-order condition for the unconstrained agent,

$$\frac{\beta^t}{c_t^1} p_t = \frac{\beta^{t+1}}{c_{t+1}^1} p_{t+1}, \quad (17)$$

holds provided that  $\eta = \frac{\beta A - B}{(1 + \beta)}$  and  $\beta A > B$ . For the constrained agent, the first-order condition requires that the left-hand side in (17) is greater than the right-hand side, and it holds. Transversality condition (see (45) in the Appendix) holds, too. If date-0 endowments are  $w_0^1 = B + \frac{1}{3}\eta$  and  $w_0^2 = A - \frac{1}{3}\eta$ , then this is an equilibrium.

Event prices associated with equilibrium prices (16) are  $q_t = 1$  for every  $t$ . The present value of the aggregate endowment  $\sum_{t=0}^{\infty} q_t \bar{w}_t$  is infinite.  $\square$

A sufficient condition for finite present value of the aggregate endowment is that there exist portfolio  $\theta \in \mathfrak{R}_+^J$  and date  $T$  such that

$$\bar{w}(s_t) \leq \theta x(s_t), \quad (18)$$

for every  $s_t \in F_t$  and every  $t \geq T$ . This follows from the fact that present value of dividend stream  $x_t^j$  is finite for every asset  $j$ , see Proposition 1. Another sufficient condition - for standard utility functions - is that the equilibrium allocation

be Pareto optimal and interior. Needless to say, neither one of those sufficient conditions holds in Example 1.

## 5. Bubbles under Endogenous Debt Constraints.

An important class of debt constraints that may lead to equilibria with infinite present value of the aggregate endowment and price bubbles on assets in strictly positive supply are endogenous debt constraints.

Endogenous debt constraints are induced by limited commitment to market transactions. We have assumed thus far that agents are fully committed to repay any debt incurred by their portfolio decisions. We shall relax this assumption now. Agents may consider defaulting on the payoff of a portfolio at any date. Whether an agent would want to default or not depends on gains and losses that such action would present to him. Once those gains and losses of default are precisely specified, endogenous (or self-enforcing) debt constraints can be defined as a sequence of debt bounds such that the agent is unwilling to default even if his indebtedness is at the maximum allowed level.

We proceed now to a formal definition of endogenous debt constraints. We assume throughout this section that agents' utility functions have the time-separable expected utility representation. The continuation utility in event  $s_t$  at date  $t$  is

$$u_{s_t}^i(c) = \sum_{\tau=t}^{\infty} \beta^{\tau-t} E[v^i(c_\tau)|s_t], \quad (19)$$

where  $0 < \beta < 1$ . Let  $B_{s_t}^i(p, D^i, \Phi^i(s_t))$  denote the budget set in event  $s_t$  at date  $t$  under debt constraints with bounds  $D^i$  when initial financial wealth (or debt) at  $s_t$  is  $\Phi^i(s_t)$ . Specifically, this set consists of all consumption plans and portfolio holdings for events in  $S^{t+}$  satisfying budget constraints (2) and debt constraints (3) for  $s_\tau \in S^{t+}$ . Further, let  $U_{s_t}^{*i}(p, D^i, \Phi^i(s_t))$  be the maximum event- $s_t$  continuation utility (19) over all consumption plans in the budget set  $B_{s_t}^i(p, D^i, \Phi^i(s_t))$ .

Gains and losses of default are described by a sequence of reservation utility levels that agent  $i$  can obtain if she defaults. We denote this sequence by  $\bar{V}_d^i = \{\bar{V}_d^i(s_t)\}$  and call it default utilities. We focus on two specifications of default utilities that have been proposed in recent literature. They are

$$\bar{V}_d^i(s_t) = u_{s_t}^i(w^i), \quad (20)$$

and

$$\bar{V}_d^i(s_t) = U_{s_t}^{*i}(p, 0, 0). \quad (21)$$

Under the first specification (20), default results in permanent exclusion from the markets so that the agent is forced to consume her endowment from  $s_t$  on (see Alvarez and Jermann (2000)). Under the second specification (21), default results in prohibition from taking any debt from  $s_t$  on, but the agent continues to participate in the markets under zero-debt constraints (see Hellwig and Lorenzoni (2009)). Note that default utilities (21) depend on asset prices  $p$ .

Debt bounds  $D^i$  are *self-enforcing* for agent  $i$  at  $p$  given default utilities  $\bar{V}_d^i$  if

$$U_{s_t}^{*i}(p, D^i, -D^i(s_t)) \geq \bar{V}_d^i(s_t). \quad (22)$$

Debt bounds  $D^i$  are *not too tight* for agent  $i$  at  $p$  given default option  $\bar{V}_d^i$  if (22) holds with equality. Equilibrium with not-too-tight debt constraints is an equilibrium with any debt bounds  $D^i$  such that  $D^i$  are not too tight at equilibrium price  $p$  for every  $i$ .

The property of being not too tight does not determine debt constraints in a unique way. This is explained as follows. We say that a real-valued process  $\{\kappa_t\}$  lies in the asset span if

$$\kappa(s_t^+) \in \text{span}\{p_j(s_t^+) + x_j(s_t^+) : j \in J\} \quad (23)$$

for every  $s_t$ . Process  $\{\kappa_t\}$  is a *discounted martingale*, if there exists a strictly positive event price process  $q$  such that the discounted martingale property (12) holds for  $\{\kappa_t\}$  with respect to  $q$ .

The relation  $\simeq_c$  between any two sets of consumption-portfolio plans indicates that consumption plans in those sets are the same. We have

**Lemma 1:** *If  $\{\kappa_t\}$  is a discounted martingale and lies in the asset span, then*

$$B_{s_t}^i(p, D^i, \Phi^i(s_t)) \simeq_c B_{s_t}^i(p, D^i + \kappa, \Phi^i(s_t) - \kappa(s_t)) \quad (24)$$

for every  $s_t$ .

PROOF: We prove (24) for  $s_0$ . Let  $(c, h) \in B_0^i(p, D^i, \Phi^i(s_0))$ . Since  $\{\kappa_t\}$  lies in the asset span, there is a portfolio strategy  $\hat{h}$  such that  $[p(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) = \kappa(s_{t+1})$

for every  $s_{t+1} \subset s_t$  and every  $s_t$ . Since  $\kappa$  is a discounted martingale, it follows that  $p(s_t)\hat{h}(s_t) = \kappa(s_t)$ . It is easy to see now that  $(c, h + \hat{h}) \in B_0^i(p, D^i + \kappa, \Phi^i(s_0) - \kappa(s_0))$ . The same argument shows that if  $(c, \tilde{h}) \in B_0^i(p, D^i + \kappa, \Phi^i(s_0) - \kappa(s_0))$ , then  $(c, \tilde{h} - \hat{h}) \in B_0^i(p, D^i, \Phi^i(s_0))$ . This concludes the proof.  $\square$

Lemma 1 implies that if debt bounds  $D^i$  are not too tight, then  $D^i + \kappa$  are not too tight as well, for any discounted martingale  $\kappa$  that lies in the asset span. Indeed, it follows that  $U_{s_t}^{*i}(p, D^i, -D^i(s_t)) = U_{s_t}^{*i}(p, D^i + \kappa, -(D^i(s_t) + \kappa(s_t)))$ .

For the default option (21), zero bounds  $D^i \equiv 0$  are not too tight. This implies that  $D^i = \kappa$  is not too tight with respect to (21) for every discounted martingale that lies in the asset span. Hellwig and Lorenzoni (2009, Theorem 1, see also Bejan and Bidian (2011)) show that the converse holds, too, if markets are complete: If debt bounds  $D$  are not too tight with respect to (21), then  $D$  is a discounted martingale. In Example 1, event prices are equal to one and debt bounds are constant, which implies that they are discounted martingale. Consequently, this equilibrium has not-too-tight debt constraints with respect to (21).

Next, we present a method of injecting price bubbles on infinitely-lived assets. Let  $\{\epsilon_t\}$  by a  $\mathfrak{R}^J$ -valued process. We say that  $\{\epsilon_t\}$  is *asset-span preserving* if

$$span\{p_j(s_t^+) + x_j(s_t^+) : j \in J\} = span\{p_j(s_t^+) + \epsilon_j(s_t^+) + x_j(s_t^+) : j \in J\} \quad (25)$$

for every  $s_t$ . The property of asset-span preservation (25) and the spanning condition (23) are closely related. If  $\epsilon_t$  is asset-span preserving, then  $\epsilon_t^j$  lies in the asset span at  $p$  for every  $j$ . The converse holds for almost every  $\epsilon_t$ . We elaborate on this in the Appendix, see also Bejan and Bidian (2010). It is important to note that the set of asset-span preserving processes is always non-empty. If markets are complete at  $p$ , then almost every  $\{\epsilon_t\}$  is asset-span preserving.

The  $\mathfrak{R}^J$ -valued process  $\{\epsilon_t\}$  is a discounted martingale, if  $\{\epsilon_t^j\}$  is a discounted martingale for every  $j$ . We have

**Lemma 2:** *If  $\{\epsilon_t\}$  is a positive asset-span preserving discounted martingale, then*

$$B_0^i(p, D^i, \Phi^i(s_0)) \simeq_c B_0^i(p + \epsilon, D^i, \Phi^i(s_0)) \quad (26)$$

for every  $\Phi^i(s_0)$ .

PROOF: Let  $(c, h) \in B_0^i(p, D^i, \Phi^i(s_0))$ . Since  $\{\epsilon_t\}$  is asset-span preserving, there exists portfolio strategy  $\hat{h}$  such that

$$[p(s_{t+1}) + x(s_{t+1})]h(s_t) = [p(s_{t+1}) + \epsilon(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) \quad (27)$$

for every  $s_{t+1} \subset s_t$  and every  $s_t$ . Further, since  $\{\epsilon_t\}$  is a discounted martingale, it follows

$$p(s_t)h(s_t) = [p(s_t) + \epsilon(s_t)]\hat{h}(s_t) \quad (28)$$

It is easy to see now that  $(c, \hat{h}) \in B_0^i(p + \epsilon, D^i, \Phi^i(s_0))$ . The same argument shows that if  $(c, \hat{h}) \in B_0^i(p + \epsilon, D^i, \Phi^i(s_0))$ , then  $(c, h) \in B_0^i(p, D^i, \Phi^i(s_0))$ . This concludes the proof.  $\square$

It follows from Lemmas 1 and 2 that if a  $\mathfrak{R}^J$ -valued process  $\{\epsilon_t\}$  is an asset-span preserving discounted martingale, then

$$B_0^i(p, D^i, p(s_0)\alpha_0^i) \simeq_c B_0^i(p + \epsilon, D^i - \alpha_0^i\epsilon, [p(s_0) + \epsilon(s_0)]\alpha_0^i) \quad (29)$$

Observation (29) leads to the following

**Theorem 3:** *Let  $p$  and  $\{c^i\}$  be an equilibrium with not-too-tight debt constraints  $D^i$ . For every positive asset-span preserving discounted martingale  $\{\epsilon_t\}$  such that  $\alpha_0^i\epsilon_t \leq D_t^i$ , price process  $p + \epsilon$  and consumption allocation  $\{c^i\}$  are an equilibrium with not-too-tight debt constraints, too.*

PROOF: It follows from (29) that  $p + \epsilon$  and  $\{c^i\}$  are an equilibrium under debt constraints with positive bounds  $D^i - \alpha_0^i\epsilon$ . Debt bounds  $D^i - \alpha_0^i\epsilon$  are not too tight by Lemma 1. Further, default utilities (20) and (21) remain the same at prices  $p + \epsilon$ . For (21), this is so because  $U_{s_t}^{*i}(p, 0, 0) = U_{s_t}^{*i}(p + \epsilon, 0, 0)$  by Lemma 2.  $\square$

Results similar to Theorem 3 can be found in Bejan and Bidian (2010) and, for complete markets, in Kocherlakota (2008).

Of course, if the hypothesis of Theorem 3 holds and assets are in strictly positive supply, then the present value of the aggregate endowment must be infinite. Condition  $\alpha_0^i\epsilon_t \leq D_t^i$  in Theorem 3 guarantees that the adjusted debt bounds  $D^i - \alpha_0^i\epsilon$  are positive. Debt constraints with negative bounds force agents to hold minimum savings, and have been excluded from consideration. A sufficient condition guaranteeing that there exists  $\epsilon_t$  such that  $\alpha_0^i\epsilon_t \leq D_t^i$  for every  $i$  is that the discounted

value of debt bounds, i.e.,  $\rho_t D_t^i$  is bounded away from zero for every agent  $i$  whose initial portfolio  $\alpha_0^i$  is non-zero, and the risk-free payoff lies in the one-period asset span for every  $s_t$ . Bejan and Bidian (2011) provide further sufficient conditions.

Hellwig and Lorenzoni (2009, Example 1) presented an example of an equilibrium with not-too-tight debt constraints for default utilities (21) such that debt limits are bounded away from zero and present value of the aggregate endowment is infinite. Theorem 3 implies that there are equilibria with price bubbles in that example. We demonstrate such equilibria in Example 2.

**Example 2:** Uncertainty is described by a binomial event-tree with the two successor events of every  $s_t$  indicated by *up* and *down*. The (Markov) transition probabilities are

$$\text{Prob}(up|s_t) = 1 - \alpha, \quad \text{Prob}(down|s_t) = \alpha, \quad (30)$$

whenever  $s_t = (s_{t-1}, up)$ , and

$$\text{Prob}(up|s_t) = \alpha, \quad \text{Prob}(down|s_t) = 1 - \alpha, \quad (31)$$

for  $s_t = (s_{t-1}, down)$ , where  $0 < \alpha < 1$ . Initial event is  $s_0 = up$ .

There are two consumers with utility functions (19) with the same logarithmic function  $v$  and discount factor  $0 < \beta < 1$ . Endowments depend only on the current state and are given by  $w^1(up) = A$ ,  $w^2(up) = B$ , and  $w^1(down) = B$ ,  $w^2(down) = A$ . It is assumed that  $A > B > 0$ . Note that there is no aggregate risk.

The market structure consists of one-period Arrow securities at every date-event and an infinitely-lived asset with zero dividends (fiat money). Arrow securities are in zero supply. Fiat money is in strictly positive supply with each agent holding one share at date 0.

We first find an equilibrium with zero price of the fiat money. There exists a stationary Markov equilibrium such that, at every event, the debt constraint is binding for the agent who receives high endowment. The equilibrium is as follows: Prices of Arrow securities are

$$p_c(s_t) = 1 - \beta(1 - \alpha), \quad p_{nc}(s_t) = \beta(1 - \alpha), \quad (32)$$

where subscript  $c$  stands for “change” of state (for example, from up to down) and  $nc$  for “no change.” Consumption allocation is

$$c^1(s_t) = \bar{c}, \quad c^2(s_t) = \underline{c} \quad (33)$$

whenever  $s_t = (s_{t-1}, up)$ , and

$$c^1(s_t) = \underline{c}, \quad c^2(s_t) = \bar{c} \quad (34)$$

whenever  $s_t = (s_{t-1}, down)$ . Consumption plans  $\bar{c}$  and  $\underline{c}$  are such that  $\bar{c} + \underline{c} = A + B$  and

$$1 - \beta(1 - \alpha) = \beta\alpha \frac{\bar{c}}{\underline{c}}. \quad (35)$$

The solution must satisfy  $\bar{c} < A$ , which is a restriction on parameters  $\alpha$  and  $\beta$ . Debt bounds are

$$D(s_t) = \bar{D} = \frac{A - \bar{c}}{2[1 - \beta(1 - \alpha)]}, \quad (36)$$

constant over time and across events.

Equilibrium holdings of Arrow securities are such that agent 1 always holds the maximum possible short position  $-\bar{D}$  in the Arrow security that pays in the up-event (where she gets high endowment  $A$ ) and long position  $\bar{D}$  in the Arrow security that pays in the down-event. The opposite holds for agent 2. Transversality condition (see (45) in the Appendix) holds. Date-0 endowments must be  $w^1(s_0) = A - \bar{D}$  and  $w^2(s_0) = B + \bar{D}$ .

Event prices are products of Arrow securities prices (32). They have the property that, at any date, the sum of event prices for all date- $t$  events equals one. In other words, there is no discounting (i.e.,  $\rho(s_t) = 1$ ). Consequently, the present value of the aggregate endowment is infinite. Further, constant debt bounds  $\bar{D}$  are a martingale, and therefore they are not too tight.

Since debt bounds (36) are strictly positive, we can take any positive martingale  $\{\epsilon_t\}$  such that  $\epsilon_t \leq \bar{D}$  and “inject” it as a bubble on fiat money as described in Theorem 3. For simplicity, we take a deterministic process  $\epsilon_t = \bar{\epsilon}$  for  $\bar{\epsilon} \leq \bar{D}$ . Price process for fiat money given by  $p_t = \bar{\epsilon}$  together with consumption allocation  $\{c^i\}$  given by (35) and debt bounds  $\bar{D} - \bar{\epsilon}$  for each agent constitute an equilibrium with non-too-tight debt constraints.

## 6. Bubbles on Assets in Zero Supply

Theorem 3 implies that if assets are in zero supply, then there exist equilibria with not-too-tight debt constraints with price bubbles on infinitely-lived assets. This result extends to equilibria under debt constraints with arbitrary bounds.

**Theorem 4:** *Suppose that all infinitely-lived assets are in zero supply. If  $p$  and  $\{c^i\}$  are an equilibrium under debt constraints, then  $p + \epsilon$  and  $\{c^i\}$  are an equilibrium, too, for every positive asset-span preserving discounted martingale  $\epsilon$ .*

PROOF: Lemma 2 implies that

$$B_0^i(p, D^i, 0) \simeq_c B_0^i(p + \epsilon, D^i, 0)$$

Since  $p(s_0)\alpha_0^i = [p(s_0) + \epsilon(s_0)]\alpha_0^i = 0$ , it follows that  $p + \epsilon$  and  $\{c^i\}$  are an equilibrium.  $\square$

Equilibria with price bubbles of Theorem 4 have the same asset span (and the same consumption allocation) as their no-bubble counterparts. There may exist equilibria with price bubbles on assets in zero supply such that the injection of price bubble changes the asset span and leads to different consumption allocation. We present an example which is a variation of Example 1.

**Example 3:** Consider the economy of Example 1. The Arrow-Debreu equilibrium in this economy has time-independent consumption plans  $c_t^i = \bar{c}^i$  for  $i = 1, 2$  and event prices equal to the discount factor, that is,

$$q_t = \beta^t, \tag{37}$$

for every  $t$ . Consumption plans  $\bar{c}^i$  are given by

$$\bar{c}^i = \sum_{t=0}^{\infty} \beta^t w_t^i, \tag{38}$$

that is  $\bar{c}^1 = \frac{1}{1-\beta^2}[A\beta + B] + \eta$  and  $\bar{c}^2 = \frac{1}{1-\beta^2}[A + B\beta] - \eta$ .

The Arrow-Debreu equilibrium allocation can be implemented by trading fiat money in zero supply under the natural debt constraints. The natural debt constraints have bounds equal to the present value of current and future agent's endowments, that is,  $D_t^i = \frac{1}{q_t} \sum_{\tau=t}^{\infty} q_{\tau} w_{\tau}^i$ . The price of fiat money equals bubble  $\sigma_t$

that must satisfy the discounted-martingale property (12), that is

$$q_t \sigma_t = q_{t+1} \sigma_{t+1}. \quad (39)$$

We take  $\sigma_t = \beta^{-t}$  so that the price of fiat money is

$$p_t = \beta^{-t}. \quad (40)$$

The present value of agent 1 future endowments at event prices (37) is  $\frac{1}{1-\beta^2}[A\beta + B]$  for even dates, and  $\frac{1}{1-\beta^2}[A + B\beta]$  for odd dates. The reverse holds for agent 2. Debt bounds are equal to those present values.

## 7. Speculative Bubbles

In this section we discuss an alternative definition of the fundamental value of an asset's future dividends that leads to a different notion of price bubbles, the so-called *speculative bubble*. To avoid confusion, we refer to the bubble in the sense of (10) as rational pricing bubble.

Throughout this section we assume that utility functions are differentiable. Consider equilibrium asset prices  $p$  and consumption allocation  $\{c^i\}$  under debt constraints. Let  $\frac{\partial_{s_\tau} u^i}{\partial_{s_t} u^i}$  denote agent's  $i$  marginal rate of substitution between consumption in events  $s_\tau$  and  $s_t$  taken at the equilibrium consumption plan  $c^i$ , assumed *interior* (i.e.,  $c^i(s_t) > 0$ ,  $\forall s_t$ ). The marginal value of buying an additional share of asset  $j$  at  $s_t$  and holding it forever is

$$V_j^i(s_t) = \sum_{\tau=t}^{\infty} \sum_{s_\tau \in s_t} \frac{\partial_{s_\tau} u^i}{\partial_{s_t} u^i} x_j(s_\tau) \quad (41)$$

It follows from first-order conditions for the optimal consumption-portfolio choice under debt constraints (see (44) in the Appendix) that

$$p_j(s_t) \geq V_j^i(s_t), \quad (42)$$

for every event  $s_t$  and every agent  $i$ . If agent's  $i$  portfolio strategy is such that debt constraint is binding in some future event that is a successor of  $s_t$ , then (42) holds with strict inequality for every security.<sup>5</sup>

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<sup>5</sup>With a slight abuse of terminology we take binding constraint to mean that the corresponding Lagrange multiplier is strictly positive.

We say that there is *speculative bubble* on asset  $j$  in event  $s_t$  if

$$p_j(s_t) > \max_i V_j^i(s_t). \quad (43)$$

Definition (43) has been used in the literature in the case of short sales constraints. If every agent's portfolio strategy is such that debt constraint is binding in some future event, then there is speculative bubble on every asset. Since binding debt constraint involves necessarily selling some assets, this gives some sense of speculative trade albeit weaker than under short sales constraints.<sup>6</sup>

Binding debt constraints at future dates are a sufficient but not a necessary condition for speculative bubbles. If there is rational price bubbles, then there is speculative bubble regardless of whether debt constraints are binding or not. In equilibrium of Example 3 (with zero asset supply) there is rational price bubble, and hence speculative bubble as well, but debt constraints never bind. In contrast, equilibria of Examples 1 and 2 have rational price bubble and binding debt constraints. Further, if the rational pricing bubble is zero, i.e.,  $\sigma_j(s_t) = 0$ , and there is speculative bubble on asset  $j$  at  $s_t$ , then there is speculative trade.

Neither infinite present value of future endowments nor zero supply of assets are necessary for the existence of speculative bubbles. Harrison and Kreps (1978) provided an example of an equilibrium under short sales constraints with speculative bubble at every date and every event. In their single-asset setting, short sales constraints are equivalent to debt constraints. The asset is in strictly positive supply and the present value of the aggregate endowment is finite. The key feature of Harrison and Kreps example are persistent heterogeneity of agents' beliefs and risk-neutral utilities. Marginal value of an asset (41) is then equal to the discounted expected value of future dividends under the agent's probability beliefs. Slawski (2008) provides general conditions for existence of speculative bubbles under heterogeneous beliefs and with learning, see also Morris (1996).

## 8. Concluding Remarks

We presented theoretical foundations of rational asset pricing bubbles under

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<sup>6</sup>Harrison and Kreps' (1978, pg 323) definition of speculative trade under short sales constraints is in the following quotation: "... *investors exhibit speculative behavior if the right to resell the stock makes them willing to pay more for it than they would if obliged to hold forever.*"

debt constraints. The standard no-bubble theorem known to hold under borrowing constraints extends to debt constraints including endogenous debt constraints. The no-bubble theorem leaves two possibilities for price bubbles to arise in an equilibrium: infinite present value of aggregate resources, and zero supply of assets. We argued that equilibria with endogenous debt constraints are prone to generate infinite present value of aggregate endowments. Two examples were given to illustrate that assertion. Further, we showed that there always exist equilibria with price bubbles on assets in zero supply.

## Appendix

### First-Order and Transversality Conditions.

Assuming that the utility function  $u^i$  is differentiable, the necessary first-order conditions for an interior solution to the consumption-portfolio choice problem under debt constraints are

$$p(s_t) = \sum_{s_{t+1} \subset s_t} [p(s_{t+1}) + x(s_{t+1})] \left[ \frac{\partial_{s_{t+1}} u^i}{\partial_{s_t} u^i} + \frac{\gamma^i(s_{t+1})}{\partial_{s_t} u^i} \right]. \quad (44)$$

for all  $s_t$ , where  $\gamma^i(s_t) \geq 0$  is the Lagrange multiplier associated with debt constraint (3).

First-order conditions (44) together with transversality condition are sufficient to determine an optimal consumption-portfolio choice for concave utility function. For the discounted time-separable expected utility (19) with concave period-utility  $v$ , the transversality condition for  $(c, h)$  is

$$\lim_{t \rightarrow \infty} \sum_{s_t \in F_t} \beta^t \pi(s_t) v'(c(s_t)) [(p(s_t) + x(s_t)) h(s_t^-) + D(s_t)] = 0. \quad (45)$$

**Proof of Theorem 2.** Let  $(p, \{c^i, h^i\})$  be the equilibrium. First we observe that

$$p_0 = \sum_{t=1}^{\infty} \sum_{s_t \in F_t} q(s_t) x(s_t) + \lim_{T \rightarrow \infty} \sum_{s_T \in F_T} q(s_T) p(s_T). \quad (46)$$

It follows that  $\sigma_0 = 0$  if and only if

$$\lim_{T \rightarrow \infty} \sum_{s_T \in F_T} q(s_T) p(s_T) = 0. \quad (47)$$

Let  $\gamma^i$  be as implied by (A1) with  $0 \leq \gamma^i < 1$ . We claim that

$$(1 - \gamma^i) p(s_t) h^i(s_t) \leq \hat{w}(s_t), \quad (48)$$

for every  $s_t$ , every  $i$ .

To prove (48), suppose that there exists  $s_t$  such that

$$(1 - \gamma^i) p(s_t) h^i(s_t) > \hat{w}(s_t), \quad (49)$$

for some  $i$ . Consider consumption plan

$$\tilde{c}^i = (c^i(-S^t), c^i(s_t) + (1 - \gamma^i)p(s_t)h^i(s_t), \gamma^i c^i(S^{t+})). \quad (50)$$

Note that portfolio  $\tilde{h}^i = (h^i(-S^t), \gamma^i h^i(S^t))$  finances  $\tilde{c}^i$  and satisfies debt constraints. By assumption (A1),  $u^i(\tilde{c}^i) > u^i(c^i)$  which is a contradiction. This proves (48).

From (48) it follows that

$$(1 - \bar{\gamma})p(s_t)\alpha_0 \leq I\hat{w}(s_t), \quad (51)$$

for every  $s_t$ , where  $\bar{\gamma} = \max \gamma^i$ .

From (51), we obtain

$$\sum_{s_T \in F_T} q(s_T)p(s_T)\alpha_0 \leq \frac{I}{(1 - \bar{\gamma})} \sum_{s_T \in F_T} q(s_T)\hat{w}(s_T). \quad (52)$$

From the assumptions of the theorem, it follows that

$$\lim_{T \rightarrow \infty} \sum_{s_T \in F_T} q(s_T)\hat{w}(s_T) = 0. \quad (53)$$

Using (52) and (47), we obtain

$$\sigma_0\alpha_0 = 0, \quad (54)$$

and consequently  $\sigma_0 = 0$ .  $\square$

### Asset-span preserving process $\{\epsilon_t\}$ .

Here, we discuss the properties of asset-span preservation and lying in the asset span introduced in Section 5. Let  $\{\epsilon_t\}$  be a  $\mathfrak{R}^J$ -valued process. First, it is easy to see that if  $\{\epsilon_t\}$  is asset span preserving at  $p$ , then  $\{\epsilon_t^j\}$  lies in the asset span at  $p$  for every  $j$ . The converse holds under a minor rank condition which we explain next.

Suppose that  $\{\epsilon_t^j\}$  lies in the asset span at  $p$ . For any  $s_t$ , let  $h^j(s_t)$  be a portfolio such that  $\epsilon(s_t^+) = [p(s_t^+) + x(s_t^+)]h^j(s_t)$ . It follows that  $p(s_t^+) + \epsilon(s_t^+) + x(s_t^+) = [p(s_t^+) + x(s_t^+)](h^j(s_t) + I^j)$ , where  $I^j$  denotes a portfolio consisting of one share of asset  $j$ . Let  $H(s_t)$  be the  $J \times J$  matrix with rows  $h^j(s_t)$  for all  $j$  and  $I$  be the

$J \times J$  identity matrix. If matrix  $H(s_t) + I$  is invertible, then  $\{\epsilon_t\}$  is asset span preserving at  $p$ .

Summing up, if  $\{\epsilon_t^j\}$  lies in the asset span at  $p$  for every  $j$ . and  $H(s_t) + I$  is invertible for every  $s_t$ , then  $\{\epsilon_t\}$  is asset span preserving at  $p$ .

We conclude with construction of an asset-span preserving process: For each asset  $j$ , consider a strategy that starts with buying 1 share of asset  $j$  at date 0 and consists of rolling over the payoff at every event after date 0. This may be called a reverse Ponzi scheme on asset  $j$ . Formally it is a real-valued process  $\gamma_j$  defined by

$$p_j(s_t)\gamma_j(s_t) = [p_j(s_t) + x_j(s_t)]\gamma_j(s_t^-)$$

for every  $s_t$ , and  $\gamma_j(s_0) = 1$ . Note that  $\gamma_j$  is positive.

Let  $\epsilon_t$  be an  $\mathfrak{R}^J$ -valued process defined by

$$\epsilon_j(s_t) = [p_j(s_t) + x_j(s_t)]\gamma_j(s_t^-)$$

Since

$$p(s_t) + \epsilon(s_t) + x(s_t) = [p(s_t) + x(s_t)](I + \gamma(s_t^-))$$

it follows that  $\epsilon_t$  is asset-span preserving. Further, it is a discounted martingale.

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