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A Property of Solutions to Linear Monopoly Problems*

Gregory Pavlov

Abstract

We extend the “no-haggling” result of Riley and Zeckhauser (1983) to the class of linear multiproduct monopoly problems when the buyer’s valuations are smoothly distributed. In particular, we show that there is no loss for the seller in optimizing over mechanisms such that all allocations belong to the boundary of the feasible set. The set of potentially optimal mechanisms can be further restricted when the costs are sufficiently low: the optimal mechanisms use only allocations from the “north-east” boundary of the feasible set and the null allocation.

KEYWORDS: multidimensional screenin, optimal selling strategies, mechanism design

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1 Introduction

Suppose that a seller who owns several goods meets a buyer who is privately informed about his valuations. If the seller had only one good, the optimal selling strategy would be simple: just post a single "take-it-or-leave-it" price (Riley and Zeckhauser, 1983). But when the seller has several goods, the optimal selling mechanism for the general case is not known, and the situation appears to be complicated. For example, we know that, in addition to the individually priced goods, the seller often benefits from offering bundles of several goods (Adams and Yellen, 1976; McAfee et al., 1989). Further, we know that the seller often benefits from offering lotteries in addition to individually priced goods and bundles of goods (Manelli and Vincent, 2006, 2007; Thanassoulis, 2004).

We consider the following multiproduct monopoly model with one buyer and one seller. The buyer's utility is $\theta \cdot p - T$, where $\theta \in \mathbb{R}^n$ describes the buyer's private valuations for each of the n available goods, $p \in \mathbb{R}^n$ describes the allocation of the goods, and $T \in \mathbb{R}$ is the buyer's payment to the seller. The seller's utility is $T - c \cdot p$, where $c \in \mathbb{R}^n$ describes the seller's costs for each of the goods. In Section 2 we discuss a few settings to which this model applies. For example, if we suppose that the seller has n indivisible goods, and all goods are desirable from the point of view of the buyer, then p_i is the probability that the buyer gets good i, and the feasible set for allocations p is $[0,1]^n$.

For the case of one good, the "no-haggling" result of Riley and Zeckhauser (1983) says that the optimal mechanism must use only allocations p from the boundary of the feasible set $[0,1]$: either provide the good for sure ($p = 1$), or provide no good ($p = 0$). We show that the optimal selling strategy for the case of several goods is a natural counterpart of the optimal selling strategy for the case of one good. Proposition 1 shows that there is no loss for the seller in optimizing over mechanisms that use only the allocations from the boundary of the feasible set when the buyer's valuations are smoothly distributed. In the indivisible goods setting described above, this means that for every offered allocation p there exists good i, which is either provided for sure ($p_i = 1$), or not provided at all ($p_i = 0$).

The set of potentially optimal mechanisms can be further restricted when the costs are sufficiently low, in the sense that it is common knowledge that there are gains from trading each of the n goods. In this case, Proposition 2 shows that the optimal mechanisms use only allocations from the "north-east" boundary of the feasible set and the null allocation. In the indivisible goods setting, this means that for every offered allocation p, such that $p \neq \mathbf{0}$, there exists good i, which is provided for sure ($p_i = 1$).

The rest of the paper is organized as follows. The model is introduced in Section 2. The main result is in Section 3. In Section 4 we show that the set of potentially optimal mechanisms can be further restricted in the case of low marginal costs. Concluding comments are in Section 5. All proofs are in the Appendix.

2 Model

There is one buyer and one seller who owns $n \geq 1$ varieties of goods.[1] The buyer values good i at θ_i which is known only to him. A vector of valuations $\theta = (\theta_1, ..., \theta_n)$ is distributed according to an almost everywhere positive bounded differentiable density f on the support $\Theta = \times_{i=1}^{n} \left[\underline{\theta}_i, \overline{\theta}_i\right] \subset \mathbb{R}_+^n$. This distribution is common knowledge.

All players have linear utilities. The buyer's utility is $\theta \cdot p - T$, where $p = (p_1, ..., p_n)$ is the vector of allocations of each of the goods, and T is the buyer's payment to the seller. The seller's utility is $T - c \cdot p$, where $c = (c_1, ..., c_n) \in \mathbb{R}_+^n$ is the vector of constant marginal costs for each of the goods. We assume that there are potential gains from trading each good: $\overline{\theta}_i > c_i$ for every $i = 1, ..., n$. The feasible set of the allocations Σ is assumed to be a convex compact subset of $\mathbb{R}_+^n$ with nonempty interior such that it contains the origin and satisfies the following property: if $p \in \Sigma$ and $\mathbf{0} \leq p' \leq p$ then $p' \in \Sigma$. Denote the boundary of the feasible set by $\partial\Sigma$.

Here are a few settings to which this model applies.

1. *Indivisible goods* (McAfee and McMillan, 1988; McAfee et al., 1989; Manelli and Vincent, 2006, 2007). The seller has n indivisible goods, and all goods are desirable from the point of view of the buyer. In this case p_i is the probability that the buyer gets good i, and the feasible set is $\Sigma = [0,1]^n$.

2. *Substitutable goods* (Thanassoulis, 2004; Balestrieri and Leao, 2008). The seller has n indivisible goods, and the buyer can consume just one unit of any good. In this case p_i is the probability that the buyer consumes good i, and the feasible set is $\Sigma = \left\{p \in \mathbb{R}_+^n \mid \sum_{i=1}^{n} p_i \leq 1\right\}$.[2]

3. *Divisible goods* (Armstrong, 1996; Rochet and Chone, 1998). The seller has n varieties of divisible goods, and all goods are desirable from the point

[1] Throughout the paper we use masculine pronouns for the buyer and feminine pronouns for the seller.

[2] Note that the seller never benefits from assigning to the buyer more than one good, because then the buyer would consume only the good that he values most. Thus we can denote by p_i the probability that good i (and only good i) is assigned to the buyer.

of view of the buyer. In this case p_i is the (expected) quantity of good i that the buyer gets. The feasible set is $\Sigma = \left\{p \in \mathbb{R}^n_+ \mid Q(p) \leq 0\right\}$, where Q describes the seller's production capacity and is assumed to be a continuous quasi-convex function increasing in each coordinate.

4. *Product line design* (Lancaster, 1971). The seller designs a menu of goods, each of which is viewed by the buyer as a bundle of n characteristics. Every good is described by a vector p, where p_i is the intensity of the ith characteristic possessed by this good. The feasible set is $\Sigma = \left\{p \in \mathbb{R}^n_+ \mid Q(p) \leq 0\right\}$, where Q describes the technological constraints the seller faces when designing goods, and is assumed to be a continuous quasi-convex function increasing in each coordinate.

5. *Auction design in the presence of "well-coordinated" cartel* (Gruyer, 2009). The seller has a single good and can prohibit reallocations of the good between n bidders. The bidders form a cartel that allows them to behave as a single buyer, maximizing the sum of the bidders' payoffs. In this case p_i is the probability that bidder i gets the good, and the feasible set is $\Sigma = \left\{p \in \mathbb{R}^n_+ \mid \sum_{i=1}^n p_i \leq 1\right\}$.

By the revelation principle we can without loss of generality assume that the seller offers a direct mechanism, which consists of a set Θ of type reports, an allocation rule $p : \Theta \to \Sigma$, and a payment rule $T : \Theta \to \mathbb{R}$.[3] The seller's problem is stated below.

$\max_{(p,T)} E\left[T(\theta) - c \cdot p(\theta)\right]$ subject to

Feasibility: $p(\theta) \in \Sigma$ for every $\theta \in \Theta$;

Incentive Compatibility: $\theta \cdot p(\theta) - T(\theta) \geq \theta \cdot p(\theta') - T(\theta')$ for every $\theta, \theta' \in \Theta$;

Individual Rationality: $\theta \cdot p(\theta) - T(\theta) \geq 0$ for every $\theta \in \Theta$.

We call a mechanism (p, T) *admissible* if it satisfies the above constraints. Denote the equilibrium utility of the buyer of type θ by $U(\theta) = \theta \cdot p(\theta) - T(\theta)$.

Alternatively, one can view the problem of the seller as the one of choosing a nonlinear price schedule that specifies which feasible allocations the buyer can purchase at what price. The fact that each mechanism is equivalent to some price schedule, and each price schedule implements some mechanism is known as the

[3] The seller never benefits from randomized payments because the payoffs are linear in money. Thus there is no loss of generality in restricting attention to deterministic payment rules.

"taxation principle".[4] We will make use of this principle and in particular of a concept of a menu of a given mechanism.

Definition 1 *A* menu of a mechanism (p,T) *is the set of allocations that the buyer can achieve in this mechanism:*

$$p(\Theta) := \{\widetilde{p} \in \Sigma \mid \textit{there exists } \theta \in \Theta \textit{ such that } \widetilde{p} = p(\theta)\}.$$

We require the distribution to satisfy a version of a "hazard rate condition" which is standard in the multidimensional mechanism design literature.[5]

Condition 1 *The density f and the cost c satisfy*

$$(n+1)f(\theta) + (\theta - c)\cdot \nabla f(\theta) \geq 0 \textit{ for every } \theta \in \Theta,$$

where ∇f is the gradient of f.

To get some intuition for this condition, note that in case $n = 1$ it requires

$$\frac{d}{d\theta}\left[\left(\theta - c - \frac{1-F(\theta)}{f(\theta)}\right)f(\theta)\right] = 2f(\theta) + (\theta - c)\frac{\partial f(\theta)}{\partial \theta} \geq 0.$$

The expression in the square brackets is the "virtual valuation" function weighted by the density, and it represents the marginal profit from selling the good to the buyer of type θ (provided that all buyer's types above θ are already being served).[6] The requirement that this function is nondecreasing amounts to assuming that the distribution of the taste parameters is sufficiently smooth. This requirement is not likely to be satisfied if there are exist distinct "market segments", like in the case of multimodal continuous distributions or discrete distributions.

3 The main result

In this section we show that under Condition 1 there is no loss for the seller in optimizing over mechanisms with the menus that belong to the boundary of the feasible set.

Proposition 1 *Suppose Condition 1 is satisfied. Consider an admissible mechanism (p,T) with a menu $p(\Theta)$ that contains allocations from the interior of the feasible set, $p(\Theta) \nsubseteq \partial\Sigma$. Then there exists an admissible mechanism $\left(\widehat{p},\widehat{T}\right)$ with a menu $\widehat{p}(\Theta)$ that contains only boundary points, $\widehat{p}(\Theta) \subseteq \partial\Sigma$, and it brings at least as much profit.*

[4] See for example Rochet (1985).

[5] See for example McAfee and McMillan (1988), Manelli and Vincent (2006).

[6] See for example Myerson (1981), Riley and Zeckhauser (1983).

The new mechanism takes the menu associated with the original mechanism and removes all point contracts other than the ones that were chosen by the types from the surface of the set Θ. Each remaining point contract is adjusted so that the allocation lies on the boundary of the feasible set Σ, and the payment is adjusted to make sure that the utility of the type that was previously choosing this point contract remains the same as in the original mechanism. The new mechanism performs better than the original for two reasons. First, it improves the allocation of the good among the types by raising the consumption of the high types while reducing it for the low types. Second, by removing from the menu all options other than the extreme ones, the seller is able to extract more surplus from the buyers who choose to purchase.

More formally, let us show the idea of the proof for the case of one good. Let $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$ and $\Sigma = [0,x]$. Take some admissible mechanism (p,T) such that its menu contains allocations other than 0 and x. Introduce a new (indirect) mechanism that consists of just two options: the buyer can get allocation x at a price $T\left(\overline{\theta}\right) + \overline{\theta}\left(x - p\left(\overline{\theta}\right)\right)$, or he can get allocation 0 at a price $T\left(\underline{\theta}\right) - \underline{\theta} p\left(\underline{\theta}\right)$. Notice that the utility of the buyer of type θ from the new mechanism ($\widehat{U}(\theta)$) is no greater than in the original mechanism ($U(\theta)$):

$$
\begin{aligned}
\widehat{U}(\theta) &= \max\left\{\theta x - \left(T\left(\overline{\theta}\right) + \overline{\theta}\left(x - p\left(\overline{\theta}\right)\right)\right), -\left(T\left(\underline{\theta}\right) - \underline{\theta} p\left(\underline{\theta}\right)\right)\right\} \\
&= \max\left\{U\left(\overline{\theta}\right) - \left(\overline{\theta} - \theta\right)x, U\left(\underline{\theta}\right)\right\} \\
&\leq \max\left\{U\left(\overline{\theta}\right) - \int_{\theta}^{\overline{\theta}} p\left(\widetilde{\theta}\right) d\widetilde{\theta}, U\left(\underline{\theta}\right)\right\} \\
&= \max\left\{U(\theta), U\left(\underline{\theta}\right)\right\} = U(\theta).
\end{aligned}
\tag{1}
$$

The inequality follows from feasibility of the original mechanism ($p\left(\widetilde{\theta}\right) \leq x$ for every $\widetilde{\theta}$), and the third equality is from the envelope formula for the buyer's utility. Note that the inequality holds as equality for the highest type $\overline{\theta}$ as well as for the lowest type $\underline{\theta}$. The new mechanism is individually rational because $\widehat{U}(\theta) \geq \widehat{U}\left(\underline{\theta}\right)$ for every $\theta \in \Theta$, and in turn, $\widehat{U}\left(\underline{\theta}\right) = U\left(\underline{\theta}\right) \geq 0$, because the original mechanism is assumed to be individually rational.

Next we can represent the expected profit in terms of the utility schedule:

$$
\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \left(T(\theta) - cp(\theta)\right) f(\theta)\, d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \left((\theta - c)\, p(\theta) - U(\theta)\right) f(\theta)\, d\theta \\
= \quad & U\left(\overline{\theta}\right)\left(\overline{\theta} - c\right) f\left(\overline{\theta}\right) - U\left(\underline{\theta}\right)\left(\underline{\theta} - c\right) f\left(\underline{\theta}\right) - \\
&\int_{\underline{\theta}}^{\overline{\theta}} U(\theta)\left(2f(\theta) + (\theta - c)\frac{\partial f(\theta)}{\partial \theta}\right) d\theta
\end{aligned}
\tag{2}
$$

where the first equality follows from substituting out the payments ($T(\theta) = \theta \cdot p(\theta) - U(\theta)$), and the second equality uses the Envelope theorem ($\frac{d}{d\theta}U(\theta) = p(\theta)$ for a.e. θ) and integration by parts. Recall that the utilities of the highest and the lowest types in both mechanisms are the same. Hence, the difference in the expected profits from the new mechanism relative to the original mechanism is

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(U(\theta) - \widehat{U}(\theta)\right)\left(2f(\theta) + (\theta - c)\frac{\partial f(\theta)}{\partial \theta}\right) d\theta \geq 0.$$

The inequality holds because of (1) and Condition 1.[7]

In case of multiple goods, we also replace the original mechanism with a mechanism that uses only allocations from the boundary of the feasible set so that the utilities of the types from the surface of the set Θ are unchanged, and the utilities of all the other types are lowered. An analogue of equation (2) can be obtained using integration by parts or the divergence theorem.[8]

For the one dimensional case, it has long been known that the optimal menu must use only boundary points. Riley and Zeckhauser (1983) refer to this property as the “no-haggling” result: the seller’s optimal mechanism when dealing with a risk-neutral buyer is to quote a single “take-it-or-leave-it” price. In the setting with indivisible goods, Manelli and Vincent (2006) argue that a natural analogue of the “no-haggling” property in the multidimensional case is for the seller to set a price for each possible bundle. In other words, the optimal mechanism should not include any lotteries.[9] However, the conditions on the distribution of valuations provided in Manelli and Vincent (2006) for optimality of deterministic mechanisms are rather stringent, and the authors acknowledge that the optimal mechanisms must often include lotteries. In this respect our “boundary” property of the menu is a better extension of the “no-haggling” property to the case of multiple goods. Our property preserves the “bang-bang” flavor of the one dimensional “no-haggling” property, but it is also flexible enough to allow lotteries to be part of menus.

Similarly, in the setting with substitutable goods it is natural to conjecture that an appropriate extension of the “no-haggling” property is for the seller to set an individual price for each good. However, Thanassoulis (2004) shows that such deterministic menus are often suboptimal, and the seller’s optimal menu must in-

[7]A strict improvement in the seller’s expected profit can be guaranteed if the following two conditions are satisfied: (i) in the original mechanism there is a subset of the buyer’s types of positive measure that choose allocations from the interior of the feasible set Σ; and (ii) a strict version of Condition 1 holds.

[8]See for example McAfee and McMillan (1988), Rochet and Chone (1998).

[9]McAfee and McMillan (1988) claim that the optimal mechanism in the case of two goods is deterministic, but Thanassoulis (2004) shows that their claim is not accurate.

clude lotteries. Hence, our "boundary" property of the menu seems like a correct extension of the "no-haggling" property in this environment as well.

In the setting with indivisible goods, Manelli and Vincent (2007) present necessary conditions for an admissible mechanism to be optimal for some arbitrary distribution of the buyer's valuations. They find that the class of potentially optimal mechanisms is very large and includes mechanisms that use allocations from the interior of the feasible set.[10] Since their proof is not constructive, it is hard to judge what kind of irregular distributions are required to rationalize mechanisms which do not satisfy our "boundary" property.

We believe that the property of the optimal mechanisms established in Proposition 1 holds beyond the class of distributions considered here. For example, it is straightforward to extend the proof of the result for any convex Θ. It is also likely that Condition 1 can be somewhat relaxed, but the next example demonstrates that it is not possible to dispense with it altogether. Though this example is with a discrete distribution, it should be possible to construct a similar example with a multimodal continuous distribution.

Example 1 *Consider the* indivisible goods *setting with* $n = 2$ *and* $c_1 = c_2 = 0$. *There are four equally likely types:* $(1,0)$, $(0,1)$, $\left(\frac{1}{2},\frac{1}{2}\right)$, *and* $(2,2)$. *In the optimal mechanism, type* $(1,0)$ *gets the first good at a price* 1, *type* $(0,1)$ *gets the second good at a price* 1, *type* $(2,2)$ *gets both goods at a price* 3, *and type* $\left(\frac{1}{2},\frac{1}{2}\right)$ *gets each good with probability* $\frac{1}{3}$ *at a price* $\frac{1}{3}$. *Thus this menu contains interior points of the feasible set.*[11]

In the one dimensional case, a reduction in the allocation for the lower types always allows the extraction of more money from the higher types. This is not necessarily the case when there are multiple dimensions. In this example all three downward incentive constraints for type $(2,2)$ are binding at the optimum. A reduction in the allocation for type $\left(\frac{1}{2},\frac{1}{2}\right)$ does not allow the extraction of more money from type $(2,2)$, since he can still mimic types $(1,0)$ and $(0,1)$.

4 The case of low marginal costs

In this section we strengthen the "boundary" property of the optimal mechanisms for the case when the marginal costs are low: $\underline{\theta}_i \geq c_i$ for every $i = 1,...,n$. We show that in such a case the search for the optimal menu can be restricted to the subset of

[10] See Theorem 9 and Lemma 12 in Manelli and Vincent (2007).

[11] This example is similar in spirit to the example in Section 7 in Baron and Myerson (1982). The calculations are in the Appendix.

the boundary of the feasible set that contains the null option **0** and the "north-east" part of the boundary, which is defined as follows:

$$\overline{\partial\Sigma} = \left\{p \in \Sigma \mid \nexists p' \in \Sigma \text{ such that } p' \gg p\right\}$$

For example, if the seller has n indivisible goods, then $\overline{\partial\Sigma}$ consists of all allocations p that provide at least one of the goods for sure: $p_i = 1$ for some $i = 1,...,n$. If the seller has n substitutable goods, then $\overline{\partial\Sigma}$ includes all allocations p that provide consumer with a good for sure: $\sum_{i=1}^{n} p_i = 1$.

Proposition 2 *Suppose Condition 1 is satisfied, and $\underline{\theta}_i \geq c_i$ for every $i = 1,...,n$. Consider an admissible mechanism (p,T) with a menu $p(\Theta)$ such that $p(\Theta) \nsubseteq \overline{\partial\Sigma} \cup \{\mathbf{0}\}$. Then there exists an admissible mechanism $\left(\widehat{p},\widehat{T}\right)$ with a menu $\widehat{p}(\Theta)$ such that $\widehat{p}(\Theta) \subseteq \overline{\partial\Sigma} \cup \{\mathbf{0}\}$, and it brings at least as much profit.*

The idea of the proof is similar to the one in Proposition 1. Let $n = 1$, $\Theta = \left[\underline{\theta},\overline{\theta}\right]$ and $\Sigma = [0,x]$. Take some admissible mechanism (p,T) such that its menu contains allocations other than 0 and x. Introduce a new (indirect) mechanism that consists of two options: the buyer can get allocation x at a price $T\left(\overline{\theta}\right) + \overline{\theta}\left(x - p\left(\overline{\theta}\right)\right)$, or he can get allocation 0 for free. As in the discussion following Proposition 1, the utility of the buyer of any type θ in the new mechanism ($\widehat{U}(\theta)$) is no greater than in the original mechanism ($U(\theta)$), and the utility of the highest type $\overline{\theta}$ is the same in both mechanisms. However, unlike before, the utility of the lowest type $\underline{\theta}$ is not necessarily the same. The new mechanism is individually rational because the new menu contains the null option, which gives each type of the buyer zero utility.

Using the expression for the expected profit (2), the difference in the expected profits from the new mechanism relative to the original mechanism is

$$\left(U(\underline{\theta}) - \widehat{U}(\underline{\theta})\right)(\underline{\theta} - c) f(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left(U(\theta) - \widehat{U}(\theta)\right)\left(2f(\theta) + (\theta - c)\frac{\partial f(\theta)}{\partial\theta}\right) d\theta \geq 0. \tag{3}$$

The inequality holds because $U(\theta) \geq \widehat{U}(\theta)$ for every θ, as well as because of Condition 1 and the assumption that the marginal costs are low ($\underline{\theta} \geq c$).

Note that the first term in the expression (3) may be negative if the marginal cost is above the lowest possible valuation. This is the technical reason why Proposition 2 requires that it is common knowledge that there are gains from trading each of the goods. The following example provides a more intuitive explanation.

Example 2 *Consider the* indivisible goods *setting with $n = 2$. Let $c_1 = c_2 = c$ and (θ_1, θ_2) be uniformly distributed on $[0,1]^2$. The buyer has the following options: get both goods (allocation $(1,1)$) at a price $\frac{3}{4}$; get second good with probability $\frac{1}{2}$ (allocation $\left(0, \frac{1}{2}\right)$) at a price $\frac{1}{4}$; get nothing (allocation $(0,0)$) for free. It is straightforward to verify that measure $\frac{11}{16}$ of the buyer's types choose the first option, measure $\frac{1}{16}$ of the buyer's types choose the second option, and the rest choose the null option. Hence, the expected profit is $\frac{11}{16}\left(\frac{3}{4} - 2c\right) + \frac{1}{16}\left(\frac{1}{4} - \frac{1}{2}c\right)$.*

While this menu uses only boundary points, allocation $\left(0, \frac{1}{2}\right)$ does not belong to the "north-east" boundary. If we apply the procedure from the proof of Proposition 2, then we must remove the second option from the menu. As a result, measure $\frac{23}{32}$ of the buyer's types choose the first option, and the rest choose the null option. Hence, the expected profit is $\frac{23}{32}\left(\frac{3}{4} - 2c\right)$.

The difference in the expected profits from the new mechanism relative to the original mechanism is

$$\frac{1}{32}\left(\frac{3}{4} - 2c\right) - \frac{1}{16}\left(\frac{1}{4} - \frac{1}{2}c\right) = \frac{1}{32}\left(\frac{1}{4} - c\right),$$

which is negative for $c > \frac{1}{4}$.[12]

The removal of allocation $\left(0, \frac{1}{2}\right)$ from the menu forces the buyer's types, who were formerly choosing this allocation, to self-select into the two remaining options. Half of them choose to buy allocation $(1,1)$, and half choose the null option. When the marginal costs are low, this change increases the social surplus $E\left[(\theta - c) \cdot p(\theta)\right]$ and decreases the buyer's ex ante utility $E\left[U(\theta)\right]$. Hence, the seller's profit goes up. However, when the marginal costs are high, this change may decrease the social surplus $E\left[(\theta - c) \cdot p(\theta)\right]$ because those buyers who have switched to allocation $(1,1)$ value the first good below its cost. Hence, the seller's profit in such a case may go down.

Note that we do not claim that the selling mechanism presented in this example is optimal. It may be that the optimal menu satisfies the property described in Proposition 2, but proving this would require a different approach.

5 Conclusion

We have extended the "no-haggling" result of Riley and Zeckhauser (1983) to the class of linear multiproduct monopoly problems when the buyer's valuations are smoothly distributed. In particular, we have shown that there is no loss for the seller

[12] The detailed calculations are available upon request.

in optimizing over mechanisms such that all allocations belong to the boundary of the feasible set. The class of potentially optimal mechanisms can be further restricted when marginal costs are sufficiently low: the optimal mechanisms use only allocations from the "north-east" boundary of the feasible set and the null allocation.

These results restrict the set of potentially optimal mechanisms, and we hope they will be useful for solving for the optimal selling mechanism in the general case. In Pavlov (2010) we use the results of this paper to characterize the optimal selling mechanism for the case of two goods both in the setting with the substitutable goods and in the setting with the indivisible goods. We also hope an approach similar to the one used here will be useful for studying optimal multi-unit auctions and related problems.

6 Appendix

Proof of Proposition 1. Consider an admissible mechanism (p,T) that generates the utility schedule U, and its menu includes interior points. Define the "north-east" and the "south-west" subsets of the boundary of the feasible set:

$$\begin{aligned}\overline{\partial\Sigma} &= \left\{p \in \Sigma \mid \nexists p' \in \Sigma \text{ such that } p' \gg p\right\},\\ \underline{\partial\Sigma} &= \left\{p \in \Sigma \mid \text{there is } i \in \{1,..,n\} \text{ such that } p_i = 0\right\}.\end{aligned}$$

Note that $\overline{\partial\Sigma} \cup \underline{\partial\Sigma} = \partial\Sigma$. Next, partition the surface of the type set $\partial\Theta$ into the "north-east" and the "south-west" subsets:

$$\begin{aligned}\overline{\partial\Theta} &= \left\{\theta \in \Theta \mid \text{there is } h \in \{1,..,n\} \text{ such that } \theta_h = \overline{\theta}_h\right\},\\ \underline{\partial\Theta} &= \partial\Theta \backslash \overline{\partial\Theta} = \left\{\theta \in \Theta \backslash \overline{\partial\Theta} \mid \text{there is } l \in \{1,..,n\} \text{ such that } \theta_l = \underline{\theta}_l\right\}.\end{aligned}$$

For every type θ from $\overline{\partial\Theta}$ let $h(\theta)$ be an index such that $\theta_{h(\theta)} = \overline{\theta}_{h(\theta)}$. If several such indices exist, then we set $h(\theta)$ to be the lowest one. Similarly, for every type θ from $\underline{\partial\Theta}$ let $l(\theta)$ be the lowest index such that $\theta_{l(\theta)} = \underline{\theta}_{l(\theta)}$.

Introduce a new indirect mechanism, which consists of a set M of message reports, an allocation rule $\widehat{p} : M \to \Sigma$, and a payment rule $\widehat{T} : M \to \mathbb{R}$. Let the set of message reports be $M = \partial\Theta$, and $\widehat{p}$ and $\widehat{T}$ for every $\theta \in \overline{\partial\Theta}$ be defined as follows

$\widehat{p}(\theta) \in \overline{\partial\Sigma}$ such that $\widehat{p}_i(\theta) = p_i(\theta)$ for every $i \neq h(\theta)$ and $\widehat{p}_{h(\theta)}(\theta)$ is as high as possible,

$\widehat{T}(\theta) = T(\theta) + \overline{\theta}_{h(\theta)} \left(\widehat{p}_{h(\theta)}(\theta) - p_{h(\theta)}(\theta)\right).$

For every $\theta \in \underline{\partial\Theta}$ let

$$\widehat{p}(\theta) \in \underline{\partial\Sigma} \text{ such that } \widehat{p}_i(\theta) = p_i(\theta) \text{ for every } i \neq l(\theta) \text{ and } \widehat{p}_{l(\theta)}(\theta) = 0,$$
$$\widehat{T}(\theta) = T(\theta) - \underline{\theta}_{l(\theta)} p_{l(\theta)}(\theta).$$

The new mechanism takes the menu associated with the original mechanism and removes all point contracts other than the ones that were chosen by the types from the surface of the set Θ. Each remaining point contract previously chosen by some $\theta \in \overline{\partial\Theta}$ is further adjusted as follows. The allocation of the good $h(\theta)$ is increased as much as feasibility allows, and the payment is adjusted so that the utility of type θ remains the same as in the original mechanism. The adjustment for each remaining point contract chosen by some $\theta \in \underline{\partial\Theta}$ is as follows. The allocation of the good $l(\theta)$ is reduced to 0, and the payment is adjusted so that the utility of type θ remains the same as in the original mechanism.

The utility of an arbitrary type $\theta \in \Theta$ in the new mechanism is:

$$\widehat{U}(\theta) = \max_{m \in M} \left\{ \sum_{i=1}^{n} \theta_i \widehat{p}_i(m) - \widehat{T}(m) \right\} = \max_{\widetilde{\theta} \in \partial\Theta} \left(\sum_{i=1}^{n} \theta_i \widehat{p}_i\left(\widetilde{\theta}\right) - \widehat{T}\left(\widetilde{\theta}\right) \right). \tag{4}$$

Notice that for every message $\widetilde{\theta} \in \overline{\partial\Theta}$ we have

$$\begin{aligned} & \sum_{i=1}^{n} \theta_i \widehat{p}_i\left(\widetilde{\theta}\right) - \widehat{T}\left(\widetilde{\theta}\right) = \sum_{i=1}^{n} \theta_i p_i\left(\widetilde{\theta}\right) - T\left(\widetilde{\theta}\right) - \\ & \left(\overline{\theta}_{h(\widetilde{\theta})} - \theta_{h(\widetilde{\theta})}\right)\left(\widehat{p}_{h(\widetilde{\theta})}\left(\widetilde{\theta}\right) - p_{h(\widetilde{\theta})}\left(\widetilde{\theta}\right)\right) \\ \leq \; & \sum_{i=1}^{n} \theta_i p_i\left(\widetilde{\theta}\right) - T\left(\widetilde{\theta}\right) \leq \sum_{i=1}^{n} \theta_i p_i(\theta) - T(\theta) = U(\theta). \end{aligned} \tag{5}$$

The first equality holds because of the way the new mechanism is defined, the first inequality is because $\widehat{p}_{h(\widetilde{\theta})}\left(\widetilde{\theta}\right)$ is no smaller than $p_{h(\widetilde{\theta})}\left(\widetilde{\theta}\right)$, and the second inequality is by the fact that the original mechanism is incentive compatible. Note that both inequalities hold as equalities when $\theta = \widetilde{\theta}$.

Similarly notice that for every message $\widetilde{\theta} \in \underline{\partial\Theta}$ we have

$$\begin{aligned} & \sum_{i=1}^{n} \theta_i \widehat{p}_i\left(\widetilde{\theta}\right) - \widehat{T}\left(\widetilde{\theta}\right) = \sum_{i=1}^{n} \theta_i p_i\left(\widetilde{\theta}\right) - T\left(\widetilde{\theta}\right) - \left(\theta_{l(\widetilde{\theta})} - \underline{\theta}_{l(\widetilde{\theta})}\right) p_{l(\widetilde{\theta})}\left(\widetilde{\theta}\right) \\ \leq \; & \sum_{i=1}^{n} \theta_i p_i\left(\widetilde{\theta}\right) - T\left(\widetilde{\theta}\right) \leq \sum_{i=1}^{n} \theta_i p_i(\theta) - T(\theta) = U(\theta). \end{aligned} \tag{6}$$

The first equality holds because of the way the new mechanism is defined, and the second inequality is by the fact that the original mechanism is incentive compatible. Note that both inequalities hold as equalities when $\theta = \widetilde{\theta}$.

Combining (4) with (5) and (6) we get

$$\widehat{U}(\theta) \leq U(\theta) \text{ for every } \theta \in \Theta \tag{7}$$

and

$$\widehat{U}(\theta) = U(\theta) \text{ for every } \theta \in \partial\Theta. \tag{8}$$

To see that the new mechanism is individually rational note that for any type $\theta \in \Theta$ we have

$$\widehat{U}(\theta) \geq \widehat{U}(\underline{\theta}_1, \theta_{-1}) = U(\underline{\theta}_1, \theta_{-1}) \geq 0.$$

The first inequality is due to the fact that, if type θ mimics the choice of type $(\underline{\theta}_1, \theta_{-1})$, then its utility will be at least as large as that of type $(\underline{\theta}_1, \theta_{-1})$, the equality follows from (8), and the second inequality holds because the original mechanism is assumed to be individually rational.

Next we represent the seller's expected profit in terms of the buyer's utility schedule. Substituting out the payments ($T(\theta) = \theta \cdot p(\theta) - U(\theta)$), and making use of the Envelope theorem ($\nabla U(\theta) = p(\theta)$ for a.e. θ), we can rewrite the seller's profit as follows:[13]

$$E[T(\theta) - c \cdot p(\theta)] = E[(\theta - c) \cdot \nabla U(\theta) - U(\theta)]$$

Using integration by parts (or the divergence theorem), we can further rewrite the expression for the expected profit:[14]

$$\begin{aligned} &\sum_{i=1}^{n} \int_{\Theta_{-i}} U\left(\overline{\theta}_i, \theta_{-i}\right)\left(\overline{\theta}_i - c_i\right) f\left(\overline{\theta}_i, \theta_{-i}\right) d\theta_{-i} \\ &- \sum_{i=1}^{n} \int_{\Theta_{-i}} U(\underline{\theta}_i, \theta_{-i})(\underline{\theta}_i - c_i) f(\underline{\theta}_i, \theta_{-i}) d\theta_{-i} \\ &- \int_{\Theta} U(\theta)\left[(n+1) f(\theta) + (\theta - c) \cdot \nabla f(\theta)\right] d\theta, \end{aligned} \tag{9}$$

where $\Theta_{-i} = \times_{j \neq i} \left[\underline{\theta}_j, \overline{\theta}_j\right]$. Using (8), we can find the difference in the expected profits from the new mechanism relative to the original mechanism:

$$\int_{\Theta} \left(U(\theta) - \widehat{U}(\theta)\right) \left[(n+1) f(\theta) + (\theta - c) \cdot \nabla f(\theta)\right] d\theta \geq 0$$

The inequality follows from (7) and Condition 1. Hence, the new mechanism brings at least as much profit as the original mechanism. ■

[13] See for example McAfee and McMillan (1988).

[14] See for example McAfee and McMillan (1988), Rochet and Chone (1998).

Calculations for Example 1.

First note that, without loss of generality, we can restrict attention to symmetric mechanisms:[15]

$$\begin{aligned}
&p_1(0,1) = p_2(1,0), p_2(0,1) = p_1(1,0), T(0,1) = T(1,0) \\
&p_1\left(\frac{1}{2},\frac{1}{2}\right) = p_2\left(\frac{1}{2},\frac{1}{2}\right), p_1(2,2) = p_2(2,2).
\end{aligned}$$

Let us ignore the incentive constraints for $(1,0)$, $(0,1)$, $\left(\frac{1}{2},\frac{1}{2}\right)$ and the individual rationality constraint for $(2,2)$. Then it is optimal to set

$$p_2(1,0) = 0,\ T(1,0) = p_1(1,0) =: p, \text{ and } T\left(\frac{1}{2},\frac{1}{2}\right) = p_1\left(\frac{1}{2},\frac{1}{2}\right) =: p'.$$

In addition, it is optimal to assign efficient allocation to type $(2,2)$: $p_1(2,2) = p_2(2,2) = 1$. Thus the incentive constraints for $(2,2)$ are

$$\begin{aligned}
4 - T(2,2) &\geq 2p - T(1,0) = p, \\
4 - T(2,2) &\geq 4p' - T\left(\frac{1}{2},\frac{1}{2}\right) = 3p'.
\end{aligned}$$

Hence, it is optimal to set $T(2,2) = 4 - \max\{p, 3p'\}$. The problem of the seller reduces to

$$\max_{0\leq p,p'\leq 1} \frac{1}{4}\left(2p + p' + 4 - \max\{p, 3p'\}\right)$$

Notice that

$$2p + p' - \max\{p,3p'\} = \begin{cases} 2p - 2p' < 2p - \frac{2}{3}p \leq \frac{4}{3} & \text{if} \quad 3p' > p \\ p + p' < p + \frac{1}{3}p \leq \frac{4}{3} & \text{if} \quad 3p' < p \\ \frac{4}{3}p \leq \frac{4}{3} & \text{if} \quad 3p' = p \end{cases}.$$

Hence, $p = 1$, $p' = \frac{1}{3}$ and $T(2,2) = 3$. It is easy to check that the ignored constraints are satisfied.

Proof of Proposition 2. By Proposition 1 we can restrict attention to mechanisms which satisfy the "boundary" property. Consider an admissible mechanism (p,T) that generates the utility schedule U, and its menu uses points that do not belong to $\overline{\partial\Sigma} \cup \{\mathbf{0}\}$.

Introduce a new indirect mechanism, which consists of a set M of message reports, an allocation rule $\widehat{p} : M \to \Sigma$, and a payment rule $\widehat{T} : M \to \mathbb{R}$. Let the set of

[15] See for example Section 1 in Maskin and Riley (1984).

message reports be $M = \overline{\partial\Theta} \cup \{\emptyset\}$, and $\widehat{p}$ and $\widehat{T}$ be defined as follows

$$\begin{aligned}
\widehat{p}(\theta) &\in \overline{\partial\Sigma} \text{ and } \widehat{p}_i(\theta) = p_i(\theta) \text{ for every } i \neq h(\theta), \\
\widehat{T}(\theta) &= T(\theta) + \overline{\theta}_{h(\theta)}\left(\widehat{p}_{h(\theta)}(\theta) - p_{h(\theta)}(\theta)\right) \text{ for every } \theta \in \overline{\partial\Theta}, \\
\widehat{p}(\emptyset) &= \mathbf{0}, \widehat{T}(\emptyset) = 0,
\end{aligned}$$

where $h(\theta)$ is defined as in the proof of Proposition 1. The new mechanism takes the menu associated with the original mechanism, and removes all point contracts other than the ones that were chosen by the types from the "north-east" surface $\overline{\partial\Theta}$. Each of the remaining point contracts (indexed by $\theta \in \overline{\partial\Theta}$) is further adjusted as follows. The allocation of good $h(\theta)$ is increased as much as feasibility allows, and the payment is adjusted so that the utility of type θ remains the same as in the original mechanism. The new mechanism contains a message $\emptyset$, which results in the null allocation and no payment.

The utility of an arbitrary type $\theta \in \Theta$ in the new mechanism is:

$$\widehat{U}(\theta) = \max_{m \in M}\left\{\sum_{i=1}^{n} \theta_i \widehat{p}_i(m) - \widehat{T}(m)\right\} = \max\left\{\max_{\widetilde{\theta} \in \overline{\partial\Theta}}\left(\sum_{i=1}^{n} \theta_i \widehat{p}_i\left(\widetilde{\theta}\right) - \widehat{T}\left(\widetilde{\theta}\right)\right), 0\right\}, \tag{10}$$

where the second equality reflects the fact that sending message $\emptyset$ results in a null contract which gives zero utility to every type of buyer. Note that this implies the new mechanism is individually rational.

Using an argument as in the proof of Proposition 1, we can show

$$\widehat{U}(\theta) \leq U(\theta) \text{ for every } \theta \in \Theta \tag{11}$$

and

$$\widehat{U}(\theta) = U(\theta) \text{ for every } \theta \in \overline{\partial\Theta}. \tag{12}$$

Using (9) and (12), the difference in the expected profits from the new mechanism relative to the original mechanism is

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\Theta_{-i}} \left(U(\underline{\theta}_i, \theta_{-i}) - \widehat{U}(\underline{\theta}_i, \theta_{-i})\right)(\underline{\theta}_i - c_i) f(\underline{\theta}_i, \theta_{-i})\, d\theta_{-i} \\
&+ \int_{\Theta} \left(U(\theta) - \widehat{U}(\theta)\right)[(n+1) f(\theta) + (\theta - c) \cdot \nabla f(\theta)]\, d\theta \geq 0
\end{aligned}$$

The inequality holds because of (11), Condition 1, and the assumption that $\underline{\theta}_i \geq c_i$ for every $i = 1, \ldots, n$. Hence, the new mechanism brings at least as much profit as the original mechanism. ■

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