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Optimal Mechanism for Selling Two Goods*

Gregory Pavlov

Abstract

We solve for the optimal mechanism for selling two goods when the buyer's demand characteristics are unobservable. In the case of substitutable goods, the seller has an incentive to offer lotteries over goods in order to charge the buyers with large differences in the valuations a higher price for obtaining their desired good with certainty. However, the seller also has a countervailing incentive to make the allocation of the goods among the participating buyers more efficient in order to increase the overall demand. In the case when the buyer can consume both goods, the seller has an incentive to underprovide one of the goods in order to charge the buyers with large valuations a higher price for the bundle of both goods. As in the case of substitutable goods, the seller also has a countervailing incentive to lower the price of the bundle in order to increase the overall demand.

KEYWORDS: multidimensional screening, price discrimination, optimal selling strategies, mechanism design

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1 Introduction

In this paper we solve for the optimal strategy for selling two heterogeneous goods when the buyer's demand characteristics are unobservable. While it is well known that the optimal strategy for selling a single good is to post a "take-it-or-leave-it" price (Riley and Zeckhauser, 1983), solving for the case of several goods proved to be much harder because of the multidimensional nature of the problem.[1] The main insights into economics of multiproduct price discrimination are the following: (i) the seller generally benefits from excluding a subset of the buyer's types from purchasing any goods (Armstrong, 1996); (ii) the seller generally benefits from offering bundles of goods at a discount in addition to the individually priced goods (Adams and Yellen, 1976; McAfee et al., 1989); and (iii) unlike in the case of a single good, the seller often benefits from using lotteries as a part of the optimal selling mechanism (Thanassoulis, 2004; Manelli and Vincent, 2006, 2007).

We consider two different settings: the case of substitutable goods and the case of indivisible goods. In the case of substitutable goods, the buyer can consume only a single unit of a good, and thus it is never optimal to give the buyer a bundle of two goods. The optimal mechanism in this case is a result of the interplay between the optimal use of stochastic contracts and the incentive to exclude some buyers. In the case of indivisible goods, the buyer can consume both goods, and the optimal mechanism is a result of the interplay between all three tools of multiproduct price discrimination: exclusion, bundling, and stochastic contracts.

The starting point of our analysis of the model of substitutable goods is the result in Pavlov (2010) that says there is no loss for the seller in optimizing over mechanisms where the buyer either gets a good for sure or gets no good.[2] In the former case, however, the seller may find it optimal to provide lotteries that determine whether the buyer receives the first good or the second good. Thus, each buyer who decides not to choose the null option is guaranteed to get at least the less desirable good of the two available. The willingness to pay for getting the more desirable good is the difference in the buyer's valuations between the two goods, which becomes a natural screening variable in the seller's problem. Note that it is efficient to assign to each type of buyer his most preferred good with certainty. However, the seller is inclined to assign lotteries to the buyers with small differences in the valuations in order to charge the buyers with large differences in the valuations a higher price for the option of getting their most preferred good with certainty. This is not

[1] See Rochet and Stole (2003) for a survey of recent literature.

[2] This property can be viewed as a natural extension of the "no-haggling" result of Riley and Zeckhauser (1983) to the case of multiple goods.

the end of the story, however, because the offered menu of options determines the size and shape of the exclusion region. Other things being equal, the share of the participating types is larger if the buyer receives his most preferred good with certainty rather than some lottery. Hence, the seller's incentive to use lotteries in order to extract extra payments from the buyer's types with high differences in the valuations comes into conflict with an incentive to make the allocation more efficient in order to expand the share of participating types. We explicitly calculate the optimal selling mechanism when the buyer's types are uniformly distributed on a square, and we discuss how the seller's conflicting incentives are resolved depending on the support of the distribution.

In the model of indivisible goods, the seller can optimize over mechanisms in which the buyer either gets no goods or gets the more preferred good for sure and the less preferred good with some probability (Pavlov, 2010). Therefore, the probability of assigning the less preferred good becomes a natural screening variable among the participating buyer's types. Note that it is efficient to assign the bundle of both goods to each type of buyer who chooses to participate. However, the seller is inclined to reduce the assignment of the less preferred good for some of the buyer's types in order to charge a higher price for the bundle. As in the model of substitutable goods, the seller's desire to price discriminate is mitigated by an incentive to improve the overall efficiency of the allocation by offering just the bundle of two goods at a reduced price in order to raise the overall demand. We explicitly calculate the optimal selling mechanism when the buyer's types are uniformly distributed on a square and discuss how the seller's conflicting incentives are resolved, depending on the parameters.

The rest of the paper is organized as follows. The model is presented in Section 2. The analyses of the case of substitutable goods and the case of indivisible goods are in Section 3 and 4, respectively. Conclusion is in Section 5. Long proofs and calculations for examples are in the Appendix.

2 Model

There is one buyer and one seller who owns two indivisible goods.[3] The buyer values good i at θ_i, which is known only to him. A pair of valuations $\theta = (\theta_1, \theta_2)$ is distributed according to an almost everywhere positive bounded differentiable density f on the support $\Theta = [\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2] \subset \mathbb{R}^2_+$. This distribution is common knowledge.

[3] Throughout the paper we use masculine pronouns for the buyer and feminine pronouns for the seller.

All players have linear utilities. The buyer's utility is $\theta_1 p_1 + \theta_2 p_2 - T$, where $p = (p_1, p_2)$ is the vector of allocations of each of the goods, and T is his payment to the seller. The seller's utility is T.

We study the following two scenarios:

1. *Substitutable goods* (Thanassoulis, 2004; Balestrieri and Leao, 2008). The buyer can consume just one unit of any good. In this case p_i is the probability that the buyer consumes good i, and the feasible set is $\Sigma = \left\{p \in \mathbb{R}_+^2 \mid p_1 + p_2 \leq 1\right\}$.[4]

2. *Indivisible goods* (McAfee and McMillan, 1988; McAfee et al., 1989; Manelli and Vincent, 2006, 2007). All goods are desirable from the point of view of the buyer. In this case p_i is the probability that the buyer gets good i, and the feasible set is $\Sigma = \left\{p \in \mathbb{R}_+^2 \mid 0 \leq p_1, p_2 \leq 1\right\}$.

By the revelation principle we can without loss of generality assume that the seller offers a direct mechanism, which consists of a set Θ of type reports, an allocation rule $p : \Theta \to \Sigma$, and a payment rule $T : \Theta \to \mathbb{R}$.[5] The seller's problem is stated below.

Program I : $\quad \max_{(p,T)} E[T(\theta)] \quad$ subject to

Feasibility: $p(\theta) \in \Sigma$ for every $\theta \in \Theta$;

Incentive Compatibility: $\theta_1 p_1(\theta) + \theta_2 p_2(\theta) - T(\theta) \geq \theta_1 p_1(\theta') + \theta_2 p_2(\theta') - T(\theta')$ for every $\theta, \theta' \in \Theta$;

Individual Rationality: $\theta_1 p_1(\theta) + \theta_2 p_2(\theta) - T(\theta) \geq 0$ for every $\theta \in \Theta$.

We call a mechanism (p, T) *admissible* if it satisfies the above constraints. Denote the equilibrium utility of the buyer of type θ by $U(\theta) = \theta_1 p_1(\theta) + \theta_2 p_2(\theta) - T(\theta)$.

We require the distribution to satisfy a version of a "hazard rate condition" that is standard in the multidimensional mechanism design literature.[6]

[4] Note that the seller never benefits from assigning to the buyer a bundle of two goods, because then the buyer would consume only the good that he values most. Thus, we can denote by p_i the probability that good i (and only good i) is assigned to the buyer.

[5] The seller never benefits from randomized payments because the payoffs are linear in money. Thus, there is no loss of generality in restricting attention to deterministic payment rules.

[6] The condition for the case of n goods says that the density f satisfies

$$(n+1)f(\theta) + \theta \cdot \nabla f(\theta) \geq 0 \text{ for every } \theta \in \Theta,$$

where ∇f is the gradient of f. See for example McAfee and McMillan (1988), Manelli and Vincent (2006).

Condition 1 *The density f satisfies*

$$3f(\theta_1,\theta_2)+\theta_1\frac{\partial f(\theta_1,\theta_2)}{\partial\theta_1}+\theta_2\frac{\partial f(\theta_1,\theta_2)}{\partial\theta_2}\geq 0.$$

3 Substitutable goods

3.1 Reformulation of the seller's problem

First, we simplify the seller's problem in the case of substitutable goods using the following result.

Proposition 1 *Under Condition 1 there is no loss for the seller in optimizing over mechanisms that for every $\theta\in\Theta$ satisfy:*

$$p_1(\theta)+p_2(\theta)\in\{0,1\}.$$

Proof. See Proposition 2 in Pavlov (2010).[7] ■

This result states that in the optimal mechanism the buyer either gets a good for sure ($p_1+p_2=1$), or gets no good ($p_1+p_2=0$). One can view this result as an extension of the "no-haggling" result of Riley and Zeckhauser (1983). For the case of one good, they have shown that the seller's optimal mechanism, when dealing with a risk-neutral buyer, is to quote a single "take-it-or-leave-it" price; so that the buyer either gets the good for sure or gets no good. Note that Proposition 1 does not rule out lotteries over goods as a part of the optimal mechanism since there is no restriction $p_1,p_2\in\{0,1\}$. As will be shown in the next section, the seller often finds it optimal to offer lotteries as a part of the optimal mechanism.

Consider the buyer of type (θ_1,θ_2) and suppose $\theta_1\geq\theta_2$. If he chooses to purchase some non-null allocation (p_1,p_2) at a price T, then his utility is

$$\theta_1p_1+\theta_2p_2-T=(\theta_1-\theta_2)p_1+\theta_2-T$$

where the equality is due to $p_1+p_2=1$. Thus, the buyer is guaranteed to get at least the value of the less preferred good (θ_2), and his willingness to pay for a higher probability of the more preferred good (p_1) depends just on the difference in the valuations ($\theta_1-\theta_2$). Moreover, note that any buyer of type $(\widetilde{\theta}_1,\widetilde{\theta}_2)$, such that $\widetilde{\theta}_1-\widetilde{\theta}_2=\theta_1-\theta_2$, will choose the same contract as type (θ_1,θ_2) if $\widetilde{\theta}_2$ is sufficiently high (unless there exists another contract that gives him the same payoff), and will choose the null allocation $(0,0)$ at zero price if $\widetilde{\theta}_2$ is low enough.

[7] Balestrieri and Leao (2008) also provide this property for the case of two substitutable goods. In their model, however, the buyer's private information is one-dimensional.

Hence, it is natural to conjecture that there is no loss for the seller in optimizing over the set of mechanisms in which the screening is performed only on the differences in the valuations conditional on participation. The next proposition shows that this is indeed the case.[8]

Denote the difference in the valuations by $\delta = \theta_1 - \theta_2$. The set of possible differences in the valuations is the interval $[\underline{\delta}, \overline{\delta}] = [\underline{\theta}_1 - \overline{\theta}_2, \overline{\theta}_1 - \underline{\theta}_2]$. Assume the seller offers a mechanism that consists of a set of messages $M = [\underline{\delta}, \overline{\delta}] \cup \{\emptyset\}$, an allocation rule $\alpha : [\underline{\delta}, \overline{\delta}] \to [0,1]$, and a payment rule $t : [\underline{\delta}, \overline{\delta}] \to \mathbb{R}$. The set of messages includes all possible differences in the valuations, and a special message $\emptyset$ that indicates the buyer is not willing to participate and thus receives the null allocation and no payment. The allocation rule α associates with each message report (other than $\emptyset$) an allocation, $\alpha(\delta)$ and $1 - \alpha(\delta)$ being the probabilities that the buyer is assigned good 1 and 2, respectively, when the message is δ. The payment rule t associates with each message report (other than $\emptyset$) a payment, $t(\delta)$ being the payment that the buyer pays when the message is δ. The seller's problem is stated below.

Program II : $\max\limits_{(\alpha,t)} E[t(\delta)]$ subject to

Feasibility: $\alpha(\delta) \in [0,1]$ for every $\delta \in [0,1]$;

Incentive Compatibility: $\delta\alpha(\delta) - t(\delta) \geq \delta\alpha(\delta') - t(\delta')$ for every $\delta, \delta' \in [\underline{\delta}, \overline{\delta}]$.

Note that every such mechanism is individually rational, because message $\emptyset$ gives each type of the buyer zero utility.

Proposition 2 *Suppose mechanism (α,t) solves Program II. Then there exists mechanism (p,T) that is outcome equivalent to mechanism (α,t) and solves Program I.*

Let $u(\delta) = \delta\alpha(\delta) - t(\delta)$. The payoff of the buyer of type (θ_1, θ_2) with a difference in the valuation δ is $u(\delta) + \theta_2$ if he chooses to participate, and is 0 if he chooses message $\emptyset$. Each type of buyer participates only if the payoff from participation is nonnegative. The profit from the buyer of type θ is $t(\delta) = \delta\alpha(\delta) - u(\delta)$ whenever $u(\delta) + \theta_2 \geq 0$, and 0 otherwise. Let us denote the measure of the participating types for a given δ by

$$g(u(\delta), \delta) = \int_{\substack{\theta:\ \theta_1 - \theta_2 = \delta, \\ u(\delta) + \theta_2 \geq 0}} f(\theta) d\theta.$$

[8]The proof of this result is similar to the proof of a similar property in Gruyer (2009).

This allows us to rewrite the seller's problem:

Lemma 1 *Program II is equivalent to Program II′.*

Program II′: $\max_{(\alpha,u)} \int_{\underline{\delta}}^{\overline{\delta}} (\delta\alpha(\delta) - u(\delta))g(u(\delta),\delta)d\delta$ subject to

$F : \alpha(\delta) \in [0,1]$ for every $\delta \in [\underline{\delta}, \overline{\delta}]$;

IC : (i) α is nondecreasing; (ii) $u(\delta) = u(0) + \int_0^{\delta} \alpha(\widetilde{\delta})d\widetilde{\delta}$ for every $\delta \in [\underline{\delta}, \overline{\delta}]$.

Proof. Note that

$$E[t(\delta)] = \int_{\underline{\delta}}^{\overline{\delta}} (\delta\alpha(\delta) - u(\delta))g(u(\delta),\delta)d\delta.$$

Using a standard argument, it is possible to show that the set of incentive compatibility constraints in Program II is equivalent to IC constraints in Program II′.[9] ■

The problem of the seller can be further simplified when the distribution of the valuations is symmetric.

Lemma 2 *Suppose the distribution* (Θ, f) *is symmetric, i.e. (i)* $[\underline{\theta}_1, \overline{\theta}_1] = [\underline{\theta}_2, \overline{\theta}_2] = [\underline{\theta}, \overline{\theta}]$*; (ii)* $f(\theta_1,\theta_2) = f(\theta_2,\theta_1)$ *for every* (θ_1,θ_2)*. Then Program II is equivalent to Program II″.*

Program II″: $\max_{(\alpha,u)} \int_0^{\overline{\delta}} (\delta\alpha(\delta) - u(\delta))g(u(\delta),\delta)d\delta$ subject to

$F : \alpha(\delta) \in [\frac{1}{2}, 1]$ for every $\delta \in [0, \overline{\delta}]$;

IC : (i) α is nondecreasing; (ii) $u(\delta) = u(0) + \int_0^{\delta} \alpha(\widetilde{\delta})d\widetilde{\delta}$ for every $\delta \in [0, \overline{\delta}]$.

Proof. In a symmetric environment there is no loss of generality in restricting attention to symmetric mechanisms.[10] In symmetric mechanisms we have $\alpha(\delta) = 1 - \alpha(-\delta)$ for every $\delta \in [0, \overline{\delta}]$. Since α is nondecreasing, we must have $\alpha(\delta) = 1 - \alpha(-\delta) \geq 1 - \alpha(\delta)$, which implies $\alpha(\delta) \geq \frac{1}{2}$ for every $\delta \in [0, \overline{\delta}]$. ■

For the rest of the paper we restrict attention to the symmetric case.

[9] See for example Myerson (1981).

[10] See for example Section 1 in Maskin and Riley (1984).

3.2 Properties of the optimal mechanism

In this section we discuss the properties of the solution to the seller's problem. In the Appendix we formulate Program II" as an optimal control problem and provide the necessary conditions for optimality.[11]

A non-standard feature of Program II" is that for every δ the measure of participating types $g(u(\delta),\delta)$ depends on $u(\delta)$, and thus on the mechanism offered by the seller. To fix ideas, let us first consider a simpler problem, where the measure of participating types is given by $h(\delta)$, which is independent of u. In this case, the marginal contribution of allocation $\alpha(\delta)$ to the profit is given by $W(\delta) = \delta h(\delta) - \int_{\delta}^{\overline{\delta}} h(\widetilde{\delta})d\widetilde{\delta}$. This expression illustrates the standard "rent extraction effect": if we increase $\alpha(\delta)$, then we can charge type δ a higher price, but we will also have to leave higher informational rents to all types above δ.[12]

Note that $W(0) < 0 \leq W(\overline{\delta})$. If the marginal profit function W is continuous and crosses zero from below only once, then it is optimal to assign the lowest possible allocation (here $\alpha = \frac{1}{2}$) to the types below the crossing point and the highest possible allocation ($\alpha = 1$) to the types above the crossing point. If W crosses zero from below more than once, then one has to use the "ironing technique".[13] In any case, the optimal allocation α is determined by the exogenously given marginal profit function W.

In Program II$''$ the marginal contribution of allocation $\alpha(\delta)$ to the profit is as follows:[14]

$$V(\delta) = \underbrace{\delta g(u(\delta),\delta) - \int_{\delta}^{\overline{\delta}} g(u(\widetilde{\delta}),\widetilde{\delta})d\widetilde{\delta}}_{\text{rent extraction effect}} + \underbrace{\int_{\delta}^{\overline{\delta}} (\widetilde{\delta}\alpha(\widetilde{\delta}) - u(\widetilde{\delta}))\frac{\partial}{\partial u} g(u(\widetilde{\delta}),\widetilde{\delta})d\widetilde{\delta}}_{\text{participation effect}} \quad (1)$$

[11] The proofs of sufficiency of the necessary conditions and uniqueness of the solution are availble in the earlier version of this paper (Pavlov, 2006).

[12] The marginal contribution of allocation $\alpha(\delta)$ to the profit is often presented in a different way:

$$W(\delta) = (\delta - \frac{\int_{\delta}^{\overline{\delta}} h(\widetilde{\delta})d\widetilde{\delta}}{h(\delta)})h(\delta)$$

where the expression in the brackets is called the "virtual valuation". See for example Myerson (1981), Riley and Zeckhauser (1983).

[13] The optimality conditions in this case are roughly as follows. The marginal profit function W must cross zero from below at every point where α changes its value. If on a given interval (δ_1,δ_2) we have $\alpha = \frac{1}{2}$, then $\int_{\delta_1}^{\delta_2} W(\delta)d\delta \leq 0$; if $\alpha \in (\frac{1}{2},1)$, then $\int_{\delta_1}^{\delta_2} W(\delta)d\delta = 0$; and if $\alpha = 1$, then $\int_{\delta_1}^{\delta_2} W(\delta)d\delta \geq 0$. See for example Myerson (1981), Riley and Zeckhauser (1983), Guesnerie and Laffont (1984).

[14] For details see equation (8) in the Appendix.

The first collection of terms is the "rent extraction effect" illustrated above. The last term is the effect on the profit of the allocation at δ through the participation decisions of the types above δ. Increasing $\alpha(\delta)$ raises the informational rents for all types $\widetilde{\delta} \geq \delta$ and thus increases the measure of the participating types by $\frac{\partial}{\partial u} g(u(\widetilde{\delta}), \widetilde{\delta})$. Every new participant of type $\widetilde{\delta}$ brings an extra profit of $\widetilde{\delta}\alpha(\widetilde{\delta}) - u(\delta)$. Hence, unlike W, the marginal profit V endogenously depends on the mechanism offered by the seller, and this complicates the problem.[15]

The solution retains some similarity to the solution to the simple problem without participation effects. The seller might find it optimal to assign an inefficient allocation $\alpha(\delta) < 1$ to a given type δ in order to reduce the informational rents to all types above δ. This concern is (nearly) absent when δ is close to $\overline{\delta}$, and thus it is optimal to assign efficient allocations to such types. Since the "participation effect" is always nonnegative, it can only reinforce the incentive to have "no distortion at the top".

Proposition 3 *In the optimal mechanism there exists $\delta^* \in [0, \overline{\delta})$ such that $\alpha(\delta) = 1$ for every $\delta \in (\delta^*, \overline{\delta}]$.*

Riley and Zeckhauser (1983) have shown that in the problem without participation effects the optimal allocation always takes a simple two-step form: there exists $\delta^* \in [0, \overline{\delta})$ such that all types below δ^* get the lowest possible allocation (here $\alpha = \frac{1}{2}$) and all types above δ^* get the highest possible allocation ($\alpha = 1$). In our problem the seller sometimes strictly benefits from assigning interior allocations $\alpha \in (\frac{1}{2}, 1)$ to a subset of types.

Example 1 *Let the distribution of the valuations be uniform on $\Theta = [c, c+1]^2$ where $c \geq 0$. The optimal mechanism is as follows.*

(i) When $c \in [0,1]$:

$$(\alpha(\delta), t(\delta)) = \left(1,\ \frac{2}{3}c + \frac{1}{3}\sqrt{c^2+3}\right) \textit{ for every } \delta \in [0,1]$$

(ii) When $c \in (1, \overline{c})$ (where $\overline{c} \approx 1.372$):

$$(\alpha(\delta), t(\delta)) = \begin{cases} \left(\frac{27}{32} + (\frac{9}{32} - \frac{1}{4}c)\sqrt{16c+9},\ \frac{1}{3}c + \frac{3}{8} + \frac{1}{8}\sqrt{16c+9}\right) & \textit{if} \quad \delta \in [0, \frac{1}{3}) \\ \left(1,\ \frac{1}{3}c + \frac{41}{96} + (\frac{1}{12}c + \frac{1}{32})\sqrt{16c+9}\right) & \textit{if} \quad \delta \in (\frac{1}{3}, 1] \end{cases}$$

[15] Incidentally, the mathematical structure of the resulting problem is very similar to the model of Rochet and Stole (2002), who study the problem of nonlinear pricing when the buyers have heterogeneous outside options. The main difference is that their model has quadratic costs. The solutions to these two models are qualitatively different; in Rochet and Stole (2002) the optimal allocation is (for the most part) separating, while in our model there is a significant amount of pooling.

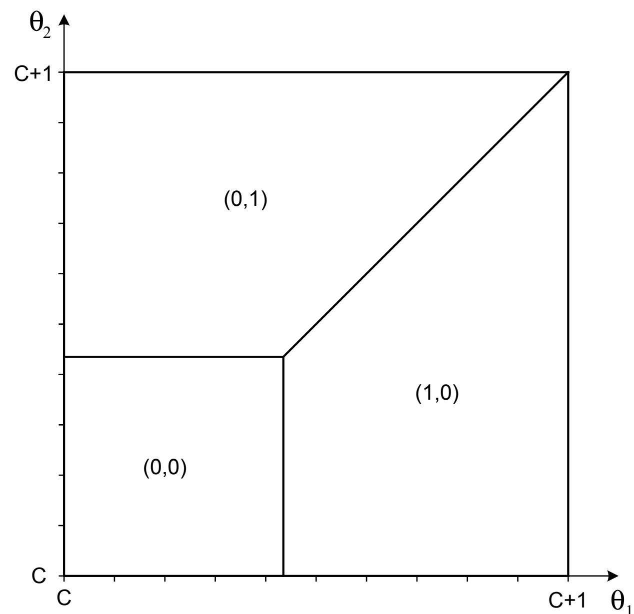

Figure 1: Optimal allocation in Example 1 when $c \in [0,1]$.

(iii) When $c \in [\bar{c}, +\infty)$*:*

$$(\alpha(\delta), t(\delta)) = \begin{cases} (\frac{1}{2},\ \frac{2}{3}c + \frac{1}{3}\sqrt{c^2 + \frac{3}{2}}) & \textit{if} \quad \delta \in [0, \frac{1}{3}) \\ (1,\ \frac{1}{6} + \frac{2}{3}c + \frac{1}{3}\sqrt{c^2 + \frac{3}{2}}) & \textit{if} \quad \delta \in (\frac{1}{3}, 1] \end{cases}$$

When c is small, it is optimal to offer the buyer an option to purchase any good he likes at a given price (see Figure 1). The reason why the seller does not gain from offering lotteries is as follows. First, note that it is not too costly to exclude buyers since c is small, and, by doing so, the seller can raise the prices across the board on all options she plans to offer. Second, since the size of the exclusion region is relatively large, the seller can attract many new buyer's types by offering them efficient allocation rather than lotteries. In other words, the "participation effect", which pushes towards a more efficient allocation, dominates the "rent extraction effect".

When c is large, it is optimal to offer a fair lottery $(\frac{1}{2}, \frac{1}{2})$ over the goods (at a discount), in addition to the option of purchasing any good at a given price (see Figure 2). The exclusion region in this case is relatively small because all the buyer's types can be charged a high price. There is little to be gained by making

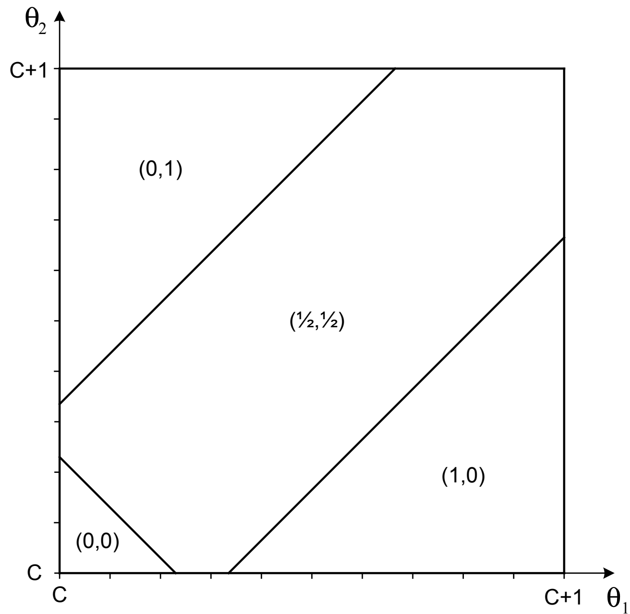

Figure 2: Optimal allocation in Example 1 when $c \in [\overline{c}, +\infty)$.

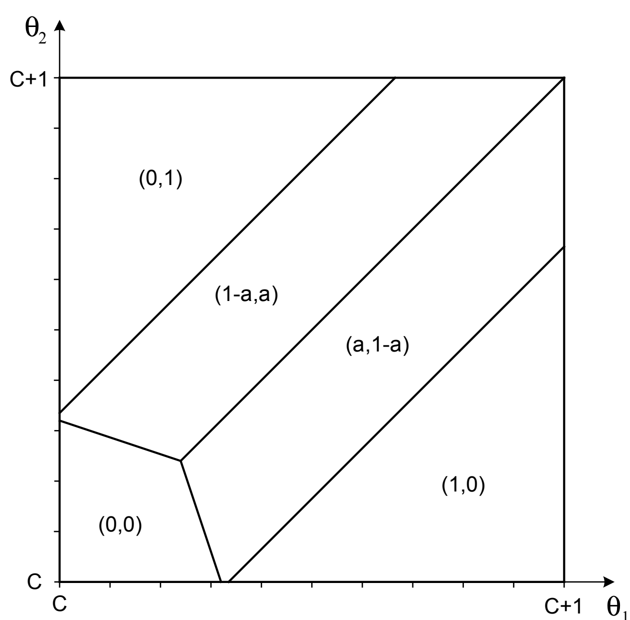

Figure 3: Optimal allocation in Example 1 when $c \in (1, \overline{c})$.

the allocation more efficient, since not so many of the buyer's types were left out. Hence, the "participation effect" is dominated in this case by the "rent extraction effect", which pushes towards offering an inefficient allocation (a fair lottery) to the buyers with small difference in the valuations in order to charge the other buyers a higher price for the option of getting their preferred good for sure.

When c is in the intermediate range, it is optimal to offer biased lotteries $(\alpha, 1-\alpha)$ and $(1-\alpha, \alpha)$ over the goods (at a discount), in addition to the option of purchasing any good for sure at a given price (see Figure 3). Neither the "participation effect", nor the "rent extraction effect" is strong enough to dominate, and the form of the optimal mechanism is the result of a trade-off between them.

The optimal menus are remarkably simple in the sense of containing a very few point contracts. The technical reason for this is roughly as follows. The seller's optimal control problem is of a "bang-bang" nature in α. A number of pooling regions for α emerge due to the presence of the monotonicity constraint, but there are only very few such regions. We conjecture that generically the optimal menus are simple in this sense.[16,17]

It is interesting to compare the expected profits from the fully optimal mechanism and the best deterministic mechanism, which makes no use of the lotteries. It is possible to show that the relative gain from using a fully optimal mechanism is at most about 1.2% (see Figure 4).[18] The next example demonstrates that there are situations when deterministic mechanisms perform much worse than the fully optimal mechanisms that use lotteries. This example is with a discrete distribution, but it is possible to construct a similar example with a continuous distribution.

[16]If allocation α is strictly increasing on an interval, then by Lemma 4 in the Appendix the marginal profit function V must be equal to zero throughout this interval. Differentiating V with respect to δ we get the condition:

$$\begin{aligned}
& \dot{V}(\delta) = g(u(\delta),\delta) + \delta\alpha(\delta)(\frac{\partial}{\partial u}g(u(\delta),\delta)) + \delta(\frac{\partial}{\partial \delta}g(u(\delta),\delta)) \\
& +g(u(\delta),\delta) - (\delta\alpha(\delta) - u(\delta))(\frac{\partial}{\partial u}g(u(\delta),\delta)) \\
= \quad & 2g(u(\delta),\delta) + \delta(\frac{\partial}{\partial \delta}g(u(\delta),\delta)) + u(\delta)(\frac{\partial}{\partial u}g(u(\delta),\delta)) = 0
\end{aligned}$$

This expression depends just on the rent schedule u and the exogenously given distribution of valuations. Intuitively, it takes a very special distribution to make this condition hold on a nondegenerate interval. For more discussion of this issue, see the earlier version of this paper (Pavlov, 2006).

[17]Balestrieri and Leao (2008) show that the seller sometimes finds it optimal to offer a menu that contains a continuum of lotteries. We conjecture that this result is due to the fact that, in their model, the buyer's private information is one-dimensional.

[18]See Appendix for the formulas of the expected profits.

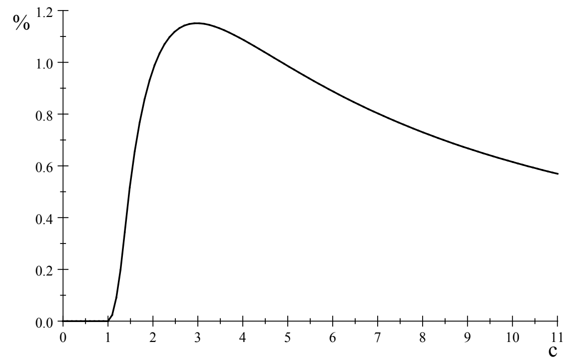

Figure 4: Profit comparison in Example 1: optimal vs. the best deterministic mechanism.

Example 2 *There are three equally likely types:* $(1,0)$*,* $(0,1)$*, and* $(\frac{1}{2},\frac{1}{2})$*. In the optimal mechanism, type* $(1,0)$ *gets the first good at a price* 1*, type* $(0,1)$ *gets the second good at a price* 1*, and type* $(\frac{1}{2},\frac{1}{2})$ *at a price* $\frac{1}{2}$ *gets a lottery that delivers a good with certainty, with probability* $\frac{1}{2}$*, it is the first good and with probability* $\frac{1}{2}$*, it is the second good. To see that this mechanism is optimal, note that the allocation is efficient, payoff of each type of the buyer is zero, and no type wants to deviate. Hence, the seller captures the whole efficient surplus (*$\frac{5}{6}$*) and cannot do any better.*

A deterministic mechanism is a pair of prices T_1 *and* T_2 *for goods* 1 *and* 2*. It is easy to see that the prices other than* $\frac{1}{2}$ *or* 1 *are dominated. If* $T_1 = T_2 = \frac{1}{2}$*, then the profit is* $\frac{1}{2}$*; if* $T_1 = T_2 = 1$ *or* $T_i = \frac{1}{2}$ *and* $T_j = 1$*, then the profit is* $\frac{2}{3}$*. Hence, the relative gain in the expected profit from using the fully optimal mechanism rather than the best deterministic mechanism is* 25%*.*[19]

4 Indivisible goods

4.1 Reformulation of the seller's problem

First, we simplify the seller's problem in the case of indivisible goods using the following result.

[19] Thanassoulis (2004) also argues in favor of using stochastic contracts, but he only provides an example where the gain in profit is 8%.

Proposition 4 *Under Condition 1 there is no loss for the seller in optimizing over mechanisms that for every $\theta \in \Theta$ satisfy:*

$$\text{If } (p_1(\theta), p_2(\theta)) \neq (0,0) \text{ then } p_i(\theta) = 1 \text{ for some } i = 1,2.$$

Proof. See Proposition 2 in Pavlov (2010). ■

When the distribution of the valuations is symmetric, we get additional restrictions on the set of potentially optimal mechanisms.

Corollary 1 *Suppose the distribution (Θ, f) is symmetric, i.e., (i) $[\underline{\theta}_1, \overline{\theta}_1] = [\underline{\theta}_2, \overline{\theta}_2] = [\underline{\theta}, \overline{\theta}]$; (ii) $f(\theta_1, \theta_2) = f(\theta_2, \theta_1)$ for every (θ_1, θ_2). There is no loss for the seller in optimizing over mechanisms that for every $\theta \in \Theta$ satisfy:*

$$\text{If } (p_1(\theta), p_2(\theta)) \neq (0,0) \text{ and } \theta_i > \theta_j, \text{ then } p_i(\theta) = 1.$$

Proof. In a symmetric environment there is no loss of generality in restricting attention to symmetric mechanisms.[20] Note that in symmetric mechanisms we have $p_2(\theta_2, \theta_1) = p_1(\theta_1, \theta_2)$, $p_1(\theta_2, \theta_1) = p_2(\theta_1, \theta_2)$, $T(\theta_2, \theta_1) = T(\theta_1, \theta_2)$.

Incentive compatibility for type (θ_1, θ_2) requires

$$\begin{aligned} \theta_1 p_1(\theta_1, \theta_2) + \theta_2 p_2(\theta_1, \theta_2) - T(\theta_1, \theta_2) &\geq \theta_1 p_1(\theta_2, \theta_1) + \theta_2 p_2(\theta_2, \theta_1) - T(\theta_2, \theta_1) \\ &= \theta_1 p_2(\theta_1, \theta_2) + \theta_2 p_1(\theta_1, \theta_2) - T(\theta_1, \theta_2) \end{aligned}$$

which implies

$$(\theta_1 - \theta_2)(p_1(\theta_1, \theta_2) - p_2(\theta_1, \theta_2)) \geq 0.$$

Hence, if $(p_1(\theta), p_2(\theta)) \neq (0,0)$ and $\theta_1 > \theta_2$, then by Proposition 4 we must have $p_1(\theta) = 1$. ■

Since the optimal mechanism is symmetric, we can solve just for the case $\theta_1 \geq \theta_2$. If the buyer of type (θ_1, θ_2) chooses to purchase some non-null allocation (p_1, p_2) at the price T, then his utility is $\theta_1 + \theta_2 p_2 - T$. Thus the buyer is guaranteed to get at least the value of his most preferred good (θ_1), and his willingness to pay for a higher probability of the less preferred good (p_2) depends just on the valuation of the second good (θ_2). Moreover, note that any buyer of type $(\widetilde{\theta}_1, \theta_2)$, such that $\widetilde{\theta}_1 \geq \theta_2$, will choose the same contract as type (θ_1, θ_2) if $\widetilde{\theta}_1$ is sufficiently high (unless there exists another contract which gives him the same payoff), and will choose the null allocation $(0,0)$ at zero price if $\widetilde{\theta}_1$ is low enough.

As in the case of substitutable goods, it is natural to conjecture that there is no loss for the seller in optimizing over a smaller set of mechanisms in which

[20] See for example Section 1 in Maskin and Riley (1984).

the screening is performed only on valuation for the less preferred good (θ_2) conditional on participation.

Assume the seller offers a mechanism that consists of a set of messages $M = [\underline{\theta}_2, \overline{\theta}_2] \cup \{\emptyset\}$, an allocation rule $\beta : [\underline{\theta}_2, \overline{\theta}_2] \to [0,1]$, and a payment rule $t : [\underline{\theta}_2, \overline{\theta}_2] \to \mathbb{R}$. The set of messages includes all possible valuations for the second good and a special message $\emptyset$ that indicates the buyer is not willing to participate and thus receives the null allocation and no payment. The allocation rule β associates with each message report (other than $\emptyset$) an allocation, 1 and $\beta(\theta_2)$ being the probabilities that the buyer is assigned good 1 and 2, respectively, when the message is θ_2. The payment rule t associates with each message report θ_2 a payment $t(\theta_2)$. The seller's problem is stated below.

Program III : $\max_{(\beta,t)} E[t(\delta)]$ subject to

Feasibility: $\beta(\theta_2) \in [0,1]$ for every $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$;

Incentive Compatibility: $\theta_2\beta(\theta_2) - t(\theta_2) \geq \theta_2\beta(\theta_2') - t(\theta_2')$ for every $\theta_2, \theta_2' \in [\underline{\theta}_2, \overline{\theta}_2]$.

Proposition 5 *Suppose mechanism (β,t) solves Program III. Then there exists mechanism (p,T) that is outcome equivalent to mechanism (β,t) and solves Program I.*

Let $u(\theta_2) = \theta_2\beta(\theta_2) - t(\theta_2)$. The payoff of the buyer of type (θ_1, θ_2) is $u(\theta_2) + \theta_1$ if he chooses to participate, and is 0 if he chooses message $\emptyset$. Each type of buyer participates only if the payoff from participation is nonnegative. The profit from the buyer of type (θ_1, θ_2) is $t(\theta_2) = \theta_2\beta(\theta_2) - u(\theta_2)$ whenever $u(\theta_2) + \theta_1 \geq 0$, and 0 otherwise. Let us denote the measure of the participating types for a given θ_2 by

$$g(u(\theta_2), \theta_2) = \int_{\substack{u(\theta_2)+\theta_1 \geq 0, \\ \theta_1 \geq \theta_2}} f(\theta_1, \theta_2) d\theta_1.$$

This allows us to rewrite the seller's problem:

Lemma 3 *Program III′ is equivalent to Program III.*

Program III′: $\max_{(\beta,u)} \int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_2\beta(\theta_2) - u(\theta_2)) g(u(\theta_2), \theta_2) d\theta_2$ subject to

F : $\beta(\theta_2) \in [0,1]$ for every $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$;

IC : (i) β is nondecreasing; (ii) $u(\theta_2) = u(\underline{\theta}_2) + \int_{\underline{\theta}_2}^{\theta_2} \beta(\widetilde{\theta}_2) d\widetilde{\theta}_2$ for every $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$.

Proof. Note that

$$E[t(\delta)] = \int_{\underline{\theta}_2}^{\overline{\theta}_2}(\theta_2\beta(\theta_2) - u(\theta_2))g(u(\theta_2), \theta_2)d\theta_2.$$

Using a standard argument, it is possible to show that the set of incentive compatibility constraints in Program III is equivalent to *IC* constraints in Program III$'$.[21]
■

4.2 Properties of the optimal mechanism

As in the case of substitutable goods, it is possible to set up the seller's problem given by Program III$'$ as an optimal control problem and obtain the necessary conditions for optimality. Formally Program III$'$ is very similar to Program II", and thus we omit the technical details and just focus on the intuition and the results.

The marginal contribution of allocation $\beta(\theta_2)$ to the profit is as follows

$$V(\theta_2) = \underbrace{\theta_2 g(u(\theta_2), \theta_2) - \int_{\theta_2}^{\overline{\theta}_2} g(u(\widetilde{\theta}_2), \widetilde{\theta}_2)d\widetilde{\theta}_2}_{\text{rent extraction effect}} + \underbrace{\int_{\theta_2}^{\overline{\theta}_2}(\widetilde{\theta}_2\beta(\widetilde{\theta}_2) - u(\widetilde{\theta}_2))\frac{\partial}{\partial u}g(u(\widetilde{\theta}_2), \widetilde{\theta}_2)d\widetilde{\theta}_2}_{\text{participation effect}} \quad (2)$$

As in the case of substitutable goods, there is the "rent extraction effect" and the "participation effect". The first effect is slightly different in this case: the lower bound of the support of θ_2 is $\underline{\theta}_2 \geq 0$, while the lower bound of the support of δ is 0. Thus there is less incentive to assign inefficient allocations (especially when $\underline{\theta}_2$ is high).

As in the case of substitutable goods, it is possible to show that there is "no distortion at the top", i.e. $\beta = 1$ when θ_2 is sufficiently high.[22] Also the seller sometimes benefits from offering lotteries as is demonstrated by the next example.

Example 3 *Let the distribution of the valuations be uniform on $\Theta = [c, c+1]^2$ where $c \geq 0$. The optimal mechanism is as follows.*

(i) When $c = 0$:

$$(\beta(\theta_2), t(\theta_2)) = \begin{cases} (0,\ \frac{2}{3}) & \text{if} \quad \theta_2 \in [0, \frac{2}{3} - \frac{1}{3}\sqrt{2}) \\ (1,\ \frac{4}{3} - \frac{1}{3}\sqrt{2}) & \text{if} \quad \theta_2 \in (\frac{2}{3} - \frac{1}{3}\sqrt{2}, 1] \end{cases}$$

(ii) When $c \in (0, \overline{c})$ (where $\overline{c} \approx 0.077$):

$$(\beta(\theta_2), t(\theta_2)) = \begin{cases} (\widetilde{\beta}(c),\ \underline{T}(c)) & \text{if} \quad \theta_2 \in [c, c+\widetilde{y}(c)) \\ (1,\ \overline{T}(c)) & \text{if} \quad \theta_2 \in (c+\widetilde{y}(c), c+1] \end{cases}$$

where $\widetilde{\beta}$ is increasing in c, $\widetilde{\beta}(0) = 0$ and $\widetilde{\beta}(\overline{c}) = 1$.

[21] See for example Myerson (1981).

[22] This result was also derived in Manelli and Vincent (2007) using a different technique.

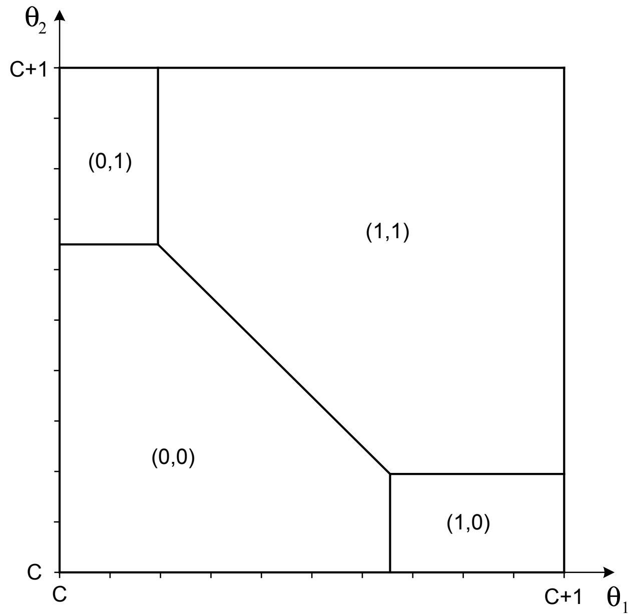

Figure 5: Optimal allocation in Example 3 when $c = 0$.

(iii) When $c \in [\bar{c}, +\infty)$*:*

$$(\beta(\theta_2), t(\theta_2)) = (1,\ \frac{4}{3}c + \frac{2}{3}\sqrt{c^2 + \frac{3}{2}})\ \textit{for every}\ \theta_2 \in [c, c+1]$$

When $c = 0$, the optimal mechanism is deterministic: the buyer can either get any one good at the price $\frac{2}{3}$ or get the bundle of two goods at the price $\frac{4}{3} - \frac{1}{3}\sqrt{2} \approx 0.862$ (see Figure 5).[23] As long as c is slightly above zero, the optimal mechanism is stochastic: the buyer can either get any one good for sure and the second good with probability β at a price $\underline{T}$ or get the bundle of the two goods at a higher price $\overline{T}$ (see Figure 6). When c is sufficiently above zero, the optimal mechanism again becomes deterministic: the buyer is only offered the bundle of the two goods (see Figure 7). As discussed above, this is possibly due to the fact that both the “rent extraction effect” and the “participation effect” push towards efficient allocations when c is sufficiently high.

[23] Manelli and Vincent (2006) give conditions for the optimality of deterministic mechanisms under the assumption that the lower bound of the support of the valuations is zero. Their results imply that the optimal mechanism is deterministic when $c = 0$, but they say nothing about the case $c > 0$.

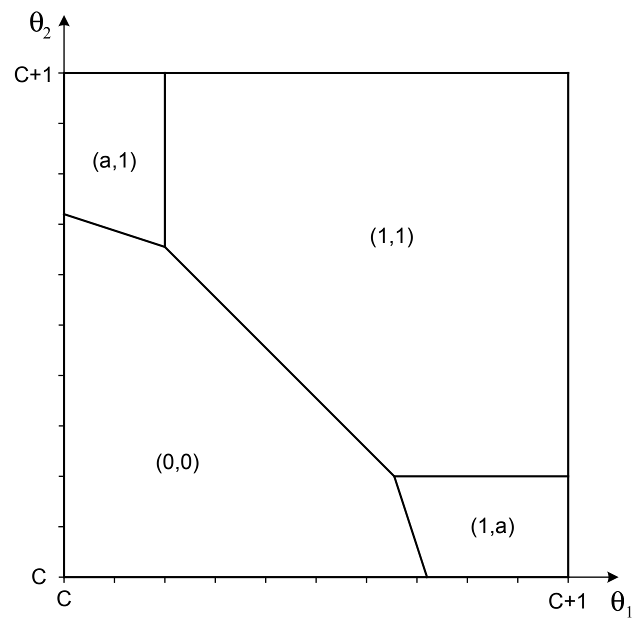

Figure 6: Optimal allocation in Example 3 when $c \in (0, \overline{c})$.

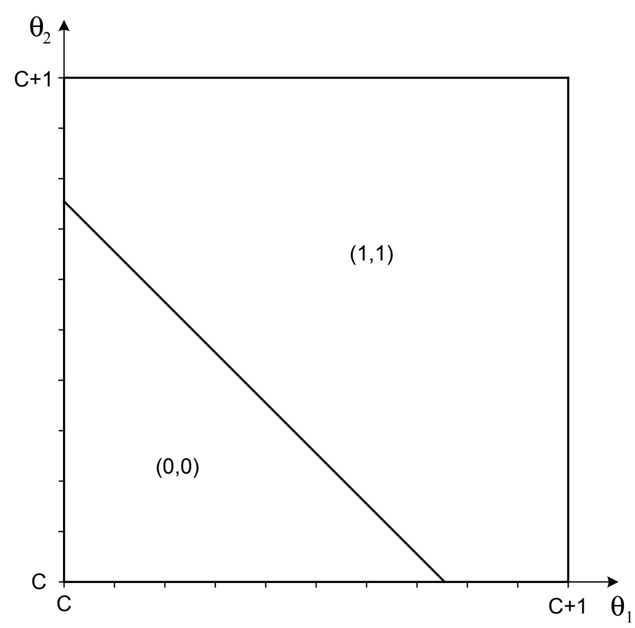

Figure 7: Optimal allocation in Example 3 when $c \in [\overline{c}, +\infty)$.

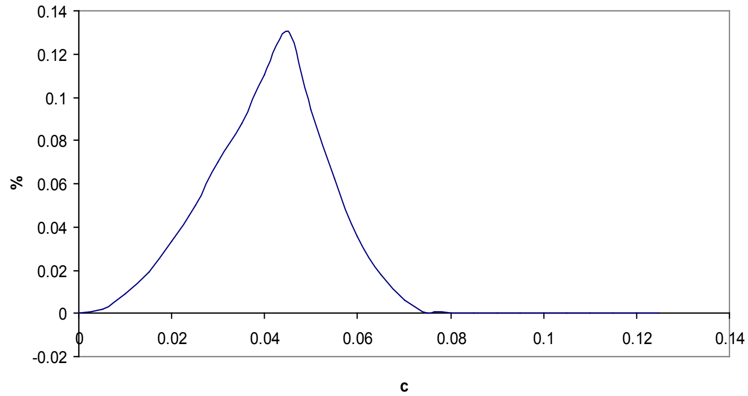

Figure 8: Profit comparison in Example 3: optimal vs. the best deterministic mechanism.

As in the case of substitutable goods, the optimal menus are very simple, and we conjecture that this must be true generically.[24] The relative profit gain from using fully optimal mechanism rather than the best deterministic mechanism in this example is very small: about 0.13% (see Figure 8).[25]

5 Conclusion

We have solved for the optimal mechanism for selling two goods when the buyer's demand characteristics are unobservable. In the case of substitutable goods, the seller has an incentive to offer lotteries over goods in order to charge the buyers with large differences in the valuations a higher price for obtaining their desired good with certainty. However, the seller also has a countervailing incentive to make the allocation of the goods among the participating buyers more efficient in order to increase the overall demand. In the case when the buyer can consume both goods, the seller has an incentive to underprovide one of the goods in order to charge the buyers with large valuations a higher price for the bundle of both goods. As in the case of substitutable goods, the seller also has a countervailing incentive to lower the price of the bundle in order to increase the overall demand.

[24] Manelli and Vincent (2007) prove that the set of potentially optimal mechanisms is very large and includes mechanisms with complicated menus. Since their proof is not constructive, it is hard to assess what kind of irregular distributions are needed to rationalize those mechanisms.

[25] See Appendix for the formulas of the expected profits.

The models and techniques considered in this paper can be applied to other settings. For example, Rochet and Stole (2002) study optimal nonlinear pricing when the buyers have heterogeneous outside options and the seller has convex costs. It is easy to address the same question when the seller has constant marginal costs with the techniques used here. Gruyer (2009) studies optimal auction design when the seller has a single good for sale, can prohibit reallocation of the good between bidders, and is bound to sell the good. The bidders are assumed to form a "well-coordinated" cartel, so that they behave as a single buyer maximizing the sum of the bidders' payoffs. Our model of substitutable goods can be used to derive the optimal auction in this setting and dispense with the assumption that the seller is bound to sell the good. We just need to reinterpret the buyer's valuation for good i to be bidder i's value for the auctioned good, and the probability of obtaining good i to be the probability that bidder i is the winner of the auction.

6 Appendix

6.1 Proofs for Section 3

Proof of Proposition 2. Suppose (p,T) solves Program I, (α,t) solves Program II, and (p,T) results in a higher profit than (α,t). Denote by U the utility schedule generated by mechanism (p,T). By Proposition 1 we can assume that $p_1(\theta)+p_2(\theta)\in\{0,1\}$ for every $\theta\in\Theta$.

Consider two types θ, $\theta'\in\Theta$ such that (i) $\theta_1-\theta_2=\theta_1'-\theta_2'=\delta$ for some $\delta\in\mathbb{R}$; and (ii) $p_1(\theta)+p_2(\theta)=p_1(\theta')+p_2(\theta')=1$. Note that

$$\begin{aligned} U(\theta') &\geq \theta_1'p_1(\theta)+\theta_2'p_2(\theta)-T(\theta)=\delta p_1(\theta)+\theta_2'-T(\theta) \\ &= \theta_1p_1(\theta)+\theta_2p_2(\theta)-T(\theta)+(\theta_2'-\theta_2)=U(\theta)+(\theta_2'-\theta_2) \end{aligned}$$

where the inequality is due to the incentive compatibility, and the first two equalities make use of (i) and (ii). Similarly

$$U(\theta)\geq U(\theta')-(\theta_2'-\theta_2).$$

Hence

$$U(\theta')=U(\theta)+(\theta_2'-\theta_2)$$

For every relevant $\delta\in\mathbb{R}$ find the type $\theta(\delta)$ that maximizes the seller's profit:

$$\max_{\theta\in\Theta} T(\theta) \qquad \text{subject to} \qquad \theta_1-\theta_2=\delta \text{ and } p_1(\theta)+p_2(\theta)=1$$

Introduce a new direct mechanism, which consists of a set Θ of message reports, an allocation rule $\widehat{p}:\Theta\to\Sigma$, and a payment rule $\widehat{T}:\Theta\to\mathbb{R}$. Let $\widehat{p}$ and $\widehat{T}$

for every $\theta \in \Theta$ such that $\theta_1 - \theta_2 = \delta$ be defined as follows

$$(\widehat{p}_1(\theta), \widehat{p}_2(\theta), \widehat{T}(\theta)) = \begin{cases} (p_1(\theta(\delta)), p_2(\theta(\delta)), T(\theta(\delta))) & \text{if } p_1(\theta) + p_2(\theta) = 1 \\ (0,0,0) & \text{if } p_1(\theta) + p_2(\theta) = 0 \end{cases}$$

Notice that the new mechanism $(\widehat{p}, \widehat{T})$ is admissible in Program II and is at least as profitable as the original mechanism (p, T). However, (α, t) solves Program II, which gives a contradiction. ■

We rewrite the seller's problem given in Program II$''$ as an optimal control problem. We deal with the monotonicity constraint in a standard way by introducing an auxiliary control variable $z : [\underline{\delta}, \overline{\delta}] \to \mathbb{R}_+$ such that $\dot{\alpha}(\delta) = z(\delta)$, and in addition allow the state variable α to have upward jumps.[26]

Program II$'''$: $\max\limits_{z,\alpha,u} \int_0^{\overline{\delta}} (\delta\alpha(\delta) - u(\delta)) g(u(\delta), \delta) d\delta$ subject to

Feasibility:	$\alpha(\delta) \geq \frac{1}{2}$	$\underline{\eta}(\delta)$
	$1 - \alpha(\delta) \geq 0$	$\overline{\eta}(\delta)$
Incentive Compatibility:	$\dot{u}(\delta) = \alpha(\delta)$	$\lambda_1(\delta)$
	$\dot{\alpha}(\delta) = z(\delta)$	$\lambda_2(\delta)$
	$z(\delta) \geq 0$	$\mu(\delta)$
Transversality conditions:	$\alpha(0)$, $\alpha(\overline{\delta})$, $u(0)$ and $u(\overline{\delta})$ are free	

Next we derive the necessary conditions for optimality.[27] Form the Lagrangian

$$L(z, \alpha, u, \overline{\eta}, \underline{\eta}, \lambda_1, \lambda_2, \mu; \delta) = (\delta\alpha - u) g(u, \delta) + \lambda_1 \alpha + \lambda_2 z + \underline{\eta}(\alpha - \tfrac{1}{2}) + \overline{\eta}(1 - \alpha) + \mu z$$

First we maximize L with respect to z.

$$L^* = (\delta\alpha - u) g(u, \delta) + \lambda_1 \alpha + \underline{\eta}(\alpha - \tfrac{1}{2}) + \overline{\eta}(1 - \alpha)$$

with the conditions

$$\mu z = 0,\ \mu = -\lambda_2 \geq 0 \text{ and } \dot{\alpha} = z \geq 0. \tag{3}$$

Next we get a system of Hamiltonian equations:

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial L^*}{\partial u} = g(u, \delta) - (\delta\alpha - u) \frac{\partial}{\partial u} g(u, \delta) \\ \dot{\lambda}_2 = -\frac{\partial L^*}{\partial \alpha} = -\delta g(u, \delta) - \lambda_1 - \underline{\eta} + \overline{\eta} \end{cases} \tag{4}$$

[26]See for example Guesnerie and Laffont (1984).
[27]See Theorem 7 in Chapter 3 in Seierstad and Sydsæter (1987).

The transversality conditions imply the following boundary requirements for λ_1 and λ_2:

$$\lambda_1(0) = \lambda_1(\overline{\delta}) = \lambda_2(0) = \lambda_2(\overline{\delta}) = 0 \tag{5}$$

The co-state variables λ_1, λ_2 are continuous throughout.[28] Moreover, λ_2 is equal to zero at the points where the state variable α jumps.[29] The remaining conditions are

$$\begin{aligned} &\underline{\eta}(\alpha - \tfrac{1}{2}) = 0,\ \underline{\eta} \geq 0 \text{ and } \alpha \geq \tfrac{1}{2}; \\ &\overline{\eta}(1-\alpha) = 0,\ \overline{\eta} \geq 0 \text{ and } \alpha \leq 1. \end{aligned} \tag{6}$$

Here is one implication of these optimality conditions:

$$\begin{aligned} 0 &= \lambda_1(\overline{\delta}) - \lambda_1(0) = \int_0^{\overline{\delta}} \dot{\lambda}_1(\widetilde{\delta})d\widetilde{\delta} \\ &= -\int_0^{\overline{\delta}} (g(u(\widetilde{\delta}),\widetilde{\delta}))d\widetilde{\delta} + \int_0^{\overline{\delta}} (\widetilde{\delta}\alpha(\widetilde{\delta}) - u(\widetilde{\delta}))\tfrac{\partial}{\partial u} g(u(\widetilde{\delta}),\widetilde{\delta})d\widetilde{\delta} \end{aligned} \tag{7}$$

where the first equality follows from (5), and the last from (4).

Define a marginal profit function as follows:

$$\begin{aligned} V(\delta) &= \delta g(u(\delta),\delta) + \lambda_1(\delta) = \delta g(u(\delta),\delta) + \lambda_1(\overline{\delta}) - \int_{\delta}^{\overline{\delta}} \dot{\lambda}_1(\widetilde{\delta})d\widetilde{\delta} \\ &= \delta g(u(\delta),\delta) - \int_{\delta}^{\overline{\delta}} (g(u(\widetilde{\delta}),\widetilde{\delta}))d\widetilde{\delta} + \int_{\delta}^{\overline{\delta}} (\widetilde{\delta}\alpha(\widetilde{\delta}) - u(\widetilde{\delta}))\tfrac{\partial}{\partial u} g(u(\widetilde{\delta}),\widetilde{\delta})d\widetilde{\delta} \end{aligned} \tag{8}$$

where the last equality follows from (4) and (5). Note that equation (7) is equivalent to $V(0) = 0$.

The next result reworks the optimality conditions into a set of requirements on the marginal profit function V. Part (i) of the result gives requirements for separation of types on an interval, and parts (ii)-(iv) are the “ironing conditions” for pooling types on an interval.[30]

Lemma 4 *Let $(z, \alpha, u, \underline{\eta}, \overline{\eta}, \lambda_1, \lambda_2, \mu)$ satisfy the necessary conditions. Then the following conditions must be satisfied.*

(i) If α is strictly increasing on (δ_1, δ_2), then $V = 0$ on this interval.

(ii) If $\alpha = \frac{1}{2}$ on (δ_1, δ_2), then $\delta_1 = 0$; $V(\delta_2) = 0$ unless $\delta_2 = \overline{\delta}$; $\int_{\delta_1}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta} = \underline{k}$ for some $\underline{k} \leq 0$. Also $\int_{\delta_1}^{\delta} V(\widetilde{\delta})d\widetilde{\delta} \geq \underline{k}$ and $\int_{\delta}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta} \leq 0$ for every δ in the interval.

[28] In general, the co-state variables may have discontinuities at the points where the state variables jump. However, this happens only when each jump in the state variable has an explicit cost, which is not the case here. See Theorem 7 in Chapter 3 in Seierstad and Sydsæter (1987).

[29] See Theorem 7 in Chapter 3 in Seierstad and Sydsæter (1987).

[30] See for example Guesnerie and Laffont (1984), Myerson (1981).

(iii) *If* $\alpha = \widehat{\alpha} \in (\frac{1}{2}, 1)$ *on* (δ_1, δ_2)*, then* $V(\delta_1) = 0$ *unless* $\delta_1 = 0$*;* $V(\delta_2) = 0$ *unless* $\delta_2 = \overline{\delta}$*;* $\int_{\delta_1}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta} = 0$*. Also* $\int_{\delta_1}^{\delta} V(\widetilde{\delta})d\widetilde{\delta} \geq 0 \geq \int_{\delta}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta}$ *for every* δ *in the interval.*

(iv) *If* $\alpha = 1$ *on* (δ_1, δ_2)*, then* $V(\delta_1) = 0$ *unless* $\delta_1 = 0$*;* $\delta_2 = \overline{\delta}$*;* $\int_{\delta_1}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta} = \overline{k}$ *for some* $\overline{k} \geq 0$*. Also* $\int_{\delta_1}^{\delta} V(\widetilde{\delta})d\widetilde{\delta} \geq 0$ *and* $\int_{\delta}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta} \leq \overline{k}$ *for every* δ *in the interval.*

Proof. (i) When α is strictly increasing, then by condition (3) we have $z > 0$ and thus $\lambda_2 = -\mu = 0$. Hence, $\dot{\lambda}_2 = 0$ on this interval. By condition (6) we also have $\underline{\eta} = \overline{\eta} = 0$ on this interval. Hence, by condition (4) $V = 0$.

(ii) By monotonicity of α we must have $\delta_1 = 0$. Also note that $\lambda_2(\delta_1) = 0$ by the transversality condition (5). By condition (6) we have $\underline{\eta} \geq 0$ and $\overline{\eta} = 0$ on this interval.

If $\delta_2 < \overline{\delta}$, then $\lambda_2(\delta_2) = 0$ since α changes its value at δ_2. Note that this implies that at the left limit of δ_2 we have

$$0 \leq \dot{\lambda}_2(\delta_2^-) \leq -V(\delta_2^-)$$

where the first inequality follows from condition (3), which requires $\lambda_2 \leq 0$, the second inequality is by condition (4) and the fact that $\underline{\eta} \geq 0$ and $\overline{\eta} = 0$ on this interval. At the right limit of δ_2 we have

$$0 \geq \dot{\lambda}_2(\delta_2^+) \geq -V(\delta_2^+)$$

where the first inequality follows from $\lambda_2 \leq 0$, the second inequality is by condition (4) and condition (6), which requires $\underline{\eta} = 0$ and $\overline{\eta} \geq 0$ outside the interval $[\delta_1, \delta_2]$. Since V is continuous, we conclude that $V(\delta_2) = 0$.

If $\delta_2 = \overline{\delta}$, then $\lambda_2(\delta_2) = 0$ by the transversality condition (5). Hence, in either case we must have

$$0 = \lambda_2(\delta_2) - \lambda_2(\delta_1) = \int_{\delta_1}^{\delta_2} \dot{\lambda}_2(\widetilde{\delta})d\widetilde{\delta}$$

Since $\underline{\eta} \geq 0$ and $\overline{\eta} = 0$ on this interval, by condition (4) we have

$$\int_{\delta_1}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta} = -\int_{\delta_1}^{\delta_2} \underline{\eta}(\widetilde{\delta})d\widetilde{\delta} =: \underline{k} \leq 0$$

Also note that

$$0 \geq \lambda_2(\delta) = \lambda_2(\delta) - \lambda_2(\delta_1) = \int_{\delta_1}^{\delta} \dot{\lambda}_2(\widetilde{\delta})d\widetilde{\delta}$$

which by condition (4) implies

$$\int_{\delta_1}^{\delta} V(\widetilde{\delta})d\widetilde{\delta} = -\int_{\delta_1}^{\delta} \underline{\eta}(\widetilde{\delta})d\widetilde{\delta} \geq \underline{k}$$

Finally, note that

$$0 \geq \lambda_2(\delta) = \lambda_2(\delta) - \lambda_2(\delta_2) = -\int_{\delta}^{\delta_2} \dot{\lambda}_2(\widetilde{\delta})d\widetilde{\delta}$$

which by condition (4) implies

$$\int_{\delta}^{\delta_2} V(\widetilde{\delta})d\widetilde{\delta} = -\int_{\delta}^{\delta_2} \underline{\eta}(\widetilde{\delta})d\widetilde{\delta} \leq 0$$

The proofs of (iii) and (iv) are similar to the proof of (ii) and therefore omitted. ■

Proof of Proposition 3. Assume that in the optimal mechanism $\alpha(\delta) < 1$ for every $\delta \in [0, \overline{\delta})$. Then by Lemma 4 we must have $\int_0^{\overline{\delta}} V(\delta)d\delta \leq 0$.

On the other hand, by condition (7) we have

$$\int_0^{\overline{\delta}} (\delta\alpha(\delta) - u(\delta)) \tfrac{\partial}{\partial u} g(u(\delta), \delta)d\delta = \int_0^{\overline{\delta}} g(u(\delta), \delta)d\delta > 0$$

Hence, the "participation effect" in the formula for $V(\delta)$ is strictly positive for a subset of types of positive measure. Thus

$$\int_0^{\overline{\delta}} V(\delta)d\delta > \int_0^{\overline{\delta}} (\delta g(u(\delta), \delta) - \int_{\delta}^{\overline{\delta}} g(u(\widetilde{\delta}), \widetilde{\delta})d\widetilde{\delta})d\delta = 0$$

where the equality follows from integration by parts. This gives a contradiction. ■

Calculations for Example 1

Let $\widehat{\delta}$ be such that

$$c + u(\widehat{\delta}) = 0. \tag{9}$$

It is straightforward to show that $\widehat{\delta} \in [0, 1]$ exists and is unique. Notice that

$$g(u(\delta), \delta) = \begin{cases} 1 - \delta + c + u(\delta) & \text{if} \quad \delta \in [0, \widehat{\delta}) \\ 1 - \delta & \text{if} \quad \delta \in (\widehat{\delta}, 1] \end{cases}.$$

Thus, if $\delta \in [0, \widehat{\delta})$, then the marginal profit (see equation (8)) is

$$\begin{aligned} V(\delta) &= \delta(c + 1 - \delta + u(\delta)) \\ &\quad - \int_{\delta}^{\widehat{\delta}} (c + u(\widetilde{\delta}))d\widetilde{\delta} - \int_{\delta}^{1} (1 - \widetilde{\delta})d\widetilde{\delta} + \int_{\delta}^{\widehat{\delta}} (\widetilde{\delta}\alpha(\widetilde{\delta}) - u(\widetilde{\delta}))d\widetilde{\delta} \qquad (10) \\ &= \delta(2(c+1) - \tfrac{3}{2}\delta + u(\delta)) - c\widehat{\delta} - \tfrac{1}{2} + \int_{\delta}^{\widehat{\delta}} (\widetilde{\delta}\alpha(\widetilde{\delta}) - 2u(\widetilde{\delta}))d\widetilde{\delta} \\ &= \delta(2(c+1) - \tfrac{3}{2}\delta + 3u(\delta)) + c\widehat{\delta} - \tfrac{1}{2} + 3\int_{\delta}^{\widehat{\delta}} \widetilde{\delta}\alpha(\widetilde{\delta})d\widetilde{\delta} \end{aligned}$$

where the last equality follows from integration by parts and equation (9).

If $\delta \in (\widehat{\delta}, 1]$, then

$$V(\delta) = \delta(1-\delta) - \int_{\delta}^{1}(1-\widetilde{\delta})d\widetilde{\delta} = \tfrac{1}{2}(1-\delta)(3\delta-1)$$

Also note that

$$\dot{V}(\delta) = \begin{cases} 2c+2+3u(\delta)-3\delta & \text{if} \quad \delta \in [0,\widehat{\delta}) \\ 2-3\delta & \text{if} \quad \delta \in (\widehat{\delta},1] \end{cases} \quad \text{and}$$

$$\ddot{V}(\delta) = \begin{cases} 3(\alpha(\delta)-1) & \text{if} \quad \delta \in [0,\widehat{\delta}) \\ -3 & \text{if} \quad \delta \in (\widehat{\delta},1] \end{cases}.$$

Hence, the marginal profit V is (weakly) concave on $[0,\widehat{\delta})$ and is concave on $(\widehat{\delta},1]$. Notice that $\dot{V}$ is discontinuous at $\widehat{\delta}$ unless $c = 0$:

$$\dot{V}(\widehat{\delta}^{-}) = 2-c-3\widehat{\delta} \leq 2-3\widehat{\delta} = \dot{V}_2(\widehat{\delta}^{+}). \tag{11}$$

Equation (7) can be rewritten as follows:

$$V(0) = c\widehat{\delta} - \tfrac{1}{2} + 3\int_0^{\widehat{\delta}} \widetilde{\delta}\alpha(\widetilde{\delta})d\widetilde{\delta} = 0. \tag{12}$$

Case 1. $\alpha(\delta) = 1$ for every $\delta \in [0,1]$.

In this case $u(\delta) = u(0)+\delta$, and $\int_0^{\delta}\widetilde{\delta}\alpha(\widetilde{\delta})d\widetilde{\delta} = \frac{1}{2}\delta^2$. Using equation (12), we get

$$\widehat{\delta} = \tfrac{1}{3}(\sqrt{c^2+3}-c)$$

and from equation (9) we find

$$u(0) = -(\tfrac{2}{3}c + \tfrac{1}{3}\sqrt{c^2+3}).$$

Thus, the marginal profit when $\delta \in [0,\widehat{\delta})$ can be rewritten as follows

$$V(\delta) = \delta(2(c+1)+\tfrac{3}{2}\delta+3u(0)) + c\widehat{\delta} - \tfrac{1}{2} + \frac{3}{2}\widehat{\delta}^2 - \tfrac{3}{2}\delta^2 = \delta(2-\sqrt{c^2+3})$$

Hence, V is nonnegative on $[0,1]$ when $c \in [0,1]$, and thus by Lemma 4 the candidate α is indeed optimal (see Figure 1). The payment for every δ is

$$t(\delta) = \delta\alpha(\delta) - u(\delta) = -u(0) = \tfrac{2}{3}c + \tfrac{1}{3}\sqrt{c^2+3} = T(c).$$

The expected profit is

$$\Pi(c) = \Pr\{\max(\theta_1,\theta_2) \geq T(c)\} \cdot T(c)$$

where

$$\Pr\{\max(\theta_1,\theta_2)\geq T(c)\}=1-\tfrac{1}{9}(\sqrt{c^2+3}-c)^2.$$

Case 2. $\alpha(\delta)$ is not identically equal to one on $[0,1]$.

In this case V is strictly concave on both $[0,\widehat{\delta})$ and $(\widehat{\delta},1]$. First, we argue that $\widehat{\delta}<\frac{1}{3}$. Assume $\widehat{\delta}\geq\frac{1}{3}$, then concavity together with (11) and the facts that $V(0)=V(1)=0$ and $V(\widehat{\delta})\geq 0$ imply that V is strictly positive almost everywhere on $(0,1)$, and thus $\alpha(\delta)\neq 1$ cannot be optimal.

Since $\delta=\frac{1}{3}$ is the only place where $V(\delta)$ crosses zero from below, by Lemma 4 we have $\alpha(\delta)$ equal to some constant $\alpha\in[\frac{1}{2},1)$ on the interval $[0,\frac{1}{3})$.

Notice that $u(\delta)=u(0)+\alpha\delta$, and $\int_0^{\delta}\widetilde{\delta}\alpha(\widetilde{\delta})d\widetilde{\delta}=\frac{1}{2}\alpha\delta^2$ for $\delta\in[0,\frac{1}{3})$. Using equation (12):

$$\tfrac{3}{2}\alpha\widehat{\delta}^2+c\widehat{\delta}-\tfrac{1}{2}=0. \tag{13}$$

and from equation (9) we find

$$u(0)=-c-\alpha\widehat{\delta}. \tag{14}$$

Thus the marginal profit when $\delta\in[0,\widehat{\delta})$ can be rewritten as follows

$$\begin{aligned}V(\delta)&=\delta(2(c+1)-\tfrac{3}{2}\delta+3u(0)+3\alpha\delta)+c\widehat{\delta}-\tfrac{1}{2}+\tfrac{3}{2}\alpha\widehat{\delta}^2-\tfrac{3}{2}\alpha\delta^2\\&=\delta(2-c-3\alpha\widehat{\delta}-\tfrac{3}{2}(1-\alpha)\delta)\end{aligned}$$

where the last equality uses equations (13) and (14). Also note that

$$\begin{aligned}\int_0^{\frac{1}{3}}V(\delta)d\delta&=\int_0^{\widehat{\delta}}\delta(2-c-3\alpha\widehat{\delta}-\tfrac{3}{2}(1-\alpha)\delta)d\delta+\int_{\widehat{\delta}}^{\frac{1}{3}}\tfrac{1}{2}(1-\delta)(3\delta-1)d\delta\\&=-\alpha\widehat{\delta}^3-\tfrac{1}{2}c\widehat{\delta}^2+\tfrac{1}{2}\widehat{\delta}-\tfrac{2}{27}=\tfrac{1}{6}(c\widehat{\delta}^2+\widehat{\delta}-\tfrac{4}{9})\end{aligned}$$

where the last equality uses equation (13).

Case 2.1. $\alpha\in(\frac{1}{2},1)$.

By Lemma 4 in this case we must have $\int_0^{\frac{1}{3}}V(\delta)d\delta=0$, which gives

$$\widehat{\delta}=\tfrac{1}{6c}(\sqrt{16c+9}-3).$$

Using equation (13), we get

$$\alpha=\tfrac{27}{32}+(\tfrac{9}{32}-\tfrac{1}{4}c)\sqrt{16c+9}.$$

Notice that α is strictly decreasing in c. Also $\alpha=1$ when $c=1$, and $\alpha=\frac{1}{2}$ when $c=\overline{c}\approx 1.372$. Every participating type $\delta\in[0,\frac{1}{3})$ chooses allocation $(\alpha,1-\alpha)$ (see Figure 2). Their payment is

$$t(\delta)=\delta\alpha(\delta)-u(\delta)=-u(0)=c+\alpha\widehat{\delta}=\tfrac{1}{3}c+\tfrac{3}{8}+\tfrac{1}{8}\sqrt{16c+9}=\underline{T}(c)$$

Every participating type $\delta \in (\frac{1}{3}, 1]$ chooses allocation $(1,0)$. Their payment is

$$\begin{aligned} t(\delta) &= \delta\alpha(\delta) - u(\delta) = -u(0) + \tfrac{1}{3}(1-\alpha) = c + \alpha\widehat{\delta} + \tfrac{1}{3}(1-\alpha) \\ &= \tfrac{1}{3}c + \tfrac{41}{96} + (\tfrac{1}{12}c + \tfrac{1}{32})\sqrt{16c+9} = \overline{T}(c) \end{aligned}$$

The expected profit is

$$\Pi(c) = \Pr\{\alpha \max\{\theta_1,\theta_2\} + (1-\alpha)\min\{\theta_1,\theta_2\} \geq \underline{T}(c)\} \cdot \underline{T}(c) +$$
$$\Pr\{|\delta| \geq \tfrac{1}{3}\} \cdot (\overline{T}(c) - \underline{T}(c))$$

where

$$\begin{aligned} & \Pr\{\alpha \max\{\theta_1,\theta_2\} + (1-\alpha)\min\{\theta_1,\theta_2\} \geq \underline{T}(c)\} \\ = \; & 1 - (\tfrac{27}{32} + (\tfrac{9}{32} - \tfrac{1}{4}c)\sqrt{16c+9})(\tfrac{1}{6c}(\sqrt{16c+9}-3))^2 \end{aligned}$$

and $\Pr\{|\delta| \geq \frac{1}{3}\} = \frac{4}{9}$.

Case 2.2. $\alpha = \frac{1}{2}$.

By Lemma 4 in this case we must have $\int_0^{\frac{1}{3}} V(\delta)d\delta \leq 0$, which gives

$$\widehat{\delta} \leq \tfrac{1}{6c}(\sqrt{16c+9}-3). \tag{15}$$

Using equation (13) we get

$$\widehat{\delta} = \tfrac{2}{3}(\sqrt{c^2 + \tfrac{3}{2}} - c)$$

It is possible to verify that inequality (15) is satisfied whenever $c \geq \overline{c}$. Every participating type $\delta \in [0, \frac{1}{3})$ chooses allocation $(\frac{1}{2}, \frac{1}{2})$ (see Figure 3). Their payment is

$$t(\delta) = \delta\alpha(\delta) - u(\delta) = -u(0) = c + \tfrac{1}{6}\widehat{\delta} = \tfrac{2}{3}c + \tfrac{1}{3}\sqrt{c^2 + \tfrac{3}{2}} = \underline{T}(c)$$

Every participating type $\delta \in (\frac{1}{3}, 1]$ chooses allocation $(1,0)$. Their payment is

$$\begin{aligned} t(\delta) &= \delta\alpha(\delta) - u(\delta) = -u(0) + \tfrac{1}{6} = c + \alpha\widehat{\delta} + \tfrac{1}{6} \\ &= \tfrac{1}{3}c + \tfrac{41}{96} + (\tfrac{1}{12}c + \tfrac{1}{32})\sqrt{16c+9} = \overline{T}(c) \end{aligned}$$

The expected profit is

$$\Pi(c) = \Pr\{\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 \geq \underline{T}(c)\} \cdot \underline{T}(c) + \Pr\{|\delta| \geq \tfrac{1}{3}\} \cdot (\overline{T}(c) - \underline{T}(c))$$

where

$$\Pr\{\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 \geq \underline{T}(c)\} = 1 - \tfrac{2}{9}(\sqrt{c^2 + \tfrac{3}{2}} - c)^2$$

and $\Pr\{|\delta| \geq \frac{1}{3}\} = \frac{4}{9}$.

The best deterministic mechanism

Note that, if the seller offers only individual goods, then the optimal price and profit are given in Case 1 above. The relative (percentage) profit gain from using the fully optimal mechanism vs the best deterministic mechanism is given in Figure 4.

6.2 Proofs for Section 4

Proof of Proposition 5.

Suppose (p,T) solves Program I, (β,t) solves Program III, and (p,T) results in a higher profit than (β,t). Denote by U the utility schedule generated by mechanism (p,T). By Corollary 1 we can assume that for every θ such that $\theta_1 > \theta_2$: if $(p_1(\theta), p_2(\theta)) \neq (0,0)$, then $p_1(\theta) = 1$.

Consider two types θ, $\theta' \in \Theta$ such that (i) $\theta_1 > \theta_2$, $\theta'_1 > \theta'_2$; (ii) $\theta_2 = \theta'_2$; and (iii) $p_1(\theta) = p_1(\theta') = 1$. Note that

$$\begin{aligned} U(\theta') &\geq \theta'_1 p_1(\theta) + \theta'_2 p_2(\theta) - T(\theta) = \theta'_1 + \theta'_2 p_2(\theta) - T(\theta) \\ &= \theta_1 + \theta_2 p_2(\theta) - T(\theta) + (\theta'_1 - \theta_1) = U(\theta) + (\theta'_1 - \theta_1) \end{aligned}$$

where the inequality is due to the incentive compatibility, and the first two equalities make use of (ii) and (iii). Similarly,

$$U(\theta) \geq U(\theta') - (\theta'_1 - \theta_1).$$

Hence,

$$U(\theta') = U(\theta) + (\theta'_1 - \theta_1)$$

For every relevant $\widehat{\theta}_2 \in \mathbb{R}$ find the type $\theta(\widehat{\theta}_2)$ that maximizes the seller's profit:

$$\max_{\theta \in \Theta, \theta_1 \geq \theta_2} T(\theta) \qquad \text{subject to} \qquad \theta_2 = \widehat{\theta}_2 \text{ and } p_1(\theta) = 1$$

Introduce a new direct mechanism, which consists of a set Θ of message reports, an allocation rule $\widehat{p} : \Theta \to \Sigma$, and a payment rule $\widehat{T} : \Theta \to \mathbb{R}$. Let $\widehat{p}$ and $\widehat{T}$ for every $\theta \in \Theta$ such that $\theta_2 = \widehat{\theta}_2$ be defined as follows

$$(\widehat{p}_1(\theta), \widehat{p}_2(\theta), \widehat{T}(\theta)) = \begin{cases} (p_1(\theta(\widehat{\theta}_2)), p_2(\theta(\widehat{\theta}_2)), T(\theta(\widehat{\theta}_2))) & \text{if } \quad p_1(\theta) = 1 \\ (0,0,0) & \text{if } \quad p_1(\theta) = p_2(\theta) = 0 \end{cases}$$

Notice that the new mechanism $(\widehat{p}, \widehat{T})$ is admissible in Program III and is at least as profitable as the original mechanism (p,T). However, (β,t) solves Program III, which gives a contradiction. ■

Calculations for Example 3

Let $\widehat{\theta}_2$ be such that

$$u(\widehat{\theta}_2)+\widehat{\theta}_2=0. \tag{16}$$

It is straightforward to show that $\widehat{\theta}_2 \in [0,1]$ exists and is unique. Notice that

$$g(u(\theta_2),\theta_2)=\begin{cases} c+1+u(\theta_2) & \text{if} \quad \theta_2 \in [c,\widehat{\theta}_2) \\ c+1-\theta_2 & \text{if} \quad \theta_2 \in (\widehat{\theta}_2, c+1] \end{cases}.$$

Thus, if $\theta_2 \in [c,\widehat{\theta}_2)$, then the marginal profit is

$$\begin{aligned} V(\theta_2) &= \theta_2(c+1+u(\theta_2)) - \textstyle\int_{\theta_2}^{\widehat{\theta}_2}(c+1+u(\widetilde{\theta}_2))d\widetilde{\theta}_2 - \int_{\widehat{\theta}_2}^{c+1}(c+1-\widetilde{\theta}_2)d\widetilde{\theta}_2 + \\ &\quad \textstyle\int_{\theta_2}^{\widehat{\theta}_2}(\widetilde{\theta}_2\beta(\widetilde{\theta}_2)-u(\widetilde{\theta}_2))d\widetilde{\theta}_2 \\ &= \theta_2(2(c+1)+u(\theta_2)) - \tfrac{1}{2}(c+1)^2 - \tfrac{1}{2}(\widehat{\theta}_2)^2 + \\ &\quad \textstyle\int_{\theta_2}^{\widehat{\theta}_2}(\widetilde{\theta}_2\beta(\widetilde{\theta}_2)-2u(\widetilde{\theta}_2))d\widetilde{\theta}_2 \\ &= \theta_2(2(c+1)+3u(\theta_2)) - \tfrac{1}{2}(c+1)^2 + \tfrac{3}{2}(\widehat{\theta}_2)^2 + 3\textstyle\int_{\theta_2}^{\widehat{\theta}_2}\widetilde{\theta}_2\beta(\widetilde{\theta}_2)d\widetilde{\theta}_2 \end{aligned}$$

where the last equality follows from integration by parts and equation (16). If $\theta_2 \in (\widehat{\theta}_2, c+1]$, then

$$V(\theta_2)=\theta_2(c+1-\theta_2)-\textstyle\int_{\theta_2}^{c+1}(c+1-\widetilde{\theta}_2)d\widetilde{\theta}_2 = \tfrac{1}{2}(c+1-\theta_2)(3\theta_2-(c+1))$$

Note that $V(\theta_2)$ is nonnegative on $(\max\left\{\widehat{\theta}_2, \frac{1}{3}(c+1)\right\}, c+1]$. Also note that

$$\dot{V}(\theta_2)=\begin{cases} 2c+2+3u(\theta_2) & \text{if} \quad \theta_2 \in [c,\widehat{\theta}_2) \\ 2c+2-3\theta_2 & \text{if} \quad \theta_2 \in (\widehat{\theta}_2, c+1] \end{cases} \quad \text{and}$$

$$\ddot{V}(\theta_2)=\begin{cases} 3\beta(\theta_2) & \text{if} \quad \theta_2 \in [c,\widehat{\theta}_2) \\ -3 & \text{if} \quad \theta_2 \in (\widehat{\theta}_2, c+1] \end{cases}.$$

Hence, V is (weakly) convex on $[c,\widehat{\theta}_2)$ and is strictly concave on $(\widehat{\theta}_2, c+1]$. Notice that $\dot{V}$ is continuous at $\widehat{\theta}_2$ by equation (16).

An analog of equation (7) in this case is

$$\begin{aligned} 0 &= -\textstyle\int_{c}^{\widehat{\theta}_2}(c+1+u(\widetilde{\theta}_2))d\widetilde{\theta}_2 - \int_{\widehat{\theta}_2}^{c+1}(c+1-\widetilde{\theta}_2)d\widetilde{\theta}_2 + \int_{c}^{\widehat{\theta}_2}(\widetilde{\theta}_2\beta(\widetilde{\theta}_2)-u(\widetilde{\theta}_2))d\widetilde{\theta}_2 \qquad (17) \\ &= c(c+1+2u(c)) - \tfrac{1}{2}(c+1)^2 + \tfrac{3}{2}(\widehat{\theta}_2)^2 + 3\textstyle\int_{c}^{\widehat{\theta}_2}\widetilde{\theta}_2\beta(\widetilde{\theta}_2)d\widetilde{\theta}_2. \end{aligned}$$

Note that equation (17) implies that $V(c) = (c+1+u(c))c \geq 0$. Hence, unless $c = 0$, there exists at most a single point on $(c, c+1)$ where V crosses zero from below.

Case 1. $\beta(\theta_2) = 1$ for every $\theta_2 \in [c, c+1]$.

In this case $u(\theta_2) = u(c) + (\theta_2 - c)$, and $\int_c^{\theta_2} \widetilde{\theta}_2 \beta(\widetilde{\theta}_2) d\widetilde{\theta}_2 = \frac{1}{2}((\widehat{\theta}_2)^2 - c^2)$. Using equation (16):

$$u(c) = u(\widehat{\theta}_2) - (\widehat{\theta}_2 - c) = -2\widehat{\theta}_2 + c$$

Equation (17) yields

$$\widehat{\theta}_2 = \tfrac{2}{3}c + \tfrac{1}{3}\sqrt{c^2 + \tfrac{3}{2}}$$

and thus

$$u(c) = -\tfrac{1}{3}c - \tfrac{2}{3}\sqrt{c^2 + \tfrac{3}{2}}.$$

The marginal profit when $\theta_2 \in [c, \widehat{\theta}_2)$ can thus be rewritten as follows

$$V(\theta_2) = \tfrac{3}{2}(\theta_2 - c)^2 + 2(\tfrac{1}{2}c + 1 - \sqrt{c^2 + \tfrac{3}{2}})(\theta_2 - c) + \tfrac{1}{3}c(2c + 3 - 2\sqrt{c^2 + \tfrac{3}{2}})$$

Note that $\widehat{\theta}_2 > \frac{1}{3}(c+1)$ and thus $V(\theta_2) \geq 0$ for $\theta_2 \in [\widehat{\theta}_2, c+1)$. By an analog of Lemma 4 it is enough to show that $\int_c^{\theta_2} V(\widetilde{\theta}_2) d\widetilde{\theta}_2 \geq 0$ for every $\theta_2 \in [c, \widehat{\theta}_2)$.

$$\int_c^{\theta_2} V(\widetilde{\theta}_2) d\widetilde{\theta}_2 = \tfrac{1}{2}((\theta_2 - c)^2 + 2(\tfrac{1}{2}c + 1 - \sqrt{c^2 + \tfrac{3}{2}})(\theta_2 - c) + \tfrac{2}{3}c(2c + 3 - 2\sqrt{c^2 + \tfrac{3}{2}}))(\theta_2 - c)$$

The quadratic polynomial $x^2 + 2(\frac{1}{2}c + 1 - \sqrt{c^2 + \frac{3}{2}})x + \frac{2}{3}c(2c + 3 - 2\sqrt{c^2 + \frac{3}{2}})$ has no real roots if

$$(\tfrac{1}{2}c + 1 - \sqrt{c^2 + \tfrac{3}{2}})^2 < \tfrac{2}{3}c(2c + 3 - 2\sqrt{c^2 + \tfrac{3}{2}})$$

which holds when $c > \overline{c} \approx 0.077$. Hence, $\int_c^{\theta_2} V(\widetilde{\theta}_2) d\widetilde{\theta}_2 \geq 0$ for every $\theta_2 \in [c, \widehat{\theta}_2)$ when $c \in [\overline{c}, +\infty)$.

Every participating type gets an allocation $(1,1)$ (see Figure 7). The payment for every θ_2 is

$$t(\theta_2) = \theta_2 \beta(\theta_2) - u(\theta_2) = c - u(c) = \tfrac{4}{3}c + \tfrac{2}{3}\sqrt{c^2 + \tfrac{3}{2}} = T(c).$$

The expected profit is

$$\Pi(c) = \Pr\{\theta_1 + \theta_2 \geq T(c)\} \cdot T(c)$$

where

$$\Pr\{\theta_1 + \theta_2 \geq T(c)\} = 1 - \tfrac{2}{9}(\sqrt{c^2 + \tfrac{3}{2}} - c)^2.$$

Case 2. $\beta(\theta_2)$ is not identically equal to one on $[c, c+1]$.

By an analog of Lemma 4 and due to the properties of V discussed above, the optimal β is a step function

$$\beta(\theta_2) = \begin{cases} \beta & \text{if} \quad \theta_2 \in [c, \theta_2^*) \\ 1 & \text{if} \quad \theta_2 \in (\theta_2^*, c+1] \end{cases}$$

where $\beta \in [0,1)$, and θ_2^* is such that $V(\theta_2^*) = 0$ and $\int_c^{\theta_2^*} V(\widetilde{\theta}_2) d\widetilde{\theta}_2 \leq 0$ (with an equality if $\beta \in (0,1)$).

Let us guess that $\theta_2^* < \widehat{\theta}_2$, which implies $\widehat{\theta}_2 > \frac{1}{3}(c+1)$. Hence,

$$u(\theta_2) = \begin{cases} u(c) + \beta(\theta_2 - c) & \text{if} \quad \theta_2 \in [c, \theta_2^*) \\ u(c) + \beta(\theta_2^* - c) + (\theta_2 - \theta_2^*) & \text{if} \quad \theta_2 \in (\theta_2^*, c+1] \end{cases} \tag{18}$$

and

$$\int_c^{\theta_2} \widetilde{\theta}_2 \beta(\widetilde{\theta}_2) d\widetilde{\theta}_2 = \begin{cases} \frac{1}{2}\beta((\theta_2)^2 - c^2) & \text{if} \quad \theta_2 \in [c, \theta_2^*) \\ \frac{1}{2}\beta((\theta_2^*)^2 - c^2) + \frac{1}{2}((\theta_2)^2 - (\theta_2^*)^2) & \text{if} \quad \theta_2 \in (\theta_2^*, c+1] \end{cases}$$

Using equation (16):

$$u(c) = -2\widehat{\theta}_2 + \beta c + (1-\beta)\theta_2^* \tag{19}$$

Equation (17) yields

$$3(\widehat{\theta}_2 - c)^2 + 2c(\widehat{\theta}_2 - c) - \frac{1}{2}(1-\beta)(3(\theta_2^* - c)^2 + 2c(\theta_2^* - c)) - \frac{1}{2} = 0 \tag{20}$$

The marginal profit when $\theta_2 \in [c, \theta_2^*)$ can thus be rewritten as follows

$$\begin{aligned} V(\theta_2) \ = \ & \frac{3}{2}\beta(\theta_2 - c)^2 + (-6(\widehat{\theta}_2 - c) + 3(1-\beta)(\theta_2^* - c) + (2-c))(\theta_2 - c) \\ & + 3(\widehat{\theta}_2 - c)^2 - \frac{3}{2}(1-\beta)(\theta_2^* - c)^2 + c - \frac{1}{2} \end{aligned} \tag{21}$$

and thus

$$\begin{aligned} & \int_c^{\theta_2^*} V(\widetilde{\theta}_2) d\widetilde{\theta}_2 = \frac{1}{2}\beta(\theta_2^* - c)^3 + \frac{1}{2}(-6(\widehat{\theta}_2 - c) + 3(1-\beta)(\theta_2^* - c) + (2-c))(\theta_2^* - c)^2 \\ & + (3(\widehat{\theta}_2 - c)^2 - \frac{3}{2}(1-\beta)(\theta_2^* - c)^2 + c - \frac{1}{2})(\theta_2^* - c) \end{aligned} \tag{22}$$

Case 2.1. $\beta \in (0,1)$.

Denote $x = \widehat{\theta}_2 - c$ and $y = \theta_2^* - c$. We can rewrite (20), $V(\theta_2^*) = 0$ (using (21)) and $\int_c^{\theta_2^*} V(\widetilde{\theta}_2) d\widetilde{\theta}_2 = 0$ (using (22)) as follows

$$\begin{cases} 3x^2 + 2cx - \frac{1}{2}(1-\beta)(3y^2 + 2cy) - \frac{1}{2} = 0 \\ 3x^2 - 6xy + \frac{3}{2}y^2 + (2-c)y + c - \frac{1}{2} = 0 \\ \frac{1}{2}(6x^2 - 6xy + \beta y^2 + (2-c)y + 2c - 1)y = 0 \end{cases}$$

We numerically check that the solution $(\widetilde{x}(c), \widetilde{y}(c), \widetilde{\beta}(c))$ is such that $\widetilde{\beta}(c)$ is increasing, $\widetilde{\beta}(0) = 0$ and $\widetilde{\beta}(\overline{c}) = 1$.

Every participating type $\theta_2 \in [c, c + \widetilde{y}(c))$ chooses an allocation $(1, \widetilde{\beta}(c))$ (see Figure 6). Using equations (18) and (19), we can compute their payment

$$t(\theta_2) = \theta_2 \beta(\theta_2) - u(\theta_2) = c(1 + \widetilde{\beta}(c)) + 2\widetilde{x}(c) - (1 - \widetilde{\beta}(c))\widetilde{y}(c) = \underline{T}(c)$$

Every participating type $\theta_2 \in (c + \widetilde{y}(c), c + 1]$ chooses an allocation $(1, 1)$. Using equations (18) and (19), we can compute their payment

$$t(\theta_2) = \theta_2 \beta(\theta_2) - u(\theta_2) = 2c + 2\widetilde{x}(c) = \overline{T}(c)$$

The expected profit is

$$\begin{aligned} \Pi(c) \;=\; & \Pr\left\{\max\{\theta_1, \theta_2\} + \widetilde{\beta}(c) \min\{\theta_1, \theta_2\} - \underline{T}(c) \geq \max\left\{0, \theta_1 + \theta_2 - \overline{T}(c)\right\}\right\} \cdot \underline{T}(c) \\ & + \Pr\left\{\theta_1 + \theta_2 - \overline{T}(c) \geq \max\left\{0, \max\{\theta_1, \theta_2\} + \widetilde{\beta}(c) \min\{\theta_1, \theta_2\} - \underline{T}(c)\right\}\right\} \cdot \overline{T}(c) \end{aligned}$$

where

$$\begin{aligned} & \Pr\left\{\max\{\theta_1, \theta_2\} + \widetilde{\beta}(c) \min\{\theta_1, \theta_2\} - \underline{T}(c) \geq \max\left\{0, \theta_1 + \theta_2 - \overline{T}(c)\right\}\right\} \\ = \;& 2\widetilde{y}(c)(1 - 2\widetilde{x}(c) + (1 - \tfrac{1}{2}\widetilde{\beta}(c))\widetilde{y}(c)) \end{aligned}$$

and

$$\begin{aligned} & \Pr\left\{\theta_1 + \theta_2 - \overline{T}(c) \geq \max\left\{0, \max\{\theta_1, \theta_2\} + \widetilde{\beta}(c) \min\{\theta_1, \theta_2\} - \underline{T}(c)\right\}\right\} \\ = \;& (1 - \widetilde{y}(c))^2 - 2(\widetilde{x}(c) - \widetilde{y}(c))^2. \end{aligned}$$

Case 2.2. $\beta = 0$.

The only possibility not covered by Cases 1 and 2.1 is when $c = 0$. Similar to Case 2.1, we can represent condition (20), $V(\theta_2^*) = 0$ (using (21)) and $\int_c^{\theta_2^*} V(\widetilde{\theta}_2) d\widetilde{\theta}_2 \leq 0$ (using (22)) as follows

$$\begin{cases} 3x^2 - \frac{3}{2}y^2 - \frac{1}{2} = 0 \\ 3x^2 - 6xy + \frac{3}{2}y^2 + 2y - \frac{1}{2} = 0 \\ \frac{1}{2}(6x^2 - 6xy + 2y - 1)y \leq 0 \end{cases}$$

The solution is: $(x, y) = (\frac{2}{3} - \frac{1}{6}\sqrt{2}, \frac{2}{3} - \frac{1}{3}\sqrt{2}) \approx (0.431, 0.195)$. The payments and the expected profit can be computed using the formulas from Case 2.1 (with $c = 0$). The allocation is given in Figure 5.

The best deterministic mechanism

Let $\underline{T}$ be the price of allocations $(1,0)$ and $(0,1)$, and $\overline{T}$ be the price of a bundle $(1,1)$. Note that, if the seller offers only the bundle, then the expressions for the optimal price and profit are given in Case 1 above. If both individual goods and bundle are offered, then the optimal prices maximize

$$2\underline{T}(c+1-\underline{T})(\overline{T}-\underline{T}-c)+\overline{T}((c+1-(\overline{T}-\underline{T}))^2-\tfrac{1}{2}(\underline{T}-(\overline{T}-\underline{T}))^2).$$

We checked numerically that it is optimal to offer just a bundle when $c > c' \approx 0.05$, and it is optimal to offer both a bundle and individual goods when $c < c' \approx 0.05$. The relative (percentage) profit gain from using the fully optimal mechanism vs the best deterministic mechanism is given in Figure 8.

7 References

Adams, W. and J. Yellen (1976) "Commodity bundling and the burden of monopoly," *Quarterly Journal of Economics* 90, 475–498.

Armstrong, M. (1996) "Multiproduct nonlinear pricing," *Econometrica* 64, 51–75.

Balestrieri, F. and J. Leao (2008) "The market for surprises: selling substitute goods through lotteries," mimeo, ISCTE.

Baron, D. and R. Myerson (1982) "Regulating a monopolist with unknown costs," *Econometrica* 50, 911–930.

Gruyer, N. (2009) "Optimal auctions when a seller is bound to sell to collusive bidders," *Journal of Industrial Economics* 57, 835–850.

Guesnerie, R. and J.-J. Laffont (1984) "A complete solution to a class of a principal agent problems with an application to the control of a self-managed firm," *Journal of Public Economics* 25, 329–369.

Lancaster, K. (1971) *Consumer Demand: A New Approach.* Columbia University Press.

Manelli, A. and D. Vincent (2006) "Bundling as an optimal mechanism by a multiple-good monopolist," *Journal of Economic Theory* 127, 1–35.

Manelli, A. and D. Vincent (2007) "Multi-dimensional mechanism design: revenue maximization and the multiple good monopoly," *Journal of Economic Theory* 137, 153–185.

Maskin, E. and J. Riley (1984) "Optimal auctions with risk averse buyers," *Econometrica* 52, 1473–1518.

McAfee, P. and J. McMillan (1988) "Multidimensional incentive compatibility and mechanism design," *Journal of Economic Theory* 46, 335–354.

McAfee, P., J. McMillan and M. Whinston (1989) "Multiproduct monopoly, commodity bundling, and correlation of values," *Quarterly Journal of Economics* 104, 371–383.

Myerson, R. (1981) "Optimal auction design," *Mathematics of Operations Research* 6, 58–73.

Pavlov, G. (2006) "Optimal mechanism for selling substitutes," mimeo, University of Western Ontario.

Pavlov, G. (2010) "A property of solutions to linear monopoly problems," mimeo, University of Western Ontario.

Riley, J. and R. Zeckhauser (1983): "Optimal selling strategies: when to haggle, when to hold firm," *Quarterly Journal of Economics* 98, 267–289.

Rochet, J.-C. (1985) "The taxation principle and multitime Hamilton-Jacobi equations," *Journal of Mathematical Economics* 14, 113–128.

Rochet, J.-C. and P. Chone (1998) "Ironing, sweeping, and multidimensional screening," *Econometrica* 66, 783–826.

Rochet, J.-C. and L. Stole (2002) "Nonlinear pricing with random participation," *Review of Economic Studies* 69, 277–311.

Rochet, J.-C. and L. Stole (2003) "The economics of multidimensional screening," in *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress*.

Seierstad, A. and K. Sydsæter (1987) *Optimal Control with Economic Applications*. Elsevier Science & Technology Books.

Thanassoulis, J. (2004) "Haggling over substitutes," *Journal of Economic Theory* 117, 217–245.