SPECIFYING NODES AS SETS OF ACTIONS

Peter A. Streufert
Department of Economics
University of Western Ontario

Abstract. The nodes of an extensive-form game are commonly specified as sequences of actions. Rubinstein calls such nodes histories. We find that this sequential notation is superfluous in the sense that nodes can also be specified as *sets* of actions. The only cost of doing so is to rule out games with absent-minded agents. Our set-theoretic analysis accommodates general finite-horizon games with arbitrarily large action spaces and arbitrarily configured information sets.

1. Introduction

In order to define an extensive-form game, one sometimes begins with a tree consisting of nodes and edges. One then uses that tree as a skeleton on which to define actions, information sets (i.e. agents), players, chance probabilities, and payoffs. By assumption, the tree must have a distinguished node, called the initial node, which is connected to every other node by exactly one path. This node-and-edge formulation can be traced to Kuhn (1953, Section 1) and it appears today in Mas-Colell, Whinston, and Green (1995, page 227).

Node-and-edge notation is complicated, even in the clean presentation of Mas-Colell, Whinston, and Green (1995). To simplify notation, Rubinstein begins with actions rather than nodes-and-edges, and then constructs each node as the sequence of actions leading to it. Accordingly, his tree is a collection of action sequences (i.e. histories) of the form $(a_1, a_2, ... a_N)$, and his initial node is the empty sequence $\{\}$. He assumes that if $(a_1, a_2, ... a_N)$ is in the tree, then $(a_1, a_2, ... a_{N-1})$ must also be in the tree. Hence he implicitly guarantees that the initial node

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is connected to every other node by exactly one path. This sequence-tree formulation appears in Osborne and Rubinstein (1994, page 200).

In this paper, we go one step further and identify each node with the *set* of actions leading to it. In particular, we define a "set tree" to be a collection of sets, which has the property that every nonempty set in the tree has a unique element whose removal results in another set of the tree. This unique element is defined to be the set's "last action."

It is incumbent upon us to demonstrate the sense in which such a set tree is equivalent to a sequence tree. Toward this end, we define an isomorphism between sequence trees and set trees: we say that a sequence tree is "isomorphic" to a set tree if there is an invertible map from sequences to sets, such that removing the last action of any sequence corresponds to removing the last action of the corresponding set. In this manner, the isomorphism formalizes the resemblance between the concatenation of sequences and the union of sets.

Finally we define "agent recall" to mean the absence of an absent-minded agent. This condition is weaker than perfect recall, and serves to rule out sequences that repeat an action. This paper's only theorem then shows that sequence-tree games with agent recall are equivalent to set-tree games. To be precise, every sequence-tree game with agent recall is isomorphic to exactly one set-tree game. Conversely, every set-tree game is isomorphic to exactly one sequence-tree game, and that sequence-tree game has agent recall. Our proofs use only basic logic and set theory.

The theorem accommodates general finite-horizon games with arbitrary action spaces and arbitrarily configured information sets. In particular, the theorem admits continuum action spaces, continuum type spaces (since a type is a chance action), and intertwined information sets that cannot be formulated within a multistage game (Myerson (1991, page 296)). The theorem is restricted to finite-horizon games because its proof contains two inductive arguments which rely upon every node consisting of only a finite number of actions.

The theorem may seem implausible because a sequence specifies order and thus has more structure than a set. In particular, first consider going from a sequence tree to a set tree. This direction starts easily because each sequence must be mapped to the set of actions that appear in the sequence. However, one must show that this map from sequences to sets is invertible, that each set has a unique last action, and that a set's last action appears as the last element of the sequence that generated the set.

Second, consider constructing a sequence tree from a set tree. This direction seems even less intuitive because both uniqueness and existence issues arise. Uniqueness seems unlikely because a given set can be ordered as a sequence in many different ways, and, to compound matters further, the theorem admits sequences that repeat actions when it admits arbitrary sequence trees that need not satisfy agent recall. Existence is also nontrivial because sequences must be assigned to sets in such a way that the concatenation of sequences is isomorphic to the union of sets, and hence, assigning a sequence to any one set places restrictions on the assignments at all the set's subsets and supersets. Essentially, the uniqueness result shows that a set tree has a surprising amount of structure, and the existence result shows that that structure is never strong enough to prevent the construction of a sequence tree.

To help develop intuition, the text considers an apparently difficult example with intertwined agents (i.e. information sets). Because the agents are intertwined, their order of play is not predetermined. Nonetheless the theorem holds. Essentially, if two actions can be played in two different orders, then there must be previous actions that determines the order in which the later actions are played.

To our knowledge, this is the first paper to formulate games by means of set trees. We believe that the availability of this alternative formulation will pay substantial dividends. One such dividend appears in Streufert (2012). There, we derive from any assessment its implied plausibility (i.e. infinite relative likelihood) relation over the game's nodes. We find that if the assessment is consistent, then its plausibility relation can be represented by a plausibility density function defined over the game's actions. This analysis is surprisingly straightforward because of an analogy with the early foundations of ordinary probability theory: actions resemble states, nodes resemble events, and a plausibility density function resembles a probability density function. Further, the two theories use exactly the same mathematics. This rich analogy grows directly out of this paper's observation that a node can be specified as a set of actions.

This paper is organized as follows. Section 2 defines set-tree games and defines what it means for a set-tree game to be isomorphic to a sequence-tree game. Section 3 contains the paper's only theorem, which shows that there is a one-to-one relationship between the collection of set-tree games and the collection of sequence-tree games having agent recall. Section 4 concludes.

2. Definitions

2.1. Reviewing Sequence-Tree Games

We begin by reviewing Osborne and Rubinstein (1994, page 200)'s formulation of an extensive-form game. For the purposes of this paper, we call their formulation a "sequence-tree game" because it incorporates the observation that each of a game's nodes can be identified with the sequence of actions leading to it. Osborne (2008, Section 3) credits Rubinstein with this observation. We take the liberty of restating their formulation using terminology upon which we can easily build.

While their formulation admits infinite-horizon games, ours does not. Accordingly, the definitions of this section assume that every node is a *finite* sequence of actions. Extending our theorem to accommodate infinite-horizon games is nontrivial because its proof contains two lengthy inductive arguments which depend upon every node having only a finite number of actions.

In every other regard, this section restates the Osborne and Rubinstein (1994) formulation in its full generality. In particular, we admit continuum action spaces. Thereby we also admit continuum type spaces, since a type is a chance action. Further, we admit arbitrarily arranged agents (i.e. information sets) which cannot be specified within the multistage formulation of Myerson (1991, page 296). Accordingly, the order in which agents move can be either exogenously or endogenously determined.

Let A be a set of *actions*. Then let $\bar{t} = \langle \bar{t}_n \rangle_{n=1}^{N(\bar{t})}$ denote a finite sequence of such actions, in which $N(\bar{t})$ is the length of the sequence. By convention, the empty set $\{\}$ is a sequence of actions of length zero. Further, for any nonempty $\bar{t}$ and any $0<m\leq N(\bar{t})$, let ${}_1\bar{t}_m$ denote the sequence $\langle \bar{t}_n \rangle_{n=1}^{m}$. By convention, ${}_1\bar{t}_0$ equals $\{\}$ regardless of $\bar{t}$.

By the way, a bar signifies that a symbol belongs to the sequence-tree formulation but not to the set-tree formulation. Accordingly, $\bar{t}$ has a bar, and its counterpart t in the set-tree formulation will not have a bar. A does not have a bar because it is common to both formulations.

Let a *sequence tree* $(A,\bar{T})$ be a set A of actions together with a set $\bar{T}$ of finite sequences $\bar{t}$ of actions such that $|\bar{T}|\geq 2$, such that

$$(\forall \bar{t}{\in}\bar{T})\ \ \bar{t}\neq\{\}\ \Rightarrow\ {}_1\bar{t}_{N(\bar{t})-1}\in\bar{T}\ , \tag{1}$$

and such that every action in A appears within at least one sequence in $\bar{T}$ (this last assumption entails no loss of generality, for if it were violated we could simply remove the superfluous actions from A). We often refer to the sequences in a sequence tree as the *nodes*[1] of the tree.

Given a sequence tree $(A,\bar{T})$, let $\bar{F}$ be the correspondence[2] from $\bar{T}$ into A that satisfies

$$(\forall \bar{t})\ \bar{F}(\bar{t}) = \{\ a \mid \bar{t}{\oplus}(a){\in}\bar{T}\ \}\ .$$

where $\oplus$ is the concatenation operator. Since every action a in $\bar{F}(\bar{t})$ can be combined with the node $\bar{t}$ to produce the new node $\bar{t}{\oplus}(a)$, the set $F(\bar{t})$ can be understood as the set of actions that are *feasible* from $\bar{t}$. Then, given this feasibility correspondence $\bar{F}$, the set of nodes $\bar{T}$ can be partitioned into the set of *terminal* nodes, $\bar{Z} = \{\ \bar{t} \mid \bar{F}(\bar{t}){=}\varnothing\ \}$, and the set of *nonterminal* nodes, $\bar{T}{\sim}\bar{Z} = \{\ \bar{t} \mid \bar{F}(t){\neq}\varnothing\ \}$.[3] Note that $\bar{F}$ and $\bar{Z}$ are derived from $(A,\bar{T})$.

A game will also specify a collection $\bar{H}\subseteq\mathcal{P}(\bar{T}{\sim}\bar{Z})$ of *agents* (i.e. information sets) $\bar{h}$ such that $\bar{H}$ partitions $\bar{T}{\sim}\bar{Z}$ and such that

$$(\forall \bar{t}^1,\bar{t}^2)\ \ [(\exists \bar{h})\{\bar{t}^1,\bar{t}^2\}{\subseteq}\bar{h}]\ \Rightarrow\ \bar{F}(\bar{t}^1){=}\bar{F}(\bar{t}^2)\ \text{ and} \tag{2a}$$

$$(\forall \bar{t}^1,\bar{t}^2)\ \ [(\nexists \bar{h})\{\bar{t}^1,\bar{t}^2\}{\subseteq}\bar{h}]\ \Rightarrow\ \bar{F}(\bar{t}^1){\cap}\bar{F}(\bar{t}^2){=}\varnothing\ . \tag{2b}$$

The first of these two implications states that the same actions are feasible from any two nodes in an agent $\bar{h}$. This assumption is standard and leads one to write $\bar{F}(\bar{h})$ for the set of actions feasible for agent $\bar{h}$.[4] The second implication states that actions are *agent-specific* in the sense that nodes from different agents must have different actions.

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[1]Osborne and Rubinstein (1994) refer to such a sequence as a "history" and denote it by "h". We reserve "h" for an agent (i.e. information set).

[2]This correspondence is usually denoted by "A". We reserve "A" for the set of all actions.

[3]As a matter of convention, we denote the empty set by $\{\}$ when it is regarded as a node and denote it by $\varnothing$ in all other contexts.

[4]As with any correspondence, the value $\bar{F}(\bar{h})$ of the correspondence $\bar{F}$ at the *set* $\bar{h}$ is defined to be $\{a|(\exists \bar{t}{\in}\bar{h})a{\in}\bar{F}(\bar{t})\}$. This construction is particularly natural here because (2a) implies that $(\forall \bar{t}{\in}\bar{h})\ \bar{F}(\bar{t}) = \bar{F}(\bar{h})$.

This assumption entails no loss of generality because one can always introduce enough actions so that agents never share actions (this is only a matter of notation).

Further, for the purposes of this paper, let a *prepartition* of a set S be a collection of disjoint sets whose union is S. Notice that $\varnothing$ can belong to a prepartition (it cannot belong to a partition).

A *sequence-tree game* $(A,\bar{T},\bar{H},\bar{I},\bar{i}^c,\bar{\rho},\bar{u})$ is a sequence tree $(A,\bar{T})$ together with (a) a collection $\bar{H}\subseteq\mathcal{P}(\bar{T}{\sim}\bar{Z})$ of agents (i.e. information sets) $\bar{h}$ such that $\bar{H}$ partitions $\bar{T}{\sim}\bar{Z}$ and satisfies (2), (b) a collection $\bar{I}\subseteq\mathcal{P}(\bar{H})$ of *players* $\bar{i}$ such that $\bar{I}$ is a prepartition of $\bar{H}$, (c) a *chance* player $\bar{i}^c\in\bar{I}$, (d) a function $\bar{\rho}:\bigcup_{\bar{h}\in\bar{i}^c}\bar{F}(\bar{h})\to(0,1]$ which assigns a positive probability to each chance action $a\in\bigcup_{\bar{h}\in\bar{i}^c}F(\bar{h})$, and (e) a function $\bar{u}:(\bar{I}{\sim}\{\bar{i}^c\})\times\bar{Z}\to\mathbb{R}$ which specifies a *payoff* $\bar{u}_{\bar{i}}(\bar{t})$ to each nonchance player $\bar{i}\in\bar{I}{\sim}\{\bar{i}^c\}$ at each terminal node $\bar{t}\in\bar{Z}$. By assumption, the chance probabilities are assumed to satisfy $(\forall\bar{h}\in\bar{i}^c)\ \Sigma_{a\in\bar{F}(\bar{h})}\bar{\rho}(a)=1$ so that they specify a probability distribution at each chance agent $\bar{h}\in\bar{i}^c$.

Note that an empty player $\bar{i}=\varnothing$ has no agents and no actions. Accordingly, a game "without chance" can be specified by setting the chance player $\bar{i}^c=\varnothing$. We assume without loss of generality that every nonchance player is nonempty.

2.2. Defining Set-Tree Games

This subsection introduces a new formulation of game in which the game's nodes are sets rather than sequences.

Given a set A of actions, let T be a collection of finite subsets of A. We call an element of T a *node* and denote it by t. Note that each node t is a subset of A, and thus nodes have been specified as sets of actions. Further, given such an (A,T), let a *last action* of a node t be any action $a\in t$ such that $t{\sim}\{a\}\in T$. Thus a last action of a node is any action in the node whose removal results in another node.

Figures 1, 2, and 3 provide three examples. In each case, the figure's caption fully defines (A,T), and accordingly, the definition is complete without the illustration itself. Each illustration links two nodes with an action-labelled line exactly when (a) that action is a last action of the larger set and (b) the smaller set is the larger set without that action. For example, f is the only last action of $\{e,f\}$ in Figure 1, and both f and g are last actions of $\{f,g\}$ in Figure 2.

A *set tree* (A, T) is a set A and a collection T of finite subsets of A such that $|T| \geq 2$, such that $A = \bigcup T$, and such that

$$\text{every nonempty } t \in T \text{ has a unique last action.} \tag{3}$$

The assumption $A = \bigcup T$ entails no loss of generality because $A \supseteq \bigcup T$ by construction and because $A \sim \bigcup T$ can be made empty by eliminating unused actions. Figure 1 fails to define a set tree because the node $\{f, g\}$ does not have a last action, and Figure 2 fails to define a set tree because the node $\{f, g\}$ has two last actions. In contrast, Figure 3 does define a set tree.

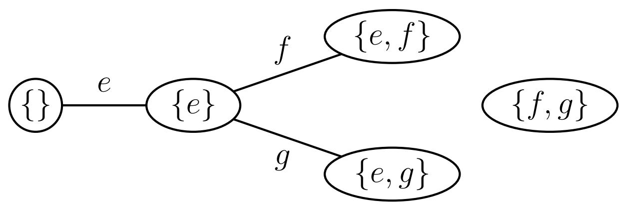

FIGURE 1. $A = \{e, f, g\}$ and $T = \{\{\}, \{e\}, \{e, f\}, \{e, g\}, \{f, g\}\}$ violate assumption (3) since $\{f, g\}$ does not have a last action.

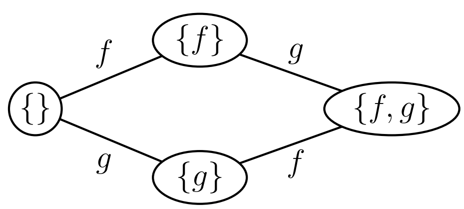

FIGURE 2. $A = \{f, g\}$ and $T = \{\{\}, \{f\}, \{g\}, \{f, g\}\}$ violate assumption (3) since $\{f, g\}$ has two last actions.

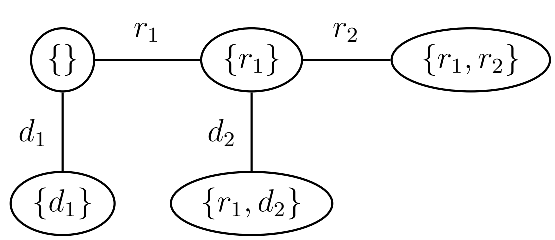

FIGURE 3. The set tree (A, T) defined by $T = \{\,\{\}, \{d_1\}, \{r_1\}, \{r_1, d_2\}, \{r_1, r_2\}\,\}$ and $A = \cup T$.

To see an analogy, recall that a topological space $(X,\mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets of X which satisfies certain properties. Similarly, a set tree (A,T) is a set A together with a collection T of subsets of A which satisfies certain properties.

Given a set tree (A,T), let F be the correspondence from T into A that satisfies

$$(\forall t)\ F(t) = \{\ a \mid a{\notin}t \text{ and } t{\cup}\{a\}{\in}T\ \}\ .$$

Since every action a in $F(t)$ can be combined with the node t to produce a new node $t{\cup}\{a\}$, the set $F(t)$ can be understood as the set of actions that are *feasible* from t. Then, given F, the set of nodes T can be partitioned into the set of *terminal* nodes, $Z = \{\ t \mid F(t){=}\varnothing\ \}$, and the set of *nonterminal* nodes, $T{\sim}Z = \{\ t \mid F(t){\neq}\varnothing\ \}$. In this fashion F and Z are derived from (A,T).

A set-tree game will also specify a collection $H \subseteq \mathcal{P}(T{\sim}Z)$ of *agents* (i.e. information sets) h such that H partitions $T{\sim}Z$ and such that

(4a) $\quad (\forall t^1,t^2)\ \ [(\exists h)\{t^1,t^2\}{\subseteq}h]\ \Rightarrow\ F(t^1){=}F(t^2)$ and

(4b) $\quad (\forall t^1,t^2)\ \ [(\nexists h)\{t^1,t^2\}{\subseteq}h]\ \Rightarrow\ F(t^1){\cap}F(t^2){=}\varnothing$.

This assumption (4) for a set-tree game is interpreted just as assumption (2) for a sequence-tree game.

Finally, a *set-tree game* (A,T,H,I,i^c,ρ,u) is a set tree (A,T) together with (a) a collection $H \subseteq \mathcal{P}(T{\sim}Z)$ of agents h such that H partitions $T{\sim}Z$ and satisfies (4), (b) a collection $I \subseteq \mathcal{P}(H)$ of *players* i such that I is a prepartition of H, (c) a *chance* player $i^c \in I$, (d) a function $\rho : \bigcup_{h \in i^c} F(h) \to (0,1]$ which assigns a positive probability to each chance action $a{\in}\bigcup_{h\in i^c} F(h)$, and (e) a function $u : (I{\sim}\{i^c\}){\times}Z \to \mathbb{R}$ which specifies a *payoff* $u_i(t)$ to each nonchance player $i{\in}I{\sim}\{i^c\}$ at each terminal node $t{\in}Z$. The chance probabilities are assumed to satisfy $(\forall h{\in}i^c)\ \Sigma_{a\in F(h)}\rho(a) = 1$ so that they specify a probability distribution at each chance agent $h{\in}i^c$. Without loss of generality, every nonchance player is assumed to be nonempty.

2.3. Defining an Isomorphism

This subsection defines a natural isomorphism between sequence-tree games and set-tree games. Accordingly, the isomorphism switches between nodes as sequences and nodes as sets.

Let R denote the function which takes a sequence $\bar{t} = (\bar{t}_1, \bar{t}_2, ... \bar{t}_{N(\bar{t})})$ of actions to a set of actions according to

$$R(\bar{t}) = \{\bar{t}_1, \bar{t}_2, ... \bar{t}_{N(\bar{t})}\} \ .$$

For example, $R((r,r,d)) = \{d,r\}$, which illustrates that neither the order of actions in the sequence nor the repetition of actions in the sequence effects the value of R. The symbol "R" is natural in several senses. First, the set $R(\bar{t})$ is the "R"ange of the sequence $\bar{t}$. Second, R "R"educes a sequence to a set. And finally, R "R"emoves the bar as "$R(\bar{t}) = t$" suggests.

A sequence tree $(A, \bar{T})$ is *isomorphic* to a set tree (A,T) if

(5a) $\qquad R|_{\bar{T}}$ is an invertible function from $\bar{T}$ onto T , and

(5b) $\qquad (\forall \bar{t}^*, a, \bar{t}) \ \ \bar{t}^* \oplus (a) = \bar{t} \ \Leftrightarrow \ a \notin R(\bar{t}^*)$ and $R(\bar{t}^*) \cup \{a\} = R(\bar{t})$.

To see an analogy, recall that two algebraic groups are "isomorphic" if there is an invertible function between the two groups which preserves the structure of each group's binary relation in the structure of the other group's binary relation. Here is something similar: $R|_{\bar{T}}$ is an invertible function between $\bar{T}$ and T which preserves the structure of $\bar{T}$'s concatenation in the structure of T's union, and conversely, preserves the structure of T's union in $\bar{T}$'s concatenation.

This isomorphism between trees has many consequences. For example, suppose that $(A, \bar{T})$ and (A,T) are isomorphic, that $\bar{F}$ is derived from $(A, \bar{T})$, and that F is derived from (A,T). Then by Lemma A.5(a) in the Appendix, we have that $\bar{F}(\bar{t}) = F(t)$ whenever $R(\bar{t}) = t$.

Next, let R_1 denote the function which takes an arbitrary set $\bar{S}_1$ of sequences into the corresponding set of sets according to[5]

$$R_1(\bar{S}_1) = \{ \ R(\bar{t}) \mid \bar{t} \in \bar{S}_1 \ \} \ .$$

For example, $R_1(\{(d,r,r),(d,s)\}) = \{\{d,r\},\{d,s\}\}$. In general, if $(A, \bar{T})$ and (A,T) are isomorphic, we have that $R_1|_{\mathcal{P}(\bar{T})}$ is an invertible function from $\mathcal{P}(\bar{T})$ onto $\mathcal{P}(T)$, that $R_1(\bar{T}) = T$, and that $R_1(\bar{Z}) =$

[5]In common parlance, if $f{:}X{\rightarrow}Y$ and $B{\subseteq}X$ then $f(B)$ is understood to be $\{f(x)|x{\in}B\}$. Thus common parlance endows the symbol $f(\cdot)$ with two meanings, one for when the argument is an element of X and the other for when the argument is a subset of X. Our introducing R_1 is like dropping the second meaning of $f(\cdot)$ (so that $f(B)$ becomes undefined) and then introducing the symbol $f_1(\cdot)$ (so that $f_1(B)$ becomes defined). We do not use the f_1 notation in general. For example, we write $F(h)$ rather than $F_1(h)$.

Z (Lemmas A.4(a) and A.5(b) in the Appendix). In the sequel, a sequence-tree agent $\bar{h}$ will be mapped to the set-tree agent $R_1(\bar{h}) = h$.

Further, let R_2 denote the function which takes an arbitrary set $\bar{S}_2$ of sets of sequences into the corresponding set of sets of sets according to

$$R_2(\bar{S}_2) = \{\ R_1(\bar{S}_1) \mid \bar{S}_1 \in \bar{S}_2\ \}\ .$$

For instance, $R_2(\ \{\{(d,r),(d,d)\},\{(x,x)\}\}\ ) = \{\{\{d,r\},\{d\}\},\{\{x\}\}\}$. In general, if $(A,\bar{T})$ and (A,T) are isomorphic, then $R_2|_{\mathcal{P}^2(\bar{T})}$ is an invertible function from $\mathcal{P}^2(\bar{T})$ onto $\mathcal{P}^2(T)$. In the sequel, a sequence-tree player $\bar{i}$ will be mapped to the set-tree player $R_2(\bar{i}) = i$.

Finally, say that $(A,\bar{T},\bar{H},\bar{I},\bar{i}^c,\bar{\rho},\bar{u})$ and (A,T,H,I,i^c,ρ,u) are *isomorphic* if $(A,\bar{T})$ and (A,T) are isomorphic,

$$\{\ R_1(\bar{h}) \mid \bar{h} \in \bar{H}\ \} = H\ , \tag{6a}$$

$$\{\ R_2(\bar{i}) \mid \bar{i} \in \bar{I}\ \} = I\ , \tag{6b}$$

$$R_2(\bar{i}^c) = i^c\ , \tag{6c}$$

$$\bar{\rho} = \rho\ ,\ \text{and} \tag{6d}$$

$$(\forall \bar{i} \neq \bar{i}^c)(\forall \bar{t} \in \bar{Z})\ \bar{u}_{\bar{i}}(\ \bar{t}\ ) = u_{R_2(\bar{i})}(\ R(\bar{t})\ )\ . \tag{6e}$$

3. Theorem

3.1. Agent Recall

Not every sequence-tree game is isomorphic to a set-tree game. For example, consider the sequence tree $(A,\bar{T})$ of Figure 4. Here $R((r)) = \{r\} = R((r,r))$, and thus $R|_{\bar{T}}$ is not an invertible function.

Examples like this one have an agent which is absent-minded in the sense of Piccione and Rubinstein (1997). Informally, an agent is absent-minded if the agent does not know whether it has already moved. Formally, an agent is *absent-minded* if there is a sequence which enters the agent more than once. In other words, an agent $\bar{h}$ is absent-minded if there exist $\bar{t}$ and $0 \leq m < n \leq N(\bar{t})$ such that $\{{}_1\bar{t}_m, {}_1\bar{t}_n\} \subseteq \bar{h}$. In the example, the agent $\bar{h}$ is absent-minded because the sequence $\bar{t} = (r)$ enters the agent twice, once at ${}_1\bar{t}_0 = \{\}$ and again at $\bar{t} = (r)$. In general, every sequence which repeats an action twice must enter the action's agent twice, and thus, the existence of a sequence repeating an action implies the existence of an absent-minded agent.

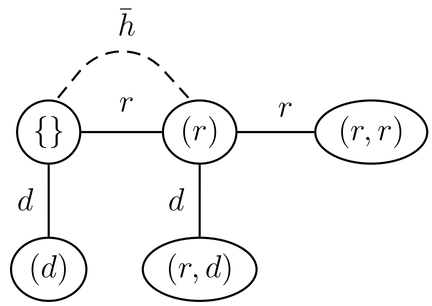

Figure 4. The sequence (r,r) repeats the action r (and thereby precludes isomorphism). Accordingly, the agent $\bar{h} = \{\{\},(r)\}$ is absent-minded, in violation of agent recall.

A sequence tree $(A,\bar{T})$ with agents $\bar{H}$ is said to have *agent recall* if it has no absent-minded agents. In other words, agent recall is the absence of absent-mindedness. Agent recall is implied by perfect recall, and perfect recall is assumed by many authors including Kreps and Wilson (1982). Specifically, they define perfect recall as the combination of their equations (2.2) and (2.3). Their equation (2.2) is equivalent to agent recall by Lemma A.6(b) in the Appendix, and their equation (2.3) might be usefully called "player recall" as opposed to "agent recall" (that additional assumption requires that players recall what actions were chosen at all of their own past agents).

3.2. Showing the Isomorphism is One-to-one

Theorem 1. *(a) Every sequence-tree game with agent recall is isomorphic to exactly one set-tree game. (b) Conversely, every set-tree game is isomorphic to exactly one sequence-tree game, and that sequence-tree game has agent recall. (Proofs A.9 and A.10 in the Appendix.)*

Thus the theorem shows that isomorphism constitutes a one-to-one correspondence between (1) the collection of sequence-tree games with agent recall and (2) the collection of set-tree games. This one-to-one correspondence is illustrated by Figure 5. Or, to put the theorem another way, the structure of a sequence-tree game with agent recall is identical to the structure of a set-tree game.

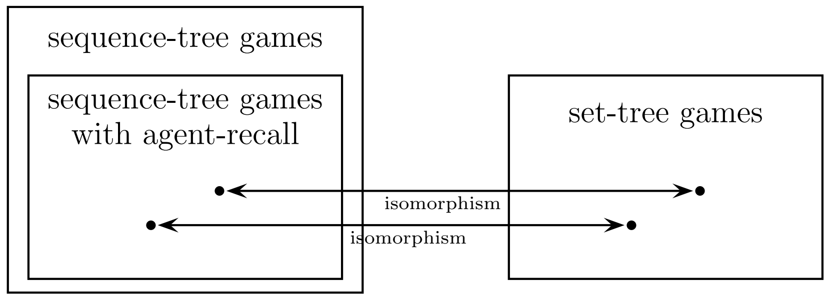

FIGURE 5

The theorem may seem implausible because an individual sequence has more structure than an individual set, since a sequence specifies order and a set does not. The following paragraphs explore this and related difficulties.

(a) Going one direction, from sequences to sets, starts simply because R determines the set tree as $T = R_1(\bar{T})$ and then determines the rest of the set-tree game by (6). Additionally, the assumption of agent recall rules out sequences that repeat actions (this was illustrated by Figure 4 above and is formally proved by the appendix's Lemma A.7).

However, substantial issues of order remain. First, is $R|_{\bar{T}}$ invertible, or could the sequence tree $\bar{T}$ have two sequences with the same actions in different orders? Second, even if $R|_{\bar{T}}$ is invertible, could a set in T have multiple last actions, as would be the case in Figure 4, where both r and d would be last actions of $R((r, d)) = \{r, d\}$? Third, even if every set in T has a unique last action, could the last action of a set be in the middle, rather than at the end, of the sequence corresponding to the set? These issues are addressed in the appendix's Proof A.9.

(b) Going the other direction, from sets to sequences, is harder in the sense that one must figure out how to define the sequence tree. Both uniqueness and existence are nontrivial.

The theorem's claim about uniqueness is strong. It claims that each set tree corresponds to no more than one sequence tree, and further, that this uniqueness stands even if the candidate sequence trees are not required to satisfy agent recall. This claim is different than the claim that $R|_{\bar{T}}$ is an invertible function for any $\bar{T}$ with agent recall. Rather, it says that for any T there is at most one $\bar{T}$ which makes $R|_{\bar{T}}$ an invertible function onto T. This is a strong statement because the many possible ways of constructing the sequences of $\bar{T}$ admit many possible ways of ordering the actions in the sets of T. Further, the possibility of defining a $\bar{T}$ without agent recall admits the further possibility of defining sequences which repeat actions (Lemma A.7). Nonetheless, the implicit structure of a set tree T precludes all this. This is proved in Step 1 of Proof A.10.

Proving existence requires finding a way to assign sequences to sets in such a way that the concatenation of sequences is isomorphic to the union of sets, as specified in (5b). This is nontrivial because assigning a sequence to a set has implications for the assignments at all the set's subsets and supersets. The solution can be found in Steps 2–5 of Proof A.10.

In summary, the uniqueness result shows that a set tree has a surprising amount of implicit structure. Then the existence result shows that that structure is never so strong that it prevents the construction of a sequence tree. Thus a sequence tree with agent recall explicitly spells out the implicit structure of a set tree.

3.3. Developing Intuition

A good way to develop intuition is to consider an example in which the order of play is determined endogenously rather than exogenously.

Imagine that two spies are racing to recover a document from a safe deposit box. En route one spy realizes that if she reaches the box first, she can install a bomb which will explode when the other spy reaches the box after her. But then she realizes that the other spy will be thinking the same thing, and hence, if she opens the box when she reaches it, she will find either the document or an exploding bomb. So, she considers blowing up the bank without opening the box in hopes of keeping the document from the other spy.

Figure 6 specifies this situation using a sequence tree. Nature determines whether Spy 1 (f_1) or Spy 2 (f_2) is first. Then the two spies either look (ℓ) in the box or chicken out (c) by blowing up the bank without looking inside. Clearly the game depends heavily on the order in which the spies move. Yet, this situation can be specified as a set tree simply by turning the figure's sequences into sets. Each set of actions can only be played in one order because any ambiguity is resolved by another action in the set. For example, the set $\{\ell_1, \ell_2, f_2\}$ can only be played in the order (f_2, ℓ_2, ℓ_1) because the set contains f_2.

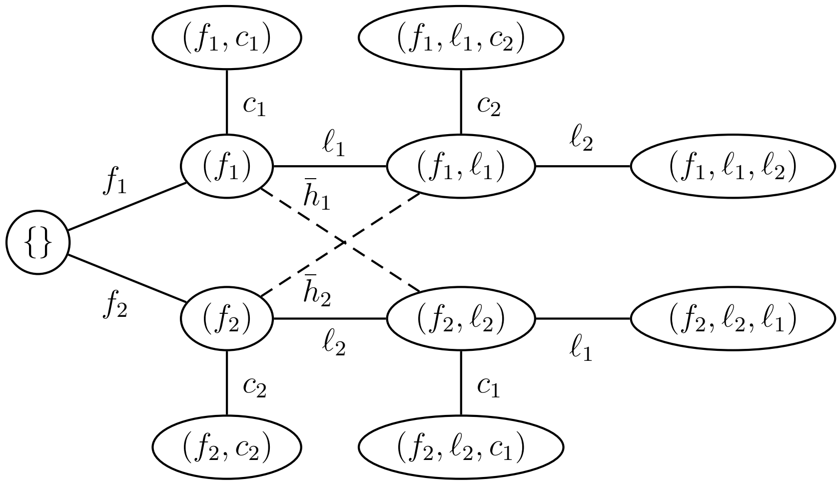

Figure 6. A sequence tree in which the order of actions appears to matter. The two agents $\bar{h}_1 = \{(f_1), (f_2, \ell_2)\}$ and $\bar{h}_2 = \{(f_2), (f_1, \ell_1)\}$ belong to the two spies.

This illustrates a general principle: A set of actions in a set tree can only be played in one order, because if that order is endogenous, it must have been determined by some action(s) in the set itself. Or, to put it another way, if two actions can be played in two different orders, then there must be earlier actions that determine the order in which the later two actions will be played.

4. Conclusion

This paper has introduced an alternative formulation for games. The innovation was to specify each node of the game tree as a set of actions rather than a sequence of actions. The paper's only theorem showed that finite-horizon set-tree games are equivalent to finite-horizon sequence-tree games with agent recall. Since agent recall is weaker than perfect recall, the theorem shows that set-tree games can formulate most of the finite-horizon sequence-tree games of interest to economists. Arbitrary action spaces, arbitrary type spaces, and arbitrarily configured information sets can all be accommodated.

This alternative formulation promises to have multiple applications. A first application was briefly discussed in the introduction: Streufert

(2012) derives a plausibility density function for every consistent assessment by drawing a remarkably straightforward analogy with the foundations of ordinary probability theory.

Appendix

A.1. Preliminaries

The five lemmas of this subsection are unsurprising but necessary components of the larger argument. The first two lemmas show how actions can be partitioned with respect to agents. The remaining three provide tools that are used to construct isomorphisms between sequence-tree games and set-tree games.

Lemma A.1. *In any sequence-tree game, $\langle \bar{F}(\bar{h})\rangle_{\bar{h}\in\bar{H}}$ is an indexed partition of A. In other words, $\{\bar{F}(\bar{h})|\bar{h}\}$ partitions A and $\bar{h}\mapsto\bar{F}(\bar{h})$ is invertible.*

Proof. We begin with three observations.

(a) Each $\bar{F}(\bar{h})$ is nonempty. To see this, note $\bar{H}$ partitions $\bar{T}\sim\bar{Z}$ by assumption, and thus each $\bar{h}$ is a nonempty set of nonterminal nodes.

(b) If $\bar{h}^1\neq\bar{h}^2$ then $\bar{F}(\bar{h}^1)\cap\bar{F}(\bar{h}^2)=\varnothing$. To see this, take any $\bar{h}^1\neq\bar{h}^2$, any $\bar{t}^1\in\bar{h}^1$, and any $\bar{t}^2\in\bar{h}^2$. Since $\bar{H}$ is a partition, we have $(\nexists\bar{h})\{\bar{t}^1,\bar{t}^2\}\subseteq\bar{h}$, and hence $\bar{F}(\bar{t}^1)\cap\bar{F}(\bar{t}^2)=\varnothing$ by (2b). This implies $\bar{F}(\bar{h}^1)\cap\bar{F}(\bar{h}^2)=\varnothing$ because $\bar{F}(\bar{t}^1)=F(\bar{h}^1)$ by $\bar{t}^1\in\bar{h}^1$ and (2a), and because $F(\bar{t}^2)=F(\bar{h}^2)$ by $\bar{t}^2\in\bar{h}^2$ and (2a).

(c) $\bigcup\{\bar{F}(\bar{h})|\bar{h}\}=A$. $\bigcup\{\bar{F}(\bar{h})|\bar{h}\}\subseteq A$ follows from the definition of $\bar{F}$. To see the converse, take any a. By assumption there exists some $\bar{t}$ and some $m\leq N(\bar{t})$ such that $\bar{t}_m=a$. By assumption (1) applied $N(\bar{t})-(m-1)$ times, both ${}_1\bar{t}_{m-1}$ and ${}_1\bar{t}_m$ are elements of $\bar{T}$. Thus since ${}_1\bar{t}_{m-1}\oplus(a)={}_1\bar{t}_m$, we have $a\in\bar{F}({}_1\bar{t}_{m-1})$. Further, since ${}_1\bar{t}_{m-1}\in T\sim Z$ and since $\bar{H}$ partitions $\bar{T}\sim\bar{Z}$ by assumption, we have some $\bar{h}$ such that ${}_1\bar{t}_{m-1}\in\bar{h}$. Thus by the last two sentences, $a\in\bar{F}(\bar{h})$.

$\{\bar{F}(\bar{h})|\bar{h}\}$ partitions A by observations (a)–(c). If $\bar{h}\mapsto\bar{F}(\bar{h})$ were not invertible, there would be $\bar{h}^1\neq\bar{h}^2$ such that $\bar{F}(\bar{h}^1)=\bar{F}(\bar{h}^2)$. Since both $\bar{F}(\bar{h}^1)$ and $\bar{F}(\bar{h}^2)$ are both nonempty by observation (a), we would then have $\bar{h}^1\neq\bar{h}^2$ such that $\bar{F}(\bar{h}^1)\cap\bar{F}(\bar{h}^2)\neq\varnothing$. This would contradict observation (b). $\square$

Lemma A.2. *In any set-tree game, $\{F(h)\}_{h \in H}$ is an indexed partition of A. In other words, $\{F(h)|h\}$ partitions A and $h \mapsto F(h)$ is invertible.*

Proof. We begin with three observations.

(a) Each $F(h)$ is nonempty. To see this, note H partitions $T{\sim}Z$ by assumption, and thus each h is a nonempty subset of nonterminal nodes.

(b) If $h^1{\neq}h^2$ then $F(h^1){\cap}F(h^2) = \varnothing$. To see this, take any $h^1{\neq}h^2$, any $t^1{\in}h^1$, and any $t^2{\in}h^2$. Since H is a partition, we have $(\nexists h)\{t^1, t^2\}{\subseteq}h$, and hence $F(t^1){\cap}F(t^2) = \varnothing$ by (4b). This implies $F(h^1){\cap}F(h^2){=}\varnothing$ because $F(t^1){=}F(h^1)$ by $t^1{\in}h^1$ and (4a), and because $F(t^2){=}F(h^2)$ by $t^2{\in}h^2$ and (4a).

(c) $\bigcup\{F(h)|h\} = A$. $\bigcup\{F(h)|h\}{\subseteq}A$ follows from the definition of F. To see the converse, take any a. By the assumption $A{=}\bigcup T$, there exists a $\hat{t}$ such that $a{\in}\hat{t}$. Since A is finite, $\hat{t}{\subseteq}A$ is finite. Thus applying assumption (3) a finite number of times yields a $t{\subseteq}\hat{t}$ such that a is the last action of t. Note $a{\in}F(t{\sim}\{a\})$. Further, since $t{\sim}\{a\}$ is nonterminal and H partitions the collection of nonterminal nodes, there is some h such that $t{\sim}\{a\}{\in}h$. Thus by the last two sentences, $a{\in}F(h)$.

$\{F(h)|h\}$ partitions A by observations (a)-(c). If $h \mapsto F(h)$ were not invertible, there would be $h^1{\neq}h^2$ such that $F(h^1){=}F(h^2)$. Since both $F(h^1)$ and $F(h^2)$ are both nonempty by observation (a), we would then have $h^1{\neq}h^2$ such that $F(h^1){\cap}F(h^2){\neq}\varnothing$. This would contradict observation (b). □

The following lemma provides a respite from this paper's notation. We use it when partitioning nodes into agents, and when prepartitioning agents into players.

Lemma A.3. *Suppose that f is an invertible function from X, and define f_1 from $\mathcal{P}(X)$ by $f_1(S) = \{\, f(x) \,|\, x{\in}S \,\}$. Then, (a) $\mathcal{S}$ is a partition of X iff $\{\, f_1(S) \,|\, S{\in}\mathcal{S} \,\}$ is a partition of $f_1(X)$. Further, (b) $\mathcal{S}$ is a prepartition of X iff $\{\, f_1(S) \,|\, S{\in}\mathcal{S} \,\}$ is a prepartition of $f_1(X)$.*

Proof. (a) This paragraph shows that if (i) f is an invertible function from X and (ii) $\mathcal{S}$ is a partition of X, then $\{\, f_1(S) \,|\, S{\in}\mathcal{S} \,\}$ is a partition of $f_1(X)$. Accordingly, suppose that X, f, and $\mathcal{S}$ satisfy (i) and (ii), and let $\mathcal{B}$ equal $\{\, f_1(S) \,|\, S{\in}\mathcal{S} \,\}$. We must show (1) that every element of $\mathcal{B}$ is nonempty, (2) that the elements of $\mathcal{B}$ are pairwise disjoint, and

(3) that $\bigcup\mathcal{B} = f_1(X)$. (1) Every element of $\mathcal{B}$ must equal $f_1(S)$ for some $S{\in}\mathcal{S}$. This S must be nonempty since $\mathcal{S}$ is a partition, and hence $f_1(S)$ is nonempty by the definition of f_1. (2) Every pair of elements from $\mathcal{B}$ must equal $\{f_1(S^1), f_1(S^2)\}$ for some $\{S^1, S^2\}{\subseteq}\mathcal{S}$. If $f_1(S^1)$ and $f_1(S^2)$ are not equal, then S^1 and S^2 are not equal, which implies that S^1 and S^2 are disjoint because $\mathcal{S}$ is a partition, which implies that $f_1(S^1)$ and $f_1(S^2)$ are disjoint because f is invertible. (3) We argue that $\bigcup\mathcal{B} = \bigcup\{\, f_1(S) \mid S{\in}\mathcal{S}\,\} = f_1(\bigcup\mathcal{S}) = f_1(X)$. The first equality holds by the definition of $\mathcal{B}$. The second holds because its two sides equal the set of values that f assumes at some member of some member of $\mathcal{S}$. The third holds because $\mathcal{S}$ is a partition of X.

The previous paragraph establishes the forward direction of part (a). To establish the converse, note that the assumed invertibility of f from X implies the invertibility of f^{-1} from $f_1(X)$. Now assume that $\mathcal{B} = \{\, f_1(S) \mid S{\in}\mathcal{S}\,\}$ is a partition of $f_1(X)$. The previous paragraph (with f^{-1} replacing f, $f_1(X)$ replacing X, and $\mathcal{B}$ replacing $\mathcal{S}$) shows that $\{\, f_1^{-1}(B) \mid B{\in}\mathcal{B}\,\}$ is a partition of $f_1^{-1}(f_1(X))$. Note that the definitions of f_1^{-1} and f_1 imply

$$\begin{aligned}(\forall S{\subseteq}X)\quad f_1^{-1}(f_1(S)) &= \{\ f^{-1}(y) \mid y{\in}f_1(S)\ \}\\ &= \{\ f^{-1}(y) \mid y{\in}\{f(x)|x{\in}S\}\ \}\\ &= \{\ f^{-1}(f(x)) \mid x{\in}S\ \}\\ &= S\ .\end{aligned}$$

Thus by the definition of $\mathcal{B}$ and the last two sentences,

$$\begin{aligned}\{\, f_1^{-1}(B) \mid B{\in}\mathcal{B}\,\} &= \{\, f_1^{-1}(B) \mid B{\in}\{\, f_1(S) \mid S{\in}\mathcal{S}\,\}\,\}\\ &= \{\, f_1^{-1}(f_1(S)) \mid S{\in}\mathcal{S}\,\}\,\}\\ &= \{\, S \mid S{\in}\mathcal{S}\,\}\,\}\\ &= \mathcal{S}\end{aligned}$$

is a partition of $f_1^{-1}(f_1(X)) = X$.

(b) Repeat the above proof for (a), but replace "partition" with "prepartition" and omit part (1) from the first paragraph. □

Lemma A.4. *The following hold when $(A, \bar{T})$ is isomorphic to (A, T).*
(a) $R_1|_{\mathcal{P}(\bar{T})}$ is an invertible function from $\mathcal{P}(\bar{T})$ onto $\mathcal{P}(T)$.
(b) $R_2|_{\mathcal{P}^2(\bar{T})}$ is an invertible function from $\mathcal{P}^2(\bar{T})$ onto $\mathcal{P}^2(T)$.

Proof. (a) Take any $\eta \in \mathcal{P}(T)$ (this η may or may not be an agent h). Since $R|_{\bar{T}}$ is an invertible function from $\bar{T}$ onto T by the assumed isomorphism, $\{\ (R|_{\bar{T}})^{-1}(t) \mid t{\in}\eta\ \}$ is the unique $\bar{\eta} \in \mathcal{P}(\bar{T})$ such that $R_1(\bar{\eta}) = \eta$.

(b) Take any $\iota \in \mathcal{P}^2(T)$ (this ι may or may not be a player i). Since $R_1|_{\mathcal{P}(\bar{T})}$ is an invertible function from $\mathcal{P}(\bar{T})$ onto $\mathcal{P}(T)$ by part (a), $\{\ (R_1|_{\mathcal{P}(\bar{T})})^{-1}(\eta) \mid \eta{\in}\iota\ \}$ is the unique $\bar{\iota} \in \mathcal{P}^2(\bar{T})$ such that $R_2(\bar{\iota}) = \iota$. □

Each of the six parts of the following lemma is used at least twice.

Lemma A.5. *Assume that $(A, \bar{T})$ is isomorphic to (A, T), that $\bar{F}$ and $\bar{Z}$ are derived from $(A, \bar{T})$, and that F and Z are derived from (A, T).*

(a) Take any $\bar{t}$. If $t = R(\bar{t})$, then $F(t) = \bar{F}(\bar{t})$.

(b) $Z = R_1(\bar{Z})$.

Further, in the following, H and $\bar{H}$ may or may not be sets of agents, and η and $\bar{\eta}$ may or may not be agents h and $\bar{h}$. Similarly, I and $\bar{I}$ may or may not be sets of players, and ι and $\bar{\iota}$ may or may not be players i and $\bar{i}$.

(c) Take any $\bar{\eta} \in \mathcal{P}(\bar{T}{\sim}\bar{Z})$. If $\eta = R_1(\bar{\eta})$, then $F(\eta) = \bar{F}(\bar{\eta})$.

(d) Take any $\bar{H} \subseteq \mathcal{P}(\bar{T}{\sim}\bar{Z})$. If $H = R_2(\bar{H})$, then H is a partition of $T{\sim}Z$ iff $\bar{H}$ is a partition of $\bar{T}{\sim}\bar{Z}$.

(e) Take any $\bar{H} \subseteq \mathcal{P}(\bar{T}{\sim}\bar{Z})$ and any $\bar{I} \subseteq \mathcal{P}(\bar{H})$. If $H = R_2(\bar{H})$ and $I = \{R_2(\bar{\iota}) | \bar{\iota}{\in}\bar{I}\}$, then I is a prepartition of H iff $\bar{I}$ is a prepartition of $\bar{H}$.

(f) Take any $\bar{I} \subseteq \mathcal{P}^2(\bar{T}{\sim}\bar{Z})$ and any $\bar{\iota}^* {\in} \bar{I}$. If $I = \{R_2(\bar{\iota}) | \bar{\iota}{\in}\bar{I}\}$ and $\iota^* = R_2(\bar{\iota}^*)$, then

$$\begin{aligned} &(\forall \iota{\in}I{\sim}\{\iota^*\})(\forall t{\in}Z)\quad u_\iota(t) = \bar{u}_{(R_2|_{\mathcal{P}^2(\bar{T})})^{-1}(\iota)}((R|_{\bar{T}})^{-1}(t)) \\ \textit{iff}\ \ &(\forall \bar{\iota}{\in}\bar{I}{\sim}\{\bar{\iota}^*\})(\forall \bar{t}{\in}\bar{Z})\quad \bar{u}_{\bar{\iota}}(\bar{t}) = u_{R_2(\bar{\iota})}(R(\bar{t}))\ . \end{aligned}$$

Proof. (a) Suppose $t{=}R(\bar{t})$. Then by the assumed equality, by the definition of F, by manipulation, by the invertibility of $R|_{\bar{T}}$ (5a), by the structure condition (5b), by manipulation, and by the definition of $\bar{F}$,

$$\begin{aligned} (\forall a)\quad & (t, a) \in F \\ \Leftrightarrow\quad & (R(\bar{t}), a) \in F \\ \Leftrightarrow\quad & a{\notin}R(\bar{t}) \text{ and } R(\bar{t}){\cup}\{a\}{\in}T \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow\ (\exists t')\ a{\notin}R(\bar t) \text{ and } R(\bar t){\cup}\{a\}{=}t' \\
&\Leftrightarrow\ (\exists \bar t')\ a{\notin}R(\bar t) \text{ and } R(\bar t){\cup}\{a\}{=}R(\bar t') \\
&\Leftrightarrow\ (\exists \bar t')\ \bar t{\oplus}(a) = \bar t' \\
&\Leftrightarrow\ \bar t{\oplus}(a) \in \bar T \\
&\Leftrightarrow\ (\bar t, a) \in \bar F
\end{aligned}$$

This is equivalent to $(\forall a)\ a{\in}F(t) \Leftrightarrow a{\in}\bar F(\bar t)$, which is in turn equivalent to $F(t) = \bar F(\bar t)$.

(b) By the definition of R_1, the definition of $\bar Z$, part (a), the invertibility of $R|_{\bar T}$ (5a), and the definition of Z,

$$\begin{aligned}
R_1(\bar Z) &= \{\ R(\bar t) \mid \bar t{\in}\bar Z\ \} \\
&= \{\ R(\bar t) \mid \bar F(\bar t){=}\varnothing\ \} \\
&= \{\ R(\bar t) \mid F(R(\bar t)){=}\varnothing\ \} \\
&= \{\ t \mid F(t){=}\varnothing\ \} \\
&= Z\ .
\end{aligned}$$

(c) Assume $\eta = R_1(\bar\eta)$. Then

$$\begin{aligned}
F(\eta) &= \bigcup\{\ F(t) \mid t{\in}\eta\ \} \\
&= \bigcup\{\ F(t) \mid t{\in}R_1(\bar\eta)\ \} \\
&= \bigcup\{\ F(t) \mid t{\in}\{R(\bar t)|\bar t{\in}\bar\eta\}\ \} \\
&= \bigcup\{\ F(R(\bar t)) \mid \bar t{\in}\bar\eta\}\ \} \\
&= \bigcup\{\ \bar F(\bar t) \mid \bar t{\in}\bar\eta\}\ \} \\
&= \bar F(\bar\eta)\ ,
\end{aligned}$$

where the third equality is the definition of $R_1(\bar\eta)$ and the fifth follows from part (a).

(d) For notational ease, let R^* denote $R|_{\bar T\sim\bar Z}$. When $\bar t \in \bar T{\sim}\bar Z$ replaces $x \in X$, R^* replaces f, and $\bar\eta \in \bar H \subseteq \mathcal{P}(\bar T{\sim}\bar Z)$ replaces $S \in \mathcal{S} \subseteq \mathcal{P}(X)$, Lemma A.3(a) becomes the following: Suppose that R^* is an invertible function from $\bar T{\sim}\bar Z$, and define R_1^* from $\mathcal{P}(\bar T{\sim}\bar Z)$ by $R_1^*(\bar\eta) = \{\, R^*(\bar t) \,|\, \bar t{\in}\bar\eta, \}$. Then for any $\bar H \subseteq \mathcal{P}(\bar T{\sim}\bar Z)$, $\bar H$ is a partition of $\bar T{\sim}\bar Z$ iff $\{\, R_1^*(\bar\eta) \,|\, \bar\eta{\in}\bar H \,\}$ is a partition of $R_1^*(\bar T{\sim}\bar Z)$.

Since $R|_{\bar T}$ is an invertible function from $\bar T$ by (5a), $R^* = R|_{\bar T\sim\bar Z}$ is an invertible function from $\bar T{\sim}\bar Z$. Thus from the version of Lemma A.3(a) quoted above, we may conclude that, for any $\bar{H} \subseteq \mathcal{P}(\bar{T}\sim\bar{Z})$, $\bar{H}$ is a partition of $\bar{T}\sim\bar{Z}$ iff $\{\, R_1^*(\bar{\eta}) \,|\, \bar{\eta}\in\bar{H} \,\}$ is a partition of $R_1^*(\bar{T}\sim\bar{Z})$.

Now take any $\bar{H} \subseteq \mathcal{P}(\bar{T}\sim\bar{Z})$. Since R^* was defined to be $R|_{\bar{T}\sim\bar{Z}}$, we have that R_1^* is $R_1|_{\mathcal{P}(\bar{T}\sim\bar{Z})}$. Thus, $\{\, R_1^*(\bar{\eta}) \,|\, \bar{\eta}\in\bar{H} \,\} = \{\, R_1(\bar{\eta}) \,|\, \bar{\eta}\in\bar{H} \,\}$. Also, $R_1^*(\bar{T}\sim\bar{Z}) = R_1(\bar{T}\sim\bar{Z}) = R_1(\bar{T})\sim R_1(\bar{Z}) = T\sim Z$, where the second equality follows from the invertibility of $R|_{\bar{T}\sim\bar{Z}}$ and the third equality follows from part (b). The last two sentences and the last sentence of the previous paragraph yield that $\bar{H}$ is a partition of $\bar{T}\sim\bar{Z}$ iff $\{\, R_1(\bar{\eta}) \,|\, \bar{\eta}\in\bar{H} \,\}$ is a partition of $T\sim Z$.

(e) Take any $\bar{H}\in\mathcal{P}(\bar{T}\sim\bar{Z})$, and for notational ease let R_1^* denote $R_1|_{\bar{H}}$. When $\bar{\eta}\in\bar{H}$ replaces $x\in X$, R_1^* replaces f, and $\bar{\iota}\in\bar{I}\subseteq\mathcal{P}(\bar{H})$ replaces $S\in\mathcal{S}\subseteq\mathcal{P}(X)$, Lemma A.3(b) becomes the following: Suppose that R_1^* is an invertible function from $\bar{H}$, and define $(R_1^*)_1$ from $\mathcal{P}(\bar{H})$ by $(R_1^*)_1(\bar{\iota}) = \{\, R_1^*(\bar{\eta}) \,|\, \bar{\eta}\in\bar{\iota} \,\}$. Then for any $\bar{I}\subseteq\mathcal{P}(\bar{H})$, $\bar{I}$ is a prepartition of $\bar{H}$ iff $\{\, (R_1^*)_1(\bar{\iota}) \,|\, \bar{\iota}\in\bar{I} \,\}$ is a prepartition of $(R_1^*)_1(\bar{H})$.

Since $R_1|_{\mathcal{P}(\bar{T})}$ is an invertible function from $\mathcal{P}(\bar{T})$ by Lemma A.4(a) and since $\bar{H}\subseteq\mathcal{P}(\bar{T}\sim\bar{Z})$ by assumption, $R_1^* = R_1|_{\bar{H}}$ is an invertible function from $\bar{H}$. Thus from the version of Lemma A.3(b) quoted above, we may conclude that, for any $\bar{I}\subseteq\mathcal{P}(\bar{H})$, $\bar{I}$ is a prepartition of $\bar{H}$ iff $\{\, (R_1^*)_1(\bar{\iota}) \,|\, \bar{\iota}\in\bar{I} \,\}$ is a prepartition of $(R_1^*)_1(\bar{H})$.

Now take any $\bar{I}\subseteq\mathcal{P}(\bar{H})$. Since R_1^* was defined to be $R_1|_{\bar{H}}$, we have that $(R_1^*)_1$ is $R_2|_{\mathcal{P}(\bar{H})}$. Thus $\{\, (R_1^*)_1(\bar{\iota}) \,|\, \bar{\iota}\in\bar{I} \,\} = \{\, R_2(\bar{\iota}) \,|\, \bar{\iota}\in\bar{I} \,\}$ and $(R_1^*)_1(\bar{H}) = R_2(\bar{H})$. Hence the last sentence of the previous paragraph yields that $\bar{I}$ is a prepartition of $\bar{H}$ iff $\{\, R_2(\bar{\iota}) \,|\, \bar{\iota}\in\bar{I} \,\}$ is a prepartition of $R_2(\bar{H})$.

(f) Assume $I = \{R_2(\bar{\iota})|\bar{\iota}\in\bar{I}\}$ and $\iota^* = R_2(\bar{\iota}^*)$. We argue

$$
\begin{aligned}
& (\forall\iota\in I\sim\{\iota^*\})(\forall t\in Z)\ u_\iota(t) = \bar{u}_{(R_2|_{\mathcal{P}^2(\bar{T})})^{-1}(\iota)}\left(\, (R|_{\bar{T}})^{-1}(t) \,\right) \\
\Leftrightarrow\ & (\forall\bar{\iota}\in\bar{I}\sim\{\bar{\iota}^*\})(\forall t\in Z)\ u_{R_2(\bar{\iota})}(t) = \bar{u}_{\bar{\iota}}\left(\, (R|_{\bar{T}})^{-1}(t) \,\right) \\
\Leftrightarrow\ & (\forall\bar{\iota}\in\bar{I}\sim\{\bar{\iota}^*\})(\forall\bar{t}\in\bar{Z})\ u_{R_2(\bar{\iota})}(R(\bar{t})) = \bar{u}_{\bar{\iota}}(\bar{t}) \\
\Leftrightarrow\ & (\forall\bar{\iota}\in\bar{I}\sim\{\bar{\iota}^*\})(\forall\bar{t}\in\bar{Z})\ \bar{u}_{\bar{\iota}}(\bar{t}) = u_{R_2(\bar{\iota})}(R(\bar{t}))\ .
\end{aligned}
$$

The first equivalence holds because of this part's assumptions, and because $R_2|_{\mathcal{P}^2(\bar{T})}$ is invertible by Lemma A.4(b). The second equivalence holds because $Z = R_1(\bar{Z})$ by part (b), and because $R|_{\bar{T}}$ is invertible by (5a). The last switches sides. □

A.2. Agent Recall

Lemma A.6. *In any sequence-tree game, each of the following is equivalent to the existence of an absent-minded agent.*
(a) There exist $\bar{h}$, $\bar{t}$, *and* $0 \leq m < n \leq N(\bar{t})$ *such that* $\{{}_1\bar{t}_m, {}_1\bar{t}_n\} \subseteq \bar{h}$.
(b) There exist $\bar{h}$, $\bar{t}$, *and* $0 \leq m < N(\bar{t})$ *such that* $\{{}_1\bar{t}_m, \bar{t}\} \subseteq \bar{h}$.
(c) There exist $\bar{h}$, $\bar{t}$, *and* $1 \leq m \leq N(\bar{t})$ *such that* $\bar{t}_m \in \bar{F}(\bar{h})$ *and* $\bar{t} \in \bar{h}$.
(d) There exist $\bar{t}$ *and* $1 \leq m < n \leq N(\bar{t})$ *such that* $\bar{t}_m = \bar{t}_n$.
(e) There exist $\bar{h}$, $\bar{t}$, *and* $1 \leq m < n \leq N(\bar{t})$ *such that* $\{\bar{t}_m, \bar{t}_n\} \subseteq \bar{F}(\bar{h})$.

Proof. By inspection, (a) is equivalent to the existence of an absent-minded agent.

(a)⇒(b). If (a) holds for $\bar{t} = \bar{t}^*$ and $n = n^*$, then (b) holds for $\bar{t} = {}_1\bar{t}^*_{n^*}$.

(b)⇒(c). If (b) holds for $m = m^*$, then (c) holds for $m = m^*{+}1$.

(c)⇒(d). Assume (c). Since $\bar{t}_m \in \bar{F}(\bar{h})$ and $\bar{t} \in \bar{h}$, it must be that $\bar{t}^* = \bar{t} \oplus (\bar{t}_m)$ belongs to $\bar{T}$. Thus (d) holds at $\bar{t} = \bar{t}^*$ because both $\bar{t}^*_m$ and $\bar{t}^*_{N(\bar{t}^*)}$ equal $\bar{t}_m$.

(d)⇒(e) Assume (d). Since $\bar{H}$ partitions $\bar{T}{\sim}\bar{Z}$, there is an $\bar{h}$ such that ${}_1\bar{t}_{m-1} {\in} \bar{h}$ and hence $\bar{t}_m {\in} \bar{F}(\bar{h})$. Since $\bar{F}(\bar{h})$ has $\bar{t}_m$ as an element, it must have the singleton $\{\bar{t}_m, \bar{t}_n\}$ as a subset. Thus (e) holds.

(e)⇒(a). If (e) holds at $m = m^*$ and $n = n^*$, then (a) holds at $m = m^*{-}1$ and $n = n^*{-}1$. □

Lemma A.7. *In any sequence-tree game, agent recall is equivalent to* $(\forall \bar{t})\ |R(\bar{t})| = N(\bar{t})$.

Proof. By Lemma A.6(d), the negation of agent recall is equivalent to the existence of a $\bar{t}$ such that $|R(\bar{t})| < N(\bar{t})$. This is equivalent to the negation of $(\forall \bar{t})\ |R(\bar{t})| = N(\bar{t})$ since $|R(\bar{t})|$ can never exceed $N(\bar{t})$. □

A.3. Reducing Sequences to Sets

Lemma A.8 (The "zipper" lemma).[6] *In any sequence-tree game with agent recall,*

$$(\forall \bar{t}, \bar{t}^*) \quad R(\bar{t}) \supseteq R(\bar{t}^*) \;\;\Rightarrow\;\; {}_1\bar{t}_{N(\bar{t}^*)} = \bar{t}^* \; .$$

[6] The lemma's two sequences are like the two sides of an unusual zipper whose sides may have different lengths. The lemma's inductive proof starts with the sequences' first actions and works its way up.

Proof. Take any $\bar{t}$ and $\bar{t}^*$ such that $R(\bar{t})\supseteq R(\bar{t}^*)$. By Lemma A.7, by $R(\bar{t})\supseteq R(\bar{t}^*)$, and by Lemma A.7 again, we have

$$N(\bar{t}) = |R(\bar{t})| \geq |R(\bar{t}^*)| = N(\bar{t}^*) \ .$$

The next two paragraphs will show by induction on $n\in\{1, 2, ... N(\bar{t}^*)\}$ that $(\forall n\leq N(\bar{t}^*))\ {}_1\bar{t}_n = {}_1\bar{t}^*_n$.

For the initial step at $n = 1$, suppose that $\bar{t}_1 \neq \bar{t}^*_1$. Let $\bar{h}$ be the agent containing the initial node $\{\}$ and note that $\{\bar{t}_1, \bar{t}^*_1\} \subseteq F(\bar{h})$ (in fact, agent recall implies that $\bar{h}$ must be $\{\{\}\}$ but this observation is superfluous here). Since $R(\bar{t})\supseteq R(\bar{t}^*)$, it must be that $\bar{t}^*_1 \in R(\bar{t})$, hence there exists a $k > 1$ such that $\bar{t}_k = \bar{t}^*_1$, and hence, by the previous sentence, there exists a $k > 1$ such that $\{\bar{t}_1, \bar{t}_k\} \subseteq \bar{F}(\bar{h})$. Thus by Lemma A.6(e) there is an absent-minded agent. This violates agent recall, and hence, it must be that $\bar{t}_1 = \bar{t}^*_1$.

For the inductive step at $n \in \{2, 3, ...N(\bar{t}^*)\}$, assume that ${}_1\bar{t}_{n-1} = {}_1\bar{t}^*_{n-1}$ and suppose that $\bar{t}_n \neq \bar{t}^*_n$. Let $\bar{h}$ be the agent containing ${}_1\bar{t}_{n-1}(= {}_1\bar{t}^*_{n-1})$ and note that $\{\bar{t}_n, \bar{t}^*_n\} \subseteq F(\bar{h})$. Since $R(\bar{t})\supseteq R(\bar{t}^*)$, it must be that $\bar{t}^*_n \in R(\bar{t})$, hence there exists a $m \neq n$ such that $\bar{t}_m = \bar{t}^*_n$, and hence, by the previous sentence, there exists a $m \neq n$ such that $\{\bar{t}_n, \bar{t}_m\}\subseteq\bar{F}(\bar{h})$. Thus by Lemma A.6(e) there is an absent-minded agent. This violates agent recall, and hence, it must be that $\bar{t}_n = \bar{t}^*_n$.

Therefore $(\forall n\leq N(\bar{t}^*))\ {}_1\bar{t}_n = {}_1\bar{t}^*_n$. In particular, at $n = N(\bar{t}^*)$, we have ${}_1\bar{t}_{N(\bar{t}^*)} = {}_1\bar{t}^*_{N(\bar{t}^*)}$. The right-hand side is $\bar{t}^*$. □

Proof A.9 (for Theorem 1(a))**.** We are to prove that every sequence-tree game with agent recall is isomorphic to exactly one set-tree game. Accordingly, let $(A, \bar{T}, \bar{H}, \bar{I}, \bar{i}^c, \bar{\rho}, \bar{u})$ be a sequence-tree game with agent recall. Then derive $\bar{F}$ and $\bar{Z}$ from $(A, \bar{T})$.

Step 1: Uniqueness. Suppose that both $(A, T, H, I, i^c, \rho, u)$ and $(A, T', H', I', (i^c)', \rho', u')$ are isomorphic to the given $(A, \bar{T}, \bar{H}, \bar{I}, \bar{i}^c, \bar{\rho}, \bar{u})$. By (5a), we have $T = T'$. Further, by (6a,b,c,d), we have $(H, I, i^c, \rho) = (H', I', (i^c)', \rho')$. Finally, by (6b,c,e) and Lemma A.5(f), we have $u = u'$.

Step 2: Two preliminary observations. This paragraph shows

(7) $\quad (\forall\bar{t}^*, a, \bar{t})\ \ \bar{t}^*\oplus(a)=\bar{t} \ \Rightarrow\ a\notin R(\bar{t}^*) \text{ and } R(\bar{t}^*)\cup\{a\}=R(\bar{t}) \ .$

Accordingly, take any $\bar{t}^*$, a, and $\bar{t}$ such that $\bar{t}^*\oplus(a) = \bar{t}$. Note

$$|R(\bar{t}^*)| + 1 = N(\bar{t}^*) + 1 = N(\bar{t}) = |R(\bar{t})| \ .$$

by Lemma A.7, by $\bar{t}^*\oplus(a) = \bar{t}$, and by Lemma A.7 again. This and $\bar{t}^*\oplus(a) = \bar{t}$ yield $a\notin R(\bar{t}^*)$, which is the first fact to be derived. Further, $\bar{t}^*\oplus(a) = \bar{t}$ also implies that $R(\bar{t}) = R(\bar{t}^*\oplus(a)) = R(\bar{t}^*)\cup\{a\}$, which is the second fact to be derived.

Conversely, this paragraph shows

(8) $\quad (\forall \bar{t}^*, a, \bar{t}) \;\; \bar{t}^*\oplus(a){=}\bar{t} \;\Leftarrow\; a\notin R(\bar{t}^*) \text{ and } R(\bar{t}^*)\cup\{a\}{=}R(\bar{t})\ .$

Accordingly, take any $\bar{t}^*$, a, and $\bar{t}$ such that $a\notin R(\bar{t}^*)$ and $R(\bar{t}^*)\cup\{a\} = R(\bar{t})$. Note

$$N(\bar{t}^*) + 1 = |R(\bar{t}^*)| + 1 = |R(\bar{t})| = N(\bar{t}).$$

by Lemma A.7, by the assumption of the previous sentence, and by Lemma A.7 again. Since $R(\bar{t}) = R(\bar{t}^*)\cup\{a\}\supseteq R(\bar{t}^*)$, the "zipper" Lemma A.8 shows that ${}_1\bar{t}_{N(\bar{t}^*)} = \bar{t}^*$. Thus by the last two sentences together, ${}_1\bar{t}_{N(\bar{t})-1} = \bar{t}^*$. Therefore, since $\{a\} = R(\bar{t})\sim R(\bar{t}^*)$, it must be that $\bar{t}_{N(\bar{t})} = a$. The last two sentences together yield $\bar{t} = \bar{t}^*\oplus(a)$.

Step 3: An isomorphic set tree. Define (A, T) by letting $T = R_1(\bar{T})$. This paragraph shows

(9) $\quad R|_{\bar{T}}$ is an invertible function from $\bar{T}$ onto T .

Since $T = R_1(\bar{T})$ by definition, we only need show that $R|_{\bar{T}}$ is injective. Accordingly, suppose that $\bar{t}$ and $\bar{t}^*$ are elements of $\bar{T}$ such that $R(\bar{t}) = R(\bar{t}^*)$. By the "zipper" Lemma A.8, we have ${}_1\bar{t}_{N(\bar{t}^*)} = \bar{t}^*$. Further, the left-hand side is $\bar{t}$ because

$$N(\bar{t}^*) = |R(\bar{t}^*)| = |R(\bar{t})| = N(\bar{t})$$

by Lemma A.7, by $R(\bar{t}) = R(\bar{t}^*)$, and by Lemma A.7 again.

Although isomorphism will follow from (7), (8), and (9), it is premature to make the claim now because we have not yet shown that (A, T) is a set tree. Toward that end, this paragraph shows that

(10) $\quad (\forall t^*, a, t)$

$$(R|_{\bar{T}})^{-1}(t^*)\oplus(a){=}(R|_{\bar{T}})^{-1}(t) \;\Leftrightarrow\; a\notin t^* \text{ and } t^*\cup\{a\}{=}t\ .$$

Accordingly, take any t^*, a, and t, and note that $(R|_{\bar{T}})^{-1}(t^*)$ and $(R|_{\bar{T}})^{-1}(t)$ are well-defined because of (9). For notational ease define $\bar{t}^* = (R|_{\bar{T}})^{-1}(t^*)$ and $\bar{t} = (R|_{\bar{T}})^{-1}(t)$. We argue

$$\begin{aligned} & (R|_{\bar{T}})^{-1}(t^*)\oplus(a){=}(R|_{\bar{T}})^{-1}(t) \\ \Leftrightarrow\;& \bar{t}^*\oplus(a) = \bar{t} \end{aligned}$$

$$\Leftrightarrow\ a{\notin}R(\bar{t}^*) \text{ and } R(\bar{t}^*){\cup}\{a\}{=}R(\bar{t})$$
$$\Leftrightarrow\ a{\notin}t^* \text{ and } t^*{\cup}\{a\}{=}t\ .$$

The first equivalence follows from the definitions of $\bar{t}^*$ and $\bar{t}$. The second follows from from (7) and (8). The third follows from the definitions of $\bar{t}^*$ and $\bar{t}$ and from the invertibility (9) of $R|_{\bar{T}}$.

We now show that (A,T) is a set tree. In particular, we must show (a) that $|T|{\geq}2$, (b) that $A=\bigcup T$, and (c) that every nonempty $t{\in}T$ has a unique last action. (a) follows from the assumption that $|\bar{T}|{\geq}2$ since $R|_{\bar{T}}$ is an invertible function from $\bar{T}$ onto T by (9). (b) follows from the assumption that every $a{\in}A$ appears in at least one $\bar{t}{\in}\bar{T}$. To see this, express the assumption as $A=\bigcup\{R(\bar{t})|\bar{t}\}$ and note that $\{R(\bar{t})|\bar{t}\}=R_1(\bar{T})=T$ by the definition of T. (c) Take any nonempty $t{\in}T$. First consider uniqueness. By (10) in the direction $\Leftarrow$, every last action of t must be the last element of the sequence $(R|_{\bar{T}})^{-1}(t)$. To see existence, define $\bar{t}=(R|_{\bar{T}})^{-1}(t)$, and then from this $\bar{t}$ derive $t^*=R({}_1\bar{t}_{N(\bar{t})-1})$ and $a=\bar{t}_{N(\bar{t})}$. Then by substitution and manipulation,

$$\begin{aligned}
&(R|_{\bar{T}})^{-1}(t^*){\oplus}(a)\\
&=(R|_{\bar{T}})^{-1}(R({}_1\bar{t}_{N(\bar{t})-1})){\oplus}(\bar{t}_{N(\bar{t})})\\
&={}_1\bar{t}_{N(\bar{t})-1}{\oplus}(\bar{t}_{N(\bar{t})})\\
&=\bar{t}\\
&=(R|_{\bar{T}})^{-1}(t)\ .
\end{aligned}$$

Since this is the left-hand side of (10), we have the right-hand side of (10), which states that this a is a last action of t.

Finally, $(A,\bar{T})$ and (A,T) are isomorphic by (7), (8), and (9).

Step 4: An isomorphic set-tree game. Derive F and Z from (A,T). Then define (H,I,i^c,ρ,u) by

(11a) $$H=\{\ R_1(\bar{h})\ |\ \bar{h}{\in}\bar{H}\ \}$$

(11b) $$I=\{\ R_2(\bar{\imath})\ |\ \bar{\imath}{\in}\bar{I}\ \}$$

(11c) $$i^c=R_2(\bar{\imath}^c)$$

(11d) $$\rho=\bar{\rho} \text{ and}$$

(11e) $$(\forall i{\neq}i^c)(\forall t{\in}Z)\ u_i(t)=\bar{u}_{(R_2|_{\mathcal{P}^2(\bar{T})})^{-1}(i)}(\,(R|_{\bar{T}})^{-1}(t)\,)\ .$$

This paragraph derives assumption (4a). Accordingly, take any t^1, t^2, and h, and define $\bar{t}^1=(R|_{\bar{T}})^{-1}(t^1)$, $\bar{t}^2=(R|_{\bar{T}})^{-1}(t^2)$, and $\bar{h}=$

$(R_1|_{\mathcal{P}(\bar{T})})^{-1}(h)$. Then

$$\begin{aligned} & \{t^1, t^2\} \subseteq h \\ \Rightarrow\ & \{\bar{t}^1, \bar{t}^2\} \subseteq \bar{h} \\ \Rightarrow\ & \bar{F}(\bar{t}^1) = \bar{F}(\bar{t}^2) \\ \Rightarrow\ & F(t^1) = F(t^2)\ , \end{aligned}$$

where the second implication follows from (2a) and from $\bar{h}{\in}\bar{H}$ by (11a), and the last implication follows from Lemma A.5(a).

We now derive assumption (4b). Accordingly, take any t^1 and t^2, and define $\bar{t}^1 = (R|_{\bar{T}})^{-1}(t^1)$ and $\bar{t}^2 = (R|_{\bar{T}})^{-1}(t^2)$. Then

$$\begin{aligned} & F(t^1){\cap}F(t^1) \neq \varnothing \\ \Rightarrow\ & \bar{F}(\bar{t}^1){\cap}\bar{F}(\bar{t}^2) \neq \varnothing \\ \Rightarrow\ & (\exists\bar{h})\{\bar{t}^1, \bar{t}^2\} \subseteq \bar{h} \\ \Rightarrow\ & (\exists h)\{t^1, t^2\} \subseteq h\ , \end{aligned}$$

where the first implication follows from Lemma A.5(a), the second from the contrapositive of (2b), and the last from (11a).

We now show $(A, T, H, I, i^c, \rho, u)$ is a set-tree game. Specifically, the next paragraph will show (a) that (A, T) is a set tree, (b) that H partitions $T{\sim}Z$ and satisfies (4), (c) that I is a prepartition of H, (d) that $(\forall h{\in}i^c)\ \Sigma_{a\in F(h)}\rho(a) = 1$, and (e) that every nonchance player is nonempty.

(a) was established in Step 3. (b) requires two steps. First H partitions $T{\sim}Z$ by the assumption that $\bar{H}$ partitions $\bar{T}{\sim}\bar{Z}$, by (11a), and by Lemma A.5(d). Second (4) follows from the last two paragraphs. (c) holds by the assumption that $\bar{I}$ is a prepartition of $\bar{H}$, by (11a,b), and by Lemma A.5(e). (d) requires considering any $h{\in}i^c$. By (11c) there exists $\bar{h}{\in}\bar{i}^c$ such that $h = R_1(\bar{h})$. Thus $\Sigma_{a\in F(h)}\rho(a) = \Sigma_{a\in\bar{F}(\bar{h})}\rho(a)$ by Lemma A.5(c), which equals $\Sigma_{a\in\bar{F}(\bar{h})}\bar{\rho}(a)$ by (11d), which equals 1 by assumption. (e) requires considering any $i{\in}I{\sim}\{i^c\}$. By (11b) there exists an $\bar{i}{\in}\bar{I}$ such that $i = R_2(\bar{i})$. Since $i{\neq}i^c$, (11c) and the invertibility of $R_2|_{\mathcal{P}(T)}$ by Lemma A.4(b) together imply $\bar{i}{\neq}\bar{i}^c$. Thus i is nonempty because $\bar{i}$ is nonempty by assumption.

Finally, we show that $(A, T, H, I, i^c, \rho, u)$ and $(A, \bar{T}, \bar{H}, \bar{I}, \bar{i}^c, \bar{\rho}, \bar{u})$ are isomorphic. Specifically, we show (a) that (A, T) and $(A, \bar{T})$ are isomorphic and (b) that (6) holds. (a) was established in Step 3. (b) is proved in two steps. First (6a–d) are identical to (11a–d). Second (6e) is implied by (11b,c,e) and Lemma A.5(f). □

A.4. Constructing Sequences from Sets

Proof A.10 (for Theorem 1(b))**.** We are to show that every set-tree game is isomorphic to exactly one sequence-tree game, and that that sequence-tree game has agent recall. Accordingly, let (A,T,H,I,i^c,ρ,u) be a set-tree game, and derive F and Z from (A,T). By assumption (3), we may let $\alpha_*{:}T{\rightarrow}A$ be the function that takes each node $t{\in}T$ to its unique last action.

Step 1: Uniqueness. Suppose that $(A,\bar{T},\bar{H},\bar{I},\bar{i}^c,\bar{\rho},\bar{u})$ and $(A,\bar{\bar{T}},\bar{\bar{H}},\bar{\bar{I}},\bar{\bar{i}}^c,\bar{\bar{\rho}},\bar{\bar{u}})$ are two sequence-tree games that are isomorphic to (A,T,H,I,i^c,ρ,u).

This and the next two paragraphs show that $\bar{T}=\bar{\bar{T}}$. Suppose not. Then because both $(A,\bar{T})$ and $(A,\bar{\bar{T}})$ satisfy isomorphism condition (5a), there must be $\bar{t}$, $\bar{\bar{t}}$, and t such that $\bar{t}{\neq}\bar{\bar{t}}$ and yet $R(\bar{t})=R(\bar{\bar{t}})=t$.

This long paragraph shows by induction that

$$(\forall k{\in}\{0,1,...\,|t|\})$$

$$\text{(12a)}\qquad {}_1\bar{t}_{N(\bar{t})-k} \neq {}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k}\ ,$$

$$\text{(12b)}\qquad R({}_1\bar{t}_{N(\bar{t})-k}) = R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k})\ ,$$

$$\text{(12c)}\qquad \text{and } |R({}_1\bar{t}_{N(\bar{t})-k})| = |t|-k\ .$$

The initial step at $k{=}0$ follows from the definition of $\bar{t}$, $\bar{\bar{t}}$, and t. Now assume that (12) holds at $k<|t|$. By the definitions of $\bar{t}$, $\bar{\bar{t}}$, and t, it must be $N(\bar{t})$ and $N(\bar{\bar{t}})$ are at least as big as $|t|$ and thus strictly bigger than k. As a result, we may write

$$\text{(13)}\qquad \begin{gathered}{}_1\bar{t}_{N(\bar{t})-k-1}\oplus(\bar{t}_{N(\bar{t})-k}) = {}_1\bar{t}_{N(\bar{t})-k}\ \text{ and}\\ {}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1}\oplus(\bar{\bar{t}}_{N(\bar{\bar{t}})-k}) = {}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k}\ .\end{gathered}$$

Thus, by applying the structure condition (5b) twice, we find

$$\text{(14)}\qquad \begin{gathered}\bar{t}_{N(\bar{t})-k}\notin R({}_1\bar{t}_{N(\bar{t})-k-1})\text{ and}\\ R({}_1\bar{t}_{N(\bar{t})-k-1}){\cup}\{\bar{t}_{N(\bar{t})-k}\} = R({}_1\bar{t}_{N(\bar{t})-k})\ ,\\ \text{and } \bar{\bar{t}}_{N(\bar{\bar{t}})-k}\notin R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1})\text{ and}\\ R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1}){\cup}\{\bar{\bar{t}}_{N(\bar{\bar{t}})-k}\} = R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k})\ .\end{gathered}$$

Thus, by applying the definition of last action twice, we find

$$\bar{t}_{N(\bar{t})-k} = \alpha_*(R({}_1\bar{t}_{N(\bar{t})-k})) \text{ and}$$
$$\bar{\bar{t}}_{N(\bar{\bar{t}})-k} = \alpha_*(R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k})) \ .$$

But by (12b), the right-hand sides of these two equalities must be equal. Thus we may define a_* to be equal to both $\bar{t}_{N(\bar{t})-k}$ and $\bar{\bar{t}}_{N(\bar{\bar{t}})-k}$, and then substitute out both of these latter terms in (13) and (14) to obtain

$$\text{(15)} \qquad \begin{aligned} &{}_1\bar{t}_{N(\bar{t})-k-1} \oplus (a_*) = {}_1\bar{t}_{N(\bar{t})-k} \text{ and} \\ &{}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1} \oplus (a_*) = {}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k} \ . \end{aligned}$$

and

$$\text{(16)} \qquad \begin{aligned} &a_* \notin R({}_1\bar{t}_{N(\bar{t})-k-1}) \text{ and } R({}_1\bar{t}_{N(\bar{t})-k-1}) \cup \{a_*\} = R({}_1\bar{t}_{N(\bar{t})-k}) \text{ and} \\ &a_* \notin R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1}) \text{ and } R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1}) \cup \{a_*\} = R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k}) \ . \end{aligned}$$

By (12a), the pair (15) implies that

$${}_1\bar{t}_{N(\bar{t})-k-1} \neq {}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1} \ .$$

The pair (16) implies that

$$R({}_1\bar{t}_{N(\bar{t})-k-1}) = R({}_1\bar{t}_{N(\bar{t})-k}) \sim \{a_*\} \text{ and}$$
$$R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1}) = R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k}) \sim \{a_*\} \ ,$$

and thus by (12b) we have that

$$R({}_1\bar{t}_{N(\bar{t})-k-1}) = R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-k-1}) \ .$$

Finally, the first half of (16) together with (12c) imply that

$$|R({}_1\bar{t}_{N(\bar{t})-k-1})| = |R({}_1\bar{t}_{N(\bar{t})-k})| - 1 = |t| - k - 1 \ .$$

The last three sentences have derived (12) at $k{+}1$.

At $k = |t|$, equations (12b) and (12c) imply that both $R({}_1\bar{t}_{N(\bar{t})-|t|})$ and $R({}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-|t|})$ are empty. Thus both ${}_1\bar{t}_{N(\bar{t})-|t|}$ and ${}_1\bar{\bar{t}}_{N(\bar{\bar{t}})-|t|}$ are empty, in contradiction to (12a) at $k = |t|$. Therefore $\bar{T} = \bar{\bar{T}}$.

Next, we show $(\bar{H}, \bar{I}, \bar{i}^c) = (\bar{\bar{H}}, \bar{\bar{I}}, \bar{\bar{i}}^c)$. Note that $R_1|_{\mathcal{P}(\bar{T})} = R_1|_{\mathcal{P}(\bar{\bar{T}})}$ since $\bar{T} = \bar{\bar{T}}$, and that this function is invertible by Lemma A.4(a). Thus since both $\bar{H}$ and $\bar{\bar{H}}$ satisfy (6a), we have

$$\text{(17a)} \qquad \bar{H} = \{(R_1|_{\mathcal{P}(\bar{T})})^{-1}(h) | h{\in}H\} = \{(R_1|_{\mathcal{P}(\bar{\bar{T}})})^{-1}(h) | h{\in}H\} = \bar{\bar{H}} \ .$$

Also note that $R_2|_{\mathcal{P}^2(\bar{T})} = R_2|_{\mathcal{P}^2(\bar{\bar{T}})}$ since $R|_{\bar{T}} = R|_{\bar{\bar{T}}}$ and that this function is invertible by Lemma A.4(b). Thus since both $\bar{I}$ and $\bar{\bar{I}}$

satisfy (6b), we have

$$\bar{I} = \{(R_2|_{\mathcal{P}^2(\bar{T})})^{-1}(i)|i{\in}I\} = \{(R_2|_{\mathcal{P}^2(\bar{\bar{T}})})^{-1}(i)|i{\in}I\} = \bar{\bar{I}}\ . \tag{17b}$$

Further since both $\bar{i}^c$ and $\bar{\bar{i}}^c$ satisfy (6c), we have

$$\bar{i}^c = (R_2|_{\mathcal{P}^2(\bar{T})})^{-1}(i^c) = (R_2|_{\mathcal{P}^2(\bar{\bar{T}})})^{-1}(i^c) = \bar{\bar{i}}^c\ . \tag{17c}$$

Finally, we show $(\bar{\rho},\bar{u}) = (\bar{\bar{\rho}},\bar{\bar{u}})$. Trivially, $\bar{\rho} = \rho = \bar{\bar{\rho}}$ since both $\bar{\rho}$ and $\bar{\bar{\rho}}$ satisfy (6d). To get at the payoff functions, begin by deriving $\bar{Z}$ from $(A,\bar{T})$ and $\bar{\bar{Z}}$ from $(A,\bar{\bar{T}})$. Then since $\bar{I}{\sim}\{\bar{i}^c\} = \bar{\bar{I}}{\sim}\{\bar{\bar{i}}^c\}$ by (17b,c), and since $\bar{Z} = \bar{\bar{Z}}$ because $\bar{T} = \bar{\bar{T}}$, we have that $(\bar{I}{\sim}\{\bar{i}^c\}){\times}\bar{Z} = (\bar{\bar{I}}{\sim}\{\bar{\bar{i}}^c\}){\times}\bar{\bar{Z}}$, or in other words, that the domain of $\bar{u}$ equals the domain of $\bar{\bar{u}}$. Then, for any $(\bar{i},\bar{t})$ in that common domain, we have

$$\bar{u}_{\bar{i}}(\,\bar{t}\,) = u_{R_2(\bar{i})}(\,R(\bar{t})\,) = \bar{\bar{u}}_{\bar{i}}(\,\bar{t}\,)$$

because both $\bar{u}$ and $\bar{\bar{u}}$ satisfy (6e) (the single bars on $\bar{i}$ and $\bar{t}$ on the right-hand side are correct). The last two sentences imply $\bar{u} = \bar{\bar{u}}$.

Step 2A: Defining $\bar{T}$. We now begin the task of constructing a sequence-tree game which is isomorphic to (A,T,H,I,i^c,ρ,u). The first job is to define $\bar{T}$.

For any $n{\geq}0$, let $T_n = \{\ t\ |\ |t|{=}n\ \}$ be the set of nodes with n elements. Because A is finite, there is some $\hat{n}$ such that $T_{\hat{n}}{\neq}\varnothing$ and $(\forall n{>}\hat{n})\ T_n{=}\varnothing$. Thus $T = \bigcup_{n=0}^{\hat{n}} T_n$. Further, let $t^{\hat{n}}$ be some element of $T_{\hat{n}}$, and for all $n{\in}\{0,1,2,...\,\hat{n}{-}1\}$, let t^n be $t^{n+1}{\sim}\{\alpha_*(t^{n+1})\}$. Since each $t^n{\in}T_n$, we have shown that $(\forall n{\leq}\hat{n})\ T_n{\neq}\varnothing$. In particular, $T_0{\neq}\varnothing$ and thus $T_0 = \{\{\}\}$.

We now define a sequence $\langle Q_n\rangle_{n=0}^{\hat{n}}$ of functions in which each function Q_n maps each node t of T_n to some finite action sequence $\bar{t}$. We do this recursively. To begin, recall $T_0 = \{\{\}\}$ from the previous paragraph and define the one-element function Q_0 by $Q_0(\{\}) = \{\}$. Thus the empty set $t = \{\}$ is mapped to the empty sequence $\bar{t} = \{\}$. Then, for any $n \in \{1,2,...\hat{n}\}$, use Q_{n-1} to define Q_n at each $t \in T_n$ by

$$Q_n(t) = Q_{n-1}(t{\sim}\{\alpha_*(t)\}){\oplus}(\alpha_*(t))\ . \tag{18}$$

Note that $Q_{n-1}(t{\sim}\{\alpha_*(t)\})$ is well-defined because $t{\sim}\{\alpha_*(t)\}$ has $n{-}1$ elements because $t \in T_n$ and $\alpha_*(t)$ is its last action.

Define $\bar{T} = \bigcup_n Q_n(T_n)$, where here, and for the remainder of the proof, we implicitly assume that n ranges over $\{0,1,...\,\hat{n}\}$.

Step 2B: Invertibility. First we show by induction that

$$(\forall n)(\forall t{\in}T_n)\ R(Q_n(t)) = t\ . \tag{19}$$

This holds at $n{=}0$ because $R(Q_0(\{\})) = R(\{\}) = \{\}$. Further, it holds at $n{\geq}1$ if it holds at $n{-}1$ because

$$\begin{aligned}(\forall t{\in}T_n)\ R(\,Q_n(t)\,) &= R\big(\,Q_{n-1}(t{\sim}\{\alpha_*(t)\}) \oplus (\alpha_*(t))\,\big)\\ &= R\big(\,Q_{n-1}(t{\sim}\{\alpha_*(t)\})\,\big) \cup R\big(\,(\alpha_*(t))\,\big)\\ &= t{\sim}\{\alpha_*(t)\} \cup \{\alpha_*(t)\}\\ &= t\ ,\end{aligned}$$

where the first equality holds by the definition (18) of Q_n, and the third holds by the inductive hypothesis.

Next we show by induction that

$$(\forall n)(\forall t{\in}T_n)\ N(Q_n(t)) = n\ . \tag{20}$$

This holds at $n = 0$ because $N(Q_0(\{\})) = N(\{\}) = 0$. Further, it holds at any $n{\geq}1$ if it holds at $n{-}1$ because

$$\begin{aligned}(\forall t{\in}T_n)\ N(\,Q_n(t)\,) &= N(\,Q_{n-1}(t{\sim}\{\alpha_*(t)\}) \oplus (\alpha_*(t))\,)\\ &= N(\,Q_{n-1}(t{\sim}\{\alpha_*(t)\})\,) + N(\,(\alpha_*(t))\,)\\ &= (n{-}1) + 1\\ &= n\ ,\end{aligned}$$

where the first equality holds by the definition (18) of Q_n, and the third by the inductive hypothesis.

This observation allows us to claim

$$(\forall n)\ \{\ \bar{t}{\in}\bar{T} \mid N(\bar{t}){=}n\ \} = Q_n(T_n)\ . \tag{21}$$

The inclusion $\supseteq$ follows from (20) at n. Conversely, if there were an element of $\{\ \bar{t}{\in}\bar{T} \mid N(\bar{t}){=}n\ \}$ that was from $Q_m(T_m)$ for some $m{\neq}n$ it would violate (20) at m.

Next define $Q = \bigcup_n Q_n$. The remainder of this paragraph shows (24) below. To begin, (19) implies that each $R|_{Q_n(T_n)}$ is the inverse of Q_n. In other words,

$$\begin{gathered}(\forall n)\ Q_n = (R|_{Q_n(T_n)})^{-1} \text{ is}\\ \text{an invertible function from } T_n \text{ onto } Q_n(T_n)\ .\end{gathered} \tag{22}$$

This implies, among other things, that the domain of Q is $T = \bigcup_n T_n$ and that its range is $\bar{T} = \bigcup_n Q_n(T_n)$. Further, T is partitioned by $\{T_n\}_n$

because of the definition of $\{T_n\}_n$, and $\bar{T}$ is partitioned by $\{Q_n(T_n)\}_n$ because of (21). Therefore (22) implies that

$$(23)\qquad Q=(R|_{\bar{T}})^{-1} \text{ is an invertible function from } T \text{ onto } \bar{T}\ .$$

This is equivalent to

$$(24)\qquad R|_{\bar{T}}=Q^{-1} \text{ is an invertible function from } \bar{T} \text{ onto } T\ .$$

Step 3A: Showing $(A,\bar{T})$ is a sequence tree. First we note that

$$(25)\qquad (\forall \bar{t}{\neq}\varnothing)\ \ {}_1\bar{t}_{N(\bar{t})-1}\in\bar{T}.$$

Take any $\bar{t}{\in}\bar{T}$. By (21), there exists $t{\in}T_{N(\bar{t})}$ such that $\bar{t}=Q_{N(\bar{t})}(t)$. Thus the definition (18) of $Q_{N(\bar{t})}$ yields that ${}_1\bar{t}_{N(\bar{t})-1}=Q_{N(\bar{t})-1}(t{\sim}\{a_*(t)\})$ $\in\bar{T}$.

Second we note that

$$(26)\qquad A=\textstyle\bigcup_{\bar{t}}R(\bar{t})\ .$$

Easily, $A\supseteq\bigcup_{\bar{t}}R(\bar{t})$ because each $R(\bar{t})$ is a set of actions. Conversely, take any a. By assumption there is some t such that $a{\in}t$. Then by construction there is some n such that $t{\in}T_n$. Thus by (19), we have $a\in t=R(Q_n(t))$. Therefore, since $Q_n(t)\in Q_n(T_n)\subseteq\bar{T}$, this $Q_n(t)$ is a $\bar{t}$ such that $a\in R(\bar{t})$.

Finally we argue that $(A,\bar{T})$ is a sequence tree. In particular, we argue (a) that $\bar{T}$ is a finite set, (b) that every $\bar{t}{\in}\bar{T}$ is a finite sequence, (c) that $|\bar{T}|{\geq}2$, (d) that (1) holds, and (e) that every action appears within at least one $\bar{t}{\in}\bar{T}$. (a) holds by (24) since $T{\subseteq}\mathcal{P}(A)$ is finite because A is finite. (b) holds by (20) and by the fact that n ranges over the finite set $\{0,1,...\hat{n}\}$ that was constructed in Step 2A. (c) follows from (24) and the assumption that $|T|{\geq}2$. (d) holds by (25). (e) holds by (26).

Step 3B: Showing isomorphism between trees. This paragraph shows

$$(27)\qquad (\forall n{\geq}1)(\forall t^*{\in}T_{n-1})(\forall a)(\forall t{\in}T_n)$$
$$Q_{n-1}(t^*){\oplus}(a)=Q_n(t)\ \Leftrightarrow\ a{\notin}t^* \text{ and } t^*{\cup}\{a\}=t\ .$$

Accordingly, take any such n, t^*, a, and t. Then

$$\begin{aligned}
&Q_{n-1}(t^*){\oplus}(a)=Q_n(t)\\
\Leftrightarrow\ &Q_{n-1}(t^*){\oplus}(a)=Q_{n-1}(t{\sim}\{\alpha_*(t)\}){\oplus}(\alpha_*(t))\\
\Leftrightarrow\ &Q_{n-1}(t^*)=Q_{n-1}(t{\sim}\{\alpha_*(t)\}) \text{ and } a=\alpha_*(t)
\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow\ t^* = t{\sim}\{\alpha_*(t)\} \text{ and } a = \alpha_*(t)\\ &\Leftrightarrow\ a{\notin}t^* \text{ and } t^*{\cup}\{a\} = t\end{aligned}$$

where the first equivalence holds by the definition of Q_n at (18), the second equivalence by breaking the vector equality into two components, the third equivalence by applying R and (24) to the first equality, and the fourth equivalence by $\alpha_*(t)$ being a last action.

Essentially, this paragraph removes the n from (27). Specifically, it shows that

$$(\forall t^*)(\forall a)(\forall t) \quad Q(t^*){\oplus}(a) = Q(t) \ \Leftrightarrow\ a{\notin}t^* \text{ and } t^*{\cup}\{a\} = t\ . \tag{28}$$

First suppose t^*, a, and t satisfy $Q(t^*){\oplus}(a) = Q(t)$ and let $n = |t|$. By (20) and the definition of Q, we have $Q(t) = Q_n(t)$ and $Q(t^*) = Q_{n-1}(t^*)$. Hence $a{\notin}t^*$ and $t^*{\cup}\{a\} = t$ by (27). Conversely, suppose t^*, a, and t satisfy $a{\notin}t^*$ and $t^*{\cup}\{a\} = t$ and let $n = |t|$. Then $n{-}1 = |t^*|$. Thus since $t{\in}T_n$ and $t^*{\in}T_{n-1}$, (27) yields that $Q_{n-1}(t^*){\oplus}(a) = Q_n(t)$. By the definition of Q, this is equivalent to $Q(t^*){\oplus}(a) = Q(t)$.

Essentially, this next paragraph quantifies (28) in terms of sequences rather than sets. Specifically, it shows that

$$(\forall \bar{t}^*, a, \bar{t}) \quad \bar{t}^*{\oplus}(a) = \bar{t} \ \Leftrightarrow\ a{\notin}R(\bar{t}^*) \text{ and } R(\bar{t}^*){\cup}\{a\} = R(\bar{t})\ . \tag{29}$$

Accordingly, take any $\bar{t}^*$, a, and $\bar{t}$, define $t^* = R(\bar{t}^*)$, and define $t = R(\bar{t})$. Then we argue

$$\begin{aligned}&\bar{t}^*{\oplus}(a) = \bar{t}\\ \Leftrightarrow\ &Q(t^*){\oplus}(a) = Q(t)\\ \Leftrightarrow\ &a{\notin}t^* \text{ and } t^*{\cup}\{a\} = t\\ \Leftrightarrow\ &a{\notin}R(\bar{t}^*) \text{ and } R(\bar{t}^*){\cup}\{a\} = R(\bar{t})\ .\end{aligned}$$

The first equivalence holds by the definitions of t^* and t and by the fact that $R|_{\bar{T}} = Q^{-1}$ by (24). The second equivalence holds by (28), and the third by the definitions of t^* and t.

Finally, (A,T) and $(A,\bar{T})$ are isomorphic by (24) and (29).

Step 4A: Defining the sequence-tree game. Derive $\bar{F}$ and $\bar{Z}$ from $(A,\bar{T})$. Then define $(\bar{H},\bar{I},\bar{\imath}^c,\bar{\rho},\bar{u})$ by

$$\bar{H} = \{\ (R_1|_{\mathcal{P}(\bar{T})})^{-1}(h) \mid h{\in}H\ \} \tag{30a}$$

(30b) $$\bar{I} = \{\ (R_2|_{\mathcal{P}^2(\bar{T})})^{-1}(i) \mid i{\in}I\ \}$$

(30c) $$\bar{i}^c = (R_2|_{\mathcal{P}^2(\bar{T})})^{-1}(i^c)$$

(30d) $$\bar{\rho} = \rho \text{ and}$$

(30e) $$(\forall \bar{i}{\neq}\bar{i}^c)(\forall \bar{t}{\in}\bar{Z})\ \ \bar{u}_{\bar{i}}(\bar{t}) = u_{R_2(\bar{i})}(R(\bar{t}))\ .$$

Since $R_1|_{\mathcal{P}(\bar{T})}$ and $R_2|_{\mathcal{P}^2(\bar{T})}$ are invertible by Lemma A.4, equations (30a,b,c) are equivalent to

(31a) $$H = \{\ R_1(\bar{h}) \mid \bar{h}{\in}\bar{H}\ \}$$

(31b) $$I = \{\ R_2(\bar{i}) \mid \bar{i}{\in}\bar{I}\ \}$$

(31c) $$\text{and } i^c = R_2(\bar{i}^c)\ .$$

This paragraph derives assumption (2a). Accordingly, take any $\bar{t}^1$, $\bar{t}^2$, and $\bar{h}$. Then

$$\begin{aligned} & \{\bar{t}^1, \bar{t}^2\} \subseteq \bar{h} \\ \Rightarrow\ & \{R(\bar{t}^1), R(\bar{t}^2)\} \subseteq R_1(\bar{h}) \\ \Rightarrow\ & F(R(\bar{t}^1)) = F(R(\bar{t}^2)) \\ \Rightarrow\ & \bar{F}(\bar{t}^1) = \bar{F}(\bar{t}^1)\ , \end{aligned}$$

where the first implication follows from the definitions of R_1, the second implication follows from assumption (4a) and the fact that $R_1(\bar{h}){\in}H$ by (31a), and the last implication comes from Lemma A.5(a).

Then we derive assumption (2b). Accordingly, take any $\bar{t}^1$ and $\bar{t}^2$. Then

$$\begin{aligned} & \bar{F}(\bar{t}^1) \cap \bar{F}(\bar{t}^2) \neq \varnothing \\ \Rightarrow\ & F(R(\bar{t}^1)) \cap F(R(\bar{t}^2)) \neq \varnothing \\ \Rightarrow\ & (\exists h)\ \{R(\bar{t}^1), R(\bar{t}^2)\} \subseteq h \\ \Rightarrow\ & (\exists \bar{h})\ \{R(\bar{t}^1), R(\bar{t}^2)\} \subseteq R_1(\bar{h}) \\ \Rightarrow\ & (\exists \bar{h})\ \{R(\bar{t}^1), R(\bar{t}^2)\} \subseteq \{\ R(\bar{t}) \,|\, \bar{t}{\in}\bar{h}\ \} \\ \Rightarrow\ & (\exists \bar{h})\ \{\ (R|_{\bar{T}})^{-1}(R(\bar{t}^1)),\ (R|_{\bar{T}})^{-1}(R(\bar{t}^2))\ \} \subseteq \{\ (R|_{\bar{T}})^{-1}(R(\bar{t})) \,|\, \bar{t}{\in}\bar{h}\ \} \\ \Rightarrow\ & (\exists \bar{h})\ \{\bar{t}^1, \bar{t}^2\} \subseteq \bar{h}\ , \end{aligned}$$

where the first implication follows from Lemma A.5(a), the second from assumption (4b), the third from (31a), the fourth from the definition of R_1, and the fifth from the invertibility of $R|_{\bar{T}}$ by (5a).

We now show $(A, \bar{T}, \bar{H}, \bar{I}, \bar{i}^c, \bar{\rho}, \bar{u})$ is a sequence-tree game. Specifically, the next paragraph will show (a) that $(A, \bar{T})$ is a sequence tree,

(b) that $\bar{H}$ partitions $\bar{T}{\sim}\bar{Z}$ and satisfies (2), (c) that $\bar{I}$ is a prepartition of $\bar{H}$, (d) that $(\forall\bar{h}{\in}\bar{i}^c)$ $\Sigma_{a\in\bar{F}(\bar{h})}\bar{\rho}(a) = 1$, and (e) that every nonchance player is nonempty.

(a) was established by Step 3A. (b) requires two steps. First $\bar{H}$ partitions $\bar{T}{\sim}\bar{Z}$ by the assumption that H partitions $T{\sim}Z$, by (31a), and by Lemma A.5(d). Next (2) follows from the last two paragraphs. (c) holds by the assumption that I is a prepartition of H, by (31b), and by Lemma A.5(e). (d) requires considering any $\bar{h}{\in}\bar{i}^c$. By (31c) there exists $h{\in}i^c$ such that $h = R_1(\bar{h})$. Thus $\Sigma_{a\in\bar{F}(\bar{h})}\bar{\rho}(a) = \Sigma_{a\in F(h)}\bar{\rho}(a)$ by Lemma A.5(c), which equals $\Sigma_{a\in F(h)}\rho(a)$ by (30d), which equals 1 by assumption. (e) requires considering any $\bar{i}{\in}\bar{I}{\sim}\{\bar{i}^c\}$. By (30b,c) there exists an $i{\in}I{\sim}\{i^c\}$ such that $i = R_2(\bar{i})$. Thus $\bar{i}$ is nonempty because i is nonempty by assumption.

Step 4B: Showing isomorphism between games. We show here that (A,T,H,I,i^c,ρ,u) and $(A,\bar{T},\bar{H},\bar{I},\bar{i}^c,\bar{\rho},\bar{u})$ are isomorphic. Specifically, we show (a) that (A,T) and $(A,\bar{T})$ are isomorphic and (b) that (6) holds. (a) was established by Step 3B. (b) follows from (31a,b,c) and (30d,e).

Step 5: Agent recall. Equation (19), the definition of T_n, and equation (20) yield that

$$(\forall n)(\forall t{\in}T_n)\ |R(Q_n(t))| = |t| = n = N(Q_n(t))\ .$$

Thus by the definition of Q,

$$(\forall t)\ |R(Q(t))| = N(Q(t))\ .$$

Since Q is an invertible function from T onto $\bar{T}$ by (23), this is equivalent to $(\forall\bar{t})$ $|R(\bar{t})| = N(\bar{t})$, which by Lemma A.7 is equivalent to agent recall. □

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