The Equilibrium of an Asymmetric First-Price Auction and Its Implication in Collusion*

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October 29, 2017

Abstract

This paper characterizes all equilibriums of a first-price auction between two bidders, one privately knowing her valuation while the other having the disadvantage that his valuation is commonly known. Despite such asymmetry, the equilibrium allocation may be fully efficient and the informationally disadvantaged bidder may have positive surplus. The game arises endogenously as the continuation play that supports bidding collusion between two bidders, each privately informed. Comparative statics between this game and those with other type-distributions results in a necessary and sufficient condition, in terms of the prior type-distributions, for existence of a collusive contract acceptable to both bidders of any realized types. This condition is more likely to hold if a bidder's prior distribution becomes more stochastically dominant.

Keywords: asymmetric auction, collusion, endogenous outside option
JEL Classification Numbers: D44, D82

*The financial support from the Social Science and Humanities Research Council of Canada, Insight Grant R4809A04, is gratefully acknowledged. I thank Rene Kirkegaard for comments.

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1 Introduction

The asymmetric auction considered here is a first-price auction of one good pursued by two bidders, one privately knowing her valuation while the other having the disadvantage that his valuation is commonly known. This is arguably the simplest model for competitions among market-power possessing rivals one of whom has an information advantage over others. Such asymmetry may occur when the former has access to some insider information while the other can only evaluate the good according to a commonly known prior (e.g., Engelbrecht-Wiggans, Milgrom and Weber [3]). The former may also gain such information advantage through his toehold or incumbent status in the industry (c.f. Bulow, Huang and Klemperer [2]). Sometimes a rent-extracting auctioneer may choose to create such ex ante asymmetry through discriminatory disclosure policies prior to the auction (c.f. Bergemann and Pesendorfer [1, Section 3]). This paper explicitly characterizes all the Bayesian Nash equilibriums of this auction game and demonstrates that, despite the ex ante asymmetry between the two bidders, the equilibriums may result in fully efficient allocations and give positive surplus to the informationally disadvantaged bidder. The equilibrium characterization is then applied to a multistage bidding collusion setting where the asymmetric auction becomes its continuation game.

It should be emphasized that this asymmetric auction game cannot be deemed negligible despite its extreme assumption that the informationally disadvantaged bidder's value is commonly known. That is because such a game, special as it is, may occur endogenously in a multistage setting such as the aforementioned example where a rent-extracting seller may deliberately keep one bidder as ignorant about the good as the prior distribution, thereby rendering his expected value of the good commonly known, and let the other bidder inspect the good to gain some private information. In this paper we will see another example where the general solution of this asymmetric game is indispensable, prescribing the continuation equilibriums that sustain bidding collusion for all possible types of two bidders.

Vickrey [12] has solved a special case of this asymmetric auction game where the informationally advantaged bidder's value is uniformly distributed on $[0, 1]$, with the other's known value strictly in between (Remark 1), but no solution for a more general case exists in the literature. Work devoted to generalize Vickrey's model, such as Griesmer, Levitan and Shubik [6], replaces the informationally disadvantaged bidder's commonly known valuation by a nondegenerate distribution, thereby avoiding the complication caused by the singularity. The solution in this paper allows for general distributions of the advantaged bidder's private value and general configurations between the distribution and the disadvantaged bidder's known value (Theorem 1). Not only does it generalize the case that confirms the properties of Vickrey's solution such as inefficiency at equilibrium and positive surplus for the disadvantaged bidder, but it also includes other cases where neither properties hold.

Engelbrecht-Wiggans et al. [3] have analyzed a similar asymmetric auction except that there are multiple, and identical, informationally disadvantaged bidders, and Martínez-Pardina [9] a similar auction except that there are multiple, ex ante identical, informationally advantaged bidders. Such multiplicity assumptions imply important differences between their models and ours. The profit of an informationally disadvantaged bidder in Engelbrecht-Wiggans et al. is reduced to zero by the competition with other disadvantaged ones, while it may be positive in our model. The ex ante homogenous advantaged bidders in Martínez-Pardina, competing among themselves, do not necessarily reduce their best-responding bids to the maximum bid of the disadvantaged bidder; by contrast, the single advantaged bidder in our two-bidder model does when his valuation is for sure above the disadvantaged one's, thereby generating an atom in his bid distribution that may render equilibrium nonexistent.

To illustrate how this asymmetric auction may arise endogenously in multistage games, this paper considers a bidding ring setup to which our solution, and especially its merit of allowing for all parameter cases, are essential. Here two players are to bid in a first-price auction for a good of common value, also commonly known to them; each bidder is privately informed at the outset of his marginal cost of monetary payments for the good. Prior to the auction, an outside neutral mediator proposes to the bidders a side transfer between them for one to bribe the other to abstain from the auction. If accepted by both, the collusive contract will be carried out, else they play the auction noncooperatively with posteriors conditional on the breakdown of the collusion. The questions are: Is it possible, among all perfect Bayesian equilibriums, for the two privately informed bidders to always agree on a collusive contract? If the answer is Yes then what is the family of prior distributions of their types that guarantee their agreement in such collusion?

The crucial step in answering these questions is obviously to calculate each bidder's type-dependent outside payoff in the off-path event where he vetoes the collusive proposal. It turns out that the worst among such outside payoffs, to the bidder-type most tempted to reject collusion, is equal to his surplus in the auction when he is believed to be of this most tempted type. Thus the crucial continuation game becomes the asymmetric auction described earlier, with the vetoer playing the role of the informationally disadvantaged bidder. Based on the solution of the asymmetric auction game, coupled with nontrivial comparative statics between this game and others with arbitrary posterior distributions allowing for atoms and gaps (Lemma 6), the exact family of collusion-guaranteeing prior distributions is identified (Theorem 2). It is proved that this family is increasing according to the ranking of first-order stochastic dominance. In other words, the prospect for both bidders to always agree on a collusive division is higher when their prior distributions become more stochastically dominant (Corollary 1).

The design of side contracts for bidders to collude in a given auction has been considered by McAfee and McMillan [10], and Marshall and Marx [8], with the premise that a bidder gets an exogenous expected payoff in any event where collusion is rejected. Whereas, this paper considers the design of collusive contracts by incorporating the signaling effects of one's response to a proposed contract, thereby endogenizing his expected payoff when collusive negotiation fails. Similar endogenous outside options for two collusive bidders have been considered by Eső and Schummer [4] and Rachmilevitch [11] based on an assumption that a collusive side transfer is proposed by one of the two bidders. This paper assumes rather that the side transfer is proposed by a mediator. While our assumption removes the informed-principal aspect of the problem, it retains the signaling effects of bidders' responses to the proposal. Such simplification allows us to solve the above-described mechanism design problem, and to solve it without the pure-strategy restriction in the last two papers.

2 Assumptions and Notations

Let us consider a first-price sealed-bid auction of a single good between two bidders. Bidder 1's valuation of the good, privately known and called his type t, is independently drawn according to a cumulative distribution function (c.d.f.) F, with support $\left[\underline{t}, \overline{t}\right]$, on which F is absolutely continuous and strictly increasing. Bidder 2's valuation of the good, by contrast, is commonly known to be a constant z. Assume that $z \geq 0$ and $\overline{t} > \underline{t} \geq 0$.

Each bidder's strategy space is the continuum $\mathbb{R}_+$. In the case of a tie, the winner is chosen by a fair coin toss. A bidder's payoff is equal to zero if he does not win, and otherwise equal to his valuation of the good subtracted by his bid. Both bidders are risk neutral.

The solution concept is Bayesian Nash equilibrium (BNE). With mixed strategies possible, any such equilibrium corresponds to a pair (G_1, G_2) such that G_i is the c.d.f. of bids submitted by bidder i, with G_1 generated by bidder 1's strategy, which associates a c.d.f. of bids to any realized type, coupled with F according to which the type is drawn. Given G_i ($i \in \{1, 2\}$), the probability $G^*_{-i}(b)$ for bidder $-i$ to win by submitting a bid b is determined:

$$G^*_{-i}(b) = \begin{cases} G_i(b) & \text{if } b \text{ is not an atom of } G_i \\ \lim_{b' \uparrow b} G_i(b') + \left(G_i(b) - \lim_{b' \uparrow b} G_i(b')\right)/2 & \text{if } b \text{ is an atom of } G_i. \end{cases} \tag{1}$$

A *serious bid* b for bidder $-i$ means $G^*_{-i}(b) > 0$. Given (G_1, G_2), by the first-price payment rule and necessary conditions for equilibrium, there exists $\overline{b} \in \mathbb{R}$ equal to the supremum of the support of G_i for both $i \in \{1, 2\}$. Their infimums, however, may differ because a player may submit non-serious bids. Thus for each i denote $\underline{b}_i$ for the infimum of the support of G_i. Note that $\overline{b}$ is a serious bid for both bidders.

3 The Equilibriums of the Auction Game

Theorem 1 *(a) A BNE exists in the above-defined game if and only if $z \neq \underline{t}$. (b) In any BNE when $z < \underline{t}$, bidder 2's surplus is zero, the allocation is ex post efficient, and bidder 1 bids z for sure. (c) In any BNE when $z > \underline{t}$, bidder 2's surplus is positive and the allocation depending on the parameters may be ex post efficient or inefficient. (d) In any BNE without weakly dominated strategies, bidder 2's surplus when $z > \underline{t}$ is equal to $\max_{b \in [\underline{t}, z]} F(b)(z - b)$.*

The rest of this section presents the proof. Subsection 3.1 proves three lemmas, Subsection 3.2 proves existence of equilibrium and characterizes all equilibriums in the case $z < \underline{t}$, and Subsection 3.3 does that in the case $z > \underline{t}$. The proof for nonexistence of equilibrium in the case $z = \underline{t}$ is omitted because it is a trivial extension of a proof in Lebrun [7, Section 1].

3.1 Lemmas

Lemma 1 *At any BNE, if $\underline{b}_2 = \overline{b}$ then $\overline{t} \leq \underline{b}_2 \leq z$ and $\underline{b}_2$ is not an atom of G_1.*

Proof Suppose $\underline{b}_2 = \overline{b}$, i.e., bidder 2 bids $\overline{b}$ for sure. Since $\overline{b}$ is a serious bid for bidder 2, whose valuation is z, we have $\overline{b} \leq z$. We claim that $\overline{b}$ is not an atom of G_1. Otherwise,

$\bar{b} = t$ for any type t of bidder 1 who is supposed to bid $\bar{b}$: If $\bar{b} > t$ then this type of bidder 1 would have negative expected payoff, $\bar{b}$ a serious bid for him since the rival bids it for sure; if $\bar{b} < t$ then this type would deviate to a bid slightly above $\bar{b}$. But $\bar{b} = t$ implies that only a single type of bidder 1 bids $\bar{b}$, which does not constitute an atom of G_1 as F is assumed atomless. Now that $\bar{b}$ is not an atom of G_1 and $\underline{b}_2 = \bar{b}$, almost every type of bidder 1 loses in the auction and gets zero payoff. Thus, $\bar{t} \leq \bar{b}$, otherwise bidder 1 of types in $(\bar{b}, \bar{t})$ would deviate to bid slightly above $\bar{b}$ thereby obtaining a positive expected payoff. ■

Lemma 2 *For any open interval $O \subseteq \mathbb{R}$, any $i \in \{1, 2\}$ and any continuous real function φ on O, if $G_i^*(b) \leq \varphi(b)$ for all $b \in O$ then $G_{-i}(b) \leq \varphi(b)$ for all $b \in O$.*

Proof By Eq. (1), $G_{-i}(b) = G_i^*(b)$ unless b is an atom of G_{-i}. Thus, by hypothesis of the lemma we have $G_{-i}(b) \leq \varphi(b)$ at all non-atom points b in O. To extend the inequality to atoms, pick any $b \in O$ that is an atom of G_{-i}. A c.d.f., G_{-i} has at most countably many atoms. Thus, there is a sequence $(b^k)_{k=1}^{\infty}$ in O such that $b^k \to_k b$ and, for each k, $b^k \geq b$ and b^k is not an atom of G_{-i}. Hence $G_{-i}(b^k) = G_i^*(b^k) \leq \varphi(b^k)$ for all k. Thus, the desired conclusion follows from upper semicontinuity of c.d.f. G_{-i} and continuity of φ:

$$G_{-i}(b) = \lim_{k \to \infty} G_{-i}(b^k) \leq \lim_{k \to \infty} \varphi(b^k) = \varphi(b). \quad ■$$

Lemma 3 *If $G_1(\underline{b}_2) = 0$ then $\bar{b} = \underline{b}_1 = z$.*

Proof Since $\bar{b}$ is a serious bid for both bidders, $\bar{b} \leq z$ by individual rationality of bidder 2. Suppose $G_1(\underline{b}_2) = 0$, then $G_2^*(\underline{b}_2) = 0$ by Eq. (1) and hence in bidding $\underline{b}_2$ bidder 2 gets zero surplus. By indifference of bidder 2 across bids in the support of his mixed strategy G_2, $(z - b)G_2^*(b) = 0$ for all $b \in [\underline{b}_2, \bar{b}]$. Thus, with $\bar{b} \leq z$, $G_2^*(b) = 0$ for all $b < \bar{b}$, hence by Lemma 2 $G_1(b) = 0$ for all $b < \bar{b}$. Thus $\underline{b}_1 = \bar{b}$, i.e., bidder 1 bids $\bar{b}$ for sure at equilibrium. Consequently, to keep bidder 2 from deviating from $\bar{b}$ to a slightly higher bid, $\bar{b} \geq z$. This coupled with the established fact $\bar{b} \leq z$ implies $\bar{b} = z$. ■

3.2 When $z < \underline{t}$

3.2.1 Necessary Conditions for any Equilibrium

By Lemma 1 and $z < \underline{t}$, $\underline{b}_2 < \bar{b}$. We claim that bidder 2 in bidding $\underline{b}_2$ gets zero surplus. Suppose not, then $\underline{b}_2$ is a serious bid for bidder 2 and $G_1(\underline{b}_2) > 0$, hence there is a positive measure of bidder 1's types, say t_*, that get zero expected payoff at the equilibrium. With $\bar{b}$ a serious bid for bidder 2, his individual rationality implies $\bar{b} \leq z$. Then the types t_* of bidder 1 would deviate to bidding z thereby winning with a positive probability (as $z \geq \bar{b}$) and, conditional on winning, getting a positive payoff because $t_* - z \geq \underline{t} - z > 0$.

Now that $G_1(\underline{b}_2) = 0$, Lemma 3 implies that bidder 1 bids $\bar{b}$ for sure and $z = \bar{b}$ is an atom of G_1. Hence z cannot be an atom of G_2. Thus, at equilibrium, bidder 1 of any type t bids z for sure and gets a payoff equal to $t - z$. To keep bidder 1 of type t from deviating to bid b below z, we need

$$t - z - G_1^*(b)(t - b) \geq 0$$

for all $b < z$. That implies, for all $b < z$, $0 \leq \underline{t} - z - G_1^*(b)\left(\underline{t} - b\right)$, i.e., $G_1^*(b) \leq \frac{\underline{t}-z}{\underline{t}-b}$. Then Lemma 2 implies

$$\forall b < z : G_2(b) \leq \frac{\underline{t} - z}{\underline{t} - b}.$$

This inequality, coupled with the fact $\underline{b}_1 = \bar{b} = z$ proved above, pins down the equilibriums, all rendering zero surplus to bidder 2 and positive payoffs to bidder 1.

Note that any equilibrium in this case is ex post efficient, as bidder 1, always the higher-value bidder in this case, wins for sure in any equilibrum.

3.2.2 An Equilibrium

With $0 \leq z < \underline{t}$, the G_2 defined by

$$G_2(b) := \frac{\underline{t} - z}{\underline{t} - b}$$

for any $b \in [0, z]$ is a c.d.f. with support $[0, z]$. An equilibrium is: Bidder 1 plays the pure strategy of bidding z, and bidder 2 plays the mixed strategy according to the G_2 defined above. Expecting bidder 1 to bid z for sure, bidder 2, whose valuation of the good equals z, gets zero payoff from submitting any bid in $[0, z]$, and negative payoff from bidding above z. Hence G_2 is a best response for bidder 2. For bidder 1 of any type $t \in \left[\underline{t}, \bar{t}\right]$, given the rival's mixed strategy G_2, a bid equal to z yields a positive payoff $t - z$, any bid above z yields a lower payoff, and any bid $b < z$ in the support of G_2 is unprofitable because

$$t - z - G_2(b)(t - b) = t - z - \frac{\underline{t} - z}{\underline{t} - b}(t - b) \geq t - z - \frac{t - z}{t - b}(t - b) = 0,$$

with the inequality due to the fact that $\frac{t-z}{t-b}$ is a strictly increasing function of t:

$$\frac{d}{dt}\left(\frac{t - z}{t - b}\right) = \frac{t - b - (t - z)}{(t - b)^2} = \frac{z - b}{(t - b)^2} > 0.$$

Thus, bidding z is a best response for bidder 1.

3.3 When $z > \underline{t}$

3.3.1 Necessary Conditions for any Equilibrium

Lemma 4 *When $z > \underline{t}$, any BNE has the following properties:*

a. *a positive mass $G_1(\underline{b}_2)$ of bidder 1's types submit non-serious bids;*

b. *bidder 1 bids below $\underline{b}_2$ iff his type is below $\underline{b}_2$, and bidder 2 gets a positive surplus equal to $F(\underline{b}_2)(z - \underline{b}_2)$;*

c. *if the BNE uses no weakly dominated strategy, then:*

   i. *$\underline{b}_2 > \underline{t} \geq \underline{b}_1$ and*

$$\underline{b}_2 = \max\left(\arg\max_{b \in [\underline{t}, z]} F(b)(z - b)\right); \tag{2}$$

   ii. *bidder 2's surplus is equal to $\max_{b \in [\underline{t}, z]} F(b)(z - b)$;*

   iii. *the allocation is not ex post efficient if $\underline{t} < z < \bar{t}$, and is ex post efficient if $\bar{t} < z$ and $F(b)(z - b) \leq z - \bar{t}$ for all $b < \bar{t}$.*

Proof If Claim (a) is not true, then by Lemma 3 bidder 1 bids z for sure, which contradicts individual rationality of the types nearby $\underline{t}$ of bidder 1, as $\underline{t} < z$. Thus Claim (a) is true.

For Claim (b), consider separately the only two possible cases, either $\underline{b}_2 < \bar{b}$ or $\underline{b}_2 = \bar{b}$.

Case 1: $\underline{b}_2 < \bar{b}$. With $G_1(\underline{b}_2) > 0$ (Claim (a)), all elements of $[\underline{b}_2, \bar{b}]$ are serious bids; by the first-price payment rule and equilibrium conditions, one readily sees that G_i for each $i \in \{1, 2\}$ has neither gap nor atom in $(\underline{b}_2, \bar{b})$. Thus, for any $b \in (\underline{b}_2, \bar{b})$, $G_1^*(b) = G_2(b)$ and a type-t bidder 1's expected payoff from bidding b is equal to $G_2(b)(t - b)$. Hence almost every bid in $b \in (\underline{b}_2, \bar{b})$ satisfies the first-order condition for a type of bidder 1 that is supposed to bid b at the equilibrium. With F gapless and atomless by assumption, one can prove[1] that there exists a unique type

$$\beta^{-1}(b) := F^{-1}(G_1(b)) \tag{3}$$

[1] That is mainly due to an observation that at any equilibrium bidder 1's strategy, possibly mixed, is *monotone* in the sense that the infimum bid played by any of his type is greater than or equal to the supremum bid played by any lower type of his. See Zheng [13, Lemma 4b] for details.

of which bidder 1 bids b. Hence for almost all $b \in \left[\underline{b}_2, \overline{b}\right]$,

$$\frac{d}{db} \ln G_2(b) = \frac{1}{F^{-1}\left(G_1(b)\right) - b}. \tag{4}$$

Note that for all such b,

$$F^{-1}\left(G_1(b)\right) > b, \tag{5}$$

otherwise bidder 1 of type $F^{-1}\left(G_1(b)\right)$, getting zero surplus had he abided by the equilibrium bid b, would deviate to bid $b' \in \left(\underline{b}_2, b\right)$ thereby obtaining a positive expected payoff. Eq. (4) implies that, for some $c \in \mathbb{R}$ and every $b \in \left[\underline{b}_2, \overline{b}\right]$,

$$\ln G_2(b) = c - \int_b^{\overline{b}} \frac{1}{F^{-1}(G_1(b')) - b'} db'.$$

By Ineq. (5), the integral in the above equation is less than ∞ when $b = \underline{b}_2$. Thus, $\ln G_2\left(\underline{b}_2\right) > -\infty$, i.e., $G_2\left(\underline{b}_2\right) > 0$. Hence $\underline{b}_2$ is an atom of G_2. Thus, for any type t of bidder 1 that is supposed to bid less than or equal to $\underline{b}_2$, $t \le \underline{b}_2$, otherwise such type would deviate to bid slightly above $\underline{b}_2$. Furthermore, with $\underline{b}_2$ a serious bid for bidder 1, if his type $t \le \underline{b}_2$ then he bids less than or equal to $\underline{b}_2$, and strictly so if $t < \underline{b}_2$. Thus, $G_1\left(\underline{b}_2\right) = F\left(\underline{b}_2\right)$, $\underline{b}_2$ is not an atom of G_1, $G_2^*\left(\underline{b}_2\right) = G_1\left(\underline{b}_2\right)$ and bidder 2's expected payoff from bidding $\underline{b}_2$ is equal to $F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right)$. This is also his equilibrium surplus, as he is playing a mixed strategy. Hence we have proved Claim (b) in the case $\underline{b}_2 < \overline{b}$.

Case 2: $\underline{b}_2 = \overline{b}$. Then by Lemma 1, $\overline{t} \le \underline{b}_2 \le z$ and $\overline{b}$ is not an atom of G_1. Thus, bidder 2, bidding $\underline{b}_2$ $(= \overline{b})$ for sure, gets the payoff $z - \underline{b}_2 = F(\underline{b}_2)(z - \underline{b}_2)$, as $\underline{b}_2 \ge \overline{t}$. Hence Claim (b) holds also in this case.

To prove Claims (c.i) and (c.ii), first consider Case 1, where $\underline{b}_2 < \overline{b}$. By Claim (b),

$$G_1(b) = \frac{F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right)}{z - b} \tag{6}$$

for all $b \in \left[\underline{b}_2, \overline{b}\right]$. Plugging Eq. (6) into Ineq. (5) we have

$$\forall b \in \left(\underline{b}_2, \overline{b}\right) : F(b)(z - b) < F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right). \tag{7}$$

To keep bidder 2 from deviating to bids $b < \underline{b}_2$, we need

$$\forall b < \underline{b}_2 : G_2^*(b)(z - b) \le F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right).$$

This, by Lemma 2, implies

$$\forall b < \underline{b}_2 : G_1(b)(z - b) \le F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right). \tag{8}$$

By the condition that no bid at equilibrium be weakly dominated,[2] a bidder never bids above his valuation. Thus, by Eq. (3), for any $b \leq \underline{b}_2$ we have $F^{-1}(G_1(b)) \geq b$, i.e., $G_1(b) \geq F(b)$. Consequently, (8) implies

$$F(\underline{b}_2)(z - \underline{b}_2) \geq F(b)(z - b)$$

for all $b \leq \underline{b}_2$. This, coupled with the strict inequality (7), implies Eq. (2). With $F(\underline{t}) = 0$ by assumption, Eq. (2) implies $\underline{b}_2 > \underline{t}$, as claimed in (c.i). The equation coupled with Claim (b) implies Claim (c.ii) in the case where $\underline{b}_2 < \overline{b}$. Second, consider Case 2, where $\underline{b}_2 = \overline{b}$. By Lemma 1, $\overline{b}$ is not an atom of G_1. Thus $\underline{b}_1 < \overline{b}$. To keep bidder 2 from deviating to bids below $\underline{b}_2$, we need $G_2^*(b)(z - b) \leq z - \underline{b}_2$ for all $b < \underline{b}_2$, which by Lemma 2 implies

$$\forall b < \underline{b}_2 : G_1(b)(z - b) \leq z - \underline{b}_2.$$

Coupled with the undominated strategy condition, $F \leq G_1$, this implies

$$\forall b < \underline{b}_2 : F(b)(z - b) \leq z - \underline{b}_2.$$

Hence $\underline{b}_2$ satisfies Eq. (2), so Claim (c.ii) also holds when $\underline{b}_2 = \overline{b}$.

For Claim (c.iii), first suppose $\underline{t} < z < \overline{t}$. Then Eq. (2) implies $\underline{t} < \underline{b}_2 < z$. Thus bidder 1 has a positive-measure set $[\underline{b}_2, z]$ of types that bid according to Eq. (6) and win with a positive probability, with valuations less than z, bidder 2's valuation. Hence misallocation occurs with a positive probability if $\underline{t} < z < \overline{t}$. Next suppose that $\overline{t} < z$ and $F(b)(z-b) \leq z - \overline{t}$ for all $b < \overline{t}$. Then Eq. (2) implies that $\underline{b}_2 = \overline{t}$. Hence all types but $\overline{t}$ of bidder 1 bid below $\underline{b}_2$, hence bidder 2 wins for sure. With $z \geq \underline{b}_2 \geq \overline{t}$, the allocation is efficient. ∎

3.3.2 An Equilibrium

Let $\underline{b}_2$ be defined by Eq. (2). Such $\underline{b}_2$ exists and is well-defined because F, a c.d.f., is upper semicontinuous. Let

$$\overline{b} := z - F(\underline{b}_2)(z - \underline{b}_2). \tag{9}$$

Define

$$G_1(b) := \begin{cases} F(b) & \text{if } b \leq \underline{b}_2 \\ F(\underline{b}_2)(z - \underline{b}_2)/(z - b) & \text{if } \underline{b}_2 \leq b \leq \overline{b}; \end{cases} \tag{10}$$

[2] Note that this condition is satisfied by the equilibrium in the case $0 \leq z < \underline{t}$.

if $b < \underline{b}_2$, define $G_2(b) := 0$; if $b \in [\underline{b}_2, \overline{b}]$, define

$$G_2(b) := \exp\left(-\int_b^{\bar{b}} \frac{1}{F^{-1}\left(F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)/(z-y)\right)-y}dy\right). \tag{11}$$

Bidder 1's equilibrium strategy is to bid his value t when $t \le \underline{b}_2$, and bid $\beta(t)$ when $t \ge \underline{b}_2$ such that the inverse of β is defined by Eq. (3), where G_1 is defined by Eq. (10); bidder 2's equilibrium strategy is to randomly submit a bid between $\underline{b}_2$ and $\bar{b}$ according to the distribution G_2 defined by Eq. (11).

Note: G_2 is a c.d.f. with support $[\underline{b}_2, \overline{b}]$. Also note that $G_1(\bar{b}) = 1$ due to Eq. (9).

For bidder 2, given the rival's strategy G_1, any bid $b \in [\underline{b}_2, \overline{b}]$ yields the same expected payoff: $G_1(b)(z-b) = F(\underline{b}_2)(z-\underline{b}_2)$; deviating to a bid $b < \underline{b}_2$ is unprofitable because

$$G_1(b)(z-b) = F(b)(z-b) \le F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right),$$

with the equality due to Eq. (10), and the inequality due to (2). Thus G_2, randomly selecting a bid in $[\underline{b}_2, \overline{b}]$, is a best response.

Consider bidder 1 with any type $t \in [\underline{t}, \underline{b}_2]$. Given the rival's strategy $G_2(b) = 0$ for all $b < \underline{b}_2$, the expected payoff from submitting any bid below $\underline{b}_2$ is zero; since $t \le \underline{b}_2$, the expected payoff from bidding above $\underline{b}_2$ is negative, and that from bidding $\underline{b}_2$ is negative unless $t = \underline{b}_2$, in which case the payoff is zero. Thus, bidding t according to G_1 is a best response.

Finally, consider bidder 1 with any type $t \in [\underline{b}_2, \bar{t}]$. Since $F(t) \in [F(\underline{b}_2), 1]$, and $F(\underline{b}_2)(z-\underline{b}_2)/(z-y)$ a continuous, strictly increasing function of $y \in [\underline{b}_2, \overline{b}]$ with range $[F(\underline{b}_2), 1]$, there exists a unique $b \in [\underline{b}_2, \overline{b}]$ such that

$$t = F^{-1}\left(\frac{F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)}{z-b}\right), \tag{12}$$

which, by Eqs. (3) and (10), means that the type-t bidder 1 is supposed to bid b. For any $b' \in [\underline{b}_2, \overline{b}]$, let $u(b', t)$ denote his expected payoff from bidding b'. By Eq. (11),

$$\frac{\partial}{\partial b'}u(b', t) = G_2(b')\left(\frac{1}{F^{-1}\left(F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)/(z-b')\right)-b'}(t-b')-1\right). \tag{13}$$

By Eq. (12), $\frac{\partial}{\partial b'}u(b,t) = 0$ when $b' = b$. Hence b satisfies the first-order condition. To verify the second-order condition, pick any $b' \in [\underline{b}_2, \overline{b}]$ such that $t > b' > b$. Then

$$F^{-1}\left(F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)/(z-b')\right)-b' > F^{-1}\left(F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)/(z-b)\right)-b' \overset{(12)}{=} t-b' > 0$$

and so by Eq. (13) $\frac{\partial}{\partial b'}u(b',t) \leq 0$. Analogously, $\frac{\partial}{\partial b'}u(b',t) \geq 0$ for any $b'' \in [\underline{b}_2, \overline{b}]$ such that $b'' < b$. Thus, no alternative bid in $[\underline{b}_2, \overline{b}]$ is a profitable deviation from b. Neither is any bid below $\underline{b}_2$ a profitable deviation, as such bids render zero winning probability for bidder 1. Thus, bidding b is a best response for bidder 1 of type t, as claimed.

Remark 1 Vickrey [12, p18, Section II] considers the special case where bidder 1's type is uniformly distributed on $[0, 1]$ and bidder 2's value is a commonly known constant in $(0, 1)$. Hence it belongs to Case 1 in Subsection 3.3, with $0 = \underline{t} < z < \overline{t} = 1$ and F the uniform distribution on $[0, 1]$. In this case, the equilibriums characterized in Subsection 3.3 specialize to those in Vickrey's Appendix III. In particular, with $F(b) = b$, Eq. (2) implies $\underline{b}_2 = z/2$ at the equilibrium in undominated strategies, which gives bidder 2 a constant surplus equal to $F(\underline{b}_2)(z - \underline{b}_2) = z^2/4$ and implies, by Eqs. (3) and (10), that the type of bidder 1 that bids $b \geq z/2$ is equal to $F^{-1}\left((z^2/4)/(z-b)\right) = z/(4(1-b/z))$, i.e., bidder 1 with type $t \geq z/2$ bids $z(1 - z/(4t))$, which is what Vickrey obtains (p18, Section II).[3]

4 An Application in Collusion

Suppose that a good of common value, commonly known to be equal to one, is pursued by players 1 and 2. For each $i \in \{1, 2\}$, player i's type t_i, privately known to i, is independently drawn from a commonly known distribution F_i, absolutely continuous and strictly increasing on its support $[a_i, z_i]$ with $0 < a_i < z_i$. The good is to be auctioned off via a first-price sealed-bid auction with zero reserve price and equal-probability tie-breaking rule. Prior to the auction, however, the two players can collude through an outside mediator, who proposes to them a *collusive division* in the form of $(v_1, v_2) \in [0, 1]^2$ such that $v_1 + v_2 = 1$. If both players accept the proposal, each commits to bidding zero in the auction and, in case of winning (at zero price), paying the other player the share according to the division, so that player i's payoff in the whole game equals v_i. Otherwise, they play the auction game noncooperatively, with player i independently submitting a bid say b_i, so that the payoff for player i is equal to $1 - b_i/t_i$ if i wins, and zero if otherwise. Both players are assumed risk neutral.

[3] Vickrey's notations $(v_1, a, x, y_i(x), k, r_2)$ correspond to $(t, z, b, \lim_{b' \uparrow b} G_i(b'), u, \underline{b}_2)$ here. The condition $x^2 - ax + k \leq 0$, based on which Vickrey determines his r_2, corresponds to our Eq. (2).

Remark 2 Within the auction game, this common value model is equivalent to the independent private value (IPV) model: In our model, a type-t_i player i's decision in the auction game, given G_{-i} the distribution of the bids submitted by the other player (and hence $G_i^*(b)$ the probability for i to win with bid b), is

$$\max_{b \in \mathbb{R}_+} G_i^*(b) \left(1 - \frac{b}{t_i}\right) = \frac{1}{t_i} \max_{b \in \mathbb{R}_+} G_i^*(b)\,(t_i - b)\,, \tag{14}$$

equivalent to the player's decision in an IPV model with valuation t_i. Thus, given any c.d.f. $\tilde{F}_1$ and $\tilde{F}_2$, (G_1, G_2) constitutes a BNE of the continuation game in this section, with $(\tilde{F}_1, \tilde{F}_2)$ being the pair of posterior distributions, if and only if (G_1, G_2) constitutes a BNE in the IPV first-price auction game given $(\tilde{F}_1, \tilde{F}_2)$ being the type-distributions with types playing the role of private values.

Remark 3 Embedded in a multistage context, however, the two auction models have an important difference. In the IPV model, a player's utility from a collusive division depends on his type, hence a collusive contract in general is type-dependent. In our model, whereas, a player's type matters only in the off-path event where collusion breaks down and he has to pay for the good; hence collusive contracts can be mutually acceptable without being type-dependent. Furthermore, one can prove that there is no loss of generality, within the class of contracts that guarantee mutual acceptance, to restrict attention to type-independent collusive divisions (c.f. Zheng [13, Lemma 2]).

Once a collusive division is proposed by the mediator, a two-stage game is defined, with perfect Bayesian equilibrium the solution concept. To this concept we add two conditions: first, no weakly dominated strategy is used; second, if rejection of the collusive division is an off-path action, then the posterior distribution of a player who has just unilaterally made such a deviation is independent of the realized type of the other player.[4] In the rest of this paper, *PBE* refers to perfect Bayesian equilibriums satisfying the two conditions. The question is What is the condition on the parameters for there to exist a *fully acceptable collusive division*, i.e., a collusive-division proposal that admits a PBE where the proposal is accepted by both players almost surely?

[4] The second condition is similar in spirit to the "no signaling what you don't know" condition of Fudenberg and Tirole [5]. See Zheng [13, Footnote 3] for explanations.

To answer this question we start with a player's endogenous outside option other than collusion. Given any BNE $G := (G_1, G_2)$ of the auction game, for any realized type t_i of player i ($i \in \{1, 2\}$), define

$$U_i(t_i|G) := \sup_{b \in \mathbb{R}_+} G_i^*(b)(v - b/t_i), \tag{15}$$

where $G_i^*(b)$ is i's winning probability derived from G according to Eq. (1), incorporating the possibility of tying at b. Hence $U_i(t_i|G)$ is the supremum among player i's expected payoffs in the first-price auction when i's bid ranges in $\mathbb{R}_+$, given his type t_i and the other player's obedience to G. In other words, $U_i(t_i|G)$ is the best that player i of type t_i can get from rejecting an otherwise mutually acceptable proposal, provided that player $-i$ abides by G if i rejects the proposal. One can prove easily that $U_i(t_i|G)$ is weakly increasing in t_i. Consequently, among all types of player i, z_i is the one most tempted to reject a collusive proposal. Thus, to prevent player i from rejecting an otherwise mutually accepted collusive proposal, it suffices to offer i a payoff above the infimum of $U_i(z_i|G)$ when G ranges among the BNEs of the auction game given i's unilateral deviation:

$$\underline{u}_i := \inf \left\{ U_i(z_i|G) : G \in \mathscr{E}_i(\tilde{F}_i); \operatorname{supp} \tilde{F}_i \subseteq \operatorname{supp} F_i \right\}, \tag{16}$$

where $\mathscr{E}_i(\tilde{F}_i)$ denotes the set of all the BNEs, without dominated strategies, of the first-price auction such that the distribution of i's type is $\tilde{F}_i$ while that of $-i$'s remains to be the prior F_{-i}. Furthermore, if player i is offered a payoff less than the infimum $\underline{u}_i$, i would reject the proposal with a positive probability: not only would the type z_i strictly prefer rejecting the proposal according to the definition of $\underline{u}_i$, but the types near z_i would also strictly prefer so, as one can prove that $U_i(t_i|G)$ is continuous in t_i.[5] Thus:

Lemma 5 *(a) If there exists a fully acceptable collusive division, then*

$$\underline{u}_1 + \underline{u}_2 \leq 1. \tag{17}$$

(b) The converse is true if $\underline{u}_i$ is attained by some G in the set in Eq. (16) for each $i \in \{1, 2\}$, or if the inequality in (17) is strict.

Proof As explained in the paragraph preceding this lemma, a player i accepts a proposal almost surely only if he is offered a payoff at least as large as $\underline{u}_i$. Hence for both players to

[5] Zheng [13, Theorem 1].

accept the proposal almost surely we need $\underline{u}_1 + \underline{u}_2$ to be no less than the common value of the good, which is equal to one by assumption. Thus Claim (a) is proved.

To prove Claim (b), suppose (17) and consider first the case where $\underline{u}_i$ is attained by some $G^i \in \mathscr{E}_i(\tilde{F}_i)$ in the set in Eq. (16) for each $i \in \{1, 2\}$. Due to (17), there is a split of the common value one that gives each i a share at least as large as $\underline{u}_i$. Propose this split to the players with the common understanding that both players will accept it for sure and, if any player i rejects the proposal while player $-i$ accepts it, then $-i$ will adopt the off-path posterior $\tilde{F}_i$ about i and abide by the BNE G^i in the auction game. The posterior F_{-i} that support G^i together with $\tilde{F}_i$ satisfies Bayes's rule since $-i$ is expected to accept the proposal for sure. Expecting G^1 and G^2 as the respective penal codes for a unilateral vetoer, neither players can profit from vetoing the proposal. Next consider the other case, where (17) holds strictly. Then there is a split of the common value one that gives each i a share $v_i > \underline{u}_i$. For each i, by definition of $\underline{u}_i$, there exists a BNE G^i in the set in Eq. (16) for which $v_i \geq U_i(z_i|G^i) \geq \underline{u}_i$. Propose (v_1, v_2) to the players with the same common understanding as in the previous case, and the desired conclusion follows as in the first case. ■

Thus, the question boils down to whether Ineq. (17) is satisfied by the prior distributions (F_1, F_2). That requires a formula to derive $\underline{u}_i$ from (F_1, F_2). To do that, by Eq. (16) the definition of $\underline{u}_i$, we characterize $\mathscr{E}_i(\tilde{F}_i)$ for all $\tilde{F}_i$, i.e., the BNEs of the auction given that player $-i$'s posterior is the prior F_{-i} while player i's posterior $\tilde{F}_i$ can be any distribution whose support is contained by the prior support $[a_i, z_i]$. Since $\tilde{F}_i$ need not be strictly monotone, $\tilde{F}_i^{-1}$ is defined to be the generalized inverse in Zheng [13]: for any $s \in [0, 1]$, let

$$\tilde{F}_i^{-1}(s) := \inf \left\{ t \in \operatorname{supp} \tilde{F}_i : \tilde{F}_i(t) \geq s \right\}.$$

Consider any BNE (G_1, G_2) in $\mathscr{E}_i(\tilde{F}_i)$, with $\left[\underline{b}_i, \overline{b}\right]$ being the support of G_i (for each $i \in \{1, 2\}$). Due to the first-price payment rule, the two supports have the same supremum $\overline{b}$, and neither G_i nor G_{-i} has gap or atom in $\left(\max\{\underline{b}_1, \underline{b}_2\}, \overline{b}\right)$. One readily sees that each player's equilibrium strategy is monotone in the sense defined in Footnote 1. With F_{-i} continuous and strictly increasing by assumption, $F_{-i}^{-1}(G_{-i}(b))$ is the unique type of player $-i$ that bids b at the equilibrium. Although $\tilde{F}_i$ need not be strictly increasing, with the monotonicity of player i's equilibrium strategy, one can prove that $\tilde{F}_i^{-1}(G_i(b))$ is a type of player i that bids b at the equilibrium.[6] Then we obtain the first-order necessary condition for the equilibrium

[6] Zheng [13, Lemma 12].

(c.f. Eq. (14)): for all $b \in \left(\max\{\underline{b}_1, \underline{b}_2\}, \overline{b}\right)$,

$$\frac{d}{db}\ln G_i(b) = \frac{1}{F_{-i}^{-1}\left(G_{-i}(b)\right) - b}, \tag{18}$$

$$\frac{d}{db}\ln G_{-i}(b) = \frac{1}{\tilde{F}_i^{-1}\left(G_i(b)\right) - b}. \tag{19}$$

Eqs. (18) and (19) lead to a nontrivial comparison between the equilibrium of the asymmetric auction characterized in the previous section, where the posterior of i is degenerate, and any BNE supported by any posterior of i:

Lemma 6 *Let $\tilde{F}_i^o$ denote the distribution whose support is $\{z_i\}$ and let $G^o := (G_i^o, G_{-i}^o)$ be the BNE in $\mathscr{E}_i(\tilde{F}_i^o)$ constructed in Section 3, with $\overline{b}^o$ the supremum of the bid distributions of G^o. For any c.d.f. $\tilde{F}_i$ with support contained in $[a_i, z_i]$, let $G := (G_i, G_{-i}) \in \mathscr{E}_i(\tilde{F}_i)$, with $\overline{b}$ the supremum of the bid distributions of G. If $z_i > a_{-i}$ then $\overline{b} \leq \overline{b^o}$.*[7]

Proof Suppose, to the contrary, that $\overline{b} > \overline{b^o}$. By Remark 2, all BNEs stated in the lemma are also BNEs in the IPV model of Section 3 with $(\tilde{F}_i, F_{-i})$ being the distributions of private values. With the hypothesis $z_i > a_{-i}$, G^o is the BNE in Section 3.3, where bidder 1 corresponds to player $-i$, and bidder 2 player i, with $(z, \underline{t})$ being (z_i, a_{-i}). For each $j \in \{i, -i\}$, let $\left[\underline{b}_j^o, \overline{b^o}\right]$ denote the support of G_j^o, and $\left[\underline{b}_j, \overline{b}\right]$ the support of G_j. In the BNE G^o, since $\underline{b}_i^o > \underline{b}_{-i}^o$ (Lemma 4.c.i), $\underline{b}_i^o = \max\{\underline{b}_1^o, \underline{b}_2^o\}$. Since the support of $\tilde{F}_i^o$ for G^o is $\{z_i\}$, Eq. (19) applied to G^o says that

$$\frac{d}{db}\ln G_{-i}^o(b) = \frac{1}{z_i - b}$$

for all $b \in \left(\underline{b}_i^o, \overline{b^o}\right)$. If such b is also above $\max\{\underline{b}_1, \underline{b}_2\}$, it also satisfies the first-order condition for BNE G, i.e., the Eq. (19) applied to G. Compare the right-hand sides of Eq. (19) for G^o and for G, note that z_i is the supremum of i's prior support, and we obtain

$$\frac{d}{db}\ln G_{-i}(b) > \frac{d}{db}\ln G_{-i}^o(b),$$

i.e., as b decreases, $G_{-i}(b)$ decreases faster than $G_{-i}^o(b)$ does. This, coupled with the supposition $\overline{b} > \overline{b^o}$, implies that $G_{-i} < G_{-i}^o$ on $\left[\underline{b}_i^o, \overline{b}\right]$ if $\underline{b}_i^o \geq \max\{\underline{b}_i, \underline{b}_{-i}\}$. By the undominated-strategy condition for all elements of $\mathscr{E}_i(\tilde{F}_i)$, bidders do not bid above their types in G, hence

[7] This lemma is substantially different from its counterpart in Zheng [13, Lemma 6], because all equilibrium bid distributions there starts with the bid zero, due to the nature of all-pay auctions, whereas here equilibrium bid distributions need not start from the same bid, due to the nature of first-price auctions.

$\underline{b}_{-i} \leq a_{-i} < \underline{b}_i^o$ (with the last inequality due to Lemma 4.c.i); thus, the supposition $\overline{b} > \overline{b^o}$ implies that $G_{-i} < G^o_{-i}$ on $\left[\underline{b}_i^o, \overline{b}\right]$ if $\underline{b}_i^o \geq \underline{b}_i$.

Suppose $\underline{b}_i^o \geq \underline{b}_i$, then $G_{-i} < G^o_{-i}$ on $\left[\underline{b}_i^o, \overline{b}\right]$. Since $G^o_{-i}(\underline{b}_i^o) = F_{-i}(\underline{b}_i^o)$ (Lemma 4.b), we have $F_{-i}(\underline{b}_i^o) = G^o_{-i}(\underline{b}_i^o) > G_{-i}(\underline{b}_i^o)$. That is, the mass of player i's types that are below $\underline{b}_i^o$ is larger than the mass of his types that *bid* below $\underline{b}_i^o$. Hence there is a positive mass of i's types that bid above their types, violating the undominated-strategy condition.

Thus, $\underline{b}_i^o < \underline{b}_i$. With $\underline{b}_{-i} < \underline{b}_i^o$ proved previously, $G_{-i}(\underline{b}_i) > 0$. By the undominated-strategy condition, $G_{-i}(\underline{b}_i) \geq F_{-i}(\underline{b}_i)$. Furthermore, $G_{-i}(\underline{b}_i) = F_{-i}(\underline{b}_i)$, otherwise some of bidder $-i$'s types that bid no more than $\underline{b}_i$ are above $\underline{b}_i$ and so they would rather bid slightly above $\underline{b}_i$, as G_i has no gap in $\left[\underline{b}_i, \overline{b}\right]$ (by $\underline{b}_i > \underline{b}_i^o > \underline{b}_{-i}$). An element of the support of $\tilde{F}_i$ for the BNE G, $\underline{b}_i$ is equal to, or arbitrarily near to, a best response to G for some type t_i of player i. Thus, the equilibrium expected payoff for t_i in G is no larger than

$$G_i^*(\underline{b}_i)\left(t_i - \underline{b}_i\right)/t_i \overset{(1)}{\leq} G_{-i}(\underline{b}_i)\left(t_i - \underline{b}_i\right)/t_i = F_{-i}(\underline{b}_i)\left(t_i - \underline{b}_i\right)/t_i.$$

We claim that with this type t_i player i would rather deviate to the lower bid $\underline{b}_i^o$. If he bids $\underline{b}_i^o$, player i wins with probability at least as large as $F_{-i}(\underline{b}_i^o)$, with player $-i$ of types below $\underline{b}_i^o$ bidding below it (undominated strategy condition), and hence obtains an expected payoff greater than or equal to $F_{-i}(\underline{b}_i^o)\left(t_i - \underline{b}_i^o\right)/t_i$. Thus, the deviation is profitable if

$$F_{-i}(\underline{b}_i^o)\left(t_i - \underline{b}_i^o\right) > F_{-i}(\underline{b}_i)\left(t_i - \underline{b}_i\right),$$

i.e.,

$$t_i\left(F_{-i}(\underline{b}_i) - F_{-i}(\underline{b}_i^o)\right) < F_{-i}(\underline{b}_i)\underline{b}_i - F_{-i}(\underline{b}_i^o)\underline{b}_i^o. \tag{20}$$

Note that $\underline{b}_i^o$ satisfies Eq. (2), where $\underline{b}_2$ corresponds to $\underline{b}_i^o$. Thus, since $\underline{b}_i^o < \underline{b}_i \leq z_i$, we have $F_{-i}(\underline{b}_i^o)\left(z_i - \underline{b}_i^o\right) > F_{-i}(\underline{b}_i)\left(z_i - \underline{b}_i\right)$ and hence

$$z_i\left(F_{-i}(\underline{b}_i) - F_{-i}(\underline{b}_i^o)\right) < F_{-i}(\underline{b}_i)\underline{b}_i - F_{-i}(\underline{b}_i^o)\underline{b}_i^o.$$

This, coupled with the fact $0 < t_i \leq z_i$, implies Ineq. (20). Thus, player i of type t_i strictly prefers to deviate from the bid prescribed by the BNE G. With expected payoffs continuous in types, this strict preference also holds for sufficiently nearby types. This contradiction implies that the supposition $\overline{b} > \overline{b^o}$ is false, as desired. ■

Lemma 6, coupled with our solution of the asymmetric auction in Section 3, leads to a formula for $\underline{u}_i$: for each $i \in \{1, 2\}$,

$$\underline{u}_i = \frac{1}{z_i}\left(\max_{\min\{a_{-i}, z_i\} \leq b \leq z_i} F_{-i}(b)(z_i - b)\right). \tag{21}$$

Proof of Eq. (21) By Remark 2, Theorem 1 is applicable to any continuation game after negotiation fails given posteriors $(\tilde{F}_i, F_{-i})$ when the posterior distribution $\tilde{F}_i$ of player i's type degenerates to a singleton, with $(i, -i, a_{-i})$ playing the role of $(2, 1, \underline{t})$ there.

Case 1: $z_i < a_{-i}$. With $z_i < a_{-i}$, Eq. (21) is equivalent to $\underline{u}_i = 0$. By player i's individual rationality in the auction game, $\underline{u}_i \geq 0$. Thus, it suffices to find a BNE of a continuation game $(\tilde{F}_i, F_{-i})$ that gives zero payoff to player i of type z_i. Such is the BNE where $\tilde{F}_i$ degenerates to z_i, in which case Theorem 1.b implies that any BNE gives zero payoff to player i of type z_i when he best responds to the BNE.

Case 2: $z_i = a_{-i}$. In this case, Eq. (21) again becomes $\underline{u}_i = 0$, and again it suffices to show $\underline{u}_i \leq 0$. To show that, pick any $\epsilon > 0$ and consider the auction game $(\tilde{F}_i, F_{-i})$ where $\tilde{F}_i$ degenerates to the point $z_i - \epsilon$. Then Theorem 1.b, where z corresponds to $z_i - \epsilon$ here, implies that a BNE of this continuation game exists such that player $-i$ bids $z_i - \epsilon$ for sure, which renders the expected payoff for player i less than ϵ. Thus, by the definition of $\underline{u}_i$, $\underline{u}_i \leq 0$.[8]

Case 3: $z_i > a_{-i}$. By Theorem 1.d, where $(z, \underline{t}, F)$ corresponds to (z_i, a_{-i}, F_{-i}) here, the continuation game $(\tilde{F}_i, F_{-i})$ with $\tilde{F}_i$ degenerated to z_i admits a BNE that gives player i of type z_i an expected payoff equal to the right-hand side of Eq. (21). Let G^o denote this BNE and $\overline{b^o}$ the supremum of the bid distributions of G^o. Hence $U_i(z_i|G^o)$ is equal to the right-hand side of Eq. (21). Thus, to prove Eq. (21), by the definition of $\underline{u}_i$, it suffices to show that, for any other posterior $\tilde{F}_i$ about i, if $\mathscr{E}_i(\tilde{F}_i)$ contains a BNE G, with $\bar{b}$ the supremum of the bid distributions of G, then $U_i(z_i|G) \geq U_i(z_i|G^o)$.

Since G^o is a BNE for the continuation game where the posterior belief is that player i's type is z_i, G_i^o prescribed by this BNE is a best response to G^o for player i of type z_i. Thus, $U_i(z_i|G^o)$ is attained by any bid in the support of G_i^o, hence $U_i(z_i|G^o) = 1 - \overline{b^o}/z_i$.

In response to the BNE G, player i of type z_i can bid just slightly above the bid supremum $\bar{b}$ of this BNE to win for sure and get a payoff just slightly below $1 - \bar{b}/z_i$. Thus, by Eq. (15) the definition of U_i, $U_i(z_i|G) \geq 1 - \bar{b}/z_i$. Now that $U_i(z_i|G^o) = 1 - \overline{b^o}/z_i$ and $U_i(z_i|G) \geq 1 - \bar{b}/z_i$, $U_i(z_i|G) \geq U_i(z_i|G^o)$ is implied by $\bar{b} \leq \overline{b^o}$, which is true by Lemma 6. ∎

Eq. (21) and Lemma 5 together imply the next theorem, which identifies the exact class of primitives that allow for collusive divisions to be accepted fully.

[8] Since $z_i = a_{-i}$, the continuation game $(\tilde{F}_i, F_{-i})$ with $\tilde{F}_i$ degenerated to z_i admits no BNE (Theorem 1.a). That is why we adopt a less straightforward approach in this case than in the previous case.

Theorem 2 *A fully acceptable collusive division exists if and only if*

$$\sum_{i=1}^{2} \frac{1}{z_i} \left(\max_{\min\{a_{-i}, z_i\} \leq b \leq z_i} F_{-i}(b)(z_i - b) \right) \leq 1. \tag{22}$$

Proof Eq. (21) and Lemma 5 combined, the only thing left to prove is that the inequality (17) is strict when $\underline{u}_i$ is not attained for some $i \in \{1, 2\}$. By the proof of Eq. (21) (c.f. Footnote 8), $\underline{u}_i$ is not attained only if $z_i = a_{-i}$. In that case, $\underline{u}_i = 0$, and $z_i = a_{-i}$ implies $z_{-i} > a_i$. Hence the calculation of $\underline{u}_{-i}$ belongs to Case 3 in the proof of Eq. (21), with the roles of i and $-i$ switched. Let $b_* \in \arg\max_{a_i \leq b \leq z_{-i}} F_i(b)(z_{-i} - b)$. Since $F_i(a_i) = 0$ by assumption, $b_* > a_i \geq 0$. Thus, by Eq. (21) with the roles of i and $-i$ switched,

$$\underline{u}_{-i} = \frac{1}{z_{-i}} F_i(b_*)(z_{-i} - b_*) \leq \frac{1}{z_{-i}} (z_{-i} - b_*) < \frac{1}{z_{-i}} z_{-i} = 1.$$

This, coupled with the fact $\underline{u}_i = 0$, implies $\underline{u}_i + \underline{u}_{-i} < 1$, as desired. ■

Corollary 1 *For any $i \in \{1, 2\}$ let F_i^d be any c.d.f. that first-order stochastically dominates F_i, with the same support $[a_i, z_i]$. Then:*

i. if Ineq. (22) is satisfied given the priors (F_1, F_2) then it is also satisfied when the priors are replaced by $\left(F_1^d, F_2^d\right)$;

ii. if Ineq. (22) is violated given the priors $\left(F_1^d, F_2^d\right)$ then it is also violated when the priors are replaced by (F_1, F_2);

Proof By Ineq. (22) and the hypothesis that F_i^d has the same support as F_i for each i, it suffices to prove that, for each $i \in \{1, 2\}$,

$$\max_{\min\{a_{-i}, z_i\} \leq b \leq z_i} F_{-i}(b)(z_i - b) \geq \max_{\min\{a_{-i}, z_i\} \leq b \leq z_i} F_{-i}^d(b)(z_i - b). \tag{23}$$

To that end, define for each $\lambda \in [0, 1]$ and any $t \in \mathbb{R}$

$$\hat{F}_{-i}(t, \lambda) := \lambda F_{-i}^d(t) + (1 - \lambda) F_{-i}(t).$$

Note that $\hat{F}_{-i}(\cdot, 0) = F_{-i}$, $\hat{F}_{-i}(\cdot, 1) = F_{-i}^d$ and $\hat{F}_{-i}(\cdot, \lambda)$ is a c.d.f. with support $[a_{-i}, z_{-i}]$ for each $\lambda \in [0, 1]$. Furthermore, for each $\lambda \in [0, 1]$ the problem

$$V(\lambda) := \max_{\min\{a_{-i}, z_i\} \leq b \leq z_i} \hat{F}_{-i}(b, \lambda)(z_i - b)$$

admits a solution, denoted by $b^*(\lambda)$, because $\hat{F}_{-i}(\cdot, \lambda)$ as a c.d.f. is upper semicontinuous, and the domain compact. By definition of $\hat{F}$, for any $t \in \mathbb{R}$,

$$\frac{\partial}{\partial \lambda} \hat{F}_{-i}(b, \lambda) = F^d_{-i}(b) - F_{-i}(b) \leq 0,$$

with the inequality due to the hypothesis that F^d_{-i} stochastically dominates F_{-i}. By the envelope theorem,

$$V(1) - V(0) = \int_0^1 (z_i - b^*(\lambda)) \left(\frac{\partial}{\partial \lambda} \hat{F}_{-i}(b, \lambda) \Big|_{b=b^*(\lambda)} \right) d\lambda \leq 0,$$

which is Ineq. (23), as desired. ■

5 Conclusion

This paper makes three contributions. First is a complete, explicit characterization of the equilibriums of an asymmetric first-price auction, which is nontrivial to analyze due to the distributional singularity and arises endogenously in multistage settings such as collusion on future bidding behaviors. It is interesting to note that these equilibriums do not necessarily have the properties that equilibriums of asymmetric first-price auctions in the literature usually have. Second, the paper applies the equilibriums of this asymmetric auction to a multistage bidding collusion setup to support a kind of collusive contracts with a striking feature of being type-independent and yet, prior type-distributions permitting, are acceptable to both privately informed bidders regardless of their types. The paper further identifies the exact class of the prior type-distributions that allow for such fully acceptable collusive contracts. Third, in identifying this class of collusion-guaranteeing primitives, the paper presents a differential equation method to analyze asymmetric first-price auctions with arbitrary type-distributions, allowing for atoms and gaps.

In addition to arising as a continuation game in a multistage bidding collusion setting, the asymmetric auction analyzed here may play a similarly important role in other multistage settings such as auctions involving information acquisition that may render some bidders informationally disadvantaged. Such possibilities may lead to fruitful future research.

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