The Equilibrium of an Asymmetric First-Price Auction and Its Implication in Collusion*

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Abstract

This is a complete characterization of the equilibriums of a first-price auction between two bidders such that one privately knows his valuation while the other's valuation is commonly known. The game despite its simplicity may render equilibrium nonexistent, and despite its asymmetry may result in a fully efficient allocation and may give positive surplus to the informationally disadvantaged bidder. This game plays an indispensable role of the continuation game that sustains bidding collusion between two bidders, each privately informed. I obtain a necessary and sufficient condition, in terms of their prior distributions, for existence of a collusive contract guaranteed to be acceptable to both bidders. This condition is more likely to hold if a bidder's prior distribution becomes more stochastically dominant.

1 Introduction

The asymmetric auction considered here is a first-price auction of one good pursued by two bidders such that one privately knows his valuation of the good while the other's valuation is commonly known. This is arguably the simplest model for competitions where one competitor has an information advantage over the other and both have some market

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power. Such asymmetry may occur when the former has access to some insider information of the good while the other can only evaluate the good according to a commonly known prior (e.g., Engelbrecht-Wiggans, Milgrom and Weber [4]). The former may also gain such information advantage through his incumbent status or toehold in the industry (c.f. Bulow, Huang and Klemperer [3]). Sometimes a rent-extracting auctioneer may choose to create such information asymmetry through discriminatory disclosure policies prior to the auction (c.f. Bergemann and Pesendorfer [2, Section 3]). In the literature on asymmetric first-price auctions are three seemingly general properties, that an equilibrium exists (Athey [1]), that the equilibrium is inefficient with a positive probability (e.g., Maskin and Riley [8]) and that an informationally disadvantaged bidder gets zero surplus (e.g., Engelbrecht-Wiggans et al. [4]). It turns out that, however, none of these properties are necessarily true when we reduce the complexity to the simple two-bidder game described above. This paper reports such a finding through a complete and explicit characterization of the Bayesian Nash equilibriums of the game. The equilibrium characterization is then applied to a bidding collusion framework where the asymmetric auction becomes its continuation game.

It should be emphasized that this asymmetric auction game cannot be deemed negligible because of its extreme assumption that the informationally disadvantaged bidder's value is commonly known. That is because such a game, special as it is, may occur endogenously in a multistage setting such as the aforementioned example where a rent-extracting seller may deliberately keep one bidder as ignorant about the good as the prior distribution and let the other bidder inspect the good thereby knowing privately the latter's valuation. In this paper we will see another example where the general solution of this asymmetric game is indispensable, playing the role of the continuation game that sustains bidding collusion.

Vickrey [10] has solved a special case of this asymmetric auction game where the informationally advantaged bidder's value is uniformly distributed on $[0, 1]$, with the other's known value strictly in between, but no solution for a more general case seems to exist in the literature.[1] Work devoted to generalize Vickrey's model, such as Griesmer, Levitan and Shubik [6], replaces the informationally disadvantaged bidder's commonly know valuation by a nondegenerate distribution, thereby avoiding the complication that this bidder's equilibrium strategy is mixed. The solution in this paper allows for general distributions of the

[1] This author would appreciate any pointer to such a work in the literature thereby sparing him the embarrassment of reinventing the wheel.

advantaged bidder's private value and general configurations between the distribution and the disadvantaged bidder's known value. Not only does our solution generalize the case that confirms the properties of Vickrey's solution such as inefficiency at equilibrium and positive surplus for the disadvantaged bidder (Section 3.2.3), but it also includes other cases where neither properties hold (Section 3.1). In addition, there is exactly one case, which might have been overlooked due to its seeming triviality, where no equilibrium exists (Section 3.3).

Engelbrecht-Wiggans et al. [4] have analyzed a similar asymmetric auction except that there are multiple, and identical, informationally disadvantaged bidders, and Martínez-Pardina [7] a similar auction except that there are multiple, ex ante identical, informationally advantaged bidders. Their multiplicity assumptions make important differences from our two-bidder model. The profit of an informationally disadvantaged bidder in Engelbrecht-Wiggans et al. is reduced to zero by the competition with other disadvantaged ones, while it may be positive in our model. The ex ante homogenous advantaged bidders in Martínez-Pardina, competing among themselves, do not necessarily reduce their best-responding bids to the maximum bid of the disadvantaged bidder, whereas the single advantaged bidder in our two-bidder model does, thereby generating an atom in his bid distribution that may render equilibrium nonexistent.

This paper also considers a bidding ring setup to which the general property of our solution that all cases are allowed is essential. Here two players are to bid in a first-price auction for a good of common value, also commonly known, to them; each bidder is privately informed at the outset of his marginal cost of monetary payments for the good. Prior to the auction, an outside neutral mediator proposes to the bidders a side transfer between them for one to bribe the other to abstain from the auction. If accepted by both, the collusive contract will be carried out, else they play the auction noncooperatively with posteriors conditional on the breakdown of the collusion. This setup is similar to those considered by Eső and Schummer [5] and Rachmilevitch [9] except that the side transfer is proposed by an exogenously designated bidder in their models, and that their subjects are perfect Bayesian equilibriums (PBE) restricted to pure strategies. With the side transfer proposed by a neutral mediator, we get to ask general, mechanism design questions: Is it possible, among all PBEs allowing for both pure and mixed strategies, for the two privately informed bidders to always agree on a collusive contract? If the answer is Yes then what is the family of distributions of their types that guarantee their mutual acceptance of such collusion?

The crucial step in answering these questions is obviously to calculate each bidder's (type-dependent) outside payoff in the off-path event where he vetoes the collusive contract. It turns out that such outside payoff is equal to the bidder's surplus in the auction game when his posterior distribution degenerates to his supremum type. Thus the continuation game becomes the asymmetric auction described earlier, with the vetoer playing the role of the informationally disadvantaged bidder. Based on the general solution of the asymmetric auction game, which allows for a general class of prior distributions, the exact family of collusion-guaranteeing prior distributions is identified (Theorem 2). Comparative statics shows that this family is increasing according to the ranking of first-order stochastic dominance. In other words, the prospect for both bidders to always agree on a collusive division is higher when their prior distributions become more stochastically dominant (Corollary 2).

2 Assumptions and Notations

Let us consider a first-price sealed-bid auction, of a single good, between two bidders. Bidder 1's valuation of the good, privately known and hence called his type t, is independently drawn according to a cumulative distribution function (c.d.f.) F, with support $\left[\underline{t}, \overline{t}\right]$, on which F is absolutely continuous and strictly increasing. Bidder 2's valuation of the good, in contrast, is commonly known to be a constant z. Assume that $z \geq 0$ and $\overline{t} > \underline{t} \geq 0$.

Each bidder's strategy space is the continuum $\mathbb{R}_+$. In the case of a tie, the winner is chosen by a fair coin toss. A bidder's payoff is equal to zero if he does not win, and otherwise equal to his valuation of the good subtracted by his bid. Both bidders are risk neutral.

The solution concept is Bayesian Nash equilibrium (BNE). With mixed strategies possible, any such equilibrium corresponds to a pair (G_1, G_2) such that G_i is the c.d.f. of bids submitted by bidder i, with G_1 generated by bidder 1's strategy, which associates a c.d.f. of bids to any realized type, coupled with F from which the type is drawn. Given G_i ($i \in \{1, 2\}$), the probability $\pi_{-i}(b)$ for bidder $-i$ to win by submitting a bid b is determined:

$$\pi_{-i}(b) = \begin{cases} G_i(b) & \text{if } b \text{ is not an atom of } G_i \\ \lim_{b' \uparrow b} G_i(b') + \left(G_i(b) - \lim_{b' \uparrow b} G_i(b')\right)/2 & \text{if } b \text{ is an atom of } G_i. \end{cases} \quad (1)$$

A *serious bid* b for bidder $-i$ means $\pi_{-i}(b) > 0$. Given (G_1, G_2), by the first-price payment rule and necessary conditions for equilibrium, there exists $\overline{b} \in \mathbb{R}$ equal to the supremum of the support of G_i for both $i \in \{1,2\}$. Their infimums, however, may differ because a player may submit non-serious bids. Thus for each i denote $\underline{b}_i$ for the infimum of the support of G_i.

3 Characterization of the Equilibriums

Lemma 1 *At any BNE, if $\underline{b}_2 = \overline{b}$ then $\overline{t} \leq \underline{b}_2 \leq z$ and $\underline{b}_2$ is not an atom of G_1.*

Proof Suppose $\underline{b}_2 = \overline{b}$, i.e., player 2 bids $\overline{b}$ for sure. Since $\overline{b}$ is a serious bid for player 2, whose valuation is z, we have $\overline{b} \leq z$. We claim that $\overline{b}$ is not an atom of G_1. Otherwise, $\overline{b} = t_1$ for any type t_1 of player 1 who is supposed to bid $\overline{b}$: If $\overline{b} > t_1$ then this type of player 1 would have negative expected payoff, $\overline{b}$ a serious bid for him since the rival bids it for sure; if $\overline{b} < t_1$ then this type would deviate to a bid slightly above $\overline{b}$. But $\overline{b} = t_1$ implies that only a single type of player 1 bids $\overline{b}$, which does not constitute an atom of G_1 as F is assumed atomless. Now that $\overline{b}$ is not an atom of G_1 and $\underline{b}_2 = \overline{b}$, almost every type of player 1 loses and gets zero payoff at the equilibrium. Thus, $\overline{t} \leq \overline{b}$, otherwise player 1 of types in $\left(\overline{b}, \overline{t}\right)$ would deviate to bid slightly above $\overline{b}$ thereby obtaining a positive expected payoff. ■

Lemma 2 *For any open interval $O \subseteq \mathbb{R}$, any $i \in \{1,2\}$ and any continuous real function φ on O, if $\pi_i(b) \leq \varphi(b)$ for all $b \in O$ then $G_{-i}(b) \leq \varphi(b)$ for all $b \in O$.*

Proof By Eq. (1), $G_{-i}(b) = \pi_i(b)$ unless b is an atom of G_{-i}. Thus, by hypothesis of the lemma we have $G_{-i}(b) \leq \varphi(b)$ at all non-atom points b in O. To extend the inequality to all atoms, pick any $b \in O$ that is an atom of G_{-i}. A c.d.f., G_{-i} has at most countably many atoms. Thus, there is a sequence $(b^k)_{k=1}^{\infty}$ in O such that $b^k \to_k b$ and, for each k, $b^k \geq b$ and b^k is not an atom of G_{-i}. Hence $G_{-i}(b^k) = \pi_i(b^k) \leq \varphi(b^k)$ for all k. As a c.d.f. is also upper semicontinuous,

$$G_{-i}(b) = \lim_{k\to\infty} G_{-i}(b^k) \leq \lim_{k\to\infty} \varphi(b^k) = \varphi(b),$$

with the last equality due to continuity of φ. ■

The next theorem summarizes the results, demonstrated in the rest of this section.

Theorem 1 *A BNE exists in the above-defined game if and only if $z \neq \underline{t}$. When $z < \underline{t}$, player 2's surplus is zero and the allocation is ex post efficient. When $z > \underline{t}$, player 2's surplus is positive and the allocation depending on the parameters may be ex post efficient or* *inefficient. In any BNE without weakly dominated strategies, player 2's surplus when* $z > \underline{t}$ *is equal to* $\max_{b\in[\underline{t},z]} F(b)(z-b)$.

3.1 When $z < \underline{t}$

3.1.1 Necessary Condition

By Lemma 1 and $z < \underline{t}$, G_2 is nondegenerate, i.e., $\underline{b}_2 < \overline{b}$. We claim that bidder 2 in bidding $\underline{b}_2$ gets zero surplus. Suppose not, then $\underline{b}_2$ is a serious bid for bidder 2 and $G_1(\underline{b}_2) > 0$, hence there is a positive measure of bidder 1's types, say t_*, that get zero expected payoff at the equilibrium. With $\overline{b}$ a serious bid for bidder 2, his individual rationality implies $\overline{b} \le z$. Then the types t_* of bidder 1 would deviate to bidding z thereby winning with a positive probability (as $z \ge \overline{b}$) and, conditional on winning, getting a positive payoff because $t_* - z \ge \underline{t} - z > 0$.

Now that $G_1(\underline{b}_2) = 0$, $\pi_2(\underline{b}_2) = 0$ and hence in bidding $\underline{b}_2$ bidder 2 gets zero surplus. By indifference of bidder 2 across bids in the support of his mixed strategy G_2, G_1 must be the Dirac measure at $\overline{b}$ and, to keep bidder 2 from deviating from $\overline{b}$ to a slightly higher bid, $\overline{b} = z$.

To keep bidder 1, of type $t \in [\underline{t}, \overline{t}]$, from deviating to bid b below z, we need

$$\forall t < z : t - z - \pi_1(b)(t-b) \ge 0,$$

which implies, for all $b < z$, $0 \le \underline{t} - z - \pi_1(b)\,(\underline{t} - b)$, i.e.,

$$\forall b < z : \pi_1(b) \le \frac{\underline{t} - z}{\underline{t} - b}.$$

Then Lemma 2 implies

$$\forall b < z : G_2(b) \le \frac{\underline{t} - z}{\underline{t} - b}.$$

The above inequality, and that G_1 is the Dirac measure at z, together pin down the equilibriums in this case, all rendering zero surplus to bidder 2 and positive payoffs to bidder 1.

Note that any equilibrium in this case is ex post efficient, as player 1, always the higher-value bidder in this case, wins for sure in any equilibrum.

3.1.2 An Equilibrium

With $0 \le z < \underline{t}$, the G_2 defined by

$$G_2(b) := \frac{\underline{t} - z}{\underline{t} - b}$$

for any $b \in [0, z]$ is a c.d.f. with support $[0, z]$. An equilibrium is: Bidder 1 plays the pure strategy of bidding z, and bidder 2 plays the mixed strategy according to the G_2 defined above. Expecting bidder 1 to bid z for sure, bidder 2, whose valuation of the good equals z, gets zero payoff from submitting any bid in $[0, z]$, and negative payoff from bidding above z. Hence G_2 is a best response for bidder 2. For bidder 1 of any type $t \in [\underline{t}, \overline{t}]$, given the rival's mixed strategy G_2, a bid equal to z yields a positive payoff $t - z$, any bid above z yields a lower payoff, and any bid $b < z$ in the support of G_2 is unprofitable because

$$t - z - G_2(b)(t - b) = t - z - \frac{\underline{t} - z}{\underline{t} - b}(t - b) \geq t - z - \frac{t - z}{t - b}(t - b) = 0,$$

with the inequality due to the fact that $\frac{t-z}{t-b}$ is a strictly increasing function of t:

$$\frac{d}{dt}\left(\frac{t - z}{t - b}\right) = \frac{t - b - (t - z)}{(t - b)^2} = \frac{z - b}{(t - b)^2} > 0.$$

Thus, bidding z is a best response for bidder 1.

3.2 When $z > \underline{t}$

3.2.1 Necessary Condition

We claim that $G_1(\underline{b}_2) > 0$. Suppose not, then $\pi_2(\underline{b}_2) = 0$ by Eq. (1); and by the indifference for bidder 2 across bids in the support of G_2, there is a sequence $(b^k)_{k=1}^{\infty}$ in the support, with $b^k \uparrow \overline{b}$, such that for all k

$$\pi_2(b^k)(z - b^k) = \pi_2(\underline{b}_2)(z - \underline{b}_2) = 0.$$

Consequently, $\lim_{b^k \uparrow \bar{b}_2} G_1(b^k) = 0$. Thus $\underline{b}_1 \geq \overline{b}$. This, by the fact $\underline{b}_1 \leq \overline{b}$, implies that player 1 bids $\overline{b}$ for sure. Then in order for bidder 2 to not deviate to tie or outbid this atom it must be true that $z \leq \overline{b}$, but that would contradict the individual rationality of the types nearby $\underline{t}$ of bidder 1, as $\underline{t} < z$. Thus, $G_1(\underline{b}_2) > 0$.

Case 1: $\underline{b}_2 < \overline{b}$ By the first-price payment rule and necessary conditions for an equilibrium, one readily sees that G_i for each $i \in \{1, 2\}$ has neither gap nor atom in $(\underline{b}_2, \overline{b})$. Thus, for any $b \in (\underline{b}_2, \overline{b})$, $\pi_1(b) = G_2(b)$ and a type-t bidder 1's expected payoff from bidding b is equal to $G_2(b)(t - b)$. Hence almost every bid in $b \in (\underline{b}_2, \overline{b})$ satisfies the first-order condition for a type of bidder 1 that is supposed to bid b at the equilibrium. With F gapless and atomless by assumption, one can prove that there exists a unique type

$$\beta^{-1}(b) := F^{-1}\left(G_1(b)\right) \tag{2}$$

of which bidder 1 bids b. Hence for almost all $b \in \left[\underline{b}_2, \overline{b}\right]$,

$$\frac{d}{db} \ln G_2(b) = \frac{1}{F^{-1}\left(G_1(b)\right) - b}. \tag{3}$$

Note that for all such b,

$$F^{-1}\left(G_1(b)\right) > b, \tag{4}$$

otherwise bidder 1 of type $F^{-1}\left(G_1(b)\right)$, getting zero surplus had he abided by the equilibrium bid b, would deviate to bid $b' \in \left(\underline{b}_2, b\right)$ thereby obtaining a positive expected payoff. Eq. (3) implies that, for some $c \in \mathbb{R}$ and every $b \in \left[\underline{b}_2, \overline{b}\right]$,

$$\ln G_2(b) = c - \int_b^{\overline{b}} \frac{1}{F^{-1}(G_1(b')) - b'} db'.$$

By Ineq. (4), the integral in the above equation is less than ∞ when $b = \underline{b}_2$. Thus, $\ln G_2\left(\underline{b}_2\right) > -\infty$, i.e., $G_2\left(\underline{b}_2\right) > 0$. This, coupled with the fact $G_1\left(\underline{b}_2\right) > 0$ derived previously, implies that the only way to avoid the contradictory tie at $\underline{b}_2$[2] is to have the mass $G_1\left(\underline{b}_2\right)$ consist only of those bids from bidder 1 that are strictly below $\underline{b}_2$. The condition that such types t of bidder 1 would rather bid below $\underline{b}_2$ than deviate to bids greater than or equal to $\underline{b}_2$ implies that $t \leq \underline{b}_2$. Since $G_2\left(\underline{b}_2\right) > 0$, $\underline{b}_2$ is a serious bid for bidder 1, hence his individual rationality implies that all types of player 1 that are below $\underline{b}_2$ bid below $\underline{b}_2$. Thus, $G_1\left(\underline{b}_2\right) = F\left(\underline{b}_2\right)$. Also note, with $\underline{b}_2$ not an atom of G_1, that $\pi_2\left(\underline{b}_2\right) = G_1\left(\underline{b}_2\right)$. Hence $\pi_2\left(\underline{b}_2\right) = F\left(\underline{b}_2\right)$ and bidder 2's expected payoff from bidding $\underline{b}_2$ is equal to $F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right)$. This also being his equilibrium surplus, we have

$$G_1(b) = \frac{F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right)}{z - b} \tag{5}$$

for all $b \in \left[\underline{b}_2, \overline{b}\right]$. Plugging Eq. (5) into Ineq. (4) we have

$$\forall b \in \left(\underline{b}_2, \overline{b}\right) : F(b)(z - b) < F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right). \tag{6}$$

To keep bidder 2 from deviating to bids $b < \underline{b}_2$, we need

$$\forall b < \underline{b}_2 : \pi_2(b)(z - b) \leq F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right).$$

[2] Such a tie is contradictory because $\underline{b}_2 < \overline{b} \leq z$ since $\overline{b}$ is a serious bid for bidder 2 given the fact $G_1\left(\underline{b}_2\right) > 0$, so bidder 2 strictly prefers to win conditional on the tie at $\underline{b}_2$.

This, by Lemma 2, implies

$$\forall b < \underline{b}_2 : G_1(b)(z - b) \leq F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right). \tag{7}$$

This condition does not pin down G_1 uniquely on the bids below $\underline{b}_2$, because such bids, not serious bids for bidder 1, may be above bidder 1's type. Hence $\underline{b}_2$ can be any point in $[\underline{t}, z]$ that satisfies (7).

However, if we add the condition that no bid at equilibrium be weakly dominated (which is satisfied by the equilibrium in the case $0 \leq z < \underline{t}$), then a bidder never bids above his valuation. Thus, by Eq. (2), for any $b \leq \underline{b}_2$ we have $F^{-1}\left(G_1(b)\right) \geq b$, i.e., $G_1(b) \geq F(b)$. Consequently, (7) implies

$$F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right) \geq F(b)(z - b)$$

for all $b \leq \underline{b}_2$. This coupled with Ineq. (6) implies that

$$\underline{b}_2 := \max\left(\arg\max_{b \in [\underline{t}, z]} F(b)(z - b)\right). \tag{8}$$

The equilibrium is then uniquely determined by Eqs. (3), (5), (8) and $G_1(b) = F(b)$ for all $b \leq \underline{b}_2$.

Case 2: $\underline{b}_2 = \overline{b}$ In this case, by Lemma 1, $\overline{t} \leq \underline{b}_2 \leq z$ and $\overline{b}$ is not an atom of G_1. Thus $\underline{b}_1 < \overline{b}$. To keep bidder 2 from deviating to bids below $\underline{b}_2$, we need $\pi_2(b)(z - b) \leq z - \underline{b}_2$ for all $b < \underline{b}_2$, which by Lemma 2 implies

$$\forall b < \underline{b}_2 : G_1(b)(z - b) \leq z - \underline{b}_2.$$

Coupled with the undominated strategy condition, $F \leq G_1$, this implies

$$\forall b < \underline{b}_2 : F(b)(z - b) \leq z - \underline{b}_2.$$

Hence $\underline{b}_2$ satisfies Eq. (8).

Summarized below are the properties of any equilibrium when $z > \underline{t}$.

Corollary 1 *When $z > \underline{t}$, in any BNE:*

a. a positive mass $G_1\left(\underline{b}_2\right)$ of bidder 1's types submit non-serious bids;

b. bidder 2 gets a positive surplus equal to $F\left(\underline{b}_2\right)\left(z - \underline{b}_2\right)$;

c. *if the BNE satisfies the undominated strategy condition, then:*

i. *bidder 2's surplus is equal to* $\max_{b \in [\underline{t}, z]} F(b)(z-b)$*;*

ii. *the allocation is not ex post efficient if* $\underline{t} < z < \overline{t}$*, and is ex post efficient if* $\overline{t} < z$ *and* $F(b)(z-b) \leq z - \overline{t}$ *for all* $b < \overline{t}$*.*

Proof Claim (a) has been established previously; so has Claim (b) in Case 1. Claim (b) in Case 2 follows directly from the fact that player 2's surplus is equal to $z - \underline{b}_2$ and $F(\underline{b}_2) = 1$, with $\underline{b}_2 \geq \overline{t}$ in that case. Claim (c.i) follows directly from Eq. (8), which holds given the undominated strategy condition. To show Claim (c.ii), first suppose $\underline{t} < z < \overline{t}$. Then Eq. (8) implies that $\underline{t} < \underline{b}_2 < z$. Thus player 1 has a positive-measure set $[\underline{b}_2, z]$ of types that bid according to Eq. (5) and win with a positive probability, where their valuations are less than z, player 2's valuation. Hence misallocation occurs with a positive probability if $\underline{t} < z < \overline{t}$. Next suppose that $\overline{t} < z$ and $F(b)(z-b) \leq z - \overline{t}$ for all $b < \overline{t}$. Then Eq. (8) implies that $\underline{b}_2 = \overline{t}$. Hence all types but $\overline{t}$ of player 1 bid below $\underline{b}_2$, hence player 2 wins for sure. With $z \geq \underline{b}_2 \geq \overline{t}$, the allocation is efficient. ■

3.2.2 An Equilibrium

Let $\underline{b}_2$ be defined by Eq. (8). Such $\underline{b}_2$ exists and is well-defined because F, a c.d.f., is upper semicontinuous. Let

$$\overline{b} := z - F(\underline{b}_2)(z - \underline{b}_2). \tag{9}$$

An equilibrium is: If $b < \underline{b}_2$ then $G_2(b) := 0$; if $b \in [\underline{b}_2, \overline{b}]$ then

$$G_2(b) := \exp\left(- \int_b^{\overline{b}} \frac{1}{F^{-1}\left(F(\underline{b}_2)(z - \underline{b}_2)/(z-y)\right) - y} dy \right); \tag{10}$$

$$G_1(b) := \begin{cases} F(b) & \text{if } b \leq \underline{b}_2 \\ F(\underline{b}_2)(z - \underline{b}_2)/(z-b) & \text{if } \underline{b}_2 \leq b \leq \overline{b}. \end{cases} \tag{11}$$

Hence bidder 2 plays a mixed strategy, randomly submitting a bid between $\underline{b}_2$ and $\overline{b}$ according to the distribution G_2; bidder 1 bids his value t when $t \leq \underline{b}_2$ and bids $\beta(t)$ when $t \geq \underline{b}_2$ such that the inverse of β is defined by Eq. (2).

Note that G_2 is a c.d.f. with support $[\underline{b}_2, \overline{b}]$. In particular, for any $b \in [\underline{b}_2, \overline{b})$, $G_2(b) < 1$ because the integrand in Eq. (10) is positive: For all $y \in (\underline{b}_2, \overline{b})$ (hence $y < z$ by Eq. (9)), by (8),

$$F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right) > F(y)(z-y) \Longrightarrow \frac{F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)}{z-y} > F(y) \Longrightarrow F^{-1}\left(\frac{F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)}{z-y}\right) > y.$$

Also note that $G_1(\bar{b}) = 1$ due to Eq. (9).

For bidder 2, given the rival's strategy G_1, any bid $b \in \left[\underline{b}_2, \bar{b}\right]$ yields the same expected payoff: $G_1(b)(z-b) = F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)$; deviating to a bid $b < \underline{b}_2$ is unprofitable because

$$G_1(b)(z-b) = F(b)(z-b) \leq F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right),$$

with the equality due to Eq. (11), and the inequality due to (8). Thus G_2, randomly selecting a bid in $\left[\underline{b}_2, \bar{b}\right]$, is a best response.

For bidder 1 with any type $t \in [\underline{t}, \underline{b}_2]$, given the rival's strategy $G_2(b) = 0$ for all $b < \underline{b}_2$, the expected payoff from submitting any bid below $\underline{b}_2$ is zero, that from any bid above $\underline{b}_2$ is negative (since $t \leq \underline{b}_2$), and that from bidding $\underline{b}_2$ is negative unless $t = \underline{b}_2$, in which case the payoff is zero. Thus, bidding t according to G_1 is a best response.

Now consider the incentive for bidder 1 with any type $t \in \left[\underline{b}_2, \bar{t}\right]$. Since $F(t) \in [F(\underline{b}_2), 1]$, and $F(\underline{b}_2)(z-\underline{b}_2)/(z-y)$ a continuous, strictly increasing function of $y \in \left[\underline{b}_2, \bar{b}\right]$ with range $[F(\underline{b}_2), 1]$, there exists a unique $b \in \left[\underline{b}_2, \bar{b}\right]$ such that

$$t = F^{-1}\left(\frac{F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)}{z-b}\right), \tag{12}$$

which by Eq. (2) means that according to G_1 the type-t bidder 1 is supposed to bid b. For any $b' \in \left[\underline{b}_2, \bar{b}\right]$, denote $U(b', t)$ for the expected payoff for the type-t bidder 1. By Eq. (10),

$$D_1 U(b', t) = G_2(b')\left(\frac{1}{F^{-1}\left(F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)/(z-b')\right) - b'}(t-b') - 1\right). \tag{13}$$

By Eq. (12), $D_1 U(b, t) = 0$. Hence b satisfies the first-order condition. To verify the second-order condition, pick any $b' \in \left[\underline{b}_2, \bar{b}\right]$ such that $t > b' > b$. Then

$$F^{-1}\left(F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)/(z-b')\right) - b' > F^{-1}\left(F\left(\underline{b}_2\right)\left(z-\underline{b}_2\right)/(z-b)\right) - b' \overset{(12)}{=} t - b' > 0$$

and so by Eq. (13) $D_1 U(b', t) \leq 0$; for any $b'' \in \left[\underline{b}_2, \bar{b}\right]$ such that $b'' < b$, by the same token, $D_1 U(b', t) \geq 0$. Thus, no alternative bid in $\left[\underline{b}_2, \bar{b}\right]$ is a profitable deviation from b. Nor is any bid below $\underline{b}_2$ a profitable deviation, as such bids renders zero winning probability for bidder 1, given that $G_2(b) = 0$ for all such bids b. Thus, bidding b is a best response for bidder 1 of type t, as claimed.

3.2.3 The Special Case Solved by Vickrey

Vickrey [10, p18, Section II] considers the special case where bidder 1's type is uniformly distributed on $[0,1]$ and bidder 2's value is a commonly known constant in $(0,1)$. Hence it belongs to Case 1 in this section, with $0 = \underline{t} < z < \overline{t} = 1$ and F the uniform distribution on $[0,1]$. In this case, the equilibriums characterized above specialize to those in Vickrey's Appendix III. In particular, with $F(b) = b$, Eq. (8) implies $\underline{b}_2 = z/2$ at the equilibrium in undominated strategies, which gives bidder 2 a constant surplus equal to $F(\underline{b}_2)(z-\underline{b}_2) = z^2/4$ and implies, by Eqs. (2) and (11), that the type of bidder 1 that bids $b \geq z/2$ is equal to $F^{-1}\left((z^2/4)/(z-b)\right) = z/(4(1-b/z))$, i.e., bidder 1 with type $t \geq z/2$ bids $z(1 - z/(4t))$, which is what Vickrey obtains (p18, Section II).[3]

3.3 Nonexistence When $z = \underline{t}$

Suppose, to the contrary, that there exists a Bayesian Nash equilibrium (G_1, G_2) in this case. With bidder 2's valuation commonly known, his expected payoff at the equilibrium is equal to a constant, denoted by u. There are only two possible cases:

Case 1: $u = 0$. Then bidder 2's expected payoff from bidding $\overline{b}$ is equal to zero, i.e.,

$$\pi_2\left(\overline{b}\right)\left(v - \overline{b}\right) = \pi_2\left(\overline{b}\right)\left(\underline{t} - \overline{b}\right) = 0. \tag{14}$$

We claim that $\pi_2\left(\overline{b}\right) > 0$. Otherwise, Eq. (1) implies $G_1\left(\overline{b}\right) = 0$, contradicting the fact that $\overline{b}$ is the supremum of the support of G_1. Now that $\pi_2\left(\overline{b}\right) > 0$, $u = 0$ implies that any bid $b < \overline{b}$ has zero probability to win for bidder 2 (otherwise bidder 2 in bidding b would have got a positive expected payoff as $\overline{b} \leq z$, contradiction). Hence $\overline{b} \leq \underline{b}_1$. By $\pi_2\left(\overline{b}\right) > 0$, $u = 0$ and Eq. (14), we also have $\overline{b} = z = \underline{t}$. It follows, from $\overline{b} = \underline{t}$ and $\overline{b} \leq \underline{b}_1$, that the type $\underline{t}$ of bidder 1 gets nonpositive expected payoff in the equilibrium. Thus $\underline{b}_2 = \overline{b}$, otherwise the type-$\underline{t}$ bidder 1 would deviate to a bid in $\left(\underline{b}_2, \overline{b}\right)$ thereby obtaining a positive expected payoff (and the same deviation incentive holds for types of player 1 sufficiently nearby $\underline{t}$, by continuity of expected payoff in types). Now that bidder 2 is playing the pure strategy $\overline{b}$ at equilibrium, bidder 1's best response does not exist: With the first-price payment rule,

[3] Vickrey's notations $(v_1, a, x, y_i(x), k, r_2)$ correspond to $(t, z, b, \lim_{b' \uparrow b} G_i(b'), u, \underline{b}_2)$ here. The condition $x^2 - ax + k \leq 0$, based on which Vickrey determined his r_2, corresponds to our Eq. (8).

bidder 1 would not bid above $\overline{b}$; however, with $\overline{b} = \underline{t}$, all types of bidder 1, except type $\underline{t}$, would rather bid slightly above $\overline{b}$: contradiction.

Case 2: $u > 0$**.** Analogous to Eq. (14) we have

$$\pi_2(\underline{b}_2)(z - \underline{b}_2) = \pi_2(\underline{b}_2)(\underline{t} - \underline{b}_2) > 0.$$

Thus, $\underline{b}_2 < \underline{t}$ and $\pi_2(\underline{b}_2) > 0$. With $\pi_2(\underline{b}_2) > 0$, Eq. (1) implies $G_1(\underline{b}_2) > 0$. Thus there is a positive measure of bidder 1's types, denoted by t_*, that are supposed to bid no higher than $\underline{b}_2$. It follows that $\underline{b}_2$ is not an atom of G_2, otherwise the types t_* of bidder 1 would deviate to a bid slightly above $\underline{b}_2$, as their payoff conditional on winning is positive due to $\underline{b}_2 < \underline{t}$. The fact that $\underline{b}_2$ is not an atom of G_2 has two implications: (i) the types t_* of bidder 1, outbid for sure, get zero expected payoff in the equilibrium, and (ii) $\underline{b}_2 < \overline{b}$. By $\underline{b}_2 < \overline{b}$ and the first-price payment rule, neither G_1 nor G_2 has a gap in $[\underline{b}_2, \overline{b}]$. Thus for any $\epsilon > 0$ the interval $(\underline{b}_2, \underline{b}_2 + \epsilon)$ has a positive G_2-measure. Consequently, the types t_* of bidder 1, which are supposed to bid no higher than $\underline{b}_2$ and get zero expected payoff at equilibrium, would deviate to a bid slightly above $\underline{b}_2$ thereby winning with a positive probability and, conditional on winning, obtaining a positive payoff since $\underline{b}_2 < \underline{t}$: again a contradiction.

Remark 1 The nonexistence problem can be avoided by raising the minimum bid to $\underline{t}$ ($= z$ in this case) with the proviso that bids below the minimum bid mean nonparticipation. Then an equilibrium is for bidder 2 to bid below $\underline{t}$ and bidder 1 to win by bidding $\underline{t}$. This fix, however, is unsatisfactory in multistage settings such as the one considered in the next section, where the roles of players 1 and 2, as well as the configuration between z and $\underline{t}$, are endogenous. Hence the auctioneer may be hard pressed to preempt the nonexistence problem via an exogenous minimum bid for an endogenous continuous game.[4]

4 An Application in Collusion

Suppose that a good of common value, commonly known to be equal to one, is pursued by players 1 and 2. For each $i \in \{1, 2\}$, player i's type t_i, privately known to i, is independently

[4] Alternatively, if a bigger departure from an otherwise simple realistic game is tolerated, one could avoid the problem by introducing an endogenous tie-breaking rule such that in case of a tie each player announces independently whether he wants to win conditional on the tie. Then an equilibrium is that both bidders bid $\underline{t}$ and, at the tie, bidder 1 says he wants to win while bidder 2 says otherwise.

drawn from a commonly known distribution F_i, absolutely continuous and strictly increasing on its support $[a_i, z_i]$ with $0 < a_i < z_i$. The good is to be auctioned off via a first-price sealed-bid auction with zero reserve price and equal-probability tie-breaking rule. Prior to the auction, however, the two players can collude through an outside mediator, who proposes to them a *collusive division* in the form of $(v_1, v_2) \in [0,1]^2$ such that $v_1 + v_2 = 1$. If both accept the proposal, both players commit to bidding zero in the auction and whoever wins the good (at zero price) pays the other player the share according to the division, so that player i's payoff in the whole game equals v_i. Otherwise, they play the auction game noncooperatively, with player i independently submitting a bid say b_i, so that the payoff for player i is equal to $1 - b_i/t_i$ if i wins, and zero if otherwise. Both players are assumed risk neutral.

Remark 2 Within the auction game, the common value model here is equivalent to the independent private value (IPV) model: In our model, a type-t_i player i's decision in the auction game, given G_{-i} the distribution of the bids submitted by the other player, is

$$\max_{b \in \mathbb{R}_+} G_{-i}(b) \left(1 - \frac{b}{t_i}\right) = \frac{1}{t_i} \max_{b \in \mathbb{R}_+} G_{-i}(b) \left(t_i - b\right), \tag{15}$$

equivalent to the player's decision in an IPV model with valuation t_i. Embedded in the collusion context, however, the two models have an important difference. In the IPV model, a player's utility from a collusive division depends on his type, hence a collusive division in general is type-dependent. In our model, whereas, a player's type matters only in the off-path event where collusion breaks down and he has to pay a positive price for the good; hence collusive divisions can be mutually acceptable without being type-dependent. Furthermore, one can prove that, even if we allow for more general forms of collusive contracts, there is no loss of generality, within the class of contracts that guarantee mutual acceptance, to restrict attention to type-independent collusive divisions (c.f. Zheng [11, Lemma 2]).

Once a collusive division is proposed by the mediator, a two-stage game is defined, with perfect Bayesian equilibrium (PBE) the solution concept. To this concept we add only the condition that the equilibrium not use weakly dominated strategies. The question is What is the condition on the parameters for there to exist a collusive proposal that admits a PBE, in undominated strategies, where the proposal is accepted by both players almost surely. The answer is stated in the next theorem, which the rest of this section proves.

Theorem 2 *A mutually acceptable collusive division exists if and only if*

$$\sum_{i=1}^{2} \frac{1}{z_i} \left(\max_{\min\{a_{-i}, z_i\} \le b \le z_i} F_{-i}(b)(z_i - b) \right) \le 1. \tag{16}$$

The main steps of the proof parallel those in the companion paper, Zheng [11], which deals with the case where the auction is all-pay instead of first-price. One can prove, from the quasilinearity of the payoff functions, that in any continuation equilibrium of the first-price auction, a player's expected payoff is weakly increasing and continuous in his type. In other words, among all types of player i, z_i is the one most tempted to reject a collusive proposal. Thus, one can prove that a mutually acceptable collusive proposal exists if and only if $\underline{u}_1 + \underline{u}_2 \le 1$, where $\underline{u}_i$ is defined to be the infimum among the equilibrium expected payoffs for the type-z_i player i in the auction game such that $-i$'s type is drawn from the prior distribution i's from a distribution that ranges across all possible posteriors. To determine $\underline{u}_i$ we need to characterize the BNE of the auction game given player $-i$'s posterior being the prior F_{-i} while player i's posterior, denoted by $\tilde{F}_i$, can be any distribution whose support is contained by the prior support $[a_i, z_i]$. Since $\tilde{F}_i$ need not be strictly monotone, we define $\tilde{F}_i^{-1}$ to be the generalized inverse defined in [11]:

$$\tilde{F}_i^{-1}(s) := \inf \left\{ t \in \operatorname{supp} \tilde{F}_i : \tilde{F}_i(t) \ge s \right\}.$$

To characterize a BNE given type-distributions $\left(\tilde{F}_i, F_{-i}\right)$, denote G_i for player i's bid distribution, and G_{-i} player $-i$'s, at the equilibrium. One can prove, through a routine reasoning based on the first-price payment rule, that the set of serious bids is a nonempty, possibly degenerate, interval $\left[\underline{b}, \overline{b}\right]$ such that $\overline{b} = \overline{b}_1 = \overline{b}_2$ and neither G_i nor G_{-i} have gap or atom in $\left(\underline{b}, \overline{b}\right)$. Furthermore, $\overline{b}$ is not an atom of G_{-i} unless $\overline{b} = \sup \operatorname{supp} \tilde{F}_i$, and the analogous statement on G_i also holds. One also readily sees that the equilibrium strategy that generates G_j ($\forall j \in \{i, -i\}$) is weakly increasing in the sense that the infimum of the bid support of a higher type is greater than or equal to the supremum of the bid support of a lower type. With F_{-i} continuous and strictly increasing by assumption, $F_{-i}^{-1}(G_{-i}(b))$ is the unique type of player $-i$ that bids b at the equilibrium. Mimicking the proof in Zheng [11], one can show that $\tilde{F}_i^{-1}(G_{-i}(b))$ is a type of player i that bids b at the equilibrium. Then we obtain the first-order necessary condition for the equilibrium (c.f. Eq. (15)): for all $b \in \left(\underline{b}, \overline{b}\right)$,

$$\frac{d}{db}\ln G_i(b) = \frac{1}{F_{-i}^{-1}\left(G_{-i}(b)\right) - b}, \tag{17}$$

$$\frac{d}{db}\ln G_{-i}(b) = \frac{1}{\tilde{F}_i^{-1}\left(G_i(b)\right) - b}. \tag{18}$$

Based on Eqs. (17) and (18), we claim that, for each $i \in \{1, 2\}$,

$$\underline{u}_i = \frac{1}{z_i}\left(\max_{\min\{a_{-i}, z_i\} \leq b \leq z_i} F_{-i}(b)(z_i - b)\right). \tag{19}$$

Case 1: $z_i < a_{-i}$ With $z_i < a_{-i}$. Eq. (19) is equivalent to $\underline{u}_i = 0$. By player i's individual rationality in the auction game, $\underline{u}_i \geq 0$. Thus, it suffices in this case to find a continuation equilibrium of the auction game that gives the type z_i player i zero expected payoff. Such an equilibrium is the one where the posterior about player i is that i's type is equal to z_i. This is the equilibrium characterized in Section 3.1.2, where the notations $(z, \underline{t})$ correspond to (z_i, a_{-i}) here. As shown there, the equilibrium surplus for player i is zero.

Case 2: $z_i = a_{-i}$ In this case, Eq. (19) again becomes $\underline{u}_i = 0$, and again it suffices to show $\underline{u}_i \leq 0$. To this end, instead of showing that when the posterior about i is "$t_i = z_i$" gives player i zero surplus at equilibrium, which we know does not exist by Section 3.3, we show that for any $\epsilon > 0$ there is a continuation equilibrium of the auction game that gives the type z_i player i a surplus less than ϵ. Hence pick any $\epsilon > 0$ and consider the auction game where the posterior about player i is that i's type is equal to $z_i - \epsilon$. At equilibrium of this continuation game, as characterized in Section 3.1.2, player $-i$ bids $z_i - \epsilon$ for sure. Thus, the expected payoff for player i of type z_i is less than ϵ, as claimed.

Case 3: $z_i > a_{-i}$ With $z_i > a_{-i}$, by Section 3.2.2, the continuation equilibrium of the auction game where the posterior about i is "$t_i = z_i$" gives player i of type z_i the expected payoff $\underline{u}_i$. We need only to show that no other continuation equilibrium can give the type z_i player i a lower expected payoff. To that end, first note that the aforementioned equilibrium is the only one admitted by the posterior "$t_i = z_i$", as other ones are eliminated by the condition of undominated strategies. Thus, denote the former equilibrium (with the degenerate posterior "$t_i = z_i$") by (G_1^*, G_2^*), with $\overline{b}_i^* = \sup \operatorname{supp} G_i^*$, and any other equilibrium (with a different posterior $\tilde{F}_i$ about i) by (G_1, G_2), with $\overline{b}_i = \sup \operatorname{supp} G_i$. As shown in Section 3.2.2,

$\overline{b}_i^*$ is equal to the supremum $\overline{b}^*$ of serious bids, and likewise $\overline{b}_i = \overline{b}$. Since $z_i > a_{-i}$, Eq. (19) implies that $\underline{u}_i > 0$. Consequently, $F_{-i}(\overline{b}_i^*)(z_i - \overline{b}_i^*) > 0$. Thus, $\overline{b}^*$, which equals $\overline{b}_i^*$, is not an atom of G^*_{-i}; otherwise player i would deviate to bid slightly above $\overline{b}^*$. Since we consider only the possibility where (G_1, G_2) renders higher surplus for player i, by the same token we see that $\overline{b}$ is not an atom of G_{-i}. Thus, the type-z_i player i's expected payoff is equal to $1 - \overline{b}^*/z_i$ at the equilibrium (G_1^*, G_2^*), and equal to $1 - \overline{b}/z_i$ at the equilibrium (G_1, G_2). Hence it suffices to show that $\overline{b}^* \geq \overline{b}$. To show that we mimic the proof of Zheng [11, Lemma 6]. The crucial step is to observe that at the equilibrium (G_1^*, G_2^*), due to its degenerate posterior "$t_i = z_i$", Eq. (18) becomes, for all $b \in \left(\underline{b}, \overline{b}\right)$,

$$\frac{d}{db} \ln G^*_{-i}(b) = \frac{1}{z_i - b}.$$

Thus, with z_i the supremum of the prior support of i's type,

$$\frac{d}{db} \ln G_{-i}(b) > \frac{d}{db} \ln G^*_{-i}(b),$$

i.e., as b decreases, $G_{-i}(b)$ decreases faster than $G^*_{-i}(b)$ does. Hence, unless $\overline{b}^* \geq \overline{b}$, we have $G_{-i} < G^*_{-i}$. If $G_{-i} < G^*_{-i}$ then for any serious bid b player $-i$'s type $F_{-i}^{-1}(G_{-i}(b))$ that bids b at equilibrium (G_1, G_2) is smaller than his type $F_{-i}^{-1}(G^*_{-i}(b))$ at equilibrium (G_1^*, G_2^*). Consequently, Eq. (17) implies, for all $b \in \left(\underline{b}, \overline{b}\right)$,

$$\frac{d}{db} \ln G_i(b) > \frac{d}{db} \ln G_i^*(b),$$

i.e., as b increases, G_i rises at a faster rate than G_i^* does. This, coupled with the fact that $G_i^*(0) = 0$ (otherwise i's surplus at equilibrium (G_1^*, G_2^*) would not have been $\underline{u}_i > 0$), implies that $\overline{b}^* \geq \overline{b}$, as desired.

Now that Eq. (19) has been established, a necessary and sufficient condition for existence of a mutually acceptable collusive division is simply Ineq. (16). The necessity of the condition follows trivially from the continuity of a player's expected payoff in the auction game with respect to his type. The sufficiency of the condition also trivially follows from the above analysis if, for each $i \in \{1, 2\}$, the calculation of $\underline{u}_i$ belongs to either Case 1 or Case 3, as in either case there exists a continuation equilibrium, in the event where player i vetoes a collusive proposal, that delivers exactly the surplus $\underline{u}_i$ to player i. Thus, we need only to consider the complementary case, where the calculation of $\underline{u}_i$, for some $i \in \{1, 2\}$, belongs to Case 2. In that case, $\underline{u}_i = 0$ and $z_i = a_{-i}$, which implies $z_{-i} > a_i$. Hence the calculation of $\underline{u}_{-i}$ belongs to Case 3, and according to Eq. (19), with the roles of i and $-i$ switched, we have, since $F_i(a_i) = 0$ by assumption, $\underline{u}_{-i} < \frac{1}{z_{-i}} z_{-i} = 1$. Thus, there exists $\epsilon > 0$ such that $\underline{u}_{-i} < 1 - \epsilon$. Consequently, the collusive division of giving ϵ to player i and $1-\epsilon$ to player $-i$ is mutually acceptable. In the event that player $-i$ vetoes the proposal, the continuation equilibrium is the one where the posterior about $-i$ is "$t_{-i} = z_{-i}$", which gives player $-i$ only $\underline{u}_{-i}$, less than his share in the collusive division. If player i vetoes the proposal, the continuation equilibrium is the one where the posterior about i is "$t_i = z_i - \epsilon$, which according to Case 2 gives player i an expected payoff at most ϵ, no more than his share in the collusive division. Thus, the proposal is mutually acceptable, as claimed.

Corollary 2 *For any $i \in \{1, 2\}$ let F_i^* be any c.d.f. that first-order stochastically dominates F_i, with the same support $[a_i, z_i]$. Then:*

i. if Ineq. (16) is satisfied given the priors (F_1, F_2) then it is also satisfied when the priors are replaced by (F_1^*, F_2^*);

ii. if Ineq. (16) is violated given the priors (F_1^*, F_2^*) then it is also violated when the priors are replaced by (F_1, F_2);

Proof By Ineq. (16) and the hypothesis that F_i^* has the same support as F_i for each i, it suffices to prove that, for each $i \in \{1, 2\}$,

$$\max_{\min\{a_{-i}, z_i\} \le b \le z_i} F_{-i}(b)(z_i - b) \ge \max_{\min\{a_{-i}, z_i\} \le b \le z_i} F_{-i}^*(b)(z_i - b). \tag{20}$$

To that end, define for each $\lambda \in [0, 1]$ and any $t \in \mathbb{R}$

$$\hat{F}_{-i}(t, \lambda) := \lambda F_{-i}^*(t) + (1 - \lambda) F_{-i}(t).$$

Note that $\hat{F}_{-i}(\cdot, 0) = F_{-i}$, $\hat{F}_{-i}(\cdot, 1) = F_{-i}^*$ and $\hat{F}_{-i}(\cdot, \lambda)$ is a c.d.f. with support $[a_{-i}, z_{-i}]$ for each $\lambda \in [0, 1]$. Furthermore, for each $\lambda \in [0, 1]$ the problem

$$V(\lambda) := \max_{\min\{a_{-i}, z_i\} \le b \le z_i} \hat{F}_{-i}(b, \lambda)(z_i - b)$$

admits a solution, denoted by $b^*(\lambda)$, because $\hat{F}_{-i}(\cdot, \lambda)$ as a c.d.f. is upper semicontinuous, and the domain compact. By definition of $\hat{F}$, for any $t \in \mathbb{R}$,

$$\frac{\partial}{\partial \lambda} \hat{F}_{-i}(t, \lambda) = F_{-i}^*(t) - F_{-i}(t) \le 0,$$

with the inequality due to the hypothesis that F^*_{-i} stochastically dominates F_{-i}. By the envelope theorem,

$$V(1) - V(0) = \int_0^1 \left(\frac{\partial}{\partial \lambda} \hat{F}_{-i}(t, \lambda) \bigg|_{t=b^*(\lambda)} \right) d\lambda \leq 0,$$

which is Ineq. (20), as desired. ■

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