Multilateral Deescalation in the Dollar Auction

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Abstract

We characterize the subgame perfect equilibriums of the dollar auction in its original form, without the constructs that the literature has added to avoid bid escalation. Contrary to Shubik's (1971) conjecture, no equilibrium can generate higher expected revenues than the value of the prize. There is a continuum of equilibriums supported by subgames where competition between only two bidders escalates to complete dissipation of their surplus. These equilibriums are Pareto dominated, in a dynamic sense, by equilibriums that always give rise to trilateral rivalry, with the lowest bidder leapfrogging the top runners, and all three retaining some surplus. There are only finitely many such trilateral-rivalry equilibriums, each corresponding to the lowest bidder's farthest lag from the current price before he quits the catchup efforts. Exactly one of such equilibrium-feasible farthest lag is undominated dynamically.

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1 Introduction

Hailed as "a paradigm for escalation" (Shubik [16]), and a favored classroom experiment (Teger [18], Haupert [4], Murnighan [12], etc.), the dollar auction is a simple setup, compelling in intuition but seemingly paradoxical in theory, to demonstrate how conflict may arise and escalate over time. Although basic game-theoretic considerations of the original game—a dollar bill being auctioned to the highest bidder through ascending bids in five-cent increments, with every bidder, winning or not, having to pay his own highest bid—would not predict expected revenues above a dollar, empirical and anecdotal evidence, consistent with Shubik's conjecture, suggests that the bidding competition does tend to escalate beyond individual rationality, profiting the auctioneer handsomely, with social psychology and behavioral dynamics such as the "sunk cost fallacy" suggested as driving forces.

At the source of the disconnect are two dynamic issues.[1] First, bidders face a time-inconsistency problem. Even if a bidder has decided at the outset a price or time at which he will drop out of the race, when that time arrives the bidder may change his mind, as the efforts he has invested previously have been sunk and he can still up his winning status with another small bid-increment just large enough to top the current frontrunner. Second, different from the usual clock model of ascending auctions, where not raising one's bid means irrevocable dropout, the dollar auction with more than two players, like most real world auctions and arms-race-like struggles, allows an underdog to leapfrog and catch up with the frontrunner. The two issues together render theoretical analyses of the original dollar auction, despite its simple rules, nontrivial and, to our knowledge, not yet available in the literature.[2] To fill the gap, this paper provides an analytical solution that captures the dynamic issues of the original game and by doing so partially reconcile game-theoretic considerations with the observed phenomena of conflict escalation.

While the dollar auction has been largely motivated by military and politico-economic conflict settings such as arms- and R&D-races of the cold war[3] without explicit seller or

[1] Setting aside issues such as participants (students) misunderstanding or auctioneer (instructor) misleading/vaguely explaining the auction rules, lack of credible payment enforcement, etc.

[2] "There is no neat game theoretic solution to apply to the dynamics of the Dollar Auction," prophesies Shubik [16, p111].

[3] For example, the "Concorde fallacy" where the United States, United Kingdoms, France and the former Soviet Union raced to develop supersonic airplanes.

mechanism designer benefiting from the sunk bids, the recent Web-2.0 development and commercialization of internet-based technologies have renewed the relevance of the dollar auction. Exemplary applications include online crowdsourcing and innovation challenges, where the fundraiser or challenger benefits from all the efforts invested by all participants, and participants may outdo one another in a dynamic process. For instance, the 2006 Netflix Prize offered one million dollars to anyone with a movie recommendation algorithm that would beat Netflix's Cinematch algorithm by at least 10% and outperform those from other contenders.[4] Two main features of these innovation-type challenges that auspiciously mimic the dollar auction include: (1) firms retain exclusive rights to all submissions thereby becoming the beneficiary of all contenders' sunk efforts; and (2) submissions and their associated performances ("bids") are dynamically and openly updated so that contenders can up their efforts and outperform the frontrunner.

Strangely enough, given the relevance and popularity of the dollar auction, the literature has not delivered an equilibrium analysis of the game in its original form. Mainly based on its time-inconsistency aspect, Shubik [16] in introducing the dollar auction conjectures that its subgame perfect equilibriums result in bid escalation generating higher revenues than the commonly known value of the prize. Subsequent studies avoid such conjectured escalation by changing the model of the game. O'Neill [14] and later Leininger [10] and Demange [3] impose a common budget constraint on all bidders so that bidding escalation is eliminated by backward induction. Leininger also considers another variant that removes the minimum requirement of bid increments so that price ascension can slow down to a halt even though bidders still keep topping each other. With the assumed budget constraint doing away with bidding escalation, Demange [3] introduces asymmetric information to explain why rivalry may arise and get escalated to a moderate degree.

Another favoured approach in the literature is to regard the dollar auction as a war of attrition. Standard treatments of wars of attrition, such as Krishna and Morgan [9], calculate a Nash equilibrium where each player decides on a stopping time at the outset. More specialized treatments include: dynamic multiplayer stopping games where players are just choosing whether to continue or irrevocably quit (Kapur [8]); removal of restrictions

[4] Details available at: http://www.netflixprize.com. In May 2017 the online real estate valuation and brokerage firm Zillow initiated a very similar challenge (focusing on home sales price predictions): https://www.kaggle.com/c/zillow-prize-1; accessed 2017-05-24.

that price ascends in exogenous increments, thereby allowing a player to start the bidding process with a sufficiently high jump bid that renders any rival's entry unprofitable (Hörner and Sahuguet [6]); or the dynamic war of attrition versions as motivated by biology settings where conflict escalation is avoided through a correlated equilibrium (Smith [17]). However, a war of attrition type of analysis tends to capture only one, but not both, of the two important dynamic issues of the dollar auction: the time-inconsistency problem for a bidder, and an underdog's possibility to leapfrog the frontrunner by bidding one increment above the current highest bid. Furthermore, much of the war of attrition literature tends to focus on two-player games. While interesting in its own right, two-player dynamics and bidding incentives do not immediately generalize to multi-player games. For one, with only two players, one's bid necessarily next to the other's by just one increment, there is no possibility of leapfrogs or catchups from a third rival. Second, two-player games cannot capture the free-rider issue among lower bidders, as the game ends once the single lower bidder decides not to counter-bid or escalate the conflict. With more than two players, by contrast, there is an incentive for a player to take an observational role for a while and let the escalation continue in the hope that the dueling opponents exhaust their resources.

By contrast, this paper characterizes the equilibriums of the dollar auction in its original form, with complete information, exogenous increments for price ascension and no budget constraint. The resulting equilibriums exhibit surprising patterns. Some equilibriums give rise to only bilateral rivalry, which may escalate to a level that dissipates the players' surplus completely, or deescalate to a level where a fortunate player receives the good essentially at marginal cost of bidding. The other equilibriums, in a three-player game that this paper focuses mainly on, give rise to trilateral rivalry, which escalates over time, and the lowest bidder tries to leapfrog to the top unless the lag between him and the frontrunner has reached a threshold, in which case the bidding escalation ends.

Contrary to Shubik's conjecture, none of the equilibriums can generate an expected revenue above the worth of the prize. In particular, the equilibrium that maximizes the bidders' surplus is a no-conflict equilibrium, where competition stops immediately after someone becomes the frontrunner by bidding the minimum increment; none others would top him by bidding two increments, expecting that should she enter the competition the subgame equilibrium would become a surplus-dissipating bilateral rivalry.[5] But then why is bidding

[5] Morone, Nuzzo and Caferra [11] have noted the on-path action of this no-conflict equilibrium as a Nash

escalation so often observed in the dollar auction? The answer is that the bidders face a tremendous coordination problem, as there is a continuum of bilateral-rivalry equilibriums, each corresponding to a probability with which a follower bids after the frontrunner has bid the initial, say, five cents. This probability can be as small as zero, corresponding to the no-conflict equilibrium, but can also be large enough to dissipate the surplus entirely. Furthermore, should someone deviate from any such bilateral-rivalry equilibrium, its punishing subgame play is Pareto dominated by another equilibrium where trilateral rivalry arises and escalates, and if the players switch to the latter equilibrium conditional on the deviation, the deviator would find the deviation profitable when he is just considering it.

Equilibriums that give rise to trilateral rivalry are compelling to consider not only for the above reason but also because real world conflicts often involve more than two parties. Thus much of the analysis of this paper is devoted to characterizing the set of trilateral-rivalry equilibriums subject to three conditions for the equilibrium strategy to be name-independent and Markov perfect. We find that there are only finitely many of such equilibriums, each corresponding to an *even* number s_*, called *dropout state*, such that the underdog (lowest bidder) tries to catch up if his lag to the frontrunner is shorter than s_* times the price increment; else he bids no longer and the rivalry collapses to a bilateral one, between the top two players, which also stops immediately. Among all equilibriums, bilateral and trilateral rivalries included, we find that exactly one equilibrium-feasible dropout state is, in a dynamic sense, not Pareto dominated by any other equilibrium-feasible ones. Specifically, the dynamically undominated equilibriums (if plural then there are at most two of such) are characterized as the ones with the highest dropout state, i.e., the trilateral rivalry that continues with the largest gap between the frontrunner and the underdog.

In finding such trilateral-rivalry equilibriums with catchups and leapfrogs, this paper contributes to auction theory an explicit analysis of an ascending auction outside the mainstream clock model, so that reentry and leapfrogging are captured. The time-inconsistency problem is incorporated by subgame perfection. Our findings about the emergence of multilateral rivalry is relevant to real-world situations where the circle of conflict includes more than two adversaries, such as the aforementioned Netflix Prize and Concorde fallacy (Footnotes 3 and 4), or may expand over time, such as the two world wars.

Empirical investigations of the "dollar auction paradox" include experiment and be-

equilibrium, though vague about whether it can also constitute a subgame perfect equilibrium.

havioral observations (Teger [18], Haupert [4], Murnighan [12], and Morone, Nuzzo and Caferra [11]), where bid escalation is attributed to psychological factors (e.g., "spiteful bidding" in Waniek, Nieścieruk, Michalak and Rahwan [21]). Our characterization of the set of equilibriums provides a theoretical basis for sharper comparisons between the theory and the lab observations. The multiplicity of equilibriums found here, especially the continuum of bilateral-rivalry ones, suggests that the observed escalation might be symptoms of just coordination failure among the bidders, consistent to the rational choice paradigm rather than requiring behavior or psychological explanations.

The dollar auction is also relevant to the recent proliferation of online pay-to-bid or penny auctions; studied by Augenblick [1], Hinnosaar [5], Kakhbod [7], Ødegaard and Anderson [13], Platt, Price and Tappen [15] and Wang and Xu [20]; see also Thaler [19]. The general online penny auction format is as follows: bidders pay a fixed amount to bid and nominally raise the price (e.g., bidders pay 75 cents to raise the price by one cent); the bidder who places the last bid wins the item and in addition to any sunk bidding cost has to pay the final price of the good.[6] Like the dollar auction, bidders in a penny auction incur a sunk cost for each bid increment they submit. The difference, however, is that the sunk cost in the dollar auction is counted as part of a bidder's eventual payment but not so in a penny auction, where the sunk bidding cost is merely a fee and a winner still needs to pay the entire price for the good. Hence the time-inconsistency problem and bidding incentives are different in penny auctions.

After defining the game in Section 2 we shall start with an intuitive presentation of the bilateral-rivalry equilibriums in Section 3, also motivating the trilateral-rivalry equilibriums. Formal characterization of the latter equilibriums, the main result of the paper, is presented in Section 4 based on three symmetric and recursive conditions that allow us to exploit the recursive structure of the game. Section 5 introduces a mild condition of Pareto perfection and shows that exactly one of the equilibriums characterized here satisfies the condition. Proofs are in the appendix in order of the appearance of the corresponding claims.

[6] The underlying premise is that the final price will be an order of magnitude less than the value of the good, while the total sunk bidding cost, from all bidders collectively, far exceeds the value.

2 The Dollar Auction

There is one indivisible good and n risk-neutral players, with $n \geq 2$. The value of the good, commonly known, is equal to v for every player. The good is to be auctioned off via an ascending-bid procedure with bid increment fixed at a positive constant δ such that $2\delta < v$. In the initial round, all players simultaneously choose whether to bid or stay put; if all stay put then the game ends with the good not sold, else one among those who bid is chosen randomly, with equal probability, to be the *frontrunner*, whose committed payment becomes δ, with everyone else's committed payment being zero, and the current price of the good becomes δ. Suppose that the game continues to any subsequent round, with p being the current price and b_i player i's committed payment ($b_i \leq p$ and strictly so unless i is the frontrunner), all players but the frontrunner simultaneously choose whether to bid or stay put. If all stay put then the game ends, the good is sold to the frontrunner, who pays the price p, and every other player i pays b_i; else the current price becomes $p + \delta$ and one among those who bid in this round is chosen randomly, with equal probability, to be the frontrunner, whose committed payment becomes $p + \delta$, with the committed payments of others unchanged. Then the game continues to the next round. If the game never ends, then each bidder pays the supremum of his committed payment, and the good is randomly assigned to one of those whose supremum committed payments equal infinity.

The payment rule in our model stipulates that every bidder pays his highest bid. A variant would be to have only the first and second highest bidders pay. Both versions have been used in actual dollar auctions such as Haupert [4], using the former, and Murnighan [12], the latter. We opt for the former because in multilateral rivalries such as arms race, political lobbying and R&D race, not only the top two runners have to pay for their investments. Further extensions are discussed in the Conclusion.

3 Coordination Failure of Bilateral Rivalry

3.1 Surplus-Dissipating Subgame of Two Bidders

We start with an observation that, within any subgame where the price p has risen to at least 2δ, with the top bidder, the frontrunner, having committed a payment p, the second-place bidder, the *follower*, having committed $p-\delta$, and all others having committed at most

$p - 2\delta$, there is a subgame perfect equilibrium where the two rivals outbid each other in alternate rounds with a probability

$$y := 1 - 2\delta/v, \tag{1}$$

and all others choosing to stay put. This subgame perfect equilibrium results in an expected surplus of zero for every player and hence is called *zero-surplus subgame equilibrium*.

To explain this subgame equilibrium, for each round denote α for the current frontrunner, and β the current follower. The strategy profile prescribes actions that depend only on a player's current position rather than his identity: β bids with probability y and every other player $i \notin \{\alpha, \beta\}$ chooses to not bid at all (α cannot bid by the rule of the game). In the event that player β ends up bidding, the current price p is incremented by δ, players α and β switch roles, and the strategy profile repeats itself with the roles exchanged. In the off-path event that any other player $i \notin \{\alpha, \beta\}$ bids and becomes the new α, the player who was the α in the previous round, now the new β, bids with probability y as if it were the on-path event where he was topped by the previous β; whereas the previous β, now 2δ below the current price, chooses to not bid at all, leaving the previous α and the deviating player competing against each other in alternate roles of α and β. Any further off-path event caused by such a unilateral deviation is responded likewise.

We verify this equilibrium in three steps. First, denote V_* for a bidder's continuation value of being the current α player, and M_* that of being the current β. Given the expectation that only the current β player bids at all,

$$V_* = (1-y)v + yM_* \overset{(1)}{=} \left(1 - 1 + \frac{2\delta}{v}\right) v + y \cdot M_* = 2\delta + y \cdot M_*. \tag{2}$$

In bidding and becoming the next round α, the current β increases his committed payment by 2δ, hence $M_* = y(-2\delta + V_*)$. This coupled with (2) implies $M_* = y\left(-2\delta + 2\delta + y \cdot M_*\right) = y^2 M_*$, hence $M_* = 0$, and so (2) implies $V_* = 2\delta$.

Second, it is a best response for the current β player to bid with probability y, and a best response for every other player $i \notin \{\alpha, \beta\}$ to not bid at all: For β, if he bids then he becomes the new α and bears a sunk cost 2δ, hence his expected payoff from bidding is equal to $V_* - 2\delta = 0$; if he does not bid then his payoff is zero as the current α wins. Hence β is indifferent, so bidding with probability y is a best response. For any $i \notin \{\alpha, \beta\}$, with committed payment $b_i \leq p - 2\delta$, the cost $p + \delta - b_i$ that i needs to incur to assume the role of α is larger than 2δ, hence the best response is not to bid at all.

Third, consider an off-path event where a player i other than the α and β in the previous round bids and gets selected to be the current α. In this event, the price committed by the β in the previous round remains to be $p - 2\delta$, with p the current price committed by the new α, and the price committed by the α in the previous round is equal to $p - \delta$. This previous α becoming the current β, the reasoning in the previous paragraph applies and hence he finds it a best response to act as the current β according to the strategy proposed above for this event. The reasoning in the paragraph regarding $i \notin \{\alpha, \beta\}$ now applies to the β in the previous round, as his committed price is 2δ below the current price. Hence his best response is to not bid at all, as in the proposed equilibrium.

Although conflict is escalated to the complete dissipation of surplus in the subgame equilibrium verified above, it does not render more expected revenue than the prize's worth v, contrary to the paradox conjectured by Shubik [16].

3.2 Continuum of Bilateral Equilibriums

With the zero-surplus subgame equilibrium acting as a penal code to deter conflict escalation, we observe that there is a continuum of subgame perfect equilibriums, each indexed by an $x \in \left[0, 1 - (\delta/v)^{1/(n-1)}\right]$, such that the equilibrium expected revenue R can be as small as δ, when $x = 0$, or as large as v, when $x = 1 - (\delta/v)^{1/(n-1)}$. At any such an equilibrium, every player bids for sure in the initial round, so one of them is selected the frontrunner, incurring a sunk cost δ and raising the price to δ. In the second round, every player other than the frontrunner bids with probability equal to x; in the event that some of them ends up bidding, the new frontrunner selected thereof commits a sunk cost 2δ, with the previous frontrunner becoming the follower; hence we enter the subgame described in the previous subsection. From this point on the zero-surplus subgame equilibrium is played, where everyone else, except the frontrunner and follower, stays put while competition between the two escalates with a probability, with the follower topping the frontrunner with probability $1 - 2\delta/v$ in any round. Thus, in the second round, anyone other than the frontrunner finds it a best response to bid with only probability x, anticipating the zero-surplus subgame equilibrium should he outbid the frontrunner. In the initial round, where the current price equals zero and no frontrunner has emerged, if player $i \in \{1, \ldots, n\}$ bids and gets selected the frontrunner, his expected payoff is equal to

$$-\delta + (1-x)^{n-1}v + \left(1-(1-x)^{n-1}\right)M_* = -\delta + (1-x)^{n-1}v,$$

which is nonnegative because $x \leq 1-(\delta/v)^{1/(n-1)}$. Hence bidding is a best response for everyone at the initial round. The equilibrium thus verified generates an expected revenue

$$R = \delta + \left(1-(1-x)^{n-1}\right)v,$$

which, depending on x, ranges from δ to v.

3.3 Emergence of Trilateral Rivalry

The above family of equilibriums with endogenously *bilateral rivalry* has two immediate implications. First, different from Demange [3] and Hörner and Sahuguet [6], where the difference between surplus dissipation and retention relies on the introduction of asymmetric information or jump bids, here the degree to which the bidders may retain the surplus depends purely on which equilibrium they happen to play. Second, with the multiplicity of a continuum, such equilibriums present a severe coordination problem to the contestants in trying to retain surplus via playing one in the continuum.

Another problem of the bilateral-rivalry equilibriums observed above is that they can be eliminated by a weak requirement of Pareto perfection. To illustrate this, suppose that one of such equilibriums with $x > 0$ is being played and the game has continued to the third round, where the price has risen to 2δ and a bilateral rivalry has emerged between two players, the frontrunner having committed 2δ and the follower having committed δ, with all other players supposed to stay put from now on. Suppose to the contrary one of these other players decides to deviate by bidding. If he gets selected the new frontrunner, the deviator commits 3δ and a trilateral rivalry is formed between him and the previous frontrunner and follower. Furthermore, the rivalry is manifested by a *consecutive configuration* where the distance of committed payments between the current frontrunner and the follower, and that between the follower and the third-place bidder, are each equal to δ. Should the players stick to the status quo, the bilateral-rivalry equilibrium, such a deviation would be unprofitable. However, the deviation could be taken as a call for switching to another equilibrium which, as we will demonstrate later, makes all three rivals better-off and none others worse-off; moreover, if the other players follow suit and switch to the new equilibrium conditional on the deviation, the deviator strictly prefers the deviation from the standpoint where he considers it. Hence the bilateral-rivalry equilibrium is not Pareto perfect: on its path there is a point where it is Pareto dominated by another equilibrium that gives rise to *trilateral rivalry*, and the "renegotiation" can be done tacitly through merely a unilateral deviation.

We will formalize the above claim in subsequent sections. Here let us illustrate it with a numerical example. It should be intuitive that bidders' expected payoff depends on the relative values of v and δ, and so for this example we focus on the case when $v/\delta \geq 35/2$ and a trilateral-rivalry equilibrium generating 4δ in expected payoff for any frontrunner in the consecutive configuration described above. At this configuration, all other players who have not bid choose to stay put (as being the next frontrunner yields at most 4δ and requires committing to pay at least 4δ), while both the follower and the third-place bidder bid for sure. If the third-place bidder gets selected the new frontrunner then the consecutive configuration repeats itself, otherwise the *gap* between the current price and the third-place bidder's committed payment widens by δ. In any subsequent round the current follower and the third-place player bid for sure unless the aforementioned gap widens to 3δ. In that event, the follower stays put and the third-place player bids with a probability, strictly between zero and one, which is pinned down by the condition that the frontrunner in the consecutive configuration has surplus 4δ; if the third-place player ends up not bidding then the current frontrunner wins, else the third-place bidder becomes the frontrunner and the trilateral rivalry is back to its consecutive configuration.

The equilibrium in this example generates a surplus of 4δ for the frontrunner, more than δ for the follower, and $\delta/2$ for the third-place player, when they are in the consecutive configuration. Whereas, at this point, any bilateral-rivalry equilibrium gives only 2δ to the frontrunner and zero to the other two. Hence it is Pareto superior for them to switch to the trilateral equilibrium conditional on the deviation. Furthermore, the switch induces a profit $4\delta - 3\delta = \delta$ for the unilateral deviator from the viewpoint of the previous round.

4 The Integral Spectrum of Equilibriums

The previous observation leads to an investigation of trilateral-rivalry equilibriums. Do they also present the bidders a similar coordination problem? To answer this question we need to characterize the set of equilibriums. To exploit the recursive structure of the game we restrict the equilibrium concept by three name-independent, Markov perfect conditions. The structure of such equilibriums turns out to be remarkably clean; there are only finitely many of them, each corresponding to an even number.

4.1 The State of the Game and the Equilibrium Concept

The *state* of the game, in any round, consists of the vector $(b_i)_{i=1}^n$ of the payments committed by the players so far, with $\max_{i=1,\ldots,n} b_i$ being the current price, and $\arg\max_{i=1,\ldots,n} b_i$ (singleton by the rule of the game) the current frontrunner. For any subgame perfect equilibrium $\mathscr{E}$ of the game and any state $(b_i)_{i=1}^n$, denote $\mathscr{E}|(b_i)_{i=1}^n$ for the continuation play of $\mathscr{E}$ in any subgame that starts with the state $(b_i)_{i=1}^n$. By *equilibrium* we mean any subgame perfect equilibrium $\mathscr{E}$ of the game that satisfies three conditions:

Symmetry For any two states $(b_i)_{i=1}^n$ and $(b'_i)_{i=1}^n$ such that $b_i = b'_{\psi(i)}$ for all $i \in \{1, \ldots, n\}$ for some permutation ψ on $\{1, \ldots, n\}$, $\mathscr{E}|(b_i)_{i=1}^n$ is isomorphic to $\mathscr{E}|(b'_i)_{i=1}^n$ given the permutation ψ.

Recursion $\mathscr{E}|(b_i)_{i=1}^n$ is equal to $\mathscr{E}|(b'_i)_{i=1}^n$ such that $b'_i = b_i - \min_{j=1,\ldots,n} b_j$ for all $i \in \{1, \ldots, n\}$.

Independence of irrelevant players For any state $(b_i)_{i=1}^n$, if, for some $k \in \{1, \ldots, n\}$ and constant c, at every state $(b'_i)_{i=1}^n$ generated on the path of $\mathscr{E}|(b_i)_{i=1}^n$ we have $b'_k = c < \max_i b'_i$, then $\mathscr{E}|(b_i)_{i=1}^n$ satisfies the previous two conditions such that $\{1, \ldots, n\}$ is replaced by $\{1, \ldots, n\} \setminus \{k\}$.

The symmetry condition requires that the strategy profile in an equilibrium be independent of players' names. The recursion condition says that a player's equilibrium strategy depends not on the amount of payments he has committed so far but rather on the distances between his and others' committed payments, i.e., past bids constitute a sunk cost. The independence condition of irrelevant players says that, if a player k drops out of the race for good according to an equilibrium, then the equilibrium strategy conditional on this subgame should not vary with the position of this player from this point on.

For tractability we specialize to the case where $n = 3$. Consequently, by the symmetry and recursion conditions, we need only to identify the three players by the relative positions of their committed payments, hence denote α for the frontrunner, whose committed payment is the current price p ($b_\alpha = p$), β the follower, whose committed payment is always just δ below the frontrunner's ($b_\beta = p - \delta$), and γ the *underdog*, whose committed payment is the lowest. The discrete state of the game can be represented by the frontrunner-underdog lag

$$s := (p - b_\gamma)/\delta,$$

i.e. $b_\gamma = p - s\delta$. Note that $s \geq 2$, and thus we extend the notation such that the state in the initial round equals zero ($s = 0$), with everyone treated as underdog, and in the second round the state equals one ($s = 1$), with all but the frontrunner being underdog. Then any equilibrium is of the form

$$\left(\pi_0, \pi_1, (\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}\right),$$

with π_0 being every bidder's probability of bidding at the initial round, π_1 the probability of bidding at the second round for everyone but the current α player, and, for every $s \geq 2$ and each $i \in \{\beta, \gamma\}$, $\pi_{i,s}$ being the probability with which the current i player bids.

4.2 The Value Functions

Let any equilibrium $\left(\pi_0, \pi_1, (\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}\right)$ be given. For every $s \geq 2$ and each $i \in \{\beta, \gamma\}$, denote $q_{i,s}$ for the probability with which the current i player becomes the α player in the next round. Note, from the uniform-probability tie-breaking rule, that at any $s \geq 2$

$$q_{i,s} = \pi_{i,s}\left(1 - \pi_{-i,s}/2\right), \tag{3}$$

with $-i$ being the element of $\{\beta, \gamma\} \setminus \{i\}$. Given this equilibrium and any state s, denote V_s for the expected payoff for the current α player, M_s the expected payoff for the current β, and L_s that for the current γ (with $M_1 = L_1$ for every non-α player at the second round). The law of motion is described below:

$$V_1 \longrightarrow \begin{cases} v & \text{prob. } (1-\pi_1)^2 \\ M_2 & \text{prob. } 1-(1-\pi_1)^2, \end{cases} \tag{4}$$

$$M_1 \longrightarrow \begin{cases} 0 & \text{prob. } (1-\pi_1)^2 \\ V_2 - 2\delta & \text{prob. } \pi_1(1-\pi_1/2) \\ L_2 & \text{prob. } \pi_1(1-\pi_1/2); \end{cases} \tag{5}$$

and, for each $s \geq 2$:

$$V_s \longrightarrow \begin{cases} v & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ M_{s+1} & \text{prob. } q_{\beta,s} \\ M_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{6}$$

$$M_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,M_*s} - q_{\gamma,s} \\ V_{s+1} - 2\delta & \text{prob. } q_{\beta,s} \\ L_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{7}$$

$$L_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ L_{s+1} & \text{prob. } q_{\beta,s} \\ V_2 - (s+1)\delta & \text{prob. } q_{\gamma,s}. \end{cases} \tag{8}$$

4.3 The Dropout State

Since v is finite, at any equilibrium V_2 is finite and hence $V_2 < s\delta$ for all sufficiently large s. Thus, for any equilibrium

$$s_* := \max \{s \in \{1, 2, 3, \ldots\} : V_2 \geq s\delta\}$$

exists and is unique. Call s_* the *dropout state* of the equilibrium. The next lemma, which follows from (8) coupled with the definition of s_*, justifies the appellation.

Lemma 1 *At any equilibrium with dropout state s_*, an underdog (γ player) (i) stays put for sure at state s if and only if $s \geq s_*$, and (ii) bids for sure at state s if $2 \leq s < s_* - 1$.*

For example, in any bilateral-rivalry equilibrium, if the state becomes $s = 2$ then it is already in the zero-surplus subgame equilibrium, where $V_2 = V_* = 2\delta$ (Section 3.1), hence $s_* = 2$. For the illustrative trilateral-rivalry equilibrium sketched previously, $V_2 = 4\delta$ (Section 3.3) and so $s_* = 4$, hence an underdog bids with positive probability as long as his lag from the frontrunner is below 4.

By Lemma 1, once the game enters the dropout state or beyond, the player currently in the underdog role will never bid to catch up and only the frontrunner and follower may remain active. A subgame equilibrium henceforth is the zero-surplus one constructed in Section 3.1. By the independence condition of irrelevant players, which deems the position of the underdog irrelevant to any equilibrium projected onto any such subgames, the zero-surplus subgame equilibrium is the only on-path outcome thereafter:

Lemma 2 *At any equilibrium with dropout state $s_* \geq 2$, if $s \geq s_*$ then $V_s = 2\delta$ and $M_s = L_s = 0$.*

Thus, the dropout state of an equilibrium can be viewed as the endogenous terminal node of the game, giving an expected payoff 2δ to the frontrunner, and zero expected payoff to the follower and the underdog.

Reasoning backward from the dropout state s_*, we see that the game does not end if it is in any state $s \leq s_* - 2$, because according to Lemma 1.ii the current underdog bids for sure trying to catch up with the frontrunner. Thus the minimum state at which the game need not continue to the next round is the state $s_* - 1$, at which the underdog need not bid for sure. Furthermore, combining (7), Lemma 2 and the definition of s_* one can show that the follower at the *critical state* $s_* - 1$ would rather be the underdog in the next round, should the game continue, than outbid the frontrunner right now thereby getting into the zero-surplus subgame equilibrium thereupon. Thus, at the critical state $s_* - 1$, the underdog solely determines whether the competition should continue or cease, asserted by the next lemma, which also implies that the frontrunner's equilibrium surplus V_2 in the consecutive configuration is necessarily an integer multiple of the bid increment δ.

Lemma 3 *At any equilibrium with dropout state $s_* \geq 3$: (i) at the critical state $s_* - 1$ the β player stays put while the γ player bids with a probability in $(0, 1)$; and (ii) $V_2 = s_*\delta$.*

4.4 Dropout States Can Only Be Even

Lemma 3 implies that on the path of any equilibrium the game ends only when the state is $s_* - 1$, at which only the underdog γ may bid. If he bids (thereby becoming the next α) then the state returns to $s = 2$, else the game ends and the current α wins the good. Thus, in order to win, a player needs to be the α player at the critical state $s_* - 1$. Consequently, if the dropout state s_* is an odd number, then on the path to winning a bidder must in the previous rounds have been the β player for all odd states $s < s_* - 1$, and the α player for all even states $s \leq s_* - 1$. An illustration for $s_* = 7$ is shown in Figure 1. Solid lines represent possible transitions if one bids, and dashed lines if he does not bid. The extra thick gray states and arrows indicate the winning path.

Thus, when s_* is odd, a player who happens to be in the β position at any even state $s < s_* - 1$ would in order to reach the winning path rather become the γ player in state

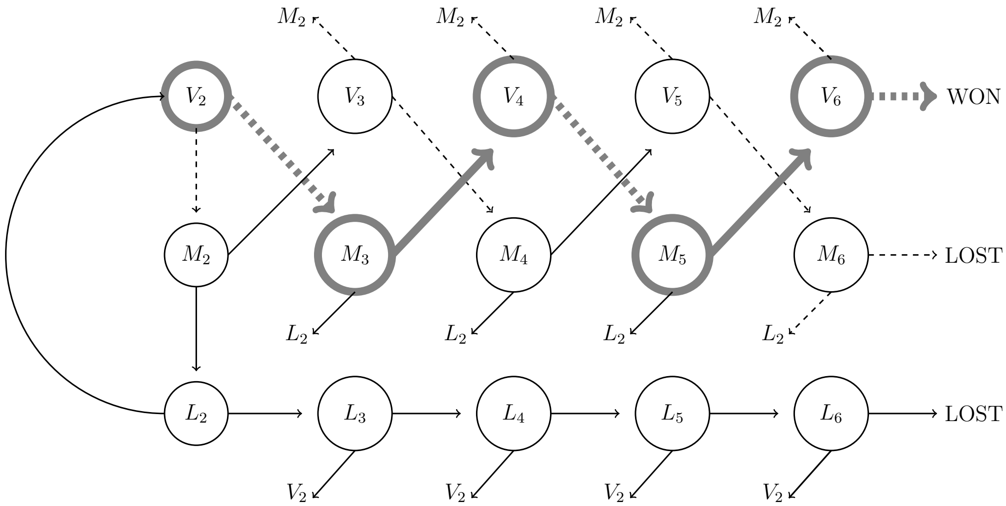

Figure 1: The law of motions and equilibrium winning path if $s_* = 7$.

$s = 2$ (through not bidding at all) than become the superfluous α player in the odd state $s+1$ at the cost of 2δ (through bidding). In particular, in state $s = 2$, the β player would never bid while the γ player would always bid; hence the state $s = 2$ repeats itself, with the players switching roles according to $\gamma \to \alpha \to \beta \to \gamma$, thereby trapping them in an infinite bidding loop. This contradiction, after being formalized, implies the first main finding—

Theorem 1 *There does not exist any equilibrium whose dropout state s_* is an odd number bigger than 2.*

When the dropout state s_* is an even number, by contrast, a β player is not in the predicament as in the previous case. First, in any even state $s < s_* - 1$ the β player wants to bid in order to stay on the winning path and become the α in the odd state $s+1$. Second, in any odd state $s < s_* - 1$ the β player would rather bid and become the α in the even state $s+1$ than stay put thereby becoming the γ player in state 2. With the former option, it takes a cost of 2δ (to become α in $s+1$) and two rounds for the player to have a chance to become the β player in state $s = 2$ thereby landing on the winning path. With the latter option, it takes a cost of 3δ and three rounds for him to have such a chance of reaching the winning path. In Figure 2, with $s_* = 6$, the situation of this odd-state β player is illustrated by the node M_3, from which the former option (becoming the next α) reaches the winning path state M_2 via $M_3 \to V_4 \to M_2$, while the latter option (being the next γ) reaches M_2 via the more roundabout route $M_3 \to L_2 \to V_2 \to M_2$.[7] Formalizing this intuition we obtain—

[7] In the more roundabout route, the last step, from V_2 to M_2, is preferable to a player because of a

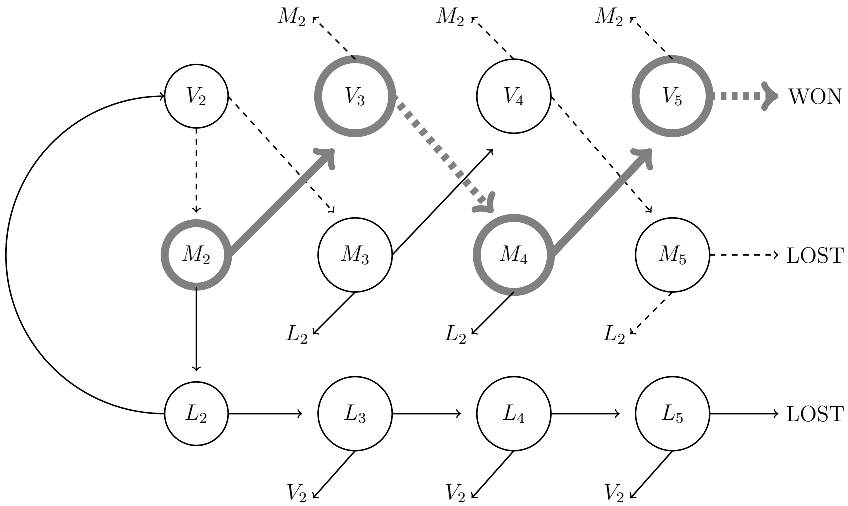

Figure 2: The law of motions and equilibrium winning path if $s_* = 6$.

Lemma 4 *At any equilibrium with dropout state s_* being an even number and $s_* \geq 4$, at any state $s \in \{1, 2, \ldots, s_* - 2\}$ the β player bids for sure.*

4.5 Characterization of the Equilibriums

Lemmas 1–4 together have mostly pinned down the strategy profile for any equilibrium with dropout state s_* above three:

(∗) s_* is an even number; at each state $s \in \{1, 2, \ldots, s_* - 2\}$ every non-α player bids for sure; at state $s_* - 1$ the β player does not bid and the γ bids with probability π_{γ, s_*-1}; at any state $s \geq s_*$, the γ player does not bid and the β bids with probability $1 - 2\delta/v$.

By Condition (∗), Eq. (3) and the equal-probability tie-breaking rule,

$$2 \leq s \leq s_* - 2 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{9}$$

Given any $\pi_{\gamma,s_*-1} \in [0, 1]$, the value functions $(V_s, M_s, L_s)_s$ associated to any strategy profile satisfying Condition (∗) can be calculated based on Eq. (9) and the law of motion, (6)–(8). The question is whether such a strategy profile constitutes an equilibrium. The crucial step in answering this question is to verify that, given Condition (∗), bidding is a best response

nontrivial Lemma 11, saying that in the consecutive configuration it is better-off to be the follower than the frontrunner.

for the β player at every state below $s_* - 1$. Verification for all such states might sound cumbersome, but it turns out that we need only to check two inequalities:

Lemma 5 *For any even number $s_* \geq 4$ and any strategy profile satisfying Condition $(*)$, bidding is a best response for the β player at state $s \in \{1, 2, \ldots, s_* - 2\}$ if either (i) s is even and $V_3 - 2\delta \geq L_2$, or (ii) s is odd and $V_{s_*-2} - 2\delta \geq L_2$.*

These two sufficient conditions, one can show, are also necessary for any equilibrium. Thus Lemma 5, combined with the previous ones, implies a necessary and sufficient condition for any equilibrium with even-number dropout state $s_* \geq 4$: that the bidding probability π_{γ,s_*-1} at the critical state is determined by the equation $V_2 = s_*\delta$ (Lemma 3.ii), with V_2 as well as other value functions derived from the law of motion (6)–(8) and Condition $(*)$, such that both $V_3 - 2\delta \geq L_2$ and $V_{s_*-2} - 2\delta \geq L_2$ are satisfied. From this necessary and sufficient condition we obtain a complete characterization of trilateral-rivalry equilibriums, equilibriums with dropout states larger than two:

Theorem 2 *Any $s_* \geq 3$ constitutes an equilibrium if and only if s_* is an even number and—*

i. either $s_* \leq 6$ and the equation

$$\begin{aligned} & \frac{3\mu_* v}{\delta}(1-x)(2-\mu_*) + (2-\mu_*)^2(s_* - 6 + \mu_*) \\ = \quad & (2(1+\mu_*) - 3\mu_* x)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\right), \quad (10) \end{aligned}$$

where $\mu_* := 2^{-s_*+3}$, admits a solution for $x \in [0,1]$;

ii. or $s_* \geq 8$ and Eq. (10) admits a solution for $x \in [0,1]$ such that

$$x \geq 1 - \frac{3(2-\mu_*)}{2(1-2\mu_*)(s_* - 4 + \mu_*)}. \quad (11)$$

The solution x to equation (10) corresponds to the γ player's bidding probability π_{γ,s_*-1} in the critical state $s_* - 1$. The bifurcated characterization in Theorem 2 is due to a fact, proved in the appendix, that at the solution for $V_2 = s_*\delta$, neither $V_3 - 2\delta \geq L_2$ nor $V_{s_*-2} - 2\delta \geq L_2$ are binding when $s_* \leq 6$, and only one of the inequalities is binding when $s_* \geq 8$.

Contrary to the case of odd-number dropout states, equilibriums with even-number dropout states exist provided that the parameter v/δ is sufficiently large:

Theorem 3 *(i) An equilibrium with* $s_* = 4$ *exists if and only if* $v/\delta > 35/2$*, and that with* $s_* = 6$ *exists if and only if* $v/\delta > 6801/120$*. (ii) For any even number* $s_* \geq 8$*, if*

$$\frac{v}{\delta} \geq \left(\frac{1}{3}s_*^2 + \frac{5}{3}s_* - 8\right) 2^{s_*-3} \tag{12}$$

then s_* *constitutes an equilibrium with dropout state equal to* s_**.*

As s_* increases from 8, the right-hand side of Ineq. (12) increases at a rate in the order of 2^{s_*}. Thus, to suffice the equilibrium feasibility of a higher dropout state s_*, Ineq. (12) requires that the parameter v/δ be higher by a magnitude in the order of 2^{s_*}.

Contrary to the bilateral-rivalry equilibriums, which constitute a continuum, there are only finitely many trilateral-rivalry ones, as the next theorem asserts. That is because the parameter v/δ, through the facts $v \geq V_2$ and $V_2 = s_*\delta$, implies an upper bound for equilibrium-feasible dropout states, which can only be integers, and given each dropout state Eq. (10) admits at most two solutions for x (i.e., π_{γ,s_*-1}), which in turn determines the equilibrium strategy profile uniquely.

Theorem 4 *There are at most finitely many equilibriums with dropout states* $s_* \geq 3$*.*

4.6 Numerical Illustration

To illustrate the formal results in Theorems 2 and 3, we fix $\delta = \$1$, vary v from \$0 to \$1,000 and consider the cases $s_* = 4, 6, 8, 10$. Figure 3 shows the γ player's (underdog) equilibrium bidding probability in the critical state $s_* - 1$ as a function of the underlying value v. The vertical lines indicate the point at which additional equilibriums for $s_* > 4$ are admitted. For instance, starting at $v = \$57$ ($\approx 6801/102$) the equilibrium corresponding to the dropout state $s_* = 6$ is permissible. We observe that within each equilibrium the bidding probability is increasing in the underlying value v (or v/δ as δ is fixed at one in this example). On the other hand, and a bit surprisingly, when a new equilibrium with a higher dropout state becomes permissible the corresponding equilibrium bidding probability drastically reduces. Furthermore, each additional equilibrium requires an order of magnitude increase in v, somewhat confirming our previous remark on the right-hand side of Ineq. (12).

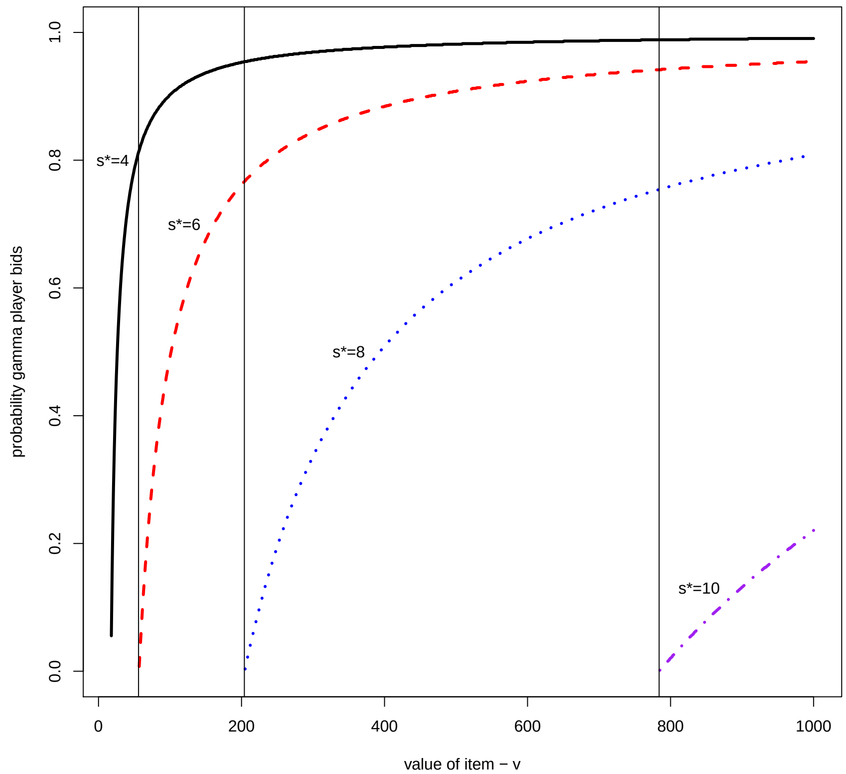

Figure 3: Equilibrium bidding probability for the underdog in the critical state $s_* - 1$; $\delta = 1$.

5 The Unique Pareto Perfect Dropout State

The previous sections have characterized the set of equilibriums in the dollar auction. It consists of a no-conflict equilibrium, where the bidding war stops for sure after the initial round; a continuum of equilibriums with only bilateral rivalry, which occurs with a probability ranging from arbitrarily close to zero to $1 - \delta/v$, all corresponding to dropout state $s_* = 2$; and a finite number of equilibriums where trilateral rivalry arises and continues until the critical state $s_* - 1$ is reached, with s_* an even number below a parametric upper bound determined by v/δ.

We reduce such multiplicity of equilibriums by introducing a mild condition of Pareto perfection to the solution concept. For any equilibriums $\mathscr{E}$ and $\bar{\mathscr{E}}$ of the dollar auction game, equilibrium $\mathscr{E}$ is said *compellingly Pareto dominated* by equilibrium $\bar{\mathscr{E}}$ if and only if there exists a state $s \in \{0, 1, 2, \ldots\}$ such that—

a. s is off the path of $\mathscr{E}$ and on the path of $\bar{\mathscr{E}}$;

b. starting from state s, each player has higher expected payoff from $\bar{\mathscr{E}}|s$ than from $\mathscr{E}|s$;

c. at $\mathscr{E}$, the state s is reached by a player's unilateral deviation from a state t that is on the path of $\mathscr{E}$, and the unilateral deviation, from the standpoint of t, is profitable for the deviating player provided that $\bar{\mathscr{E}}|s$ is the subgame play starting from s.

An equilibrium is said *weakly Pareto perfect* if and only if it is not compellingly Pareto dominated by another equilibrium.

This notion of Pareto perfection is less restrictive than the one initiated by Berheim, Peleg and Whinston [2], because we require Pareto optimality not at every node but only at a commonly known immediate deviation from the path. To switch away from a compellingly Pareto dominated equilibrium, interpretation-wise, the players do not need pre-play communication at the outset or "renegotiation" after a deviation. Rather, the choice of one equilibrium over the other can be instigated by a single player's unilateral deviation from the status quo.

Theorem 5 *If $v/\delta > 35/2$ then the weakly Pareto perfect equilibriums are exactly the trilateral-rivalry ones with a unique dropout state equal to the maximum among the even numbers s_* that satisfy condition (i) or (ii) in Theorem 2.*

In other words, among the spectrum of equilibriums, not only are the bilateral-rivalry equilibriums compellingly Pareto dominated by the trilateral-rivalry ones (which exist by the hypothesis $v/\delta > 35/2$ and Theorem 3), but also are the trilateral-rivalry equilibriums except the one(s) with the maximum dropout state. Section 3.3 has sketched how any bilateral-rivalry equilibrium with dropout state $s_* = 2$ is dominated by a trilateral-rivalry one. Here we sketch how the no-conflict equilibrium, with everyone refraining from topping the frontrunner in the second round, is dominated. The no-conflict equilibrium is dominated at the second round, when a non-frontrunner, who is supposed to not bid, deviates by committing to pay 2δ thereby topping the frontrunner. Conditional on the deviation, one can show that any trilateral-rivalry equilibrium yields positive surplus for all three players, with a surplus larger than 4δ for the deviator specifically, while the original equilibrium, now running according to the zero-surplus subgame equilibrium, yields only 2δ for the deviator-turned frontrunner and zero surplus for the other two. Provided that the equilibrium switch is made conditional on the deviation, it generates a profit at least $4\delta - 2\delta$ for the deviator from the standpoint before he deviates.

For any trilateral-rivalry equilibrium whose dropout state s_* is not the maximum one, it is dominated by another trilateral-rivalry one with a higher dropout state say s'_*. At state $s_* - 1$, when the β player is supposed to stay put and receive zero surplus at the former equilibrium, β can deviate by bidding. In the event that he deviates and becomes the next frontrunner, if the former equilibrium is played then the deviator-turned frontrunner gets 2δ and the other two get zero surplus, with the deviator and the previous-round frontrunner engaged in the surplus-dissipating bilateral rivalry, and the underdog dropping out; if they switch to the latter equilibrium, by contrast, the trilateral rivalry is prolonged since the larger dropout state has not been reached, and one can show that each gets positive surplus, which for the one who has just deviated is at least $s'_*\delta$, larger than the 2δ that he would get from the former equilibrium conditional on his deviation, and also larger than the additional payment 2δ that he commits in making the deviation.[8]

6 Conclusion and Extensions

Ever since its introduction to the literature in 1971, the dollar auction has for decades provided an intuitive illustration, yet seemingly paradoxical, of conflict escalation. Much theoretical research of the game has relied on ad hoc adjustments of the original game, including budget constraints and jump bids, to sidestep the seeming arbitrage paradox that the generated revenue exceeds the value of the good. In this paper, we return to the original form and analyze equilibrium bidding behaviors. Our results show that while some equilibrium may give rise to severe conflict escalation generating marginal or even zero (expected) bidder surplus, none generate an expected revenue greater than the value of the good. By contrast, there exist equilibriums in which bidders can extract positive surplus, including one that reduces the expected revenue to the marginal amount of one bid increment. The bidding escalation often observed in the dollar auction experiments is attributed to the bidders' coordination failure, as we have found a continuum of such equilibriums of the game.

[8] Another interpretation of this weak notion of Pareto perfection is to iteratively eliminate equilibriums by forward induction, starting from the equilibrium with the second-highest dropout state. For any equilibrium with non-maximum dropout state say s_*, conditional on a deviation at critical state $s_* - 1$, the equilibrium with the maximum dropout state $\bar{s}_*$ is the only one that Pareto dominates the one with s_*, since either $\bar{s}_* = s_* + 2$ or any equilibrium with dropout states between s_* and $\bar{s}_*$ have been eliminated in this manner. Hence the deviation can be taken as a signal that the deviator is to play the equilibrium with $\bar{s}_*$.

Furthermore, and more surprisingly, we show that if the conflict escalation is expanded to more players, specifically from bilateral to trilateral rivalries, bidders can extract surplus in a "renegotiation-proof" manner, with their coordination problem resolved tacitly via a single player's unilateral deviation.

In the discourse to characterize the trilateral rivalry we have shown that the permissible equilibriums have a particular form regarding the lag between the frontrunner and the underdog, the lowest participating bidder. We also show that at these equilibriums it is the underdog who ultimately decides whether the bidding escalation shall continue, and that this decision is made when and only when his distance to the frontrunner reaches a specific integer multiple of the bid increment. An interesting feature of these equilibriums is that at the onset of the trilateral rivalry, the ideal position is to be the follower, wedged in between the frontrunner and the underdog, rather than the frontrunner (Footnote 7).

In this paper we have mainly focused on bilateral and trilateral rivalries. Immediately one may ask: Are there also quadrilateral or pentalateral rivalry equilibriums? And if so how are they characterized? While the state of bilateral and trilateral games can be represented by a unidimensional variable, for quadrilateral and pentalateral games the state needs to be represented by a bidimensional and tridimensional vector, summarizing the lags to the frontrunner of the multiple lower bidders. Thus even just formalizing the state of the game would require further considerations.

Other interesting extensions include considering different tie-breaking and payment rules. This paper stays in line with Shubik's original model formulation. Motivated by lobbying and R&D races, however, one could consider alternative forms where in each round all active bidders simultaneously submit a sunk bid, and the game ends once no more bids are submitted. Then there could be multiple frontrunners, who may happen to drop out simultaneously thereby giving rise to ties in the end. Finally, one may also want to further enhance Shubik's framework by considering asymmetric information. We leave these and other extensions for future work with the hope that this paper may stimulate yet another fruitful research stream on the dollar auction and conflict escalation.

A Proofs

A.1 Lemmas 1 and 2

Lemma 1 By definition of L_s, the equilibrium expected payoff for an underdog whose lag from the frontrunner is s, we know that $L_s = 0$ for all $s \geq v/\delta$. Starting from any such s and use backward induction towards smaller s, together with the law of motion (8) and the fact $V_2 - (s+1)\delta < 0$ for all $s \geq s_*$ due to the definition of s_*, we observe that $L_s = 0$ for all $s \geq s_*$. At any state $s \geq s_*$, by (8), an underdog gets zero expected payoff if he does not bid; if he bids then by Eq. (3) there is a positive probability with which he gets a negative payoff $V_2 - (s+1)\delta$; hence his best response is uniquely to not bid at all. Hence

$$s \geq s_* \implies L_s = 0 \text{ and } \pi_{\gamma,s} = q_{\gamma,s} = 0, \tag{13}$$

which proves Claim (i) of the lemma. Apply backward induction to (8) starting from $s = s_*$ and we obtain

$$2 \leq s \leq s_* - 1 \Longrightarrow V_2 - (s+1)\delta \geq L_s \geq L_{s+1} \geq 0, \tag{14}$$

with the inequality $L_s \geq L_{s+1}$ being strict whenever $s < s_* - 1$. Thus, for any $s < s_* - 1$, $V_s - (s+1)\delta > L_{s+1} \geq 0$; hence Eqs. (3) and (8) together imply that an underdog's best response is uniquely to bid for sure:

$$2 \leq s < s_* - 1 \implies L_s > 0 \text{ and } \pi_{\gamma,s} = 1, \tag{15}$$

which proves Claim (ii) of the lemma. ■

Lemma 2 Take any equilibrium, with dropout state s_* and value functions V_s, M_s and L_s. By Lemma 1.i, at any state $s \geq s_*$ the player who is the current underdog stays put for all future rounds, and hence the independence condition of irrelevant players implies that in any subgame give s the equilibrium strategy profile for the remaining two players, the current frontrunner α and the follower β, satisfies the symmetry and recursion conditions as if the two constitute the entire set of players. The two conditions together imply that in any such subgame a remaining player's strategy depends only on his current role as either the α or the β, regardless of his name or the amount of his committed payment. Thus, there exist constants $(V_*, M_*, y) \in \mathbb{R}^2 \times [0,1]$ such that $V_* = V_s$, $M_* = M_s$ and $y = \pi_{\beta,s}$ for all $s \geq s_*$.

Then the Bellman equations are

$$\begin{aligned} V_* &= (1-y)v + yM_*, \\ M_* &= y(-2\delta + V_*). \end{aligned}$$

Note that $y > 0$, otherwise $V_* = v$ and $M_* = 0$; with $v > 2\delta$ by assumption, the current β player would bid for sure, so the two players are trapped in an infinite bidding loop and each get zero payoff. Also note $y < 1$, otherwise $V_* = 0$ and $M_* = -2\delta$, violating individual rationality. Now that $0 < y < 1$, the β player is indifferent about bidding, hence $M_* = 0$. This combined with the Bellman equations uniquely pins down the subgame equilibrium as $V_* = 2\delta$, $M_* = 0$ and $y = 1 - 2\delta/v$, which is exactly the zero-surplus subgame equilibrium. Hence $V_s = V_* = 2\delta$ and $M_s = M_* = 0$. Since (13) implies $L_s = 0$, the lemma is proved. ■

A.2 Lemma 3 and Theorem 1

To prove Lemma 3 we make several observations first. By (8) and (14), L_2 is a convex combination between L_3 and $V_2 - 3\delta$, with $V_2 - 3\delta \geq L_3$ when $s_* \geq 3$. Thus,

$$s_* \geq 3 \Longrightarrow L_2 \leq V_2 - 3\delta. \tag{16}$$

Lemma 2, combined with (7) and (8), implies

$$M_{s_*-1} = q_{\gamma,s_*-1} L_2 \overset{(16)}{\leq} (V_2 - 3\delta)^+ . \tag{17}$$

Lemma 6 *There does not exist an equilibrium with dropout state $s_* = 3$*

Proof Suppose, to the contrary, that $s_* = 3$. Hence $0 \leq V_2 - 3\delta < \delta$. Thus, by Eq. (17), $M_2 < \delta$. Then (4) requires that $\pi_1 < 1$, otherwise $V_1 = M_2 < \delta$, implying a contradiction that no one would bear the sunk cost δ to become the initial α player. Now consider the decision of any non-α player at the state $s = 1$, as depicted by (5). Since $V_2 - 2\delta > V_2 - 3\delta \geq L_2$, with the second inequality due to (16), each non-α player at $s = 1$ would maximize the probability of becoming the α in the next round, i.e., $\pi_1 = 1$, contradiction. ■

Lemma 7 *At any equilibrium with dropout state $s_* \geq 4$, $V_3 - 2\delta \geq M_2 \geq L_2 > 0$.*

Proof Suppose that $V_3 - 2\delta < L_2$. Then, by the fact $\pi_{\gamma,2} = 1$ (Lemma 1.ii and $s_* \geq 4$) and Eq. (3), the β player at state $s = 2$ would rather stay put than bid, hence $\pi_{\beta,2} = 0$. This, combined with (6) in the case $s = 2$ and the fact $\pi_{\gamma,2} = 1$, implies that $V_2 = M_2$. Since $V_3 - 2\delta < L_2$ coupled with (7) implies $M_2 \leq L_2$, we have a contradiction $V_2 \leq L_2 < V_2$, with the last inequality due to (8). Thus we have proved $V_3 - 2\delta \geq L_2$. Therefore, with M_2 a convex combination between $V_3 - 2\delta$ and L_2 (since $\pi_{\gamma,2} = 1$), $V_3 - 2\delta \geq M_2 \geq L_2$. Finally, to show $L_2 > 0$, note from the hypothesis $s_* \geq 4$ and definition of s_* that $V_2 - 3\delta > 0$. This positive payoff the underdog at state $s = 2$ can secure with a positive probability through bidding. Hence $L_2 > 0$ follows from (8). ■

Lemma 8 *At any equilibrium with dropout state $s_* \geq 4$, $\pi_{\gamma,s_*-1} > 0$.*

Proof Suppose, to the contrary, that $\pi_{\gamma,s_*-1} = 0$ at equilibrium. Then $M_{s_*-1} = 0$ according to (7), with $s = s_* - 1$, and the fact $V_{s_*} - 2\delta = 0$ by Lemma 2. Consequently, (6) applied to the case $s = s_* - 2$, coupled with the fact $\pi_{\gamma,s_*-2} = 1$ (Lemma 1.i), implies that $V_{s_*-2} \leq M_2$, which in turn implies, by (7) in the case $s = s_* - 3$ and the fact $\pi_{\gamma,s_*-3} = 1$, that $M_{s_*-3} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, with the last inequality due to Lemma 7. That in turn implies $V_{s_*-4} \leq M_2$ by (6) and the fact $\pi_{\gamma,s_*-4} = 1$. Thus $V_{s_*-2} \leq M_2$, $V_{s_*-4} \leq M_2$ and $M_{s_*-3} \leq M_2$.

The supposition $\pi_{\gamma,s_*-1} = 0$, coupled with the fact that $\pi_{\gamma,s} = 0$ at all $s > s_* - 1$ (Lemma 1.ii), also implies that α drops out of the race starting from the state $s_* - 1$. Thus, by the independence condition of irrelevant players, $V_{s_*-1} = V_{s_*}$, hence Lemma 2 implies $V_{s_*-1} = 2\delta$. Then (7) applied to the case $s = s_* - 2$, coupled with the fact $\pi_{\gamma,s_*-2} = 1$, implies $M_{s_*-2} \leq L_2$. Thus, by (6) and the fact $\pi_{\gamma,s_*-3} = 1$, we have $V_{s_*-3} \leq \max\{L_2, M_2\} \leq M_2$, the last inequality again due to Lemma 7. With $V_{s_*-3} \leq M_2$, (7) implies $M_{s_*-4} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$. Thus $V_{s_*-1} = 2\delta$, $V_{s_*-3} \leq M_2$, $M_{s_*-2} \leq L_2 \leq M_2$ and $M_{s_*-4} \leq M_2$.

Repeat the above reasoning on (6) and (7) for smaller and smaller s and we obtain the fact that $V_{s_*-1} = 2\delta$, $V_s \leq M_2$ and $M_s \leq M_2$ for all $s \leq s_* - 2$. Thus, $V_3 \leq \max\{M_2, 2\delta\}$, which contradicts Lemma 7. ■

Proof of Lemma 3 Since $s_* \geq 4$ by Lemma 6, $L_2 > 0$ by Lemma 7. Thus, for the β player at $s = s_* - 1$, depicted by (7), given the fact $V_{s_*} - 2\delta = 0$ by Lemma 2 and the fact that $L_2 > 0$ and $\pi_{\gamma,s_*-1} > 0$ (Lemma 8), it is the unique best response to not bid at all, i.e., $\pi_{\beta,s_*-1} = 0$. Thus, the β player stays put for sure at state $s_* - 1$, as the lemma asserts.

Next we show that $0 < \pi_{\gamma,s_*-1} < 1$. The first inequality is implied by Lemma 8 since $s_* \geq 4$. To prove $\pi_{\gamma,s_*-1} < 1$, suppose to the contrary that $\pi_{\gamma,s_*-1} = 1$. Then by the fact

$\pi_{\beta,s_*-1} = 0$ and (6) applied to the case $s = s_* - 1$, we have $V_{s_*-1} = M_2$. Consequently, by (7) applied to the case $s = s_* - 2$, $M_{s_*-2} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, with the last inequality due to Lemma 7. The supposition $\pi_{\gamma,s_*-1} = 1$ also implies $M_{s_*-1} = L_2$, which in turn implies, via (6) in the case $s = s_* - 2$, that $V_{s_*-2} \leq \max\{L_2, M_2\} \leq M_2$, the last inequality again due to Lemma 7. Then (7) for the case $s = s_* - 3$ implies $M_{s_*-3} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, and (6) implies $V_{s_*-3} \leq \max\{M_{s_*-2}, M_2\} \leq M_2$. Repeat the above reasoning on smaller s and we prove that $V_s \leq M_2$ for all $s \leq s_* - 1$. Hence $V_3 \leq M_2$, which contradicts Lemma 7. Thus we have proved that $\pi_{\gamma,s_*-1} < 1$.

With $\pi_{\gamma,s_*-1} < 1$, bidding is not the unique best response for the γ player at state $s_* - 1$, hence $V_2 \leq s_*\delta$ (otherwise the bottom branch of (8) in the case $s = s_* - 1$ is strictly positive and, by (13), is strictly larger than the middle branch, so the γ player would strictly prefer to bid). By definition of s_*, $V_2 \geq s_*\delta$. Thus $V_2 = s_*\delta$. ■

Proof of Theorem 1 Suppose, to the contrary, that there is an equilibrium with dropout state s_* an odd number. Since $s_* = 3$ is impossible by Lemma 6 and $s_* = 1$ meaningless in our model, $s_* \geq 5$. By Lemma 7, $L_2 \leq M_2 \leq V_3 - 2\delta$. Consider (6) in the case $s = s_* - 2$ together with the facts that $\pi_{\gamma,s_*-2} = 1$ (thereby ruling out $V_{s_*-2} \to v$) due to Lemma 1.ii and $s_* \geq 5$, that $M_{s_*-1} \leq L_2$ due to (17), and that $M_2 \leq V_3 - 2\delta$. Thus we have $V_{s_*-2} \leq V_3 - 2\delta$. Then consider the decision of the β player at state $s = s_* - 3$, depicted by (7), to observe that M_{s_*-3} is between L_2 and $V_3 - 4\delta$. Thus, by (6) applied to the case $s = s_* - 4$, together with the facts $\pi_{\gamma,s_*-4} = 1$ and $M_2 \leq V_3 - 2\delta$, we have $V_{s_*-4} \leq V_3 - 2\delta$. Since s_* is an odd number and $s_* \geq 5$, this procedure of backward reasoning eventually reaches V_3, i.e., $3 = s_* - 2m$ for some positive integer m. Hence we obtain the contradiction $V_3 \leq V_3 - 2\delta$. ■

A.3 Lemma 4

Lemma 4 follows from Lemmas 10 and 12, the former showing that bidding is a follower's unique best response to an equilibrium at even-number states, and the latter, odd-number states. We start with—

Lemma 9 *At any equilibrium with dropout state an even number $s_* \geq 4$, $L_2 < V_2 \leq V_3 - 2\delta$.*

Proof Since $\pi_{\beta,s_*-1} = 0$ (Lemma 3), $M_{s_*-1} \leq L_2$. Thus, since $\pi_{\gamma,s_*-2} = 1$ (Lemma 1.ii), V_{s_*-2} is a convex combination between M_{s_*-1}, which is less than L_2, and M_2, which is a convex combination between $V_3 - 2\delta$ and L_2, as $\pi_{\gamma,2} = 1$. Thus V_{s_*-2} is between L_2 and $V_3 - 2\delta$. Consequently, M_{s_*-3}, a convex combination between L_2 and $V_{s_*-2} - 2\delta$ (since $\pi_{\gamma,s_*-3} = 1$), is between L_2 and $V_3 - 2\delta$. Repeating this reasoning, with s_* being an even number, we eventually reach $2 = s_* - 2m$ for some integer $m \geq 1$, and obtain the fact that V_2 is a number between L_2 and $V_3 - 2\delta$. Thus, $L_2 < V_3 - 2\delta$, otherwise the fact $L_2 < V_2$ by (8) would be contradicted. Hence $L_2 < V_2 \leq V_3 - 2\delta$. ■

A.3.1 Bidding at Even States

Lemma 10 *At any equilibrium with any even number dropout state $s_* \geq 4$, $\pi_{\beta,s} = 1$ if $2 \leq s \leq s_* - 2$ such that s is an even number.*

Proof First, by Lemma 9, $L_2 < V_3 - 2\delta$. Thus at state $s = 2$ the β player strictly prefers to bid, i.e., $\pi_{\beta,2} = 1$. Second, pick any even number s such that $4 \leq s \leq s_* - 2$ and suppose, to the contrary of the lemma, that $\pi_{\beta,s} < 1$, which means that the β player at state s does not strictly prefer to bid. Thus $M_s \leq L_2$ (as the transition $M_s \to 0$ is ruled out by the fact $\pi_{\gamma,s} = 1$). Consequently, V_{s-1}, a convex combination between M_s and M_2, is weakly less than M_2, as $L_2 \leq M_2$ by Lemma 7. Furthermore, M_{s-2}, a convex combination between $V_{s-1} - 2\delta$ and L_2, is less than M_2, and that in turns implies $V_{s-3} \leq M_2$. Repeating this reasoning, with s an even number, we eventually obtain the conclusion that $V_3 \leq M_2$, which contradicts Lemma 7. Thus, $\pi_{\beta,s} = 1$. ■

At any equilibrium with any even number dropout state $s_* \geq 4$, since $\pi_{\gamma,s} = 1$ for all $s \leq s_* - 2$ (Lemma 1.ii), Eq. (3) and the equal-probability tie-breaking rule together imply

$$\forall s \in \{2, 3, 4, \ldots, s_* - 2\} : [\pi_{\beta,s} = 1 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2] \,. \tag{18}$$

By Lemma 10,

$$2 \leq s \leq s_* - 2 \text{ and } s \text{ is even} \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{19}$$

A.3.2 Bidding at Odd States

In the following, we extend the summation notation by defining, for any sequence $(a_k)_{k=1}^{\infty}$,

$$i > j \Longrightarrow \sum_{k=i}^{j} a_k := 0. \tag{20}$$

In particular, $\sum_{k=1}^{0} a_k = 0$ according to this notation.

Lemma 11 *At any equilibrium with any even number dropout state $s_* \geq 4$, $M_2 > V_2 + \delta/2$.*

Proof Let $m := \min\{k \in \{0, 1, 2 \dots\} : V_{2k+4} - 2\delta \leq L_2\}$. Note that m is well-defined because $s_*/2 - 2$ belongs to the set, as $V_{s_*} - 2\delta = 0 \leq L_2$ (Lemma 2). At any odd state $2k+1 \leq 2m+1$ (hence $k-1 < m$) we have $V_{2k+2} - 2\delta = V_{2(k-1)+4)} - 2\delta > L_2$, with the last inequality due to the definition of m; hence by (6) in the state $s = 2k+1$ the β player bids for sure, i.e. $\pi_{\beta,2k+1} = 1$. Thus, (18) implies that $q_{\beta,s} = q_{\gamma,s} = 1/2$ at any such odd state. Coupled with (19), that means the transition at every state s from 2 to $2m+2$ is that the current β and γ players each have probability 1/2 to become the next α player. Thus,

$$V_2 = M_2 \sum_{k=0}^{m} 2^{-2k-1} + L_2 \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} z_m \right) - 2\delta \sum_{k=1}^{m} 2^{-2k}, \tag{21}$$

where $z_m := 1$ if $2m+2 < s_* - 2$, and $z_m := 2\pi_{\gamma,s_*-1} - 1$ if $2m+2 = s_* - 2$; and the last series $\sum_{k=1}^{m}$ on the right-hand side uses the summation notation defined in (20) when $m = 0$.

To understand the term for M_2 on the right-hand side, note that M_2 enters the calculation of V_2 at the even states $s = 2, 4, 6, \dots, 2m-2$, and upon entry at state s and in every round transversing from states s to 2, the M_2 is discounted by the transition probability 1/2. The term for L_2 is similar, except that L_2 enters at the odd states $s = 3, 5, 7, \dots, 2m-1$, and that the transition probability for the L_2 at the last state $2m-1$ is equal to one if $2m-1 < s_* - 1$, and equal to π_{γ,s_*-1} if $2m-1 = s_* - 1$. That is why the last two terms within the bracket for L_2 are

$$2^{-2m-2} + 2^{-2m-2} z_m = \begin{cases} 2^{-2m-2} + 2^{-2m-2} = 2^{-2m-1} & \text{if } z_m = 1 \\ 2^{-2m-2} + 2^{-2m-2}\left(2\pi_{\gamma,s_*-1} - 1\right) = 2^{-2m-1}\pi_{\gamma,s_*-1} & \text{if } z_m = 2\pi_{\gamma,s_*-1} - 1. \end{cases}$$

The term for -2δ is analogous to that for M_2.

With $s_* \geq 4$, $V_2 - 4\delta \geq 0$. Thus, by the above-calculated transition probabilities,

$$L_2 = \frac{1}{2}(L_3 + V_2 - 3\delta) \leq \frac{1}{2}(V_2 - 4\delta + V_2 - 3\delta) = V_2 - \frac{7}{2}\delta.$$

This, combined with Eq. (21) and the fact $z_m \leq 1$ due to its definition, implies that

$$\begin{aligned} V_2 &\leq M_2 \sum_{k=0}^{m} 2^{-2k-1} + \left(V_2 - \frac{7}{2}\delta\right) \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} \right) - 2\delta \sum_{k=1}^{m} 2^{-2k} \\ &< M_2 \sum_{k=0}^{m} 2^{-2k-1} + V_2 \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} \right) - \frac{7}{8}\delta. \end{aligned}$$

Thus, the lemma is proved if

$$1 - \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} \right) = \sum_{k=0}^{m} 2^{-2k-1}, \tag{22}$$

as the left-hand side of this equation is clearly strictly between zero and one. To prove (22), we use induction on m. When $m = 0$, (22) becomes $1 - 2^{-2} - 2^{-2} = 2^{-1}$, which is true. For any $m = 0, 1, 2, \ldots$, suppose that (22) is true. We shall prove that the equation is true when m is replaced by $m + 1$, i.e.,

$$1 - \left(\sum_{k=0}^{m+1} 2^{-2k-2} + 2^{-2(m+1)-2} \right) = \sum_{k=0}^{m+1} 2^{-2k-1}. \tag{23}$$

The left-hand side of (23) is equal to

$$\begin{aligned} & 1 - \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} \right) + 2^{-2m-2} - 2^{-2(m+1)-2} - 2^{-2(m+1)-2} \\ = & \sum_{k=0}^{m} 2^{-2k-1} + 2^{-2m-2} - 2^{-2(m+1)-1} \quad \text{(the induction hypothesis)} \\ = & \sum_{k=0}^{m} 2^{-2k-1} + 2^{-2m-3}, \end{aligned}$$

which is equal to the right-hand side of (23). Thus (22) is true in general, as desired. ■

Lemma 12 *At any equilibrium with any even number dropout state $s_* \geq 4$ and at any state $1 \leq s \leq s_* - 2$ such that s is an odd number, $\pi_{\beta,s} = 1$.*

Proof Pick any odd number s such that $s \leq s_* - 2$. It suffices to prove that $V_{s+1} - 2\delta > L_2$. Since $s + 1$ is even, it follows from (19) that

$$V_{s+1} = \frac{1}{2}(M_2 + M_{s+2}) \geq \frac{1}{2}(M_2 + L_2),$$

with the inequality due to the fact $M_{s+2} \geq L_2$, which in turn is due to the fact that the β player at state $s + 2$ can always secure the payoff L_2 through not bidding at all. Thus,

$$\begin{aligned} V_{s+1} - 2\delta - L_2 & \geq \frac{1}{2}(M_2 + L_2) - 2\delta - L_2 \\ & = \frac{1}{2}M_2 - \frac{1}{2}L_2 - 2\delta \\ & = \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}L_3 + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\ & \geq \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}(V_2 - 4\delta) + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\ & = \frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta, \end{aligned}$$

with the second inequality due to the definition of L_s and the fact $V_2 - 4\delta \geq 0$ ($s_* \geq 4$). Since $\frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta > 0$ by Lemma 11, $V_{s+1} - 2\delta - L_2 > 0$, as desired. ■

A.4 Lemma 5

All lemmas in this subsection assume the hypotheses in Lemma 5, that $s_* \geq 4$ is an even number and a strategy profile $\left(\pi_0, \pi_1, (\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}\right)$ satisfying Condition $(*)$ is given, with the associated value functions $(V_s, M_s, L_s)_s$ derived from (6)–(8) and Eq. (9).

Lemma 13 *For any positive integer m such that $2m+1 \leq s_* - 1$, if $V_{2m+1} - 2\delta \leq L_2$ then $V_3 - 2\delta < L_2$.*

Proof Pick any m specified by the hypothesis such that $V_{2m+1} - 2\delta \leq L_2$. Suppose, to the contrary of the lemma, that $V_3 - 2\delta \geq L_2$. Thus, the law of motion (6) in the case $s = 2$, with $\pi_{\gamma,2} = 1$, implies that M_2 is between L_2 and $V_3 - 2\delta$, hence $V_3 - 2\delta \geq M_2 \geq L_2$. By the law of motion (7) in the case $s = 2m$, M_{2m} is a convex combination among zero, $V_{2m+1} - 2\delta$ and L_2. Thus the hypothesis implies that $M_{2m} \leq L_2$. Consequently, the law of motion (6) in the case $s = 2m-1$, together with $\pi_{\gamma,2m-1} = 1$ and $M_2 \geq L_2$, implies that $V_{2m-1} \leq M_2$ and hence $V_{2m-1} - 2\delta \leq M_2 - 2\delta$. Then (7) in the case $s = 2m-2$ implies $M_{2m-2} \leq L_2$. Repeating this reasoning backward, with 3 being odd, we eventually reach state $s = 3$ and obtain $V_3 \leq M_2$. But since $V_3 - 2\delta \geq M_2$, we have a contradiction $V_3 - 2\delta \geq M_2 \geq V_3$. ■

Lemma 14 *Denote $x := \pi_{\gamma,s_*-1}$. For any integer m such that $1 \leq m \leq s_*/2 - 1$,*

$$M_{s_*-(2m-1)} = -\delta \sum_{k=1}^{m-1} 2^{-2k+2} + M_2 \sum_{k=1}^{m-1} 2^{-2k} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)} x \right), \quad (24)$$

$$V_{s_*-2m} = -\delta \sum_{k=1}^{m-1} 2^{-2k+1} + M_2 \sum_{k=1}^{m} 2^{-2k+1} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1} x \right), \quad (25)$$

$$V_{s_*-(2m-1)} = -\delta \sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)}(1-x)v + L_2 \sum_{k=1}^{m-1} 2^{-2k} + M_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)} x \right), \quad (26)$$

$$M_{s_*-2m} = -\delta \sum_{k=0}^{m-1} 2^{-2k} + 2^{-2m+1}(1-x)v + L_2 \sum_{k=1}^{m} 2^{-2k+1} + M_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1} x \right), \quad (27)$$

$$L_2 \quad = \quad \delta\left(s_* - 4 + 2^{-s_*+3}\right). \tag{28}$$

Proof First, we prove Eqs. (24) and (25). When $m = 1$, Eq. (24), coupled with the summation notation defined in (20), becomes $M_{s_*-1} = xL_2 = \pi_{\gamma,s_*-1}L_2$, which follows from (7) and the fact that $V_s = 2\delta$ and $M_s = 0$ for all $s \geq s_*$, due to Condition ($*$). This coupled with Eq. (9) implies that

$$V_{s_*-2} = (M_{s_*-1} + M_2)/2 = M_2/2 + xL_2/2,$$

which is Eq. (25) when $m = 1$ (using again the summation notation in (20)). Suppose, for any integer m' with $1 \leq m' \leq s_*/2 - 2$, that Eqs. (24) and (25) are true with $m = m'$. By the induction hypothesis of (25) and Eq. (9),

$$\begin{aligned} M_{s_*-(2m'+1)} &= \frac{1}{2}\left(V_{s_*-2m'} - 2\delta + L_2\right) \\ &= -\delta\left(1 + \frac{1}{2}\sum_{k=1}^{m'-1} 2^{-2k+1}\right) + \frac{M_2}{2}\sum_{k=1}^{m'} 2^{-2k+1} + \frac{L_2}{2}\left(1 + \sum_{k=1}^{m'-1} 2^{-2k} + 2^{-2m'+1}x\right), \end{aligned}$$

which is Eq. (24) when $m = m' + 1$. By the above calculation of $M_{s_*-(2m'+1)}$ and Eq. (9),

$$\begin{aligned} V_{s_*-(2m'+2)} &= \frac{1}{2}\left(M_{s_*-(2m'+1)} + M_2\right) \\ &= -\frac{\delta}{2}\left(1 + \frac{1}{2}\sum_{k=1}^{m'-1} 2^{-2k+1}\right) + \frac{M_2}{2}\left(1 + \sum_{k=1}^{m'} 2^{-2k+1}\right) \\ &\quad + \frac{L_2}{4}\left(1 + \sum_{k=1}^{m'-1} 2^{-2k} + 2^{-2m'+1}x\right), \end{aligned}$$

which is Eq. (25) in the case $m = m' + 1$. Thus Eqs. (24) and (25) are proved.

Next we prove Eqs. (26) and (27). When $m = 1$, Eq. (26), coupled with the notation $\sum_{k=1}^{0} a_k = 0$, becomes $V_{s_*-1} = (1 - x)v + xM_2$, which is true by definition of x and the fact $\pi_{\beta,s_*-1} = 0$ (Condition ($*$)). Then by Eq. (9)

$$M_{s_*-2} = \left(V_{s_*-1} - 2\delta + L_2\right)/2 = \left((1 - x)v + xM_2 - 2\delta + L_2\right)/2,$$

which is Eq. (27) when $m = 1$ (again using the notation $\sum_{k=1}^{0} a_k = 0$). Suppose, for any integer m' with $1 \leq m' \leq s_*/2 - 2$, that Eqs. (26) and (27) are true with $m = m'$. By the induction hypothesis and Eq. (9),

$$\begin{aligned} V_{s_*-(2m'+1)} &= \frac{1}{2}\left(M_{s_*-2m'}+M_2\right) \\ &= -\frac{\delta}{2}\sum_{k=0}^{m'-1}2^{-2k}+2^{-1}2^{-2m'+1}(1-x)v+\frac{L_2}{2}\sum_{k=1}^{m'}2^{-2k+1} \\ &\quad +M_2\left(2^{-1}+2^{-1}\sum_{k=1}^{m'-1}2^{-2k}+2^{-1}2^{-2m'+1}x\right), \end{aligned}$$

which is Eq. (26) in the case $m = m'+1$. By the above calculation and Eq. (9),

$$\begin{aligned} M_{s_*-(2m'+2)} &= \frac{1}{2}\left(V_{s_*-(2m'+1)}-2\delta+L_2\right) \\ &= -\delta\left(1+\frac{1}{2}\sum_{k=1}^{m'}2^{-2k+1}\right)+2^{-1}2^{-2m'}(1-x)v \\ &\quad +L_2\left(\frac{1}{2}+2^{-1}\sum_{k=1}^{m'}2^{-2k}\right)+\frac{M_2}{2}\left(\sum_{k=1}^{m'}2^{-2k+1}+2^{-2m'}x\right), \end{aligned}$$

which is Eq. (27) in the case $m = m'+1$. Hence Eqs. (26) and (27) are proved.

Finally we prove Eq. (28). Applying Eq. (9) to (8) recursively we obtain, for any integer $s_* \geq 4$, that

$$\begin{aligned} L_2 &= \frac{1}{2}\left(V_2-3\delta+\frac{1}{2}\left(V_2-4\delta+\frac{1}{2}\left(\cdots+\frac{1}{2}\left(V_2-(s_*-1)\delta\right)\right)\right)\right) \\ &= \frac{\delta}{2}\left(s_*-3+\frac{1}{2}\left(s_*-4+\frac{1}{2}\left(\cdots+\frac{1}{2}\cdot 1\right)\right)\right) \\ &= \delta\left(\frac{1}{2}(s_*-3)+\frac{1}{2^2}(s_*-4)+\frac{1}{2^3}(s_*-5)+\cdots+\frac{1}{2^{s_*-3}}\right), \end{aligned}$$

which is equal to the right-hand side of (28). In the above multiline calculation, the first and second lines are due to $V_2 = s_*\delta$ (Lemma 3.ii). ■

Lemma 15 $V_{s_*-2} - 2\delta \geq L_2 \Longrightarrow \forall m \in \{1,\ldots,s_*/2-1\} : V_{s_*-2m} - 2\delta \geq L_2$.

Proof By the law of motion and Eq. (9), Eqs. (24), (25), (26), (27) and (28) hold. Denote

$$\begin{aligned} \mu(m) &:= 2^{-2m+1}, \\ \mu_* &:= 2^{-s_*+3}. \end{aligned}$$

With the fact $\sum_{k=1}^{m-1}2^{-2k} = (1-2^{-2m+2})/3$, Eq. (25) becomes

$$V_{s_*-2m} = -\delta\cdot\frac{2}{3}(1-2\mu(m))+M_2\left(\frac{2}{3}(1-2\mu(m))+\mu(m)\right)+L_2\left(\frac{1}{3}(1-2\mu(m))+\mu(m)x\right).$$

Hence

$$\begin{aligned} V_{s_*-2m} - 2\delta - L_2 &= -\delta\left(\frac{2}{3}(1-2\mu(m))+2\right) + M_2\left(\frac{2}{3}(1-2\mu(m))+\mu(m)\right) \\ &\quad -L_2\left(1-\frac{1}{3}(1-2\mu(m))-\mu(m)x\right) \\ &= -\frac{4}{3}(2-\mu(m))\delta + \frac{1}{3}(2-\mu(m))M_2 \\ &\quad -(s_*-4+\mu_*)\,\delta\left(\frac{2}{3}(1+\mu(m))-\mu(m)x\right), \end{aligned}$$

with the second equality due to (28). Thus, $V_{s_*-2m} - 2\delta \geq L_2$ is equivalent to

$$\frac{1}{3}(2-\mu(m))M_2 \geq \delta\left(\frac{4}{3}(2-\mu(m)) + (s_*-4+\mu_*)\left(\frac{2}{3}(1+\mu(m))-\mu(m)x\right)\right),$$

i.e.,

$$\frac{M_2}{\delta} \geq 4 + \frac{2(1+\mu(m))-3\mu(m)x}{2-\mu(m)}(s_*-4+\mu_*). \tag{29}$$

Since $s_* - 4 \geq 0$ by hypothesis, and

$$\begin{aligned} \frac{d}{d\mu(m)}\left(\frac{2(1+\mu(m))-3\mu(m)x}{2-\mu(m)}\right) &= \frac{(2-\mu(m))(2-3x)+2(1+\mu(m))-3\mu(m)x}{(2-\mu(m))^2} \\ &= \frac{6(1-x)}{(2-\mu(m))^2} \;\geq\; 0, \end{aligned}$$

the right-hand side of (29) is weakly increasing in $\mu(m)$, which in turn is strictly decreasing in m. Thus the right-hand side of (29) is weakly decreasing in m. Consequently, $V_{s_*-2m} - 2\delta - L_2 \geq 0$ is satisfied for all m if the inequality holds at the minimum $m = 1$, i.e., if $V_{s_*-2} - 2\delta - L_2 \geq 0$, as claimed. ■

Proof of Lemma 5 Let $s \in \{1, 2, \ldots, s_*-2\}$. If s is even and $V_3 - 2\delta \geq L_2$, then Lemma 13 implies $V_{s+1} - 2\delta > L_2$; thus, by (7) and by the fact that $\pi_{\gamma,s} = 1$ due to Condition $(*)$, the β player at s gets L_2 if he does not bid, and $\frac{1}{2}(V_{s+1} - 2\delta) + \frac{1}{2}L_2$ if he does. Hence bidding is the unique best response for β at s. If s is odd and $V_{s_*-2} - 2\delta \geq L_2$, then Lemma 15 implies that $V_{s+1} - 2\delta \geq L_2$; thus, by the same token as in the previous case, the β player at s weakly prefers to bid. ■

A.5 Theorem 2

Lemma 16 *For any even $s_* \geq 4$, if Eqs. (9) and (28) hold and $M_2 \geq V_2 = s_*\delta$, then at the initial and second rounds each player strictly prefers to bid.*

Proof First, consider the second round, which means $s = 1$. For each non-α player, becoming the next α player gives him an expected payoff $V_2 - 2\delta = (s_* - 2)\delta$ by the hypothesis $V_2 = s_*\delta$, whereas staying put gives payoff L_2, which is less than $(s_* - 3)\delta$ by Eq. (28). Thus, each non-α player strictly prefers to bid at state one, hence $s = 2$ occurs for sure given $s = 1$. Second, consider the initial state. Based on the analysis of the previous step (from $s = 1$ to $s = 2$), becoming the first α yields the expected payoff $-\delta + M_2$, whereas staying put yields $\frac{1}{2}(V_2 - 2\delta + L_2)$. Since $M_2 \geq V_2$ by hypothesis and $V_2 - 2\delta > L_2$ by the previous analysis, each player strictly prefers to become the first α player. ■

Lemma 17 *Any integer $s_* \geq 3$ constitutes an equilibrium if and only if s_* is an even number and there exists $(M_2, x, L_2) \in \mathbb{R}^3_+$ such that—*

a. $(M_2, x, L_2) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+$ and it solves simultaneously Eq. (25) in the case $m = s_*/2 - 1$, Eq. (27) in the case $m = s_*/2 - 1$ such that $V_2 = s_*\delta$, and Eq. (28);

b. $M_2 \geq s_*\delta$;

c. Ineq. (29) is satisfied in the case $m = 1$.

Proof The necessity that s_* is even for an equilibrium follows from Theorem 1. In Condition (a), the necessity of $V_2 = s_*\delta$ follows from Lemma 3, and the rest from Lemma 14, which in turn follows from Condition $(*)$, necessary due to Lemmas 1–4. With $V_2 = s_*\delta$, the condition $M_2 \geq s_*\delta$ is equivalent to $M_2 \geq V_2$; hence the necessity of Condition (b) follows from Lemma 11. The necessity of Condition (c) follows from Lemma 4, which implies the necessity of $V_{s_*-2m} - 2\delta \geq L_2$, which as shown in the proof of Lemma 15 requires Ineq. (29).

To prove that these conditions together suffice an equilibrium, pick any even number $s_* \geq 4$ and assume Conditions (a)–(c). Consider the strategy profile such that everyone bids in the initial round, each non-α player bids in the second round and, in any future round, acts according to Condition $(*)$. This strategy profile implies Eq. (9), which allows calculation of the value functions $(V_s, M_s, L_s)_{s=2}^{s_*}$ via the law of motions. By Conditions (a) and (b), $M_2 \geq V_2 = s_*\delta$, hence Lemma 16 implies that bidding at the initial round is a best response for each player, and bidding at second rounds a best response for each non-α player. The incentive for each player to abide by the strategy profile at any state $s \geq s_*$ is the same as in the two-bidder equilibrium. At the state $s_* - 1$, bidding with probability x is a best response for the γ player because he is indifferent about bidding, since $V_2 - s_*\delta = 0 = L_{s_*}$, and not bidding at all is the best response for the β player because $V_{s_*} - 2\delta = 0 < L_2$. At any state s with $2 \le s \le s_* - 2$, bidding is the best response for the γ player because $V_2 - (s+1)\delta > L_{s+1}$ (by Eq. (8)); Condition (c) by Lemma 15 suffices the incentive for the β player at every odd state to bid. To incentivize the β player at every even state $s \le s_* - 2$ to bid, Lemma 13 says that it suffices to have $V_3 - 2\delta \ge L_2$, which is equivalent to $M_2 \ge L_2$ since, by the law of motion and Eq. (9), M_2 is the midpoint between $V_3 - 2\delta$ and L_2. Since $L_2 < s_*\delta$ by Eq. (28), the condition $M_2 \ge L_2$ is guaranteed by Condition (b), $M_2 \ge s_*\delta$. ∎

Lemma 18 *For any $s_* \ge 4$, Condition (c) in Lemma 17 implies Condition (b) in Lemma 17.*

Proof Condition (c) in Lemma 17 is Ineq. (29) in the case $m = 1$, i.e., when $\mu(m) = 2^{-2m+1} = 1/2$. Hence the condition is equivalent to

$$\frac{M_2}{\delta} \ge 4 + (2 - x)(s_* - 4 + \mu_*). \tag{30}$$

To prove that this inequality implies Condition (b), i.e., $M_2/\delta \ge s_*$, it suffices to show

$$4 + (2 - x)(s_* - 4 + \mu_*) > s_*,$$

i.e.,

$$(1 - x)(s_* - 4) + \mu_*(2 - x) > 0,$$

which is true because $s_* \ge 4$, $\mu_* = 2^{-s_*+3} > 0$ and $x \le 1$. ∎

Lemma 19 *Condition (a) in Lemma 17 is equivalent to the existence of an $x \in [0,1]$ that solves Eq. (10).*

Proof Condition (a) requires existence of $(M_2, x, L_2) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+$ that satisfies Eqs. (25), (27) and (28) in the case of $m = s_*/2 - 1$ and $V_2 = s_*\delta$. Combine (25) with (28) and use the notation $\mu_* := 2^{-s_*+3}$ and the fact $\sum_{k=1}^{m-1} 2^{-2k} = (1 - 2^{-2m+2})/3$ to obtain

$$s_*\delta = V_2 = -\delta \cdot \frac{2}{3}(1 - 2\mu_*) + M_2\left(\frac{2}{3}(1 - 2\mu_*) + \mu_*\right) + \underbrace{\delta(s_* - 4 + \mu_*)}_{L_2}\left(\frac{1}{3}(1 - 2\mu_*) + \mu_* x\right),$$

i.e.,

$$\frac{M_2}{\delta} = \frac{1}{2 - \mu_*}\left(3s_* + 2(1 - 2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\right). \tag{31}$$

By the same token, (27) coupled with (28) is equivalent to

$$M_2\left(1 - \frac{1}{3}(1 - 2\mu_*) - \mu_* x\right) = -\delta\left(1 + \frac{1}{3}(1 - 2\mu_*)\right) + (1-x)\mu_* v + \delta(s_* - 4 + \mu_*)\left(\frac{2}{3}(1 - 2\mu_*) + \mu_*\right),$$

i.e.,

$$\frac{M_2}{\delta}\left(2(1+\mu_*)-3\mu_* x\right)=\frac{3\mu_* v}{\delta}(1-x)+(2-\mu_*)(s_*-6+\mu_*). \tag{32}$$

Plug (31) into (32) and we obtain Eq. (10). ∎

Lemma 20 *For any even number $s_* \geq 4$, suppose that Eq. (31) holds. Then Condition (c) in Lemma 17 is equivalent to Ineq. (11), which is implied by $x \geq 0$ if and only if $s_* \leq 6$.*

Proof Condition (c) in Lemma 17 has been shown to be equivalent to Ineq. (30). Provided that Eq. (31) is satisfied, Ineq. (30) is equivalent to

$$4+(2-x)(s_*-4+\mu_*) \leq \frac{1}{2-\mu_*}\left(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_* x)\right).$$

This inequality, given the fact $1-2\mu_* \geq 0$, is equivalent to

$$x \geq \frac{1}{2(1-2\mu_*)}\left(5-4\mu_*-\frac{3(s_*-2)}{s_*-4+\mu_*}\right),$$

i.e., Ineq (11). Given Condition (a) in Lemma 17, which implies $x \geq 0$, Ineq. (11) is redundant if and only if the right-hand side of (11) is nonpositive, i.e.,

$$\frac{3(2-\mu_*)}{2(1-2\mu_*)(s_*-4+\mu_*)} \geq 1,$$

i.e.,

$$s_* \leq 4-2^{-s_*+3}+\frac{3\left(2-2^{-s_*+3}\right)}{2\left(1-2^{-s_*+4}\right)}.$$

This inequality is satisfied when $s_* \in \{4,6\}$, as its right-hand side is equal to ∞ when $s_*=4$, and $61/8$ when $s_*=6$. The inequality does not hold, by contrast, when $s_* \geq 8$, as

$$\begin{aligned} s_* \geq 8 \;\Rightarrow\; 2^{-s_*+2} \leq 2^{-6} \;\Rightarrow\; & \frac{1-2^{-s_*+2}}{1/4-2^{-s_*+2}} \leq \frac{1-2^{-6}}{1/4-2^{-6}}=\frac{63}{15} \\ \Rightarrow\; & 4-2^{-s_*+3}+\frac{3\left(2-2^{-s_*+3}\right)}{2\left(1-2^{-s_*+4}\right)} < 4+\frac{3}{2}\cdot\frac{2}{4}\cdot\frac{63}{15} < 8 \leq s_*. \end{aligned}$$

Thus, for all even numbers $s_* \geq 4$, Ineq. (30) follows if and only if $s_* \leq 6$. ∎

Proof of Theorem 2 The theorem follows from Lemma 17, where Condition (a) has been characterized by Lemma 19, Condition (b) by Lemmas 18 can be dispensed with, and Condition (c), by Lemma 20, can be dispensed with when $s_* \leq 6$ (hence Claim (i) of the theorem) and is equivalent to Ineq (11) when $s_* > 6$ (hence Claim (ii) of the theorem). ∎

A.6 Theorem 3

Lemma 21 *If $x = 1$, the left-hand side of (10) is less than the right-hand side of (10).*

Proof When $x = 1$, the left-hand side of (10) is equal to $(2 - \mu_*)^2(s_* - 6 + \mu_*)$, and the right-hand side equal to

$$\begin{aligned} & (2(1+\mu_*) - 3\mu_*)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_*)\right) \\ = \ & (2-\mu_*)\left(2s_* + 6 - \mu_* - \mu_* s_* - \mu_*^2\right). \end{aligned}$$

Thus, the lemma follows if

$$(2-\mu_*)(s_* - 6 + \mu_*) < 2s_* + 6 - \mu_* - \mu_* s_* - \mu_*^2,$$

i.e., $9\mu_* < 18$, which is true because $\mu_* = 2^{-s_*+3}$. ■

Lemma 22 *$s_* = 4$ constitutes an equilibrium if and only if $v/\delta > 35/2$, and $s_* = 6$ constitutes an equilibrium if and only if $v/\delta > 6801/120$ $(= 56.675)$.*

Proof By Theorem 2, with $s_* \leq 6$ the necessary and sufficient condition for equilibrium is that Eq. (10) admits a solution for $x \in [0, 1]$. By Lemma 21, the left-hand side of that equation is less than its right-hand side when $x = 1$. Thus, it suffices to show that the left-hand side is greater than the right-hand side when $x = 0$, i.e.,

$$\frac{3\mu_* v}{\delta}(2-\mu_*) + (2-\mu_*)^2(s_* - 6 + \mu_*) > 2(1+\mu_*)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1-2\mu_*)\right),$$

which is equivalent to

$$\frac{v}{\delta}(2-\mu_*) > s_*(4+\mu_*) + (6-\mu_*)(2/\mu_* - 2 - \mu_*).$$

Since μ_* is equal to $1/2$ when $s_* = 4$, and equal to $1/8$ when $s_* = 6$, the above inequality is equivalent to $v/\delta > 35/2$ when $s_* = 4$, and $v/\delta > 6801/120$ when $s_* = 6$. ■

Proof of Theorem 3 Claim (i) of the theorem is just Lemma 22. To prove Claim (ii), pick any even number $s_* \in \{8, 10, 12, \ldots\}$. By Theorem 2.ii, s_* constitutes an equilibrium if Eq. (10) admits a solution for $x \in [0, 1]$ that satisfies Ineq. (11). By Lemma 21, the left-hand side of (10) is less than its right-hand side when $x = 1$. Thus, it suffices to show that the left-hand side is greater than the right-hand side when x is equal to some number greater than or equal to the right-hand side of Ineq. (11). To that end, note from $s_* \geq 8$ that $\mu_* = 2^{-s_*+3} \leq 1/32$, hence $2 - \mu_* \geq 63/32$ and $1 + \mu_* < 33/32$. Thus, the left-hand side of (10) is greater than

$$\frac{3\mu_* v}{\delta}(1-x)\frac{63}{32} + \left(\frac{63}{32}\right)^2 (s_* - 6),$$

and the right-hand side of Ineq. (11)

$$1 - \frac{3(2-\mu_*)}{2(1-2\mu_*)(s_* - 4 + \mu_*)} < 1 - \frac{3 \times 63/32}{2 \times 1 \times (s_* - 3)}.$$

Therefore, it suffices, for s_* to constitute an equilibrium, to have

$$\frac{3\mu_* v}{\delta}(1-x)\frac{63}{32} + \left(\frac{63}{32}\right)^2 (s_* - 6)$$

greater than or equal to the right-hand side of (10) when

$$x = x_* := 1 - \frac{3 \times 63}{64(s_* - 3)}.$$

To that end, denote $\phi(s_*, x)$ for the right-hand side of (10), i.e.,

$$\phi(s_*, x) := (2(1+\mu_*) - 3\mu_* x)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\right)$$

(recall that $\mu_* = 2^{-s_*+3}$). Note, from $0 < x < 1$, that $-1/32 < \mu_*(2 - 3x) < 1/16$. Hence

$$\begin{array}{rcccl} \frac{63}{32} = 2 - \frac{1}{32} & < & 2(1+\mu_*) - 3\mu_* x & < & 2 + \frac{1}{16} = \frac{33}{16}, \\ \frac{15}{16} = 1 - \frac{1}{16} & < & 1 - 2\mu_* + 3\mu_* x & < & 1 + \frac{1}{32} = \frac{33}{32}. \end{array}$$

Thus, the first factor $2(1+\mu_*) - 3\mu_* x$ of $\phi(s_*, x)$ is positive for all $x \in (0, 1)$. If the second factor of $\phi(s_*, x)$ is nonpositive when $x = x_*$ then $\phi(s_*, x_*) \leq 0$ and we are done, as the left-hand side of (10) is positive. Hence we may assume, without loss of generality, that

$$3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x_*) > 0.$$

Consequently, $\phi(s_*, x_*)$ can only get bigger if we replace its first factor by the upper bound $33/16$, and the term $1 - 2\mu_* + 3\mu_* x$ in the second factor by its lower bound $15/16$ (note that, in the second factor, $s_* - 4 + \mu_* > 0$ because $s_* \geq 8$). I.e., $\phi(s_*, x_*)$ is less than

$$\begin{aligned} \frac{33}{16}\left(3s_* + 2(1-2\mu_*) - \frac{15}{16}(s_* - 4 + \mu_*)\right) &= \frac{33}{16}\left(\frac{33}{16}s_* + \frac{23}{4} - \frac{79}{16}\mu_*\right) \\ &< \frac{33}{16}\left(\frac{33}{16}s_* + \frac{23}{4}\right) \\ &< 5s_* + 12. \end{aligned}$$

Therefore, the above observations put together, we are done if

$$\frac{3\mu_* v}{\delta}(1-x_*)\frac{63}{32}+\left(\frac{63}{32}\right)^2(s_*-6)\geq 5s_*+12$$

In other words, it suffices to have

$$\frac{3\mu_* v}{\delta}\cdot\frac{3\times 63}{64(s_*-3)}\cdot\frac{63}{32}+\left(\frac{63}{32}\right)^2(s_*-6)\geq 5s_*+12,$$

i.e.,

$$\frac{3^2\mu_* v}{\delta}\geq -2(s_*-6)(s_*-3)+\frac{32\times 64}{63^2}(5s_*+12)(s_*-3).$$

With $\frac{32\times 64}{63^2}\approx 0.516$, the above inequality holds if

$$\frac{3^2\mu_* v}{\delta}\geq -2(s_*-6)(s_*-3)+(5s_*+12)(s_*-3),$$

i.e.,

$$\frac{9\mu_* v}{\delta}\geq 3s_*^2+15s_*-72,$$

which is equivalent to Ineq. (12). Thus the theorem is proved. ■

A.7 Theorem 4

Since $v/\delta \geq s_*$ for any dropout state of any equilibrium, there are only finitely many equilibrium-feasible dropout states. Given any dropout state of any equilibrium, each player's action at every state is uniquely determined, according to Lemmas 1–4 and 16, except the γ player's bidding probability π_{γ,s_*-1} at the critical state. Thus, it suffices to prove that for each dropout state s_* there are only finitely many compatible π_{γ,s_*-1} at the equilibrium. To that end, since π_{γ,s_*-1} is determined by Eq. (10) given s_*, we need only to show that for each s_* Eq. (10) admits at most two solutions for x, the shorthand for π_{γ,s_*-1}. To show that, note that the left-hand side of Eq. (10) is a linear function of x, whereas the right-hand side is strictly convex in x: The derivative of the right-hand side with respect to x is equal to

$$\begin{aligned}&-3\mu_*\left(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_* x)\right)\\&+\left(2(1+\mu_*)-3\mu_* x\right)\left(-(s_*-4+\mu_*)3\mu_*\right),\end{aligned}$$

whose derivative with respect to x is equal to

$$3\mu_*(s_*-4+\mu_*)3\mu_*+3\mu_*(s_*-4+\mu_*)3\mu_*=18\mu_*(s_*-4+\mu_*)>0,$$

with the inequality due to the fact that $s_*-4+\mu_*=s_*-4+2^{-s_*+4}>0$ as $s_*\geq 4$ (Theorem 1). Thus, Eq. (10) admits at most two solutions for x, as desired. ■

A.8 Theorem 5

By the hypothesis $v/\delta > 35/2$ and Theorem 3, a trilateral-rivalry equilibrium exists. Hence bilateral-rivalry equilibriums are compellingly dominated by a trilateral-rivalry one, as is explained around the statement of the theorem. We need only to prove that, for any two trilateral-rivalry equilibriums with dropout states s'_* and s''_* such that $s'_* < s''_*$, the one with s'_* is compellingly Pareto dominated by the one with s''_*, while the converse is not true. (If there are multiple trilateral-rivalry equilibriums of the same dropout state, none of them is compellingly dominated by the other, because they reach exactly the same set of states.)

Hence pick any two trilateral-rivalry equilibriums with dropout states $s'_* < s''_*$. As both s'_* and s''_* are even numbers, $s'_* \leq s''_* - 2$. Label the value functions in the equilibrium with dropout state s'_* by $(V'_1, M'_1, (V'_s, M'_s, L'_s))_{s=2}^{\infty}$, and those in the equilibrium s''_* by $(V''_1, M''_1, (V''_s, M''_s, L''_s))_{s=2}^{\infty}$. First note that s''_* is not compellingly Pareto dominated by s'_*. That is because $s''_* > s'_*$, hence Condition (a) of compelling Pareto dominance is not satisfied, as any state that can be reached on the path of s'_*, i.e., any integer $s \in \{1, 2, \ldots, s'_* - 1\}$ by Condition $(*)$, can also be reached on the path of s''_*, includes any integer up to $s''_* - 1$.

Second, we show that s'_* is compellingly Pareto dominated by s''_*. To that end, consider the state s'_*. Since $s'_* \leq s''_* - 1$, it can be reached on path of the equilibrium s''_*, but cannot be reached on the path of equilibrium s'_* (since the game according to the latter equilibrium either ends or collapses back to state 2 at the critical state $s'_* - 1$; c.f. Condition $(*)$). Hence Condition (a) is met. To verify Condition (b), let us compare the players' expected payoffs from the two equilibriums conditional on state s'_* being reached.

i. For the current α player (who was the previous β and deviated at state $s'_* - 1$), the status quo equilibrium s'_* gives him a payoff equal to $V'_{s'_*}$, which is equal to 2δ (Lemma 2). Whereas the new equilibrium s''_* would give him $V''_{s'_*}$; since $s'_* \leq s''_* - 2$, the incentive condition in equilibrium s''_* implies that $V''_{s'_*} - 2\delta \geq L''_2 > 0$ (Condition $(*)$ and Lemma 7) and hence $V''_{s'_*} > 2\delta = V'_{s'_*}$.

ii. For the current β player at state s'_*, the status quo equilibrium s'_* yields a payoff $M'_{s'_*}$, which equals zero (Lemma 2); while the new equilibrium s''_* yields $M''_{s'_*}$, which by (7) and (9) is equal to the average between $V''_{s'_*+1} - 2\delta$ and L''_2, each strictly positive. Hence $M''_{s'_*} > M'_{s'_*}$.

iii. For the current γ player, the status quo equilibrium s'_* gives $L'_{s'_*} = 0$ (Lemma 2); while the new equilibrium s''_* gives him $L''_{s'_*} > 0$ because $s'_* \leq s''_* - 2$ (c.f. (15)).

Thus Condition (b) is satisfied. To verify Condition (c), note that in the status quo equilibrium s'_*, the state s'_* is reached by the unilateral deviation of the β player at state $s'_* - 1$, without whose bid the game would either end (when the γ player at state $s'_* - 1$ does not bid) or collapse back to state 2 (when γ bids). We still need to check the deviation incentive for this β player. His deviation is pivotal only when he becomes the next α player, at the state s'_*. Hence it suffices to compare his payoffs when he becomes the next α players versus otherwise. When he gets to become the next α player, his expected payoff becomes $V''_{s'_*} - 2\delta$ provided that the other two players abides by the new equilibrium s''_* from now on. When he does not get to be the next α player, the game does not reach the off-path state s'_* and hence the status quo equilibrium remains at place, which gives him a payoff $\pi'_{\gamma, s'_*-1} L'_2$. As explained previously, $V''_{s'_*} - 2\delta \geq L''_2$, and

$$\begin{aligned} L''_2 \overset{(28)}{=} \left(s''_* - 4 + 2^{-s''_*+3}\right)\delta \quad &\geq \quad \left(s'_* + 2 - 4 + 2^{-s''_*+3}\right)\delta \\ &= \quad \left(\left(s'_* - 4 + 2^{-s'_*+3}\right) + (2 - 2^{-s'_*+3} + 2^{-s''_*+3})\right)\delta \\ &\overset{(28)}{=} \quad L'_2 + \left(2 - 2^{-s'_*+3} + 2^{-s''_*+3}\right)\delta \\ &> \quad L'_2 + \delta, \end{aligned}$$

with the first inequality due to $s''_* \geq s'_* + 2$, and the last due to the fact $2^{-s'_*+3} \leq 1/2$. Thus $V''_{s'_*} - 2\delta \geq L''_2 > \pi'_{\gamma, s'_*-1} L'_2$, and Condition (c) is met. ■

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