A Necessary and Sufficient Condition for Peace*

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July 4, 2017

Abstract

This paper examines the possibility for two contestants to agree on a peace settlement thereby avoiding a contest, in which each would exert a costly effort given the posterior distributions inferred from the negotiation. I find a necessary and sufficient condition of the prior distributions for there to exist a negotiation mechanism that admits a peace-ensuring perfect Bayesian equilibrium. The finding is based on an analysis of two-player all-pay contests that unifies the methods previously separated by the difference in discrete versus continuous distributions, and handles continuation plays that may cause empty best responses and infinitesimal effort costs for all but one type of a contestant who deviated previously. When a contestant becomes ex ante stronger, the peace condition is more likely to hold, though the strengthened contestant need not gain in the peace payoff, nor does the opponent need to exert more efforts to punish the former should the former deviate. The peace condition is also robust to equilibrium refinements such as the intuitive criterion and universal divinity.

*The financial support from the Social Science and Humanities Research Council of Canada is gratefully acknowledged. I thank Beth Allen, Mariagiovanna Baccara, Fredrik Ødegaard, Scott Page, Mike Peters, John Quah, Roberto Serranos, Hanzhe Zhang and seminar participants of the 2016 NSF/NBER/CEME Mathematical Economics Conference at Johns Hopkins University, 2017 NSF/CEME Decentralization Conference at the University of Michigan, 2017 Canadian Economic Theory Conference at the University of British Columbia, the Michigan State University, the Lingnan University College of Sun YatSen University, and the Ivey School of Business at the University of Western Ontario for their comments.

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1 Introduction

Under what conditions can two contestants avoid conflicts through a peace settlement? This question is relevant to international conflicts, pre-litigation negotiations and anticompetitive collusion. In the spirit of Myerson and Satterthwaite [20] on bilateral trade, where information asymmetry impedes mutually beneficial outcomes, a natural approach to address this question is mechanism design. Following this approach, the conflict resolution literature models a conflict as the outside option of a peace negotiation mechanism and assumes that the outcome of a conflict is exogenous: its cost is parametric, and a contestant's winning probability an exogenous function of the contestants' private information (Bester and Wärneryd [4], Fey and Ramsay [9], Hörner, Morelli and Squintani [13], and Spier [23]). Thus the literature observes that a contestant would reject peace settlements if and only if the exogenous cost of conflict is sufficiently small. To provide normative guidance, however, such a framework would be hard pressed to fit its exogeneity assumption into situations where the outcome of a conflict depends not only on the contestants' private information but also on the efforts and resources put into the conflict by each side in response to those by the other. Endogenous by nature, the cost of a conflict is hard to assess at the outset. It is the innate difficulty of making predictions of a game-theoretic interaction, rather than the erroneous assessment of an exogenous process, that is more likely a force dragging into quagmires so many international conflicts that are initially viewed as cakewalks.

To capture the endogenous nature of a conflict, this paper models it as an all-pay auction, the outcome of which depends on the efforts and resources sunk into the conflict by each contestant based on his posterior belief after the failed negotiation. Given such a multistage environment, this paper presents a necessary and sufficient condition, in terms of the prior distributions of the contestants' private information, or their *types*, for there to exist a mechanism that, if employed as the negotiation protocol, admits a perfect Bayesian equilibrium that results in peace settlements almost surely.

With conflicts—the outside options of peace settlements—endogenous, our design problem necessitates new considerations. First, the revelation principle may fail because, as Celik and Peters [6] explain in a different, cartel formation context, an equilibrium-feasible mechanism need not have full participation. Second, and more importantly, even if one can restrict attention only to the class of fully participated mechanisms—which is true in this paper since we look for peace-ensuring mechanisms—whether a mechanism belongs to such a class or not depends on the posterior belief and the associated continuation equilibrium outside the mechanism. Thus, to check the incentive conditions for a mechanism, we allow for all possible posterior beliefs in the off-path event that the peace proposal is vetoed. Given any such belief system, allowing for gaps and atoms in the type-distributions, we characterize all Bayesian Nash equilibriums (BNE) of the ensuing conflict, an all-pay auction. To calculate the incentive for a contestant anticipating any such BNE as the continuation equilibrium in case of conflict, we analyze the supremum of the expected payoffs the BNE yields for a contestant given various types, including the deviating types that are not expected in, and may (due to possibilities of ties) have no best response to, the BNE.

In solving the all-pay auction games, where the type-distributions, being endogenous posteriors, may or may not be continuous, we develop a distributional method generalized from Vickrey [24, Section II]. The method, encapsulated by our Eqs. (9) and (10), unifies the previously separate approaches to two-player contests in the literature, one based on discrete or degenerate distributions, and the other, continuous, strictly increasing, and often identical distributions. The first approach is not conducive to a general formula for equilibriums, which we need for comparison among them; the second one provides general formulas but it relies on the pure strategy of an equilibrium and the invertibility thereof to map one's bid to the other's type submitting the same bid, whereas we need to handle mixed and non-invertible strategies due to type-distributions with atoms and gaps. Developing new constructs to map one's bid to his type, our method extends the second approach (e.g., Amann and Leininger [1] and Kirkegaard [15]) to include all cases handled by the first one, except when types are correlated across bidders (Krishna and Morgan [18] and Siegel [22]).[1]

Resolving the issue of empty best responses for a deviating contestant who has vetoed a peace proposal, we prove that the supremum of the expected payoffs a BNE of the ensuing conflict gives a peace vetoer is equal to his surplus in an auction where the tie-breaking rule is altered to always favor the vetoer in case of ties (Theorem 1).

Based on the general analysis, we find the minimum payoff that a peace proposal needs to offer a contestant in order to guarantee his acceptance. In the event that the

[1] Krishna and Morgan, and Siegel, allow for correlation between contestants but restrict the extent of correlation to retain monotonicity of the equilibrium strategy (c.f. Footnote 3). See Fu, Lu and Pan [10] and Kaplan and Zamir [14] for references to the vast contest literature.

contestant vetoes the proposal, there is a penalizing continuation equilibrium that gives the vetoer merely his minimum peaceful payoff, proportional to the probability with which the vetoer wins the contest by exerting only an infinitesimal amount of efforts in response to the penalizing continuation equilibrium (Theorem 3). In this off-path conflict, the vetoer gets a surplus essentially constant to his types, with all but one type incurring a zero or arbitrarily small cost, in contrast to the conflict resolution literature cited previously (Remark 4).

From the minimum peaceful payoff and its explicit relationship with the parameters comes a necessary and sufficient condition for existence of peace-ensuring, or peaceful, mechanisms (Theorem 4). It implies that peace can be guaranteed if and only if at least one player stands a sufficiently small chance to win the contest by bidding just above zero in the penalizing continuation equilibrium after vetoing peace. While the conflict resolution literature has an analogous result, it requires a sufficiently large exogenous cost of conflict to guarantee peace. In our model, by contrast, the cost of conflict plays no role and, with the cost endogenously zero or infinitesimal for all but one type of the vetoer, makes little sense.

The peace condition implies how changes in the prior distributions of the players' strength levels may affect the prospect of conflict preemption. Unlike bilateral trade, where overlapping between traders' types renders full efficiency impossible, ex ante disparity between the contestants is not necessary to preempt conflicts fully. In fact, the prospect for peace improves when both contestants become ex ante stronger in the sense of stochastic dominance within the prior support (Theorem 5). Somewhat counterintuitively, when one contestant gets ex ante stronger, his minimal peaceful payoff does not increase, though the opponent's decreases; neither does such unilateral strengthening of a contestant require more efforts from the other to deter aggression from the strengthened one (Remark 7).

Albeit a condition to guarantee peace, the peace condition is robust to small probabilities of conflict in some cases, where the condition is the limit of the feasibility conditions for a sequence of proposals whose equilibriums result in both conflict and peace with the probabilities of peace converging to one (Remark 6). Furthermore, the peace condition remains the same when we restrict off-path posteriors by equilibrium refinements such as the intuitive criterion of Cho and Kreps [7], and universal divinity of Banks and Sobel [3] (Corollary 3). If a peace settlement can leave some value of the good unallocated, our result is also robust to the ratifiability condition of Cramton and Palfrey [8] (Remark 5).

The working paper by Balzer and Schneider [2] is the only work that considers the design problem of conflict preemption in a sequential negotiation-contest setting similar to ours. They assume an identical discrete distribution for both contestants, while we allow for arbitrary distributions. While we seek conditions to fully preempt conflicts, they consider conflict-probability minimization within cases where full preemption is impossible.[2]

The concluding Section 9 remarks on related problems such as cases where the peace condition is violated, or types are correlated between contestants, or the negotiation mechanism is proposed by a contestant instead of a neutral trustworthy mediator. The appendix contains the proofs of all lemmas, theorems and corollaries, in the order of their appearance.

2 The Model

Suppose that a prize of common value v, with $v > 0$ commonly known, is pursued by players 1 and 2. For each $i \in \{1, 2\}$, player i's type t_i, privately known to i, is independently drawn from a commonly known cumulative distribution function (c.d.f.) F_i, with $[a_i, z_i]$ the convex hull of its support, $\operatorname{supp} F_i$, such that $0 \le a_i < z_i$ and $F_i(0) = 0$. The two players start by negotiating for a peaceful split of the prize according to a protocol to be defined below. If a peaceful split, in the form of $(v_1, v_2) \in [0, 1]^2$ such that $v_1 + v_2 = v$, is accepted by both players, the game ends with a payoff v_i for player i. If no peaceful split is accepted by both, or if at least one player chooses nonparticpation thereby the two skip negotiation completely, the outcome is *conflict*, an all-pay contest where the two players simultaneously exert efforts, for which they each bear a sunk cost, and the player who exerts more effort wins the prize, with ties broken by a coin toss. The game ends once the winner is determined, with the payoff for player i equal to

$$v\mathbf{1}_i - b_i/t_i$$

if $\mathbf{1}_i := 1$ when i wins the prize and $\mathbf{1}_i := 0$ otherwise, b_i denotes the quantity of efforts, or *bid*, which player i has exerted ($b_i \ge 0$), and t_i the realized type of i. (When $t_i = 0$, the notation b_i/t_i means ∞ if $b_i > 0$, and zero if $b_i = 0$.) Both players are assumed risk neutral.

In the negotiation stage, a neutral trustworthy mediator chooses a *mechanism* à la Myerson [19], which solicits a confidential message from each player and then computes a recommendation, which is either conflict or a peaceful split, and finally delivers to each player

[2] Their condition for such cases, Assumption 2, corresponds to the complement of the "then" clause of our Lemma 3 applied to the special case where the contestants have the same discrete distribution.

a confidential message that contains this recommendation, possibly accompanied with some truthful information about the message submitted by the other player. Once such a mechanism is announced, each player, already privately informed of his own type, announces independently and publicly whether to participate. If and only if both participate, the mechanism is operated; if the recommendation thereof is a peaceful split, each player independently announces, publicly, whether to accept or reject it; if it is accepted by both then the game ends with the peaceful split. In any other case, conflict ensues.

Given any mechanism, the multistage game is defined, for which the solution concept is perfect Bayesian equilibrium with an additional condition of independence: if nonparticipation in the mechanism is an off-path action, then the posterior distribution of a player who has just unilaterally made such a deviation is independent of the realized type of the other player. This independence condition is to rule out scenarios where the players' types, assumed stochastically independent at the outset, suddenly become correlated without the two having had any communication.[3] Abusing notations, I abbreviate perfect Bayesian equilibrium satisfying the independence condition as PBE.

A PBE is *peaceful* if and only if, on the path of the equilibrium, conflict occurs with zero probability relative to the prior distributions. A mechanism is *peaceful* if and only if the multistage game given the mechanism admits a peaceful PBE.

Remark 1 While we model a conflict as an all-pay auction for its war-of-attrition aspect, the auction format can be replaced by first- and second-price auctions, except that the altered framework would be more pertinent to bidding collusion rather than conflict resolution. Extension is trivial for second-price auctions, but nontrivial for first-price auctions; results analogous to those reported here are presented in a companion paper [25] by this author.

[3] The independence condition is in the same spirit as the "no signaling what you don't know" condition of Fudenberg and Tirole [12], but ours is weaker than theirs, as we require it only when the deviation is nonparticipation, where the condition is compelling because the nonparticipation decision is made without any communication with the other player and hence cannot signal any new information that the deviant might have about the latter. The independence condition ensures monotonicity of any equilibrium strategy in the contest continuation game; such monotonicity is needed for Lemma 6. Without the independence condition, if the correlation is sufficiently small, monotonicity can still be guaranteed (c.f. Footnote 1). But if the correlation is strong then monotonicity cannot be guaranteed (Rentschler and Turocy [21]) and may even be impossible at equilibrium.

Remark 2 Within the auction stage, our common value model is equivalent to the independent private value (IPV) model: In our model, a player i's decision in the auction, given any positive type t_i and distribution of the bids submitted by the other player, is equivalent to his decision in an IPV model with valuation vt_i. Embedded in the pre-contest negotiation context, however, the two models have an important difference. In the IPV model, a player's utility from a peaceful split depends on his type, hence a peaceful mechanism in general is type-dependent. In our model, by contrast, a player's type matters only in the off-path event where conflict ensues; hence peaceful splits can be mutually acceptable without being type-dependent (Section 3). We opt for the common value model to highlight the zero-sum aspect of conflicts, whereas an IPV model, albeit more complete, would mix two issues that at this stage of investigation would better be separate: how to resolve a conflict when both sides desire a good versus how to allocate the good to the party that values it more.

3 Type Independence of Peaceful Mechanisms

To the convenience of our search for peaceful mechanisms, the first observation is that any peaceful mechanism is essentially offering each player a payoff constant to his type.

Lemma 1 *Given any peaceful mechanism and its associated PBE, for any $i \in \{1, 2\}$ there exists a unique $k_i \in \mathbb{R}_+$ such that player i's on-path expected payoff is equal to k_i for almost all types of i (with respect to F_i).*

The reason of Lemma 1 is the assumption that a player's type affects his payoff only when conflict ensues. Conditional on no occurrence of conflict, his expected payoff is independent of his real type, so he would send whatever message that maximizes his peaceful share. The only twist in the proof is the need to express negotiation mechanisms as augmented revelation mechanisms, whose message space contains not only types but also "nonparticipation." Such augmentation is needed because the traditional version of the revelation principle may fail when a player's nonparticipation in negotiation is not equivalent to representing himself as certain types during negotiation, as the posterior beliefs caused by the two may differ.

Call a mechanism *type-independent proposal* if it directly proposes a peaceful split without first soliciting messages from the players, so that the outcome is the proposed peaceful split if both players accept it, and conflict if at least one rejects (or not to participate in) it. Hence the message space is null and the only action for each player is to either accept or reject the recommended type-independent split. Call a type-independent proposal *mutually acceptable* if and only if, given the proposal being the mechanism, there exists a PBE on the path of which both players accept the proposal almost surely with respect to the prior distributions. The next corollary follows from Lemma 1 and the assumption that a player can, through nonparticipation, shut down inter-player communication which could have taken place within a mechanism.

Corollary 1 *Any peaceful mechanism, coupled with its associated PBE, is almost surely (relative to the prior distributions) outcome- and payoff-equivalent to a mutually acceptable type-independent proposal.*

According to Corollary 1, a peaceful mechanism can be replicated by a type-independent, mutually acceptable, proposal. At the associated PBE of the latter, which offers the players no chance to communicate before they independently choose their responses and which is accepted by both players on path, the independence condition in our solution concept requires that the off-path posterior belief about a player who has unilaterally rejected the proposal be independent of the realized type of the other player. Thus, in the search for off-path continuation plays to deter a player from vetoing peace, there is no loss of generality to restrict attention to such independent posteriors.

4 The Most Belligerent Type

By Lemma 1, to guarantee acceptance from a player a peace proposal needs only to offer him a constant payoff that is at least as large as what almost every type of his could have obtained by vetoing the proposal. To do that, we need to locate for each player a type most tempted to veto a peace proposal. To that end, we calculate what each type of a player expects to get by vetoing a peace proposal thereby triggering the conflict.

Denote $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ for the all-pay auction defined in our model such that each player i's type is independently drawn from a distribution $\tilde{F}_i$ whose support is contained in $[a_i, z_i]$. A player i's (mixed) strategy in the game $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ corresponds to a mapping $\sigma_i : \mathbb{R}_+ \times \operatorname{supp} \tilde{F}_i \to [0, 1]$ such that $\sigma_i(\cdot, t_i)$ is a c.d.f. of i's bid given his realized type t_i.

For any strategy pair $\sigma := (\sigma_1, \sigma_2)$ of the game $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any $i \in \{1, 2\}$, define the induced distribution $H_{i,\sigma}$ of i's bids and the supremum x_σ of bids by, for any $b \in \mathbb{R}_+$,

$$H_{i,\sigma}(b) := \int_{\mathbb{R}} \int_0^b \sigma_i(dr, t_i) d\tilde{F}_i(t_i), \tag{1}$$

$$x_\sigma := \max_{i \in \{1,2\}} \sup \operatorname{supp} H_{i,\sigma}. \tag{2}$$

For any $i \in \{1, 2\}$, any realized type t_i of player i and any distribution H of bids submitted by player $-i$, define

$$U_i(t_i \mid H) := \sup_{b \in \mathbb{R}_+} vH^*(b) - b/t_i, \tag{3}$$

where, for any $b \in \mathbb{R}$, $H^*(b)$ is the winning probability incorporating the possibility of ties:

$$H^*(b) := \begin{cases} H_{-i,\sigma}(b) & \text{if } b \text{ is not an atom of } H \\ \lim_{b' \uparrow b} H(b') + \left(H(b) - \lim_{b' \uparrow b} H(b')\right)/2 & \text{if } b \text{ is an atom of } H. \end{cases} \tag{4}$$

In other words, $U_i(t_i \mid H)$ is the supremum among player i's expected payoffs in the all-pay auction given his type t_i and the distribution H of bids from the other player $-i$, when i's bid b ranges in $\mathbb{R}_+$. The operator in Eq. (3) is sup instead of max because a maximum need not exist when H has an atom, at which the equal-probability tie-breaking rule renders the objective function discontinuous. Straightforward comparative statics yields—

Lemma 2 *For any $i \in \{1, 2\}$ and any c.d.f. H, $U_i(\cdot \mid H)$ is weakly increasing on $[a_i, z_i]$.*

Thus, a peace proposal is accepted by all types of player i if and only if it offers i at least $U_i(z_i \mid H_{-i,\sigma})$ with $H_{-i,\sigma}$ the distribution of the other player $-i$'s bids generated by the continuation play σ in the event that i unilaterally vetoes the proposal. This σ constitutes a continuation equilibrium in the mind of player $-i$ when $-i$ adopts a posterior belief $\tilde{F}_i$ about the deviant i that, coupled with the prior F_{-i} of the obedient $-i$, rationalizes σ. As explained immediately after Corollary 1, there is no loss of generality to assume that the posterior belief $\tilde{F}_i$ is independent of the realized type of $-i$. Thus, the lowest possible expected payoff needed to induce acceptance from all types of player i is the infimum of $U_i(z_i \mid H_{-i,\sigma})$ when σ ranges among the Bayesian Nash equilibriums (BNE) of the auction game given i's unilateral deviation:

$$\underline{u}_i := \inf\{U_i\left(z_i \mid H_{-i,\sigma}\right) : \sigma \in \mathscr{E}_i(\tilde{F}_i); \operatorname{supp} \tilde{F}_i \subseteq \operatorname{supp} F_i\}, \tag{5}$$

where $\mathscr{E}_i(\tilde{F}_i)$ denotes the set of all the BNEs of the all-pay auction $\mathscr{G}(\tilde{F}_i, F_{-i})$ such that the posterior distribution of i's type is $\tilde{F}_i$ while that of $-i$'s remains to be the prior F_{-i}.

Lemma 3 *Suppse, for each $i \in \{1,2\}$, that $\underline{u}_i = U_i\left(z_i \mid H_{-i,\sigma_*^i}\right)$ for some σ_*^i belonging to the set in Eq. (5) and that $U_i\left(\cdot \mid H_{-i,\sigma}\right)$ is continuous for any σ belonging to the aforementioned set. Then peaceful mechanisms exist if and only if $\underline{u}_1 + \underline{u}_2 \leq v$.*

Our task thus becomes deriving $\underline{u}_i$ from the primitives and checking the continuity condition. That requires characterizing all the equilibriums of the auction game in the event that a player vetoes an otherwise mutually acceptable peace proposal. Such events being off-path, the posterior belief of the vetoing player is arbitrary. Hence we need to analyze two-player all-pay auctions given arbitrary distributions.

5 General Analysis of Two-Player Contests

A strategy $\sigma_i : \mathbb{R}_+ \times \operatorname{supp} \tilde{F}_i \to [0,1]$ is said *monotone* if and only if, for any $t, t' \in \operatorname{supp} \tilde{F}_i$,

$$t' > t \Longrightarrow \inf \operatorname{supp} \sigma_i(\cdot, t') \geq \sup \operatorname{supp} \sigma_i(\cdot, t).$$

Routine analysis of the equilibrium conditions yields—

Lemma 4 *For any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, any BNE $\sigma := (\sigma_1, \sigma_2)$ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any $i \in \{1,2\}$:*

a. the support of $H_{i,\sigma}$ is $[0, x_\sigma]$ and $H_{i,\sigma}$ has neither gap nor atom in $(0, x_\sigma]$;

b. σ_i is monotone.

With Lemma 4 we resolve the discontinuity issue caused by the uniform tie-breaking rule in the auction. After a player i has deviated by vetoing a peace proposal, as long as the opponent $-i$ abides by some continuation equilibrium σ which $-i$ believes, not necessarily correctly, that i also abides by, Lemma 4.a says that the bid distribution $H_{-i,\sigma}$ has no atom except possibly at the zero bid. Thus, unless $b = 0$, player i's probability of winning by bidding b is equal to $H_{-i,\sigma}(b)$, as if the tie-breaking rule were altered to always picking him the winner in the (zero-probability) event that the opponent also bids b. That is also true when $b = 0$ unless zero is an atom of $H_{-i,\sigma}$. When zero is an atom of $H_{-i,\sigma}$, given the uniform tie-breaking rule, player i of any positive type would rather bid slightly above zero to secure an expected payoff approximately $vH_{-i,\sigma}(0)$ than bid exactly zero to get only $vH_{-i,\sigma}(0)/2$; if, in addition, he cannot do better than $vH_{-i,\sigma}(0)$, the supremum among his expected payoffs, when his bid ranges in $\mathbb{R}_+$, is equal to $vH_{-i,\sigma}(0)$, again as if he were bidding exactly zero and the tie-breaking rule were altered to always favor him. Thus—

Theorem 1 *For any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, any BNE σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any $i \in \{1, 2\}$, $U_i(\cdot \mid H_{-i,\sigma})$ is continuous on $[a_i, z_i] \setminus \{0\}$ and, for any $t_i \in [a_i, z_i] \setminus \{0\}$,*

$$U_i(t_i \mid H_{-i,\sigma}) = \max_{b \in \mathbb{R}_+} vH_{-i,\sigma}(b) - b/t_i. \tag{6}$$

Since $H_{-i,\sigma}$ restricted to $\mathbb{R}_+$ is continuous (Lemma 4.a), and the domain for the maximization problem in Eq. (6) can be restricted without loss by $[0, vz_i]$, the correspondence from t_i to $\arg\max_{b \in \mathbb{R}_+} vH_{-i,\sigma}(b) - b/t_i$ is upper hemicontinuous and hence admits an upper semicontinuous selection function $\beta_{i,\sigma}$, which one can prove is also weakly increasing. Hence the composite $H_{-i,\sigma} \circ \beta_{i,\sigma}$ constitutes a distribution function and, with $\beta_{i,\sigma}$ ranging among selection functions, is unique except on a set of zero Lebesgue measure. Apply the envelope theorem to the problem in Eq. (6) to obtain—

Corollary 2 *For any $i \in \{1, 2\}$, and any BNE σ, $U_i(\cdot \mid H_{-i,\sigma})$ is differentiable over (a_i, z_i); if $a_i = 0$ then for any $t_i \in (a_i, z_i)$ and any upper semicontinuous function $\beta_{i,\sigma} : [a_i, z_i] \to \mathbb{R}_+$ such that $\beta_{i,\sigma}(t_i) \in \arg\max_{b \in \mathbb{R}_+} vH_{-i,\sigma}(b) - b/t_i$ for all $t_i \in [a_i, z_i] \setminus \{0\}$,*

$$\frac{d}{dt_i} U_i(t_i \mid H_{-i,\sigma}) = \frac{v}{t_i^2} \int_0^{t_i} s \, d\left(H_{-i,\sigma} \circ \beta_{i,\sigma}\right)(s). \tag{7}$$

Next we characterize the equilibriums in the contest game. By Lemma 4.a, a type-t_i player i's expected payoff $\frac{1}{t_i}\left(vt_i H_{-i,\sigma}(b) - b\right)$, as a function of his bid b, is differentiable almost everywhere in $[0, x_\sigma]$. Any such differentiable point b satisfies the first-order necessary condition for b to be a bid prescribed by σ_i to t_i:

$$\frac{d}{db} H_{-i,\sigma}(b) = \frac{1}{vt_i}. \tag{8}$$

To characterize σ based on this equation, we need to find a correspondence between the bid b and a type t_i for which b is a bid prescribed by σ_i. If $\tilde{F}_i$ has neither atom nor gap, then naturally the correspondence is given by

$$\gamma_{i,\sigma}(b) = \tilde{F}_i^{-1}\left(H_{i,\sigma}(b)\right) \tag{9}$$

for all b, so that $\gamma_{i,\sigma}(b)$ is the type whose cumulative mass is equal to the cumulative mass of the bid b. However, with more general $\tilde{F}_i$, which we cannot rule out because $\tilde{F}_i$ is endogenous here, the two cumulative masses may be impossible to be the same. Hence we generalize $\gamma_{i,\sigma}(b)$ to mean the infimum among i's types whose cumulative masses are not below that of b. More precisely, for any c.d.f. F, define the generalized inverse F^{-1} by

$$F^{-1}(s) := \inf \{t \in \operatorname{supp} F : F(t) \geq s\} \tag{10}$$

for each $s \in [0,1]$. Now define $\gamma_{i,\sigma}$ by Eq. (9) for each $b \in \mathbb{R}_+$. Then we prove that, for almost all $b \in [0, x_\sigma]$, Eq. (8) holds with $t_i = \gamma_{i,\sigma}(b)$ for each $i \in \{1,2\}$ (Lemma 12). That gives us a differential equation system for $(H_{1,\sigma}, H_{2,\sigma})$. Since $H_{i,\sigma}$ has no atom except at zero (Lemma 4.a), $H_{i,\sigma}$ can be decomposed into two parts, one absolutely continuous, the other singular with mass $H_{i,\sigma}(0)$ at zero. Integration of the differential equation yields the absolutely continuous part, to which we add the mass $H_{i,\sigma}(0)$, denoted by $c_{i,\sigma}$, at zero.[4] Hence we characterize the equilibrium bid distributions given arbitrary type-distributions.

Theorem 2 *For any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any BNE σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ there exists a unique triple $(x_\sigma, c_{1,\sigma}, c_{2,\sigma}) \in \mathbb{R}_{++} \times [0,1]^2$ such that $c_{1,\sigma} c_{2,\sigma} = 0$ and, for each $i \in \{1,2\}$ and all $b \in [0, x_\sigma]$, $H_{i,\sigma}(x_\sigma) = 1$ and*

$$H_{i,\sigma}(b) = c_{i,\sigma} + \int_0^b \frac{1}{v\tilde{F}_{-i}^{-1}\left(H_{-i,\sigma}(y)\right)} dy. \tag{11}$$

This pins down the equilibrium strategy uniquely via the next lemma.

Lemma 5 *If H is a c.d.f. that has neither gap nor atom in $(0, x]$, with $[0, x]$ its support, then for any c.d.f. F there is at most one strategy $\sigma : \mathbb{R}_+ \times \operatorname{supp} F \to [0,1]$ that is monotone and $H(b) = \int_{\mathbb{R}} \int_0^b \sigma(dr, t) dF(t)$ for all $b \in \mathbb{R}_+$.*

6 The Outside Option

The above characterization, facilitating comparison among various posterior beliefs coupled with their associated equilibriums, leads to the next theorem, which locates the worst posterior belief for the most belligerent type z_i of each player i. To describe player i's bid in response to the other player's continuation play rationalized by this posterior belief, especially when i's type is not expected by this posterior and hence his best response to the

[4] This way of solving the differential equation is a slight improvement of the usual procedure which, illustrated in Fudenberg and Tirole [11, fn 16, p234], assumes Lipschitz continuity to solve the differential equation but then haphazardly deals with the failure of Lipschitz continuity at zero.

continuation play may be empty, denote

$$\mathrm{BR}_i\left(t_i, \epsilon \mid H\right) := \left\{b \in \mathbb{R}_+ : \forall b' \in \mathbb{R}_+ \left[vH^*(b) - b/t_i \geq vH^*(b') - b'/t_i - \epsilon\right]\right\}$$

for any $t_i \in [a_i, z_i]$, any $\epsilon \geq 0$ and any c.d.f. H, with the notation H^* defined in Eq. (4).

Theorem 3 *For any $i \in \{1, 2\}$:*

a. *there exists a unique BNE, σ_*^i, in the contest game $\mathscr{G}\left(\delta_{z_i}, F_{-i}\right)$, where δ_{z_i} denotes the Dirac measure at z_i;*

b. *for any $t_i \in [a_i, z_i)$, $\lim_{\epsilon \downarrow 0} \sup \mathrm{BR}_i\left(t_i, \epsilon \mid H_{-i,\sigma_*^i}\right) = 0$, and if $H_{-i,\sigma_*^i}(0) = 0$ then $\mathrm{BR}_i\left(t_i, 0 \mid H_{-i,\sigma_*^i}\right) = \{0\}$;*

c. *$\underline{u}_i = vc_{-i,\sigma_*^i} = U_i(t_i \mid H_{-i,\sigma_*^i})$ for all $t_i \in [a_i, z_i] \setminus \{0\}$.*

In Part (a), the unique form of the equilibrium σ_*^i follows from Theorem 2 applied to the case where the type-distribution of player i is degenerate to z_i. For the existence claim, Appendix A.6.1 presents properties of the function $\gamma_{i,\sigma}$, with which Appendix A.6.4 verifies the equilibrium condition of σ_*^i.

While σ_*^i is an equilibrium, it is based on the posterior δ_{z_i} of the peace vetoer i. Expecting none but type z_i of player i, the opponent $-i$ draws her bid from the distribution H_{-i,σ_*^i}, which may have an atom at zero, rendering the best response for almost all types of i empty (Lemma 17, Appendix A.6.4), though intuitively speaking player i with such types would bid just slightly above the atom zero. Part (b) formalizes such an intuition.

Part (c) coupled with Eq. (5) implies that the posterior δ_{z_i} about i induces the lowest possible surplus for type z_i among all posteriors. One might have guessed such a result with an intuition that when i's type is degenerate to z_i this type would get zero information rent. The problem of this intuition, however, is that zero information rent merely means that type z_i does not get higher surplus than the infimum type a_i, whereas even type a_i might enjoy a high surplus as the opponent is intimidated by i, believed to be of type z_i.

Remark 3 Theorem 3.c should not be confused with observations that a bidder would rather play an auction game where his type is drawn from the same distribution as his rival's than an auction where his type, say t_i, is commonly known (e.g., Kovenock, Morath and Münster [17]). Such observations are binary comparisons between the games $\mathscr{G}(F, F)$

and $\mathscr{G}(\delta_{t_i}, F)$, and the symmetric distribution in $\mathscr{G}(F, F)$, facilitating an explicit solution, is crucial to such observations. The comparison in Theorem 3.c, by contrast, is not binary, but rather between the game $\mathscr{G}(\delta_{z_i}, F_{-i})$ and a continuum of other games $\mathscr{G}(\tilde{F}_i, F_{-i})$, with $\tilde{F}_i$ ranging among all posteriors of i. The theorem does not rely on symmetric distributions.

Crucial in the proof of Part (c) is the next lemma, saying that the type z_i of player i bids up to a higher supremum $x_{\sigma^i_*}$ in the equilibrium σ^*_i given the degenerate posterior than in any equilibrium σ given other posteriors. Since bidding such supremum levels guarantees winning, the lemma implies $U_i(z_i \mid H_{-i,\sigma^i_*}) = v - x_{\sigma^i_*}/z_i \leq v - x_\sigma/z_i = U_i(z_i \mid H_{-i,\sigma})$.

Lemma 6 *For any $i \in \{1, 2\}$, if $c_{-i,\sigma^i_*} > 0$ then $x_{\sigma^i_*} \geq x_\sigma$ for any posterior distribution $\tilde{F}_i$ of player i's type and any $\sigma \in \mathscr{E}_i(\tilde{F}_i)$.*

The lemma results from a subtle linkage between the two players' marginal costs of bids. By our definition of a player's payoff, a player i's marginal cost of bids is equal to $1/t_i$ when t_i is supposed to be his type that submits the bid. At the equilibrium σ^i_*, with i's type degenerate to the type supremum z_i, his marginal cost $1/z_i$ is less than his marginal cost $1/t_i$ at any equilibrium say σ given other posteriors. Thus, his marginal revenue of bids at equilibrium σ^i_* is less than that at equilibrium σ. Again by definition of a player's payoff, his marginal revenue of bids is proportional to the slope of his opponent $-i$'s bid-distribution function. Hence player $-i$'s bid distribution H_{-i,σ^i_*} at σ^i_* is less steep than her bid distribution $H_{-i,\sigma}$ at σ. Consequently, unless $x_{\sigma^i_*} \geq x_\sigma$, $H_{-i,\sigma}$ first-order stochastically dominates H_{-i,σ^i_*}; thus, for any bid, the type of $-i$ that submits it at equilibrium σ^i_* is higher than the type of $-i$ that submits it at σ. In other words, player $-i$'s marginal cost of bids, and hence her marginal revenue, are lower at equilibrium σ^i_* than at equilibrium σ. Thus, since her marginal revenue is proportional to the slope of her opponent i's bid distribution, player i's bid distribution H_{i,σ^i_*} rises at a lower rate at equilibrium σ^i_* than the bid distribution $H_{i,\sigma}$ does at σ. Since $H_{i,\sigma^i_*}(0) = 0$ (due to the hypothesis $c_{-i,\sigma^i_*} > 0$ of the lemma and the fact that the zero bid cannot be an atom for both players), H_{i,σ^i_*} stochastically dominates $H_{i,\sigma}$, which implies that at the supremums of their supports, $x_{\sigma^i_*} \geq x_\sigma$.

Remark 4 Theorem 3 characterizes the limit of off-path expected payoffs not only for type z_i of player i, but also for all other positive types, which are not expected in, and may have no best response to, the continuation equilibrium σ^i_* (Lemma 17.b, Appendix A.6.4): this off-path surplus is identically vc_{-i,σ_*^i} for all positive types of i. By contrast, in the conflict resolution literature cited in the Introduction, a player's expected payoff from the conflict, modeled as an exogenous lottery there, varies with the player's type. Furthermore, in our model all the types $t_i \in [a_i, z_i)$ incur a zero or arbitrarily small cost in the conflict (Part (b) of the theorem), whereas in that literature the cost is an exogenous, positive constant.

7 The Condition for Peace

The outside option vc_{-i,σ_*^i} identified in Theorem 3 is determined by the primitives quite straightforwardly according to the next lemma, which uses the notation F_i^{-1} of generalized inverses and, for notational cleanliness, switches the roles between i and $-i$ in c_{-i,σ_*^i}.

Lemma 7 *For each $i \in \{1, 2\}$,*

$$c_{i,\sigma_*^{-i}} = c_i^* := \inf \left\{ c_i \in [0,1] : z_{-i} \int_{c_i}^{1} \frac{1}{F_i^{-1}(s)} ds \leq 1 \right\}. \tag{12}$$

Theorem 3, combined with Lemmas 3 and 7, implies the main result, where the condition (13) for peace is purely about the prior distributions (F_1, F_2) according to Eq. (12):

Theorem 4 *Peaceful mechanisms exist if and only if*

$$c_1^* + c_2^* \leq 1. \tag{13}$$

Although Theorem 4 characterizes the possibility of peaceful mechanisms without restriction on off-path posterior beliefs, such possibility of peace does not shrink at all when we restrict the posteriors by equilibrium refinements:

Corollary 3 *Theorem 4 remains true when the equilibrium concept is refined by the intuitive criterion and universal divinity.*

This corollary is due to the fact that, among all types of a player i, type z_i is most tempted to veto a peace proposal (Lemma 2). Thus, conditional on i's deviation of vetoing peace, it is reasonable to adopt the posterior belief δ_{z_i} that i's type is z_i, which is precisely the one that induces the maximal possibility for the existence of peaceful mechanisms.

Remark 5 If the peaceful payoff for player i is equal to $\underline{u}_i$, one can show that the posterior belief δ_{z_i} in the event of i's unilateral deviation is a credible veto belief in the sense of

Cramton and Palfrey [8], and that the continuation equilibrium σ_*^i induced by δ_{z_i} makes type z_i, the support of δ_{z_i}, indifferent between vetoing and ratifying the peace proposal. Thus, the peaceful payoff $\underline{u}_i$ is ratifiable in the sense of Cramton and Palfrey. Consequently, if the mediator can propose peaceful allocations that do not exhaust the value of the prize, then when Condition (13) is satisfied the peaceful allocation $(\underline{u}_1, \underline{u}_2)$ is ratifiable, albeit causing a deadweight loss $v - \underline{u}_1 - \underline{u}_2$ from the players' viewpoint. Without such flexibility, however, ratifiability of peaceful allocations other than $(\underline{u}_1, \underline{u}_2)$ is an open question.[5]

Remark 6 We can think of Condition (13) as the limit of the conditions for negotiation mechanisms that result in conflict with positive but arbitrarily small probabilities. For simplicity, suppose that the prior distributions are atomless, gapless and $a_i = 0$ for both i. Consider a type-independent peaceful split offering payoff v_i to player i, for each $i \in \{1, 2\}$, and let ψ_i be the ex ante probability with which player i accepts it at equilibrium. Denote $\sigma(A, R)$ for the continuation equilibrium after i accepts it and $-i$ rejects it, $\sigma(R, A)$ after i rejects it and $-i$ accepts it, and $\sigma(R, R)$ after both reject it. For any type t_i of player i, the expected payoff from accepting the proposal is equal to

$$\psi_{-i} v_i + (1 - \psi_{-i}) U_i \left(t_i \mid H_{-i,\sigma(A,R)} \right),$$

and that from rejecting it equal to

$$\psi_{-i} U_i \left(t_i \mid H_{-i,\sigma(R,A)} \right) + (1 - \psi_{-i}) U_i \left(t_i \mid H_{-i,\sigma(R,R)} \right).$$

Hence accepting the proposal is a best response for t_i if and only if $v_i \geq W_i(t_i)$, where

$$W_i(t_i) := U_i \left(t_i \mid H_{-i,\sigma(R,A)} \right) + \frac{1 - \psi_{-i}}{\psi_{-i}} \left(U_i \left(t_i \mid H_{-i,\sigma(R,R)} \right) - U_i \left(t_i \mid H_{-i,\sigma(A,R)} \right) \right). \tag{14}$$

By Eq. (7), $\frac{d}{dt_i} W_i(t_i) = \frac{v}{t_i^2} \int_0^{t_i} s d\pi_i(s)$, where π_i is a signed measure: for any $s \in [a_i, z_i]$,

$$\pi_i(s) := H_{-i,\sigma(R,A)} \left(\beta_{i,\sigma(R,A)}(s) \right) + \frac{1 - \psi_{-i}}{\psi_{-i}} \left(H_{-i,\sigma(R,R)} \left(\beta_{i,\sigma(R,R)}(s) \right) - H_{-i,\sigma(A,R)} \left(\beta_{i,\sigma(A,R)}(s) \right) \right),$$

[5] The complication stems from a fixed-point condition that Cramton and Palfrey require of a credible posterior belief $\tilde{F}_i$ in the off-path event that i vetoes the peace proposal. Their condition requires that $\tilde{F}_i$ be Bayesian-consistent to i's optimal choice between ratifying the peace proposal versus vetoing it and then best responding to the off-path continuation equilibrium induced by $\tilde{F}_i$. To check ratifiability of a peace proposal, one needs to characterize the entire set of such fixed points.

with $\beta_{i,\sigma}$ specified in Corollary 2. Thus, if π_i is a positive measure, which is true if ψ_{-i} is sufficiently small and $H_{-i,\sigma(R,A)} \circ \beta_{i,\sigma(R,A)} \geq H_{-i,\sigma(A,R)} \circ \beta_{i,\sigma(A,R)}$, then W_i is strictly increasing on $[a_i, z_i]$, hence there exists $\tau_i \in [a_i, z_i]$ such that $v_i \geq W_i(t_i)$ if and only if $t_i \leq \tau_i$; i.e., $[\tau_i, z_i]$ is the posterior support of player i's type in the event that i rejects the proposal. Apply the same reasoning to player $-i$ and we have, if π_1 and π_2 are positive measures,

$$v \geq v_1 + v_2 \geq W_1(\tau_1) + W_2(\tau_2). \tag{15}$$

Now suppose there is a sequence of such peace proposals whose corresponding sequence of (ψ_1, ψ_2) converges to $(1, 1)$, then the corresponding sequence of (τ_1, τ_2) converges to (z_1, z_2), and that of $\sigma(R, A)$ converges to the continuation equilibrium σ_*^i, which gives surplus $c_{-i}^* v$ to player i. In short, if $(\psi_1, \psi_2) \to (1, 1)$ then $(W_1(\tau_1), W_2(\tau_2)) \to (c_2^* v, c_1^* v)$. Then Ineq. (15) implies $1 \geq c_2^* + c_1^*$, which is the peace condition (13).

The next corollary and examples illustrate the tractability of the peace condition.

Corollary 4 *If $F_1 = F_2 = F$ for some c.d.f. F with z the supremum of $\operatorname{supp} F$, there exists a unique $c_* \in [0, 1)$ such that*

$$z \int_{c_*}^{1} \frac{1}{F^{-1}(s)} ds = 1, \tag{16}$$

and peaceful mechanisms exist if and only if $c_* \leq 1/2$.

Example 1 If $F_1 = F_2 = F$ and F is the uniform distribution on $[a, z]$, then peaceful mechanisms exist. To see that, note $F^{-1}(s) = a + (z - a)s$ for any $s \in [0, 1]$. Hence the left-hand side of Eq. (16) is equal to

$$z \int_{c_*}^{1} (a + (z - a)s)^{-1} \, ds = \frac{z}{z - a} \ln \frac{z}{a + (z - a)c_*}.$$

Thus Eq. (16) implies

$$c_* = \frac{e^{-1+a/z} - a/z}{1 - a/z}.$$

We claim that $c_* \leq 1/2$, which, by the above equation and the fact $a \leq z$, is equivalent to

$$2e^{-1+r} - r \leq 1$$

for all $r \in [0, 1]$. Since the left-hand side of this inequality is convex in r, it attains its maximum at either $r = 0$ or $r = 1$. When $r = 0$, $2e^{-1+r} - r = 2/e < 1$; when $r = 1$, $2e^{-1+r} - r = 1$. Thus, $2e^{-1+r} - r \leq 1$ for all $r \in [0, 1]$, as claimed.

Example 2 The peace condition $c_* \le 1/2$ is satisfied when $F(t) = \sqrt{t}$ for all $t \in [0, 1]$, as $c_* = 1/2$ by Eq. (16). By contrast, for the distribution $F(t) = t^{1/3}$ for all $t \in [0, 1]$, Eq. (16) becomes $\int_{c_*}^1 s^{-3} ds = \left(c_*^{-2} - 1\right)/2 = 1$, i.e., $c_* = 1/\sqrt{3} > 1/2$, violating the peace condition.

Example 3 To underscore the applicability of our result to both continuous and discrete distributions, suppose that the type of each player is independently drawn from the same binary distribution F, supported by $\{a, z\}$ with $a < z$, such that $F(a) = \theta$ for some $\theta \in (0, 1)$. By Eq. (10), the generalized inverse of F is

$$F^{-1}(s) = \begin{cases} z & \text{if } s \in (\theta, 1] \\ a & \text{if } s \in [0, \theta]. \end{cases}$$

If $c > \theta$ then $z \int_c^1 \frac{1}{F^{-1}(s)} ds = (z/z)(1 - c) < 1$; if $c \in [0, \theta]$,

$$z \int_c^1 \frac{1}{F^{-1}(s)} ds \le 1 \iff \frac{z}{z}(1 - \theta) + \frac{z}{a}(\theta - c) \le 1 \iff c \ge \left(1 - \frac{a}{z}\right) \theta.$$

Thus, $c_* = \left(1 - \frac{a}{z}\right) \theta$ by Eq. (16), and peaceful mechanisms exist if and only if $\left(1 - \frac{a}{z}\right) \theta \le \frac{1}{2}$, which is satisfied if and only if the probability θ of being the weak type a is sufficiently small.

8 The Effect of Ex Ante Strength

Obviously the peace condition Ineq. (13) is satisfied if at least one of c_1^* and c_2^*, the low ends of the contestants' prior distributions that are minimally excluded via Eq. (12), is sufficiently small. For c_i^* to be small, it suffices to have a prior distribution F_i that ranks high in stochastic dominance:

Lemma 8 *For each $i \in \{1, 2\}$, if F_i becomes more stochastically dominant while z_{-i} is either unchanged or lowered, then c_i^* becomes weakly smaller than before.*

Thus peace is guaranteed if at least one contestant becomes ex ante sufficiently strong in the sense of stochastic dominance. To formalize that, for any two distributions F and $\widehat{F}$, write $\widehat{F} \triangleright F$ if and only if $\widehat{F}$ first-order stochastically dominates F and $\operatorname{supp} \widehat{F} = \operatorname{supp} F$.

Theorem 5 *Given any prior distributions (F_1, F_2) of the contestants' types:*

a. if $\widehat{F}_i \triangleright F_i$ for each $i \in \{1, 2\}$ then:

i. if $(F_i)_{i=1}^2$ *admits peaceful mechanisms, so does* $(\widehat{F}_i)_{i=1}^2$*;*

ii. if $(\widehat{F}_i)_{i=1}^2$ *does not admit peaceful mechanisms, neither does* $(F_i)_{i=1}^2$*;*

b. there exists $(\widehat{F}_i)_{i=1}^2$ *that admits peaceful mechanisms and* $\widehat{F}_i \triangleright F_i$ *for each* $i \in \{1, 2\}$*.*

Part (b) of the theorem says that peace is guaranteed if at least one of the contestants becomes ex ante strong. It is worth noting that such guarantee for peace does not require that the other contestant be ex ante weaker than the ex ante strong one. Rather, the prospect of peace can only improve when both contestants become ex ante strong including when they become equally so. One can easily prove a corollary of Theorem 5: For any prior distributions (F_1, F_2) such that $\operatorname{supp} F_1 = \operatorname{supp} F_2$, there exists $\widehat{F}$ such that $\widehat{F} \triangleright F_i$ for each $i \in \{1, 2\}$ and, when $\widehat{F}_1 = \widehat{F}_2 = \widehat{F}$, $(\widehat{F}_i)_{i=1}^2$ admits peaceful mechanisms.

That peace can be ensured without ex ante disparity between the contestants highlights the contrast between the mechanism design problem in conflict resolution and that in bilateral trade, as the Myerson-Satterthwaite theorem on the latter says that full efficiency is impossible when traders' type-supports overlap. Disparity is unnecessary in our setting because the minimal payoff that guarantees acceptance from a contestant is assessed not according to a competition between the prior distributions of the players, nor that between their realized types, but rather between an a priori fixed type of the contestant and the prior distribution of his opponent (Theorem 3). Hence the ex ante strength of one player suffices to deter conflict triggered by the other.[6]

Remark 7 The ex ante strengthening of a contestant i does not necessarily benefit i, nor require any more efforts from $-i$ to punish the former should i veto peace. To see that, let the prior distribution F_i of contestant i be replaced by some $\widehat{F}_i$ with $\widehat{F}_i \triangleright F_i$, while that of $-i$ unchanged. The replacement shrinks $c_{i,\sigma_*^{-i}}$ according to Eq. (12), whereas the c_{-i,σ_*^i} for contestant i, determined by Eq. (12) with the roles of i and $-i$ exchanged, remains the same. By Theorem 3.c, therefore, replacing F_i by $\tilde{F}_i$ merely reduces the other contestant $-i$'s minimal peaceful payoff $\underline{u}_{-i}$ without increasing i's. Furthermore, in the event that player i

[6] While ex ante disparity is not necessary for peace, peace is guaranteed when such disparity is sufficiently large. One can prove that, for any $i \in \{1, 2\}$ with $a_i > 0$, there exists $\xi_i \in (a_i, z_i)$ such that whenever $z_{-i} \leq \xi_i$ we have $c_i^* = 0$, thereby satisfying the peace condition Ineq. (13) because $c_{-i}^* \in [0, 1]$ by definition. Even in this case, however, with z_{-i} allowed to be above a_i, such disparity does not require non-overlapping type-supports, nor even that one's type-support be lower in strong-set order than the other's.

vetoes peace so that the other contestant $-i$ punishes i through the continuation play σ_*^i, the bid distribution H_{-i,σ_*^i} of the punisher, determined by Eq. (26), remains unchanged before and after the replacement, as it alters neither z_i nor c_{-i,σ_*^i}.

9 Concluding Remarks

Fundamental to humanity is the question whether conflicts can be preempted by peace settlements. Like its counterpart in bilateral trade, the question concerns the possibility for opposite sides to achieve a socially optimal outcome despite their information asymmetry. Yet the role of private information is different in conflict preemption than in bilateral trade. Whereas a trader's private information determines her on-path payoff, a contestant's private information affects his off-path payoff through a continuation equilibrium of the contest after negotiation fails. Examining a general model of such a relatively new structure, this paper develops a general method to solve the two-player contest game and as a result presents an exact characterization of the possibility of conflict preemption. The peace condition implies that the prospect of social optimums in conflict preemption is not as bleak as in bilateral trade. It also implies explicit predictions regarding the possibilities of conflict preemption given various distributions of private information.

While this paper, motivated by issues about conflict resolution, focuses on all-pay contests, the method developed here can handle cases where the all-pay contest is replaced by other formats of auctions, hence applicable to the study of bidding collusion (Remark 1). In a similar spirit, though relevant to different contexts, is to investigate how various contest mechanisms in the conflict phase may affect the prospect of conflict preemption. This problem is particularly germane, and anticipated by Spier [23], when the conflict is litigation, where the fraction of the winner's fees that the loser needs to pay varies with the litigation system. While Klemperer [16] argues that such fee-shifting rules are irrelevant when the revenue equivalence theorem applies, the theorem is inapplicable to our continuation game because the posteriors, being endogenous, need not be identical between contestants.

A second problem is to calculate the negotiation mechanisms that maximize the social surplus in cases where conflict ensues with strictly positive probability. The primitives for such cases have been characterized by Theorem 4. In those cases, while the posterior beliefs during on-path conflicts are determined differently from those in this paper, our general solution of the contest game given arbitrary posteriors will still be useful (c.f. Remark 6).

A potential direction for extension is to consider types correlated between the contestants at the outset. A major change due to such correlation is that an equilibrium in the contest continuation game is no longer necessarily monotone. Moreover, there exist cases with strong correlations that render monotone equilibrium nonexistent.[7]

Another possible extension concerns unmediated environments, where one of the contestants instead of a neutral mediator proposes a negotiation mechanism. The necessity of the peace condition can be easily extended to such cases. The sufficiency of the condition, whereas, is no longer guaranteed. One can construct cases where the condition is satisfied and yet peaceful equilibriums do not exist, as the proposing contestant, like an undersupplying monopolist, would rather undercut the offer to the opponent at the risk of conflict.

A Proofs

A.1 Lemma 1 and Corollary 1

A.1.1 Augmented Revelation Mechanisms

For each contestant $i \in \{1,2\}$ let $T_i := \operatorname{supp} F_i$, which we identify as i's type space. An *augmented revelation mechanism (ARM)* is denoted by a tuple $\left(q, (p_i, d_i)_{i=1}^2\right)$ of functions,

$$q : \Pi_{j=1}^2 \left(T_j \cup \{\text{out}\}\right) \to [0,1],$$
$$p_i : \Pi_{j=1}^2 T_j \to [0,v],$$
$$d_i : \Pi_{j=1}^2 \left(T_j \cup \{\text{out}\}\right) \to 2^{T_i} \setminus \{\varnothing\},$$

such that, for any $i \in \{1,2\}$ and any $(t_i, t_{-i}) \in \Pi_{j=1}^2 \left(T_j \cup \{\text{out}\}\right)$,

$$q(\text{out}, t_{-i}) = 1,$$
$$q(t_i, t_{-i}) < 1 \Rightarrow \textstyle\sum_{i=1}^2 p_i(t_1, t_2) = v,$$
$$t_i \neq \text{out} \neq t_{-i} \Rightarrow t_i \in d_i(t_i, t_{-i}) \subseteq \{t_i' \in T_i : q(t_i', t_{-i}) > 0\}.$$
$$d_i(\text{out}, t_{-i}) = d_i(\text{out}) = d_i(\text{out}, t_{-i}') \quad \forall t_{-i}' \in T_{-i} \cup \{\text{out}\}.$$

[7] Nicholas Bedard and this author in a joint work in progress have found a case where monotone equilibrium does not exist. Despite such correlation, we have constructed within a subcase a (non-monotone) equilibrium where peace is ensured.

The interpretation is:

i. each player i either announces that he does not participate ("out"), or participates and reports a type confidentially to the mechanism;

ii. $q(t_1, t_2)$ is equal to the probability with which conflict is the outcome;

iii. if the outcome is peace then $(p_1(t_1, t_2), p_2(t_1, t_2))$ is the split of the prize;

iv. if both players participate in the mechanism and if the outcome is conflict, then for each $i \in \{1, 2\}$ the fact that player i's message belongs to $d_i(t_i, t_{-i})$ is confidentially disclosed to player $-i$ before the contest begins;

v. what others can infer about a player from his nonparticipation cannot depend on the information revealed by the other player, as nonparticipation shuts down the mechanism for the players to communicate with each other.

Remark 8 "Out" is included in the message space due to the dynamic nature of the negotiation-contest game, in which the action "participate in the mechanism and reject all peace recommendations" is not equivalent to nonparticipation. By the former action the player may obtain through the mechanism some signal from the rival, which may affect the continuation play in the contest; whereas the latter action simply shuts down the mechanism and hence all communication channels. Thus, different from the standard revelation principle, it may lose generality to replace nonparticipation with reporting a type for which nonparticpation is optimal to the player.

Remark 9 Different from a mechanism defined in the model, an ARM does not offer a player an ex post option to reject a peace recommendation. Hence an ARM is binding.

Given any ARM $\left(q, (p_i, d_i)_{i=1}^2\right)$ and any pair $(\hat{t}_1, \hat{t}_2) \in \Pi_{i=1}^2 (T_i \cup \{\text{out}\})$ of reports, if the outcome is conflict then the continuation game, denoted by $\mathscr{C}(\hat{t}_1, \hat{t}_2)$, is defined by the all-pay contest where player $-i$'s posterior belief of player i's type is derived from the prior F_i conditional on the event $d_i(\hat{t}_1, \hat{t}_2)$ disclosed to player $-i$, for each $i \in \{1, 2\}$. Any BNE $\sigma(\hat{t}_1, \hat{t}_2)$ of $\mathscr{C}(\hat{t}_1, \hat{t}_2)$ determines the surplus $U_i\left(t_i \mid H_{-i,\sigma(\hat{t}_1,\hat{t}_2)}\right)$ for player i of type t_i according to Eq. (3). Expecting any such a mapping $\sigma(\cdot)$ from messages to continuation equilibriums, if player $-i$ participates and is truthful almost surely (relative to F_{-i}), player i's surplus from reporting $\hat{t}_i \in T_i \cup \{\text{out}\}$, given true type t_i, is equal to

$$u_i^{\sigma}(\hat{t}_i \mid t_i) := \mathbb{E}\left[q\left(\hat{t}_i, \boldsymbol{t}_{-i}\right) U_i\left(t_i \mid H_{-i,\sigma(\hat{t}_i,\boldsymbol{t}_{-i})}\right)\right] + \mathbb{E}\left[\left(1 - q\left(\hat{t}_i, \boldsymbol{t}_{-i}\right)\right) p_i\left(\hat{t}_i, \boldsymbol{t}_{-i}\right)\right], \tag{17}$$

where $\mathbb{E}[g(\boldsymbol{t}_{-i})]$ denotes the expected value of function g of random variable $\boldsymbol{t}_{-i}$, its boldface signifying the randomness, distributed according to the prior F_{-i}.[8] With continuation plays prescribed by $\sigma(\cdot)$, the ARM is *incentive compatible* for player i if and only if

$$u_i^{\sigma}(t_i \mid t_i) \geq u_i^{\sigma}(\hat{t}_i \mid t_i)$$

for all $t_i, \hat{t}_i \in T_i$. The ARM is *individually rational* for player i if and only if

$$u_i^{\sigma}(t_i \mid t_i) \geq u_i^{\sigma}(\text{out} \mid t_i)$$

for all $t_i \in T_i \setminus S_i$ such that S_i is of zero probability according to F_i and if $S_i \neq \varnothing$ then $S_i = d_i(\text{out})$.

An ARM is said *incentive feasible* if and only if (i) for any $(\hat{t}_1, \hat{t}_2) \in \Pi_{j=1}^{2} T_j$ there exists a BNE $\sigma(\hat{t}_1, \hat{t}_2)$ in the contest game $\mathscr{C}(\hat{t}_1, \hat{t}_2)$ and (ii) with continuation plays prescribed by $\sigma(\cdot)$ the ARM is incentive compatible and individually rational for each player i.

Lemma 9 *Suppose that, given mechanism M and the prior belief $(F_i)_{i=1}^2$, the continuation game admits a PBE $\mathscr{S}$ where both players participate in M almost surely on path. Then there exists an incentive feasible ARM such that, when this ARM replaces M, the ARM coupled with participation and truthtelling is outcome- and payoff-equivalent to the pair $(M, \mathscr{S})$.*

Proof First, modify the original $(M, \mathscr{S})$ by collapsing every player's entire sequence of actions, which includes announcing whether to participate, submitting messages and responding to recommendations, into a one-shot action in which the player submits a contingent message together with his response contingent on the recommendation and the accompanying disclosed information. The modification preserves the equilibrium condition because of the confidentiality of the recommendation and disclosed information. Second, conditional on participation, before playing in accord with the equilibrium in the modified mechanism,

[8] The types of player $-i$ that report "out" are omitted because they constitute a set of zero probability, as this definition is based on the hypothesis that player $-i$ participates and is truthful almost surely. By the same token, in the following we do not bother to spell out the condition for $d_i(t_i, \text{out})$.

each player i could have inputted a $\hat{t}_i \in T_i$ into a proxy for the proxy to play the equilibrium on his behalf as if his true type were $\hat{t}_i$. Thus for each player the message space is equivalent to the player's type space augmented with "out" (nonparticipation). The information disclosed or inferred on the path of the equilibrium induces the disclosure policy d_i for each player. Thus an ARM $\left(q, (p_i, d_i)_{i=1}^2\right)$ is defined. By a player's revealed preference in sending messages to his proxy and the original equilibrium condition, truthtelling is a best response in this ARM. By the hypothesis of full participation in the original $(M, \mathscr{S})$, the ARM is also individually rational, and hence incentive feasible. ■

A.1.2 Proof Lemma 1

Let M be a peaceful mechanism, and $\mathscr{S}$ the associated PBE. Since $(M, \mathscr{S})$ is peaceful, both players participate in M almost surely (relative to the prior) on path of $\mathscr{S}$. Thus Lemma 9 applies; we need only to consider any ARM $\left(q, (p_i, d_i)_{i=1}^2\right)$ that is incentive feasible and, with its associated continuation equilibrium prescribed by a mapping $\sigma(\cdot)$ in case of conflict, induces peace settlement on path almost surely. With the expectation operator $\mathbb{E}$ defined after (17), let

$$\begin{aligned} \overline{U}_i\left(\hat{t}_i, t_i\right) &:= \mathbb{E}\left[q\left(\hat{t}_i, \boldsymbol{t}_{-i}\right) U_i\left(t_i \mid H_{-i,\sigma(\hat{t}_i,\boldsymbol{t}_{-i})}\right)\right], \\ \overline{P}_i\left(\hat{t}_i\right) &:= \mathbb{E}\left[\left(1 - q\left(\hat{t}_i, \boldsymbol{t}_{-i}\right)\right) p_i\left(\hat{t}_i, \boldsymbol{t}_{-i}\right)\right]. \end{aligned}$$

Hence player i's surplus in reporting $\hat{t}_i$ in this ARM, given true type t_i, is equal to

$$u_i^\sigma\left(\hat{t} \mid t_i\right) = \overline{U}_i\left(\hat{t}_i, t_i\right) + \overline{P}_i\left(\hat{t}_i\right).$$

Since the ARM is peaceful, for any $i \in \{1, 2\}$ there exists an $A_i \subseteq \operatorname{supp} F_i$, with full probability measure relative to F_i, such that, for any $t_i \in A_i$, the type t_i of player i weakly prefers reporting t_i over "out" in the ARM, and $q(t_i, \cdot) = 0$ almost surely (relative to F_{-i}). Thus, for any $\hat{t}_i \in A_i$, $q(\hat{t}_i, \cdot) = 0$ almost surely, hence the definition of $\overline{U}_i\left(\hat{t}_i, t_i\right)$ implies

$$\forall \hat{t}_i \in A_i \; \forall t_i \in \operatorname{supp} F_i : u_i^\sigma(\hat{t}_i \mid t_i) = \overline{P}_i(\hat{t}_i). \tag{18}$$

To complete the proof, suppose $\overline{P}_i(t_i') > \overline{P}_i(t_i'')$ for some types $t_i', t_i'' \in A_i$. By Eq. (18), $u_i^\sigma(t_i' \mid t_i'') = \overline{P}_i(t_i') > \overline{P}_i(t_i'') = u_i^\sigma(t_i'' \mid t_i'')$, hence the type-$t_i''$ player i would rather misrepresent his type as t_i', a contradiction. Hence $\overline{P}_i$ is equal to some constant $k_i \in \mathbb{R}$ on A_i. ■

A.1.3 Proof of Corollary 1

Let M be a peaceful mechanism, and $\mathscr{S}$ its associated PBE. By Lemma 1, for each player i there exists an $A_i \subseteq \operatorname{supp} F_i$ and constant $k_i \in \mathbb{R}_+$ such that A_i is of full F_i-probability measure and i's equilibrium expected payoff is equal to k_i on A_i. For each $i \in \{1, 2\}$ pick any $t_i^* \in A_i$. With the notations p_i and $\overline{P}_i$ defined in the proof of Lemma 1,

$$k_i = \mathbb{E}\left[p_i(t_i^*, \boldsymbol{t}_{-i})\right] = \int_{A_i} \mathbb{E}\left[p_i(t_i^*, \boldsymbol{t}_{-i})\right] dF_i(t_i) = \int_{A_i} \mathbb{E}\left[p_i(t_i, \boldsymbol{t}_{-i})\right] dF_i(t_i) = \mathbb{E}\left[p_i(\boldsymbol{t}_i, \boldsymbol{t}_{-i})\right],$$

where the first and third equalities are due to the fact $\overline{P}_i = k_i$ on A_i, and the second and the last equalities due to the fact that A_i is of full probability measure. Hence

$$k_1 + k_2 = \mathbb{E}\left[p_1(\boldsymbol{t}_1, \boldsymbol{t}_2)\right] + \mathbb{E}\left[p_2(\boldsymbol{t}_1, \boldsymbol{t}_2)\right] = \mathbb{E}\left[p_1(\boldsymbol{t}_1, \boldsymbol{t}_2) + p_2(\boldsymbol{t}_1, \boldsymbol{t}_2)\right] = v.$$

Thus, the on-path payoff allocation of the original $(M, \mathscr{S})$ is almost surely equal to a type-independent proposal of a peaceful split (k_1, k_2). To complete the proof it suffices to show that if M is replaced by this proposal (k_1, k_2) then there is a PBE where all types of A_i, for both players i, accept the proposal. To this end, for each $i \in \{1, 2\}$ let the continuation equilibrium in the event that player i unilaterally rejects the proposal be the continuation equilibrium $\sigma(\hat{t}_i = \text{out})$ according to $\mathscr{S}$ in the event that player i chooses "out" unilaterally. Since $(M, \mathscr{S})$ is individually rational for i, for any $t_i \in A_i$ we have

$$u_i^\sigma(\text{out} \mid t_i) \leq u_i(t_i \mid t_i) \overset{(18)}{=} \overline{P}_i(t_i) = k_i.$$

Thus, given our construction of the continuation equilibrium for the type-independent proposal, the type-t_i player i's surplus from rejecting the proposal does not exceed his share k_i of the prize. Therefore, for each player i and given any type in A_i, which is of full probability measure, player i weakly prefers having the peaceful payoff k_i rather than rejecting the proposal. This being true for each i, the proposal is mutually acceptable, as desired. ■

A.2 Lemmas 2 and 3

A.2.1 Lemma 2

Let $t_i' > t_i$. If $t_i = 0$ then bidding zero is the best response for player i (see the definition of the cost of zero bid given zero type in the model) and so $U_i(t_i \mid H) = 0$. Thus $U_i(t_i' \mid H) \geq$

$U_i(t_i \mid H) = 0$ because type t_i' can alway ensure zero payoff by bidding zero. Hence assume that $t_i > 0$. Then $vH(b) - b/t_i' \geq vH(b) - b/t_i$ for any $b \in \mathbb{R}_+$. Consequently,

$$U_i(t_i' \mid H) = \sup_{b \in \mathbb{R}_+} vH(b) - b/t_i' \geq \sup_{b \in \mathbb{R}_+} vH(b) - b/t_i = U_i(t_i \mid H). \quad \blacksquare$$

A.2.2 Lemma 3

Lemma 10 *Suppose the hypotheses of Lemma 3. Given any peaceful mechanism (coupled with its associated PBE) and any $i \in \{1, 2\}$, there exists a unique $k_i \in \mathbb{R}_+$ such that player i's on-path expected payoff is equal to k_i on an F_i-probability-one subset of* $\operatorname{supp} F_i$ *and $k_i \geq \underline{u}_i$.*

Proof The constant k_i and the for-sure constancy of player i's on-path expected payoff follow Lemma 1. To prove the rest of the claim, suppose, to the contrary, that $k_i < \underline{u}_i$ for some $i \in \{1, 2\}$. According to the associated PBE, consider the continuation equilibrium, say σ, in the event where player i deviates to nonparticipation. With the mechanism peaceful, the other player $-i$ participates almost surely and hence the posterior of player $-i$'s type remains to be the prior. Thus, player i in making such deviation expects a distribution $H_{-i,\sigma}$ of bids from the rival $-i$, hence the supremum among i's expected payoffs from the deviation, with his bids ranging in $\mathbb{R}_+$, is equal to $U_i(t_i \mid H_{-i,\sigma})$. By definition of $\underline{u}_i$, $U_i(z_i \mid H_{-i,\sigma}) \geq \underline{u}_i$. Hence $U_i(z_i \mid H_{-i,\sigma}) > k_i$. Thus, with $U_i(\cdot \mid H_{-i,\sigma})$ continuous by hypothesis of the lemma, this strict inequality is preserved when z_i is replaced by t_i sufficiently near to z_i. Given any such type t_i, by definition of U_i in Eq. (3), player i can submit a bid against $H_{-i,\sigma}$ such that his expected payoff is arbitrarily close to $U_i(t_i \mid H_{-i,\sigma})$ and hence strictly greater than k_i. Hence player i with type t_i strictly prefers to deviate by nonparticipation. Thus, there is a set of i's types, of strictly positive F_i-probability, that strictly prefer to deviate: If z_i is an atom of F_i then $\{z_i\}$ is such a set; else, with $z_i = \sup \operatorname{supp} F_i$, $(z_i - \delta, z_i]$ is such a set for some $\delta > 0$. Hence we have obtained the desired contradiction. $\blacksquare$

Proof Lemma 3 To prove the "only if" part of the lemma, consider any peaceful mechanism. By Lemma 10, for each player i there exists a constant $k_i \in \mathbb{R}_+$ such that $k_i \geq \underline{u}_i$, and player i's on-path expected payoff in the mechanism is equal to k_i almost surely. By feasibility of peaceful splits, $k_1 + k_2 = v$. Thus, $v \geq \underline{u}_1 + \underline{u}_2$.

To prove the "if" part of the lemma, suppose that $\underline{u}_1 + \underline{u}_2 \leq v$. Then there exists (k_1, k_2) such that $k_1 + k_2 = v$ and $k_i \geq \underline{u}_i$ for each i. By hypothesis of the lemma, for each $i \in \{1, 2\}$

let $\underline{u}_i$ be attained by an equilibrium σ_*^i of the contest game, with $\tilde{F}_i^i$ the posterior distribution of i's type. Consider the game where the mechanism is the type-independent proposal that offers payoff $k_i \geq \underline{u}_i$ to player i, for each $i \in \{1, 2\}$. It suffices to construct a PBE of this game where every type of each player best replies by accepting the proposal. To that end, for each $i \in \{1, 2\}$ let the posterior belief in the off-path event where player i unilaterally rejects the proposal be the $\tilde{F}_i^i$, and σ_*^i the continuation equilibrium. By Lemma 2, for any $t_i \in \operatorname{supp} F_i$, $t_i \leq z_i$ and hence

$$U_i(t_i \mid H_{-i,\sigma_*^i}) \leq U_i(z_i \mid H_{-i,\sigma_*^i}) = \underline{u}_i \leq k_i.$$

Thus, by definition of U_i in (3), no type of player i can profit from vetoing the proposal. ■

A.3 Lemma 4: The No-Gap, No-Atom and Monotone Arguments

Claim (a): The supremum of the support of $H_{i,\sigma}$ exists by individual rationality, with the size v of the prize finite. By the payment rule of an all-pay auction and the equilibrium condition, this supremum is the same between the two players, and $H_{i,\sigma}$ has no gap in $[0, x_\sigma]$. To prove the no-atom claim, pick any $b \in (0, x_\sigma]$. We have noted that $H_{-i,\sigma}$ has no gap, hence for any $\epsilon > 0$ there exists a strictly positive mass of player $-i$'s equilibrium bids belonging to $(b - \epsilon, b)$. Thus, if b is an atom of $H_{i,\sigma}$, those types t_{-i} of $-i$ that submit such bids would deviate from such bids to a bid slightly above b when ϵ is sufficiently small, as the incremental revenue $v\left(H_{i,\sigma}(b + \epsilon) - H_{i,\sigma}(b - \epsilon)\right)$ outweighs the incremental cost $2\epsilon/t_{-i}$. This contradiction to the equilibrium condition implies that b is not an atom of $H_{i,\sigma}$.

Claim (b): We need only to prove that, for any $t_i', t_i'' \in \operatorname{supp} \tilde{F}_i$, if $t_i' < t_i''$, $b' \in \operatorname{supp} \sigma_i(\cdot, t_i')$ and $b'' \in \operatorname{supp} \sigma_i(\cdot, t_i'')$, then $b' \leq b''$. Since $b' \in \operatorname{supp} \sigma_i(\cdot, t_i')$, b' best replies $H_{-i,\sigma}$ for the type t' of player i. Thus, by revealed preference and Eq. (3),

$$vH_{-i,\sigma}^*(b') - b'/t_i' \geq vH_{-i,\sigma}^*(b'') - b''/t_i'.$$

The same reasoning applied to $b'' \in \operatorname{supp} \sigma_i(\cdot, t_i'')$ yields

$$vH_{-i,\sigma}^*(b'') - b''/t_i'' \geq vH_{-i,\sigma}^*(b') - b'/t_i''.$$

Sum these two inequalities to obtain $\left(b'' - b'\right)/t_i' \geq \left(b'' - b'\right)/t_i''$. Thus, $b'' - b' < 0 \Rightarrow 1/t_i' \leq 1/t_i'' \Rightarrow t_i' \geq t_i''$, which is the contrapositive of the claim. ■

A.4 Theorem 1 and Corollary 2

Theorem 1 With σ an equilibrium of the contest game, Lemma 4.a says that the bid distribution $H_{-i,\sigma}$ has no atom except possibly at the zero bid. Thus, by Eq. (4), $H^*_{-i,\sigma}(b) = H_{-i,\sigma}(b)$ for all $b \in \mathbb{R}_+ \setminus \{0\}$, and by the uniform tie-breaking rule $H^*_{-i,\sigma}(0) = H_{-i,\sigma}(0)/2$. Thus, for any $t_i \in [a_i, z_i] \setminus \{0\}$,

$$\begin{aligned} U_i(t_i \mid H_{-i,\sigma}) &\overset{(3)}{=} \sup_{b\in\{0\}\cup\mathbb{R}_{++}} vH^*_{-i,\sigma}(b) - b/t_i \\ &= \max\left\{vH_{-i,\sigma}(0)/2, \lim_{b\downarrow 0} vH_{-i,\sigma}(b) - b/t_i, \sup_{b\in\mathbb{R}_{++}} vH_{-i,\sigma}(b) - b/t_i\right\} \\ &= \max\left\{vH_{-i,\sigma}(0), \sup_{b\in\mathbb{R}_{++}} vH_{-i,\sigma}(b) - b/t_i\right\} \\ &= \sup_{b\in\mathbb{R}_+} vH_{-i,\sigma}(b) - b/t_i. \end{aligned} \tag{19}$$

Since $H_{-i,\sigma}$, a c.d.f., is upper semicontinuous and its only possible discontinuous point is zero (Lemma 4.a), $H_{-i,\sigma}$ restricted to $\mathbb{R}_+$ is continuous. This, combined with the fact that the domain for b in the problem (19) can be bounded without loss by $[0, vz_i]$, implies that the maximum in (19) is attained. Thus $U_i(t_i \mid H_{-i,\sigma}) = \max_{b\in\mathbb{R}_+} vH_{-i,\sigma}(b) - b/t_i$ for all $t_i \in [a_i, z_i] \setminus \{0\}$. Since $\max_{b\in\mathbb{R}_+} vH_{-i,\sigma}(b) - b/t_i$ is continuous in t_i for all $t_i \in \mathbb{R}_{++}$ by the theorem of maximum, $U_i(t_i \mid H_{-i,\sigma})$ is continuous in t_i for all $t_i \in [a_i, z_i] \setminus \{0\}$. ■

Corollary 2 Let β_i be any upper semicontinuous function $\beta_{i,\sigma} : [a_i, z_i] \to \mathbb{R}_+$ such that $\beta_{i,\sigma}(t_i) \in \arg\max_{b\in\mathbb{R}_+} vH_{-i,\sigma}(b) - b/t_i$ for all $t_i \in [a_i, z_i] \setminus \{0\}$. For any $t_i \in [a_i, z_i]$, define

$$\tilde{U}_i(t_i) := \max_{\hat{t}_i\in[a_i,z_i]} vt_iH_{-i,\sigma}\left(\beta_i(\hat{t}_i)\right) - \beta_i(\hat{t}_i). \tag{20}$$

By the envelope theorem, $\tilde{U}_i(t_i) = \int_{a_i}^{t_i} vH_{-i,\sigma}(\beta_i(s))\, ds + \tilde{U}_i(a_i)$ for any $t_i \in [a_i, z_i]$. Consequently, for every $t_i > 0$, Theorem 1 implies $U_i(t_i \mid H_{-i,\sigma}) = \tilde{U}_i(t_i)/t_i$ and hence

$$U_i(t_i \mid H_{-i,\sigma}) = \frac{1}{t_i}\left(\int_{a_i}^{t_i} vH_{-i,\sigma}(\beta_i(s))\, ds + \tilde{U}_i(a_i)\right). \tag{21}$$

Differentiability of $U_i(\cdot \mid H_{-i,\sigma})$ follows from Eq. (21), as $vH_{-i,\sigma}$ in the integrand is uniformly bounded. To prove the rest of the claim, denote $\pi_i(s_i) := H_{-i,\sigma}(\beta_i(s_i))$. One readily sees that β_i is weakly increasing. Thus, with β_i also upper semicontinuous, π_i is a distribution, whose support is contained in $[a_i, z_i]$. Now pick any $t_i \in (a_i, z_i)$. By Eq. (21),

$$\frac{d}{dt_i}U_i(t_i \mid H_{-i,\sigma}) = \frac{1}{t_i}\left(v\pi_i(t_i) - \frac{1}{t_i}\tilde{U}_i(a_i) - \frac{1}{t_i}\int_{a_i}^{t_i} v\pi_i(s)ds\right).$$

To the last integral on the right-hand side, apply integration by parts (Border [5, Corollary 8]), which is valid because both mappings π_i and $s \mapsto s$ are distributions, with the latter also continuous, hence

$$\begin{aligned}\frac{d}{dt_i}U_i(t_i \mid H_{-i,\sigma}) &= \frac{1}{t_i}\left(v\pi_i(t_i) - \frac{1}{t_i}\tilde{U}_i(a_i) - \frac{vt_i\pi_i(t_i) - v\pi_i(a_i)a_i}{t_i} + \frac{1}{t_i}\int_{a_i}^{t_i} vsd\pi_i(s)\right)\\ &= \frac{v}{t_i^2}\int_{a_i}^{t_i} sd\pi_i(s) + \frac{1}{t_i^2}\left(\pi_i(a_i)va_i - \tilde{U}_i(a_i)\right).\end{aligned}$$

When $a_i = 0$, the second term on the last line equals zero, with $\tilde{U}_i(0) = 0$ by Eq. (20). Hence Eq. (7). ■

A.5 Theorem 2 and Lemma 5

A.5.1 Preparation for Theorem 2

Lemma 11 says that $\gamma_{i,\sigma}(b)$, defined in Eq. (9), is essentially the supremum of player i's types whose bids do not exceed b at equilibrium σ. This cutoff type, Lemma 12 further observes, is almost surely the unique type for i to bid b at σ. Thus the first-order condition, Eq. (8), becomes a differential equation that yields the absolutely continuous part of the equilibrium bid distribution. Lemma 13 then justifies assembling this absolutely continuous part with the possible atom at zero, thereby obtaining the equilibrium bid distribution.

Lemma 11 *Given any c.d.f. F and $\sigma : \mathbb{R}_+ \times \operatorname{supp} F \to [0,1]$, let $H(b) := \int_{\mathbb{R}}\int_0^b \sigma(dr,t)dF(t)$ and $\gamma(b) := F^{-1}(H(b))$ for all $b \in \mathbb{R}_+$. If H has neither gap nor atom in $(0,x]$, with $[0,x]$ its support, and σ is monotone, then for any $b \in [0,x]$ and any $t,t' \in \operatorname{supp} F$ such that $t < \gamma(b) < t'$:*

a. for any $b \in [0,x]$, $F(\gamma(b)) \geq H(b)$;

b. $\sup\operatorname{supp}\sigma(\cdot,t) \leq b \leq \inf\operatorname{supp}\sigma(\cdot,t')$;

c. if $b \in \operatorname{supp}\sigma(\cdot,t')$, then $(\gamma(b),t')$ is a gap of F;

d. $b \in \operatorname{supp}\sigma(\cdot,\gamma(b))$.

Proof Claim (a): By Eq. (10) and the definition $\gamma(b) := F^{-1}(H(b))$ $(\forall b)$,

$$\gamma(b) = \inf\{\tau \in \operatorname{supp} F : F(\tau) \geq H(b)\}. \tag{22}$$

Thus Claim (a) follows from the upper semicontinuity of any distribution.

Claim (b): By Eq. (22) and $t < \gamma(b)$, $F(t) < H(b)$. If $\sup\operatorname{supp}\sigma(\cdot,t) > b$, then by monotonicity of σ no type above t would bid b, hence $H(b) \le F(t)$, contradiction. To prove the second inequality of Claim (b), suppose, to the contrary, that $b > \inf\operatorname{supp}\sigma(\cdot,t') =: b'$. By monotonicity of σ, no type below t' bids above b', hence $H(b') \ge \lim_{\tau\uparrow t'} F(\tau)$. Thus

$$H(b) > H(b') \ge \lim_{\tau\uparrow t'} F(\tau) \ge F\left(\gamma(b)\right) \ge H(b),$$

with the strict inequality due to the gapless H, the second last inequality due to $\gamma(b) < t'$, and the last due to Claim (a). The contradiction implies Claim (b).

Claim (c): First, we note that $b \le \sup\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right)$. Otherwise, $b > \sup\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right)$. This, combined with the monotonicity of σ and the second inequality in Claim (b) for all $t' > \gamma(b)$, means that H has a gap between $\sup\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right)$ and b, contradiction. Thus, if $b \in \operatorname{supp}\sigma(\cdot,t')$, and hence $b \ge \inf\operatorname{supp}\sigma(\cdot,t')$, we have for any $\tau \in (\gamma(b),t')$

$$b \le \sup\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right) \le \inf\operatorname{supp}\sigma(\cdot,\tau) \le \sup\operatorname{supp}\sigma(\cdot,\tau) \le \inf\operatorname{supp}\sigma(\cdot,t') \le b$$

by monotonicity of σ. Consequently,

$$\lim_{\tau\uparrow t'} F(\tau) \le H(b) \overset{(22)}{\le} F\left(\gamma(b)\right) \le \lim_{\tau\uparrow t'} F(\tau).$$

Thus $\lim_{\tau\uparrow t'} F(\tau) = F\left(\gamma(b)\right)$ for all $\tau \in (\gamma(b),t')$. I.e., $(\gamma(b),t')$ is a gap of F.

Claim (d): Note $b \ge \inf\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right)$, which, like $b \le \sup\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right)$ proved above, follows from the first inequality in Claim (b), the gapless H and monotone σ. Thus

$$\inf\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right) \le b \le \sup\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right).$$

Thus, $\operatorname{supp}\sigma\left(\cdot,\gamma(b)\right)$, convex due to the gapless H and monotone σ, contains b. ■

Lemma 12 *For any $\mathscr{G}(\tilde{F}_1,\tilde{F}_2)$, any BNE σ of $\mathscr{G}(\tilde{F}_1,\tilde{F}_2)$ and any $i \in \{1,2\}$, for almost every $b \in [0,x_\sigma]$, $H_{-i,\sigma}$ is differentiable at b and $\{\gamma_{i,\sigma}(b)\} = \{t_j \in [a_j,z_j] : b \in \operatorname{supp}\sigma_j(\cdot,t_j)\}$.*

Proof Denote $\Gamma_{i,\sigma}(b) := \{t_j \in [a_j,z_j] : b \in \operatorname{supp}\sigma_j(\cdot,t_j)\}$. A monotone function, $H_{-i,\sigma}$ is differentiable almost everywhere on $[0,x_\sigma]$; with zero not an atom of F_{-i} by assumption, $\{0\} \not\supseteq \Gamma_{i,\sigma}(b)$ for almost all such differentiable points b. For any such b, let $t_i \in \Gamma_{i,\sigma}(b) \setminus \{0\}$; the first-order necessary condition for b to be a best reply for the type t_i of player i implies that the derivative of his expected payoff at b is

$$vH'_{-i,\sigma}(b) - 1/t_i \ge 0,$$

which in turn implies, for any $t'_i > t_i$, that $vH'_{-i,\sigma}(b) - 1/t'_i > 0$ and hence b cannot be a best reply for the type t'_i. Thus, $\Gamma_{i,\sigma}(b)$ cannot contain both t_i and t'_i. Hence the singleton $\Gamma_{i,\sigma}(b)$ can only contain $\gamma_{i,\sigma}(b)$, which belongs to $\Gamma_{i,\sigma}(b)$ by Lemma 11.d. ■

Lemma 13 *For any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, any BNE σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any $i \in \{1, 2\}$, there exists a unique $c_{i,\sigma} \in [0, 1]$ and a unique function $\tilde{H}_{i,\sigma} : [0, x_\sigma] \to [0, 1]$ such that $\tilde{H}_{i,\sigma}$ is absolutely continuous, $\tilde{H}_{i,\sigma}(0) = 0$ and, for any $b \in [0, x_\sigma]$,*

$$H_{i,\sigma}(b) = \tilde{H}_{i,\sigma}(b) + c_{i,\sigma}; \tag{23}$$

furthermore, $c_{1,\sigma}c_{2,\sigma} = 0$.

Proof Since $H_{i,\sigma}$ is a c.d.f., its corresponding probability measure, say μ, can be uniquely decomposed into the sum of two measures ν_0 and ν_1, with $\mu = \nu_0 + \nu_1$, such that ν_1 is absolutely continuous, and ν_0 singular, to the Lebesgue measure (Lebesuge decomposition theorem). By Lemma 4.a, the only possible atom of $H_{i,\sigma}$ is $\{0\}$, which is therefore the only possible support of the singular measure ν_0, and $\nu_0(\{0\})$ is equal to the mass of $\{0\}$ assigned by $H_{i,\sigma}$. Denote $c_{i,\sigma}$ for $\nu_0(\{0\})$, and $\tilde{H}_{i,\sigma}$ for the c.d.f. derived from ν_1. Then Eq. (23) follows from $\mu = \nu_0 + \nu_1$, and $\tilde{H}_{i,\sigma}$ is absolutely continuous since ν_1 is so. With ν_0 and ν_1 mutually singular, $\nu_1(\{0\}) = 0$, hence $\tilde{H}_{i,\sigma}(0) = 0$. Finally, $c_{1,\sigma}c_{2,\sigma} = 0$ because $\{0\}$ cannot be an atom of both players' equilibrium bid distributions, otherwise such nonzero measure of either player's zero-bidding types would deviate to a bid slightly above zero. ■

A.5.2 Proofs of Theorem 2 and Lemma 5

Theorem 2 Given equilibrium σ, the supremum x_σ of bids and each player i's bid distribution $H_{i,\sigma}$ are uniquely defined by Eqs. (1) and (2). By definition of x_σ, $H_{i,\sigma}(x_\sigma) = 1$. By Lemma 12, at almost every $b \in (0, x_\sigma)$, $H_{i,\sigma}$ is differentiable and $b \in \operatorname{supp} \sigma_i (\cdot, \gamma_{i,\sigma}(b))$ for each $i \in \{1, 2\}$. Thus, for any such b, the first-order necessary condition for b to best reply $H_{i,\sigma}$ for the type $\gamma_{-i,\sigma}(b)$ of player $-i$ holds. This condition is

$$\frac{d}{db}H_{i,\sigma}(b) = \frac{d}{db}\tilde{H}_{i,\sigma}(b) = \frac{1}{v\gamma_{-i,\sigma}(b)},$$

by Eq. (23) and the fact that the player's expected payoff is equal to $vH_{i,\sigma}(b') - b'/\gamma_{-i,\sigma}(b)$ if he bids b' nearby b. From the second equality in the above-displayed equation and the fact that $\tilde{H}_{i,\sigma}$ is absolutely continuous and $\tilde{H}_{i,\sigma}(0) = 0$ (Lemma 13), we have

$$\tilde{H}_{i,\sigma}(b) = \int_0^b \frac{1}{v\gamma_{-i,\sigma}(y)} dy$$

for all $b \in [0, x_\sigma]$. Then Eq. (11) follows from (23). The rest of the theorem, $c_{1,\sigma}c_{2,\sigma} = 0$, is simply the last statement of Lemma 13. ∎

Lemma 5 Define $\gamma(b) := F^{-1}(H(b))$ for all $b \in \mathbb{R}_+$. Then Eq. (22) holds and, by Lemma 11.a, $F(\gamma(b)) \geq H(b)$. Hence $H(b) = \int_{\mathbb{R}} \int_0^b \sigma(dr, t) dF(t)$ becomes

$$\begin{aligned} H(b) &= \int_{\mathbb{R}} \left(\mathbf{1}_{t<\gamma(b)} + \mathbf{1}_{t=\gamma(b)} \sigma(b,t) \right) dF(t) \\ &= \lim_{t \uparrow \gamma(b)} F(t) + \sigma(b, \gamma(b)) \left(F(\gamma(b)) - \lim_{t \uparrow \gamma(b)} F(t) \right). \end{aligned}$$

By monotonicity of σ, $\lim_{t \uparrow \gamma(b)} F(t) = H(\beta_*(b))$ where $\beta_*(b) := \inf \operatorname{supp} \sigma(\cdot, \gamma(b))$, hence

$$H(b) = H(\beta_*(b)) + \sigma(b, \gamma(b)) \left(F(\gamma(b)) - H((\beta_*(b))) \right). \tag{24}$$

If $\gamma(b)$ is an atom of F, its mass is equal to $F(\gamma(b)) - H(\beta_*(b))$ and $\sigma(b, \gamma(b))$ is uniquely determined by Eq. (24); else Eq. (24) is reduced to $H(b) = H(\beta_*(b))$, which by strict monotonicity of H means $\beta_*(b) = b$ and $\operatorname{supp} \sigma(\cdot, \gamma(b)) = \{\beta_*(b)\} = \{b\}$.

To pin down σ completely, consider those $t \in \operatorname{supp} F \setminus \operatorname{range} \gamma$. Since H is a continuous, one-to-one and onto function from $[0, x]$ to $[H(0), 1]$, for any such t, either (i) $F(t) < H(0)$ or (ii) $F(t) = H(b)$ for a unique $b \in [0, x]$. In Case (i), $t < \gamma(0)$ ($F(\gamma(0)) \geq H(0)$ by Lemma 11.a), hence monotonicity of σ implies that $\sigma(\cdot, t)$ is the Dirac measure at zero. In Case (ii), the fact $t \neq \gamma(b)$ implies, by Eq. (22), that $t > \gamma(b)$. Then Lemma 11.a implies

$$F(\gamma(b)) \geq H(b) = F(t) \geq F(\gamma(b)),$$

hence t is not an atom of F. Thus, with H gapless, $\operatorname{supp} \sigma(\cdot, t)$ is singleton. Consequently, σ cannot prescribe to t a bid $b' < b$; otherwise, some types above t would be prescribed to bid in (b', b) since H has no gap and σ is monotone, but then $H(b) > F(t)$, contradiction. By the same token, σ cannot prescribe to t a bid above b. Thus, $\sigma(\cdot, t)$ is the Dirac measure at b. All t in $\operatorname{supp} F$ considered, strategy σ is thus uniquely pinned down. ∎

A.6 Theorem 3 and Lemma 6

A.6.1 Preparation: Properties of Strategy σ and its Generalized Inverse $\gamma_{i,\sigma}$

Lemma 14 *Suppose that H is a c.d.f. that has neither gap nor atom in $(0, x]$, with $[0, x]$ its support. For any c.d.f. F let $\gamma(b) := F^{-1}(H(b))$ for all $b \in [0, x]$. Then—*

a. γ is weakly increasing on $[0, x]$;

b. $[\gamma(b) = t = \gamma(b')$ and $b \neq b'] \Leftrightarrow [t$ is an atom of $F]$;

c. if there is a unique $b \in [0, x]$ such that $\gamma(b) = t$, then $F(t) = H(b)$;

d. if $t \in \operatorname{supp} F \setminus \operatorname{range} \gamma$ then either (i) $F(t) < H(0)$ and $t < \gamma(0)$, or (ii) there exists a unique $b \in [0, x]$ such that $F(t) = H(b)$ and $F(\gamma(b)) = F(t)$.

Proof Claim (a): By the definition of γ and Eq. (10), Eq. (22) holds. Let $b' > b$. By Lemma 11.a, $F(\gamma(b')) \geq H(b')$; hence $F(\gamma(b')) \geq H(b)$ as H is increasing. Thus $\gamma(b') \geq \gamma(b)$ by Eq. (22).

Claim (b): Let $b' \geq b$ and $\gamma(b) = t = \gamma(b')$. By Lemma 11.a, $F(t) \geq H(b')$. By Eq. (22), $F(t') < H(b)$ for any $t' < t$ such that $t' \in \operatorname{supp} F$, hence $\lim_{t' \uparrow t:\, t' \in \operatorname{supp} F} F(t') \leq H(b)$. Thus,

$$F(t) - \lim_{t' \uparrow t:\, t' \in \operatorname{supp} F} F(t') \geq H(b') - H(b).$$

Since H has no gap, $H(b') - H(b) > 0 \Leftrightarrow b' > b$. Thus, $b' > b$ if and only if $F(t) - \lim_{t' \uparrow t:\, t' \in \operatorname{supp} F} F(t') > 0$, i.e., t is an atom of F.

Claim (c): Since $\gamma(b) = t$, $F(t) \geq H(b)$ (Lemma 11.a). Suppose $F(t) > H(b)$. Then there exists $b' > b$ for which $F(t) \geq H(b')$, as H has no gap. Thus, for any $t' \in \operatorname{supp} F$ such that $t' < t$, $F(t') < H(b')$, otherwise $F(t') \geq H(b') \geq H(b)$ and hence by Eq. (22) $\gamma(b) \neq t$, contradiction. Now that $F(t') < H(b')$ for all such t', by definition of $\gamma(b')$ we have $\gamma(b') = t$, contradicting the uniqueness of b. Hence $F(t) \leq H(b)$, as desired.

Claim (d) is implied by the second paragraph of the proof of Lemma 5. ■

Given any c.d.f. F, a c.d.f. $H : \mathbb{R}_+ \to [0, 1]$ is said *generated by strategy* $\sigma : \mathbb{R}_+ \times \operatorname{supp} F \to [0, 1]$ if and only if $H(b) = \int_{\mathbb{R}} \int_0^b \sigma(dr, t) dF(t)$ for any $b \in \mathbb{R}_+$.

Lemma 15 *Suppose that H is a c.d.f. that has neither gap nor atom in $(0, x]$, with $[0, x]$ its support. For any c.d.f. F let $\gamma(b) := F^{-1}(H(b))$ for all $b \in [0, x]$. Then—*

a. there exists a monotone strategy $\sigma : \mathbb{R}_+ \times \operatorname{supp} F \to [0,1]$ *that generates* H*;*

b. if H *is generated by a monotone strategy* $\sigma : \mathbb{R}_+ \times \operatorname{supp} F \to [0,1]$*, then—*

i. $[\operatorname{supp} \sigma(\cdot, t)$ *is not singleton*$] \iff t$ *is an atom of* F*;*

ii. for any $t \in \operatorname{supp} F$*, if* $t \geq \gamma(0)$ *then either* $\operatorname{supp} \sigma(\cdot, t) \subseteq \{b : \gamma(b) = t\}$ *or* $\operatorname{supp} \sigma(\cdot, t) = \{b\}$ *for which* $F(t) = H(b)$ *(and* $(\gamma(b), t)$ *is a gap of* F*); if* $t < \gamma(0)$ *then* $\operatorname{supp} \sigma(\cdot, t) = \{0\}$.

Proof Claim (a): First, construct a strategy σ: For any $t \in \operatorname{supp} F$ such that $t = \gamma(b)$ for some $b \in \mathbb{R}_+$, if t is an atom of F then define a c.d.f. $\sigma(\cdot, t)$ according to Eq. (24); else define $\sigma(\cdot, t)$ to be the Dirac measure at b. For any $t \in \operatorname{supp} F$ that does not belong to the range of γ, if $t < \gamma(0)$ then let $\sigma(\cdot, t)$ be the Dirac measure at 0; else there exists a unique $b \in [0, x]$ for which $F(t) = H(b)$, and we let $\sigma(\cdot, t)$ be the Dirac measure at b.

We show that the σ constructed above is monotone. Since γ is weakly increasing (Lemma 14.a), σ restricted to the range of γ is monotone by construction (note that when t is an atom of F the support of $\sigma(\cdot, t)$, by Eq. (24), is the closure of $\{b : \gamma(b) = t\}$). To show that monotonicity is preserved when the types $t \in \operatorname{supp} F \setminus \operatorname{range} \gamma$ are also included, pick any such t. By Lemma 14.d, either (i) $F(t) < H(0)$ and $t < \gamma(0)$, in which case our σ prescribes to t the zero bid, or (ii) $t > \gamma(b)$ and $F(t) = F(\gamma(b))$ for a unique b, in which case σ prescribes the bid b. In Case (i), as zero is the lowest possible bid and $t < \gamma(0)$, $\sigma(\cdot, t)$ does not violate monotonicity. In Case (ii), σ prescribes to t the same bid b as to $\gamma(b)$ and $(\gamma(b), t)$ is a gap of F. Then there is no type between them for which σ needs to prescribe a bid; any type below $\gamma(b)$ that belongs to $\operatorname{range} \gamma$ is prescribed by σ a bid no higher than b, by monotonicity of σ restricted to $\operatorname{range} \gamma$; likewise, any type above t belonging to $\operatorname{range} \gamma$ is prescribed by σ a bid higher than or equal to $\sup \operatorname{supp} \sigma(\cdot, \gamma(b))$, which is at least as high as b. Thus, the monotonicity of σ is preserved when such t is included. This being true for all $t \in \operatorname{supp} F \setminus \operatorname{range} \gamma$, we have extended the monotonicity of σ to the entire $\operatorname{supp} F$.

Finally, note from monotonicity of σ that Eq. (24) holds for all b. Again by monotonicity of σ, the proof of Lemma 5 has shown that Eq. (24) is equivalent to $H(b) = \int_{\mathbb{R}} \int_0^b \sigma(dr, t) dF(t)$, i.e., the σ constructed above generates H. Hence Claim (a) is proved.

Claim (b): By Lemma 5, any monotone strategy that generates H is equal to the σ constructed above. Thus properties (i)–(ii) in the claim follow by construction of σ. ■

A.6.2 The Equilibrium σ_*^i Given the Degenerate Posterior at z_i

For any $i \in \{1,2\}$, let δ_{z_i} denote the Dirac measure at z_i, and $\mathscr{E}_i(\delta_{z_i})$ the set of all BNEs of the all-pay contest game where the distribution of i's type is δ_{z_i}, and that of $-i$'s type is the prior F_{-i}.

Lemma 16 *For each $i \in \{1,2\}$, $\mathscr{E}_i(\delta_{z_i})$ is singleton and, with σ_*^i being its unique element,*

$$\begin{aligned}
\forall b \in [0, x_{\sigma_*^i}]: \quad H_{i,\sigma_*^i}(b) &= c_{i,\sigma_*^i} + \frac{1}{v}\int_0^b \left(F_{-i}^{-1}\left(\frac{b'}{vz_i} + c_{-i,\sigma_*^i}\right)\right)^{-1} db', && (25)\\
\forall b \in [0, x_{\sigma_*^i}]: \quad H_{-i,\sigma_*^i}(b) &= \frac{b}{vz_i} + c_{-i,\sigma_*^i}, && (26)\\
c_{i,\sigma_*^i} c_{-i,\sigma_*^i} &= 0, && (27)\\
x_{\sigma_*^i}/z_i &= v(1 - c_{-i,\sigma_*^i}), && (28)\\
1 - c_{i,\sigma_*^i} &= z_i \int_{c_{-i,\sigma_*^i}}^1 \frac{1}{F_{-i}^{-1}(s)} ds. && (29)
\end{aligned}$$

The lemma is proved in two steps. Section A.6.3 proves the uniqueness of the equilibrium in $\mathscr{E}_i(\delta_{z_i})$, and Section A.6.4, its existence.

A.6.3 The Uniqueness Proof for Lemma 16

Pick any $\sigma \in \mathscr{E}_i(\delta_{z_i})$ and denote $(H_{i,\sigma}, H_{-i,\sigma}, c_{i,\sigma}, c_{-i,\sigma}, x_\sigma)$ for the associated tuple of bid distributions, masses of $\{0\}$ and bid supremum. We shall show that σ is unique. By definition of δ_{z_i}, supp $\tilde{F}_i = \{z_i\}$ if $\tilde{F}_i$ denotes the c.d.f. corresponding to δ_{z_i}. Hence $\gamma_{i,\sigma} = z_i$ on $[0, x_\sigma]$ by Eq. (9). Thus, Eq. (11), where the role of i is played by $-i$ here, implies that

$$H_{-i,\sigma}(b) = \frac{b}{vz_i} + c_{-i,\sigma}$$

for all $b \in [0, x_\sigma]$, i.e., Eq. (26) is satisfied. By definition of $\gamma_{-i,\sigma}$, for all $b \in [0, x_\sigma]$,

$$\gamma_{-i,\sigma}(b) = F_{-i}^{-1}\left(H_{-i,\sigma}(b)\right) = F_{-i}^{-1}\left(\frac{b}{vz_i} + c_{-i,\sigma}\right).$$

Then again Eq. (11) implies that, for all $b \in [0, x_\sigma]$,

$$H_{i,\sigma}(b) = c_{i,\sigma} + \frac{1}{v}\int_0^b \left(F_{-i}^{-1}\left(\frac{b'}{vz_i} + c_{-i,\sigma}\right)\right)^{-1} db'.$$

Hence Eq. (25) follows. Eq. (27) is also satisfied due to Lemma 13. Apply Eq. (26) to the supremum x_σ of the bid distribution $H_{-i,\sigma}$ to obtain $1 = x_\sigma/(vz_i) + c_{-i,\sigma}$, i.e., Eq. (28). And apply Eq. (25) to the supremum x_σ to get

$$1 - c_{i,\sigma} = \frac{1}{v} \int_0^{x_\sigma} \left(F_{-i}^{-1} \left(\frac{b'}{vz_i} + c_{-i,\sigma} \right) \right)^{-1} db' = z_i \int_{c_{-i,\sigma}}^{x_\sigma/(vz_i)+c_{-i,\sigma}} \left(F_{-i}^{-1}(s) \right)^{-1} ds,$$

with the second equality due to the change of variables $s := b/(vz_i) + c_{-i,\sigma}$. This equation coupled with Eq. (28) gives Eq. (29), which for any $c_{i,\sigma}$ admits at most one solution for $c_{-i,\sigma}$, with the right-hand side strictly decreasing in $c_{-i,\sigma}$. Consequently, Eqs. (27) and (29) together determine uniquely $(c_{i,\sigma}, c_{-i,\sigma})$, hence Eq. (28) determines x_σ uniquely, and so $H_{i,\sigma}$ and $H_{-i,\sigma}$ are each uniquely determined by Eqs. (25) and (26). Note from Eqs. (25)–(27) that both bid distributions $H_{i,\sigma}$ and $H_{-i,\sigma}$ are gapless and atomless on $(0, x_\sigma]$. This coupled with the monotonicity of any BNE of the contest game (Lemma 4.b) implies that Lemma 5 is applicable. Hence σ is unique as $(H_{i,\sigma}, H_{-i,\sigma})$ is unique. ■

A.6.4 The Existence Proof for Lemma 16

Step 1: Construction Clearly, Eqs. (27)–(29) together admit a unique solution for $(c_{i,\sigma_*^i}, c_{-i,\sigma_*^i}, x_{\sigma_*^i}) \in [0,1]^2 \times \mathbb{R}_+$. Plugging this solution into Eq. (25)–(26), we obtain a pair $(H_{i,\sigma_*^i}, H_{-i,\sigma_*^i})$, each being a c.d.f. on $[0, x_{\sigma_*^i}]$ due to Eqs. (28) and (29). Let us suppress the symbol σ_*^i in the subscripts and write the tuple as $(H_i, H_{-i}, c_i, c_{-i}, x)$, which we shall prove constitutes an equilibrium in $\mathscr{E}_i(\delta_{z_i})$. Let $\gamma_{-i}(b) := F_{-i}^{-1}(H_{-i}(b))$ for all b.

Lemma 17 *For any $t_i \in [a_i, z_i] \backslash \{0\}$: (a) $U_i(t_i|H_{-i}) = vc_{-i}$; (b) if $c_{-i} = 0$ then $b = 0$ is a best reply to H_{-i}; if $c_{-i} > 0$ then player i's best reply to H_{-i} is null when $t_i < z_i$; (c) if $t_i = z_i$ then any bid in $(0, x]$ is a best reply; (d) if $c_{-i} > 0$ and $t_i < z_i$ then $\lim_{\epsilon \downarrow 0} \sup \mathrm{BR}_i \left(t_i, \epsilon | H_{-i,\sigma_*^i} \right) = 0$.*

Proof Player i has no incentive to bid outside the support $[0, x]$ of H_{-i}. In bidding any $b \in (0, x]$, given any type $t_i \in [a_i, z_i] \setminus \{0\}$, player i's expected payoff is equal to, by Eq. (26),

$$v \left(\frac{b}{vz_i} + c_{-i} \right) - \frac{b}{t_i} = vc_{-i} + b \left(\frac{1}{z_i} - \frac{1}{t_i} \right) \leq vc_{-i},$$

where the weak inequality becomes equality if $t_i = z_i$. If he bids zero, because of the equal-probability tie-breaking rule, i's expected payoff is equal to $vc_{-i}/2$. Thus, Claims (b) and (c) are true. Taking the limit of the above equation when $b \downarrow 0$, coupled with the definition of $U_i(t_i \mid H_{-i})$ in Eq. (3), we also obtain Claim (a). To prove (d), simply note that $\mathrm{BR}_i \left(t_i, \epsilon | H_{-i,\sigma_*^i} \right) = (0, \epsilon / (1/t_i - 1/z_i)]$ for any $t_i < z_i$ and any $\epsilon > 0$. ■

Lemma 18 *For any* $t_{-i} \in \operatorname{supp} F_{-i}$*, define for any* $b \in [0, x]$

$$u_{-i}(b, t_{-i}) := \begin{cases} vt_{-i}H_i(b) - b & \text{if } b > 0 \\ vt_{-i}H_i(0)/2 & \text{if } b = 0. \end{cases} \tag{30}$$

Then $u_{-i}(\cdot, t_{-i})$ *is concave on* $(0, x]$ *and, if* $H_i(0) = 0$*, also on* $[0, x]$*; furthermore, if* $u_{-i}(\cdot, t_{-i})$ *is differentiable at* b*, then* $\frac{\partial}{\partial b}u_{-i}(b, t_{-i})$*, denoted by* $D_1 u_{-i}(b, t_{-i})$*, satisfies*

$$D_1 u_{-i}(b, t_{-i}) = \frac{t_{-i}}{\gamma_{-i}(b)} - 1. \tag{31}$$

Proof By the definition of u_{-i}, to prove concavity of $u_{-i}(\cdot, t_{-i})$ on $(0, x]$ it suffices to show that H_i is concave on $[0, x]$. By Eq. (25), H_i restricted to $[0, x]$ is absolutely continuous; thus, we need only to show that the left-derivative of H_i is never smaller than the right-derivative and that, whenever the two coincide, the derivative is weakly decreasing. To that end, pick any $b \in [0, x)$. By Eq. (25), the right-derivative of H_i at b is

$$D_+H_i(b) = \frac{1}{v} \lim_{b'' \downarrow b} \frac{1}{b'' - b} \int_b^{b''} \left(F_{-i}^{-1} \left(\frac{b'}{vz_i} + c_{-i} \right) \right)^{-1} db' = \frac{1}{v} \lim_{b'' \downarrow b} \left(F_{-i}^{-1} \left(\frac{\xi(b, b'')}{vz_i} + c_{-i} \right) \right)^{-1},$$

with $b \le \xi(b, b'') \le b''$ by the intermediate-value theorem. Thus, with F_{-i}^{-1} weakly increasing,

$$D_+H_i(b) \le \frac{1}{v} \left(F_{-i}^{-1} \left(\frac{b}{vz_i} + c_{-i} \right) \right)^{-1}.$$

Analogously, the left-derivative at any $b \in (0, x]$ is

$$D_-H_i(b) = \frac{1}{v} \lim_{b'' \uparrow b} \frac{1}{b - b''} \int_{b''}^{b} \left(F_{-i}^{-1} \left(\frac{b'}{vz_i} + c_{-i} \right) \right)^{-1} db' \ge \frac{1}{v} \left(F_{-i}^{-1} \left(\frac{b}{vz_i} + c_{-i} \right) \right)^{-1}.$$

Thus, for any $b \in (0, x)$, $D_-H_i(b) \ge D_+H_i(b)$ and, when they coincide,

$$\frac{d}{db}H_i(b) = D_+H_i(b) = D_-H_i(b) = \frac{1}{v} \left(F_{-i}^{-1} \left(\frac{b}{vz_i} + c_{-i} \right) \right)^{-1} = \frac{1}{v\gamma_{-i}(b)},$$

with the last equality due to the definition of γ_{-i} and Eq. (26). Thus, by Lemma 14.a, $\frac{d}{db}H_i(b)$ is weakly decreasing. Hence $u_{-i}(\cdot, t_{-i})$ is concave on $(0, x]$, and Eq. (31) holds. If $H_i(0) = 0$ then $u_{-i}(\cdot, t_{-i})$ by Eq. (30) is continuous at zero, hence also concave on $[0, x]$. ■

Step 2: Verification By Parts (b) and (c) of Lemma 17, as well as the fact that $H_i(0) = 0$ unless $c_{-i} = 0$, bidding according to H_i is a best response for the type-z_i player i. Thus the rest of the proof concerns the best response for player $-i$. By Lemmas 5 and 15.a, there is a unique strategy σ_{-i} that generates H_{-i} given F_{-i}. We shall show that σ_{-i} is player $-i$'s best response to H_i for all types but a set of zero F_{-i}-measure. Since $F_{-i}(0) = 0$ by assumption, we may assume without loss that $t_{-i} > 0$. Thus, player $-i$'s decision in the contest is equivalent to maximizing $u_{-i}(\cdot, t_{-i})$, defined in Eq. (30).

Hence pick any $t_{-i} \in \operatorname{supp} F_{-i}$ such that $t_{-i} > 0$. To prove that $\sigma_{-i}(\cdot, t_{-i})$ is a best response for player i of type t_{-i}, it suffices to prove that every element of $\operatorname{supp} \sigma_{-i}(\cdot, t_{-i})$, except a set of zero $\sigma_{-i}(\cdot, t_{-i})$-measure, is such a best response. Thus, consider any $b \in \operatorname{supp} \sigma_{-i}(\cdot, t_{-i})$ with the condition that either $b > 0$ or "$b = 0$ and $H_i(0) = 0$" holds. This condition of b causes no loss of generality because if $H_i(0) > 0$ then by the construction in Step 1 we have $H_{-i}(0) = 0$, which means that either the type t_{-i} belongs to the zero-measure set of types that bid zero according to σ_{-i} and hence can be omitted, or the bid zero is assigned zero weight according to $\sigma_{-i}(\cdot, t_{-i})$ and hence can be omitted.

First, consider the case where $b > 0$. With $b \in \operatorname{supp} \sigma_{-i}(\cdot, t_{-i})$ and σ_{-i} monotone (Lemma 4.b), $F_{-i}(t_{-i}) \geq H_{-i}(b) > H_{-i}(0)$. Thus by Eq. (22) $t_{-i} \geq \gamma_{-i}(0)$. By Lemma 15.b.ii, either (A) $\gamma_{-i}(b) = t_{-i}$ or (B) $F_{-i}(t_{-i}) = H_{-i}(b)$ and $(\gamma_{-i}(b), t_{-i})$ is a gap of F_{-i}. Pick any $b'' > b$. In Case (A), monotonicity of γ_{-i} (Lemma 14.a) implies $\gamma_{-i}(b'') \geq \gamma_{-i}(b) = t_{-i}$. In Case (B), by Lemma 11.a and the gapless H_{-i},

$$F_{-i}(\gamma_{-i}(b'')) \geq H_{-i}(b'') > H_{-i}(b) = F_{-i}(t_{-i}),$$

hence $\gamma_{-i}(b'') \geq t_{-i}$. Thus $\gamma_{-i}(b'') \geq t_{-i} \geq \gamma_{-i}(b)$ in each case. This, again coupled with the monotonicity of γ_{-i}, implies that in each case

$$b' < b < b'' \Longrightarrow \gamma_{-i}(b') \leq t_{-i} \leq \gamma_{-i}(b'').$$

Thus, for any $b' < b < b''$ such that $u_{-i}(\cdot, t_{-i})$ is differentiable at b' and b'', Eq. (31) implies

$$D_1 u_{-i}(b', t_{-i}) = \frac{t_{-i}}{\gamma_{-i}(b')} - 1 \geq 0 \geq \frac{t_{-i}}{\gamma_{-i}(b'')} - 1 = D_1 u_{-i}(b'', t_{-i}). \tag{32}$$

Hence, since $u_{-i}(\cdot, t_{-i})$ is concave on $(0, x]$ and, by Eq. (30), $u_{-i}(0, t_{-i}) \leq \lim_{b' \downarrow 0} u_{-i}(b', t_{-i})$, b is a global maximum of $u_{-i}(\cdot, t_{-i})$.

Second, consider the case where $b = 0$. Thus, as explained above, $H_i(0) = 0$. Hence $u_{-i}(\cdot, t_{-i})$ is concave on $[0, x]$ (Lemma 18). For any $b'' > 0$ at which $u_{-i}(\cdot, t_{-i})$ is differentiable, $t_{-i} \leq \gamma_{-i}(b'')$ by monotonicity of σ_{-i} (Lemma 14.a) and the fact $b'' \in \operatorname{supp} \sigma_{-i}(\cdot, \gamma_{-i}(b''))$ (Lemma 11.d), and Eq. (31) implies

$$D_1 u_{-i}(b'', t_{-i}) = \frac{t_{-i}}{\gamma_{-i}(b'')} - 1 \leq 0.$$

This, combined with the fact that zero is the left corner of the domain of bids and that $u_{-i}(\cdot, t_{-i})$ concave on $[0, x]$, implies that zero ($= b$) is a best response for type t_{-i}. All cases considered, σ_{-i} best responds H_i for all types of player $-i$. ■

A.6.5 Proof of Lemma 6: Linkage between the Players' Marginal Costs of Bids

Given any c.d.f. $\tilde{F}_i$ with support contained in $[a_i, z_i]$, pick any equilibrium $\sigma \in \mathscr{E}_i(\tilde{F}_i)$. We shall prove that $x_\sigma \leq x_{\sigma_*^i}$. It suffices to prove $x_\sigma \geq x_{\sigma_*^i} \Rightarrow x_\sigma \leq x_{\sigma_*^i}$, as its contrapositive implies that $x_\sigma > x_{\sigma_*^i}$ leads to a contradiction. Hence suppose $x_\sigma \geq x_{\sigma_*^i}$. Define $\gamma_{i,\sigma}$ by Eq. (9), and likewise for γ_{i,σ_*^i}, $\gamma_{-i,\sigma}$ and γ_{-i,σ_*^i}. Since i's type in the equilibrium σ_*^i is degenerate to z_i, $\gamma_{i,\sigma_*^i} = z_i$. Eq. (11), with the roles of i and $-i$ switched, implies

$$\frac{d}{db} H_{-i,\sigma}(b) = \frac{1}{v\gamma_{i,\sigma}(b)} \geq \frac{1}{vz_i} = \frac{1}{v\gamma_{i,\sigma_*^i}(b)} = \frac{d}{db} H_{-i,\sigma_*^i}(b)$$

for almost every b. This, combined with the fact $H_{-i,\sigma}(x_\sigma) = 1 = H_{-i,\sigma_*^i}(x_{\sigma_*^i})$, the supposition $x_\sigma \geq x_{\sigma_*^i}$, and the absolute continuity of H_{-i} except at zero, implies $H_{-i,\sigma} \leq H_{-i,\sigma_*^i}$. Since the generalized inverse F^{-1} of any c.d.f. F, defined in Eq. (10), is weakly increasing,

$$\gamma_{-i,\sigma}(b) = F_{-i}^{-1}\left(H_{-i,\sigma}(b)\right) \leq F_{-i}^{-1}\left(H_{-i,\sigma_*^i}(b)\right) = \gamma_{-i,\sigma_*^i}(b)$$

for all b. Thus, Eq. (11) implies that, for almost every b,

$$\frac{d}{db} H_{i,\sigma}(b) = \frac{1}{v\gamma_{-i,\sigma}(b)} \geq \frac{1}{v\gamma_{-i,\sigma_*^i}(b)} = \frac{d}{db} H_{i,\sigma_*^i}(b).$$

With the hypothesis $c_{-i,\sigma_*^i} > 0$, $H_{i,\sigma_*^i}(0) = c_{i,\sigma_*^i} = 0$ by Eq. (27). Hence $H_{i,\sigma}(0) \geq H_{i,\sigma_*^i}(0)$. Then the above-displayed inequality implies $H_{i,\sigma}(x_{\sigma_*^i}) \geq H_{i,\sigma_*^i}(x_{\sigma_*^i}) = 1$. Thus, $x_\sigma \leq x_{\sigma_*^i}$. We have hence proved $x_\sigma \geq x_{\sigma_*^i} \Rightarrow x_\sigma \leq x_{\sigma_*^i}$, as desired. ■

A.6.6 Proof of Theorem 3

Part (a) of the theorem is Lemma 16, and Part (b) a summary of Claims (b) and (d) of Lemma 17. For Part (c), note that the second equality there, $vc_{-i,\sigma_*^i} = U_i(t_i \mid H_{-i,\sigma_*^i})$ for all $t_i \in [a_i, z_i]$, is simply Lemma 17.a. This equality also implies $\underline{u}_i \leq vc_{-i,\sigma_*^i}$ by definition of $\underline{u}_i$. Thus, we need only to prove $\underline{u}_i \geq vc_{-i,\sigma_*^i}$, i.e., for any c.d.f. $\tilde{F}_i$ and any BNE σ in $\mathscr{E}_i(\tilde{F}_i)$, $U_i(z_i \mid H_{-i,\sigma}) \geq vc_{-i,\sigma_*^i}$. The case where $c_{-i,\sigma_*^i} = 0$ is trivial, as $U_i(z_i \mid H_{-i,\sigma}) \geq 0$ by individual rationality of i. Hence suppose $c_{-i,\sigma_*^i} > 0$. By Lemma 6, $x_\sigma \leq x_{\sigma_*^i}$. With $H_{-i,\sigma}$ atomless at its supremum x_σ (Lemma 4.a), by bidding x_σ player i wins for sure, hence

$$U_i(z_i \mid H_{-i,\sigma}) \geq v - x_\sigma/z_i \geq v - x_{\sigma_*^i}/z_i = vc_{-i,\sigma_*^i},$$

with the last equality due to Eq. (28). Thus, $\underline{u}_i \geq vc_{-i,\sigma_*^i}$, as desired. ■

A.7 Lemma 7, Theorem 4 and Corollaries 3 and 4

Lemma 7 For each $i \in \{1,2\}$ denote $\phi_i(c_i) := z_{-i}\int_{c_i}^1 \frac{1}{F_i^{-1}(s)}ds$. Note that ϕ is continuous and strictly decreasing on $[0,1]$, with $\phi_i(1) = 0$. Thus, the set in Eq. (12) is nonempty, $c_i^* \in [0,1]$, and $\phi_i(c_i) = 1$ admits at most one solution for c_i. To prove the lemma, consider first the case $c_{i,\sigma_*^{-i}} = 0$. Eq. (29), with the roles of i and $-i$ switched, implies

$$1 \geq 1 - c_{-i,\sigma_*^{-i}} = z_{-i}\int_{c_{i,\sigma_*^{-i}}}^1 \left(1/F_i^{-1}(s)\right)ds = z_{-i}\int_0^1 \left(1/F_i^{-1}(s)\right)ds,$$

which by Eq. (12) implies $c_i^* = 0 = c_{i,\sigma_*^{-i}}$. Next consider the other case, where $c_{i,\sigma_*^{-i}} > 0$. We have $c_{-i,\sigma_*^{-i}} = 0$ by Eq. (27). Thus, Eq. (29) implies

$$1 = 1 - c_{-i,\sigma_*^{-i}} = z_{-i}\int_{c_{i,\sigma_*^{-i}}}^1 \left(1/F_i^{-1}(s)\right)ds,$$

i.e., $\phi_i(c_{i,\sigma_*^{-i}}) = 1$. Since $\phi_i(c) = 1$ admits at most one solution for c, we have $c_i^* = c_{i,\sigma_*^{-i}}$. ■

Theorem 4 It follows directly from Lemmas 3 and 7 and Theorem 3. ■

Corollary 3 By Lemma 3, it suffices to show, for each $i \in \{1,2\}$, that the equilibrium refinement does not increase the lowest deviation payoff $\underline{u}_i$ for type z_i of contestant i in the off-path event E^i where i deviates to vetoing the peace proposal. As $\underline{u}_i$ is generated by the off-path posterior δ_{z_i}, the Dirac measure at z_i, coupled with the continuation equilibrium σ_*^i (Theorem 3), we need only to show that δ_{z_i} satisfies both refinement criteria.[9]

To verify universal divinity, recall from Lemma 2 that given any distribution H of bids from the opponent $-i$ in event E^i the supremum expected payoff $U_i(z_i \mid H)$ for type z_i is highest among $U_i(t_i \mid H)$ for all types t_i of i. Thus, for any posterior belief $\tilde{F}_i$ of i and any continuation equilibrium σ induced by $\tilde{F}_i$ in the off-path event E^i, if the off-path assessment

[9] A convenient reference to these criteria is Fudenberg and Tirole[11, p448, p452].

$(\tilde{F}_i, \sigma)$ makes some type of i strictly prefer to deviate (to veto peace) then it also makes type z_i strictly prefer to deviate. Hence z_i survives any iterative elimination by the divinity criterion. Thus δ_{z_i}, whose support is $\{z_i\}$, passes the universal divinity test.

To verify the intuitive criterion, pick any peaceful equilibrium $\mathscr{S}_*$ with σ^i_* being the continuation equilibrium in the off-path event E^i. By Lemmas 1 and 10, there is no loss of generality to assume that the peaceful equilibrium $\mathscr{S}_*$ gives a constant payoff k_i to every type t_i of contestant i and $k_i \geq \underline{u}_i$. This, coupled with Lemma 2, implies that if type z_i strictly prefers the peaceful outcome k_i to deviation then so does every type of i. Thus, type z_i is not excluded from the family $\mathscr{F}$ of reasonable posterior beliefs in the off-path event E^i. Hence δ_{z_i} belongs to $\mathscr{F}$. Since δ_{z_i} rationalizes player $-i$'s bid distribution H_{-i,σ^i_*} in the continuation equilibrium σ^i_* given event E^i, and since $\underline{u}_i = U_i(t_i \mid H_{-i,\sigma^i_*})$ for any type t_i of player i (Theorem 3.c),

$$k_i \geq \underline{u}_i = U_i(t_i \mid H_{-i,\sigma^i_*}) \geq \inf\left\{U_i(t_i \mid H_{-i,\sigma}) : \sigma \in \mathscr{E}_i(\tilde{F}_i); \tilde{F}_i \in \mathscr{F}\right\}$$

for any $t_i \in [a_i, z_i]$. It follows that $\mathscr{S}_*$ satisfies the intuitive criterion. ■

Corollary 4 With $F_1 = F_2 = F$, Eq. (12) becomes $c^*_i := \inf\{c_i \in [0,1] : \phi(c_i) \leq 1\}$ such that $\phi(c) = z\int_c^1 \frac{1}{F^{-1}(s)}ds$. Note that ϕ is continuous and strictly decreasing, and that $\phi(1) = 0$, and $\phi(0) = z/F^{-1}(\xi)$ for some $\xi \in [0,1]$, hence $\phi(0) \geq 1$ since $F^{-1}(\xi) \leq z$. Thus, a solution in $\phi(c) = 1$ for c exists and, by the strict monotonicity of ϕ, is unique, hence denote it by c_*. Thus, $c^*_i = c_*$ for each $i \in \{1,2\}$, and the conclusion follows from Theorem 4. ■

A.8 Lemma 8 and Theorem 5

Lemma 19 *If $\widehat{F}$ and F are each a c.d.f. and $\widehat{F}(t) \leq F(t)$ for all $t \in \mathbb{R}$, then $\widehat{F}^{-1}(s) \geq F^{-1}(s)$ for all $s \in [0,1]$.*

Proof For any $s \in [0,1]$, apply the hypothesis $F \geq \widehat{F}$ to the type $\widehat{F}^{-1}(s)$ to obtain

$$F\left(\widehat{F}^{-1}(s)\right) \geq \widehat{F}\left(\widehat{F}^{-1}(s)\right) \geq s,$$

with the second inequality due to the definition of $\widehat{F}^{-1}(s)$, Eq. (10), and the upper semi-continuity of any c.d.f. Now that $F\left(\widehat{F}^{-1}(s)\right) \geq s$, Eq. (10) applied to F implies that $\widehat{F}^{-1}(s) \geq F^{-1}(s)$. ■

Proof of Lemma 8 Let the prior F_i be replaced by another $\widehat{F}_i$ that dominates F_i, and the supremum z_{-i} replaced by a $\hat{z}_{-i} \leq z_{-i}$. By definition of dominance and Lemma 19, $\widehat{F}_i^{-1}(s) \geq F_i^{-1}(s)$ for all $s \in [0,1]$. Hence for any $c_i \in [0,1]$

$$\int_{c_i}^{1} \frac{1}{\widehat{F}_i^{-1}(s)} ds \leq \int_{c_i}^{1} \frac{1}{F_i^{-1}(s)} ds.$$

Since $\hat{z}_{-i} \leq z_{-i}$, we have, for all $c_i \in [0,1]$,

$$\hat{z}_{-i} \int_{c_i}^{1} \frac{1}{\widehat{F}_i^{-1}(s)} ds \leq z_{-i} \int_{c_i}^{1} \frac{1}{F_i^{-1}(s)} ds.$$

Thus, by Eq. (12), the c_i^* given $\left(\widehat{F}_i, \hat{z}_{-i}\right)$ is weakly smaller than the c_i^* given (F_i, z_{-i}). ∎

Proof of Theorem 5 To prove Claim (a), recall from Theorem 4 that the necessary and sufficient condition for peaceful mechanisms to exist is $c_1^* + c_2^* \leq 1$. Thus, the claim follows from the implication of Lemma 8 that $\widehat{F}_i \triangleright F_i$ for both $i \in \{1,2\}$ implies, for each $i \in \{1,2\}$, c_i^* cannot increase when $(F_i)_{i=1}^2$ is replaced by $(\widehat{F}_i)_{i=1}^2$.

For Claim (b), pick the $i \in \{1,2\}$ for whom $z_i \geq z_{-i}$. Note from Eq. (12) that $c_{-i}^* < 1$. To satisfy the peaceful condition $c_1^* + c_2^* \leq 1$ it suffices to replace F_i by some $\widehat{F}_i$ such that

$$z_{-i} \int_{1-c_{-i}^*}^{1} \frac{1}{\widehat{F}_i^{-1}(s)} ds \leq 1 \tag{33}$$

and $\widehat{F}_i \triangleright F_i$. To satisfy Ineq. (33), note from $c_{-i}^* < 1$ that there exists $\epsilon > 0$ for which

$$\frac{z_{-i}}{z_{-i} - \epsilon} c_{-i}^* < 1.$$

Pick any c.d.f. F_i^* with $\operatorname{supp} F_i^* = \operatorname{supp} F_i$ such that $1 - F_i^*(z_{-i} - \epsilon) > 1 - c_{-i}^*$, which is compatible with $\operatorname{supp} F_i^* = \operatorname{supp} F_i$ because $z_i \geq z_{-i}$ by the choice of i. Let $\widehat{F}_i := \min\{F_i, F_i^*\}$ pointwise. Then $\widehat{F}_i \triangleright F_i$ and, by the definition of the generalized inverse $\widehat{F}_i^{-1}$, the left-hand side of (33) is less than or equal to

$$z_{-i}(1 - 1 + c_{-i}^*) \cdot \frac{1}{z_{-i} - \epsilon} < 1.$$

Thus, with $\hat{c}_i^*$ defined by Eq. (12) where F_i is replaced by $\widehat{F}_i$ here, $\hat{c}_i^* < 1 - c_{-i}^*$, hence (33) is satisfied. Since $F_{-i} \triangleright F_{-i}$, the pair $(\widehat{F}_i, \widehat{F}_{-i})$, with $\widehat{F}_{-i} := F_{-i}$, is what Claim (b) needs. ∎

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