The Category of Node-and-Choice Preforms for Extensive-Form Games

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THE CATEGORY OF NODE-AND-CHOICE PREFORMS FOR EXTENSIVE-FORM GAMES

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Abstract. It would be useful to have a category of extensive-form games whose isomorphisms specify equivalences between games. Since working with entire games is too large a project for a single paper, I begin here with preforms, where a "preform" is a rooted tree together with choices and information sets.

My first contribution is to introduce a compact preform specification called a "node-and-choice" preform. This specification's compactness allows me to introduce tractable morphisms which map one node-and-choice preform to another. I incorporate these morphisms into a category called the "category of node-and-choice preforms". Finally, I characterize the isomorphisms of this category.

1. Introduction

Category theory has been used to systematize many subjects in mathematics and elsewhere. For example, there is the category of graphs whose morphisms allow one to systematically compare graphs. There, morphisms can be used to state that one graph is embedded within another. Further, isomorphisms can be used to state that two graphs are equivalent.

Similarly, it would be useful to have a category of extensive-form games whose morphisms would allow one to systematically compare extensive-form games. As yet, little has been done.[1] Lapitsky (1999)

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[1] Extensive-form games are not readily comparable with the games defined in the theoretical computer-science literature. Categories of such games are developed in McCusker (2000), Abramsky, Jagadeesan, and Malacaria (2000), and Hyland and Ong (2000).

and Jiménez (2014) define categories of normal-form games. Machover and Terrington (2014) defines a category of simple voting games. Finally, Vannucci (2007) defines categories of various kinds of games, but in its category of extensive-form games, every morphism merely maps a game to itself.

Building a category of extensive-form games with nontrivial morphisms is a large project because each extensive-form game has so many components: each is a rooted tree with choices, information sets, players, chance probabilities, and preferences. Accordingly, this paper takes a small necessary step: it builds a category of preforms, where a "preform" is a rooted tree with choices and information sets (Streufert (2015a, 2015b)).[2]

This paper's first contribution is to introduce a compact preform specification. In particular, Section 2 defines a "node-and-choice" preform to consist of nodes, choices, and an operator $\otimes$. The operator $\otimes$ is new. It maps node-and-choice pairs to nodes. In particular, each node-and-choice pair in the operator's domain is mapped to the node that follows the pair's node by way of the pair's choice. This operator is sufficient to determine the tree, the information-set collection, and a number of other derivative entities.

Section 3 uses this compact preform specification to define tractable preform morphisms. Each such morphism maps an "old" preform to a "new" preform. More precisely, each morphism takes old nodes to new nodes, and old choices to new choices, in such a way that the structure of the old operator is preserved within the new operator. Theorem 1 incorporates these morphisms into the "category of node-and-choice preforms". Then, Theorem 2 characterizes the isomorphisms of this category.

Because each preform's operator $\otimes$ determines its tree, the existence of a morphism or isomorphism between two preforms implies a relationship between their two trees. Similarly, it implies a relationship between their two information-set collections. Relatedly, it implies relationships between their root nodes, their feasibility correspondences, their immediate-predecessor functions, their stage functions,

[2] A "form" is understood to be a rooted tree with choices, information sets, and players.

their precedence relations, their decision-node sets, their finite-play collections, and their infinite-play collections. All these relationships and a few more are derived by the three propositions in Section 3.

Work is currently underway to prove that node-and-choice preforms are general enough to encompass the formulations used in the extensive-form games of von Neumann and Morgenstern (1944), Kuhn (1953), Osborne and Rubinstein (1994), Alós-Ferrer and Ritzberger (2013), and Streufert (2015a).[3] When that work is completed, fundamental equivalences across these various formulations can be stated formally as isomorphisms within the category of node-and-choice preforms.

In addition, the obvious sequel to this paper is to develop a category of forms[2] built upon this paper's category of preforms. Work on that is also underway.

2. Preforms

2.1. Definition

Let T be a set and call $t \in T$ a *node*. Let C be a set and call $c \in C$ a *choice*. A *(node-and-choice)*[4] *preform* $\mathit{\Pi}$ is a triple $(T, C, \otimes)$ such that

$$\text{(1a)} \quad (\exists F{\subseteq}T{\times}C)(\exists t^o{\in}T)$$

$$\otimes \text{ is a bijection from } F \text{ onto } T{\smallsetminus}\{t^o\} \ ,$$

$$\text{(1b)} \quad (T,p) \text{ is a tree oriented toward } t^o$$

$$\text{where } p := \{(t^\sharp,t)|(\exists c)(t,c,t^\sharp){\in}{\otimes}\} \ , \text{ and}$$

$$\text{(1c)} \quad \mathcal{H} \text{ partitions } F^{-1}(C)$$

$$\text{where } \mathcal{H} := \{F^{-1}(c)|c\} \ .$$

Call $\otimes$ the *node-and-choice operator*. Note that equation (1) derives F, t^o, p, and $\mathcal{H}$ from $(T, C, \otimes)$. Call F the *feasibility* correspondence. Call t^o the *root* node. Call p the *immediate-predecessor* function. Call $\mathcal{H}$ the collection of *information sets*.

The remainder of this Section 2 discusses definition (1). Roughly, the remainder of this Section 2.1 focuses on (1a). Then Section 2.2 focuses on nodes and (1b). Finally Section 2.3 focuses on choices and (1c). Incidentally, Section 3.1 provides a pair of example preforms.

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[3] Differential games, and the non-discrete games of Alós-Ferrer and Ritzberger (2005, 2008), are beyond the scope of node-and-choice preforms.

[4] The modifier "node-and-choice" is redundant after this point in this paper. However, less abstract kinds of preforms appear in Streufert (2015a, 2015b).

(1a) states that the operator $\otimes$ is a function from $F \subseteq T{\times}C$ into T. Accordingly, let the statement $t{\otimes}c = t^\sharp$ be equivalent to the statement $(t,c,t^\sharp) \in \otimes$. Call $t{\otimes}c$ the *result* of the node-and-choice pair (t,c).

(1a) also states that the range of $\otimes$ is $T{\smallsetminus}\{t^o\}$. This determines the root node t^o as the only node t that is not in the range of $\otimes$. Hence T has no superfluous elements: every node t other than the root node t^o is the result of some node-and-choice pair.

(1a) also states that the domain of $\otimes$ is $F \subseteq T{\times}C$. Thus

$$F = \{\ (t,c) \mid (\exists t^\sharp)(t,c,t^\sharp){\in}{\otimes}\ \}\ . \tag{2}$$

Since F is a subset of $T{\times}C$, F can be regarded as a (nonempty-valued) correspondence whose domain is some subset of T and whose range is some subset of C. Accordingly, let the statement $c \in F(t)$ be equivalent to the statement $(t,c) \in F$. Thus by (2), $c \in F(t)$ iff $t{\otimes}c$ exists. Accordingly, $F(t)$ is called the set of choices that are *feasible* from the node t.

Now consider the range of F. This set consists of those choices c that are feasible from some node. By (1c) and the fact that a partition consists of nonempty sets, each inverse image $F^{-1}(c)$ is nonempty. Thus the range of F is all of C. Hence C has no superfluous elements: each choice c is feasible from at least one node.

Finally, note that the domain of F is $F^{-1}(C)$. This set consists of those nodes with at least one feasible choice. Accordingly, the elements of $F^{-1}(C)$ are called the *decision nodes*.[5] Note that all the nodes might be decision nodes (this happens, for example, in an infinitely repeated stage game).

2.2. Nodes

(1b) defines p as a set. Since $\otimes$ is a bijection onto $T{\smallsetminus}\{t^o\}$ by (1a), p is a function from $T{\smallsetminus}\{t^o\}$. Call $p(t)$ the *immediate predecessor* of $t{\neq}t^o$, and note that $p(t^o)$ is undefined. An elementary argument shows that the range $p(T{\smallsetminus}\{t^o\})$ of p equals the set $F^{-1}(C)$ of decision nodes (Lemma A.1(a)).

[5] I avoid the term "nonterminal node" because I avoid the term "terminal node". I avoid the latter because it is natural to expect that the set of "terminal nodes" would be in one-to-one correspondence with the set of plays. This does not happen because the definition of a node-and-choice preform does not provide nodes that correspond to infinite plays. For more details, see Proposition 2.1(b) below.

(1b) uses the terminology of Diestel (2010, pages 13–15 and 28). It requires that (T,p) is a tree with root t^o in which every edge is oriented toward the root. This is equivalent to (T,p) being an arborescence converging to t^o, in the sense of Tutte (1984, page 127). To be explicit, (1b) is equivalent to there being a function $k{:}T{\rightarrow}\mathbb{N}_0$ such that

$$k(t^o) = 0 \text{ and } (\forall t{\neq}t^o)\ p^{k(t)}(t) = t^o\ . \tag{3}$$

This requires that t^o can be reached from any $t{\neq}t^o$ by iterating p a finite number of times. Setting $k(t^o) = 0$ is not restrictive. Further, note that $k(t)$ is determined for any $t{\neq}t^o$ because $p(t^o)$ is not defined. Accordingly, call k the *stage function* and call $k(t)$ the *stage* of node t.

Define the *(strict) precedence relation* $\prec$ on T by

$$t^1 \prec t^2 \quad \text{iff} \quad (\exists m{\geq}1)\ t^1 = p^m(t^2)\ . \tag{4}$$

Say that t^1 *(strictly) precedes* t^2 iff $t^1{\prec}t^2$. Equivalently, say that t^2 *(strictly) succeeds* t^1. An elementary argument shows that a node has a successor iff it is a decision node (Lemma A.1(b)).

Define the *weak precedence relation* $\preccurlyeq$ on T by

$$t^1 \preccurlyeq t^2 \quad \text{iff} \quad t^1 \prec t^2 \text{ or } t^1 = t^2\ . \tag{5}$$

Notice that the term "precedence" without the modifier "weak" refers to strict precedence. As the notation suggests, $\prec$ is the asymmetric part of $\preccurlyeq$ (Lemma A.2(b)). Further, it easily shown that $(T,\preccurlyeq)$ is a partially ordered set (Lemma A.2(d)). But (1b) makes it a rather special sort of partially ordered set. In contrast, Alós-Ferrer and Ritzberger (2005, 2008) define games over more general partially ordered sets. They would use the term "discreteness" to describe a restriction like (1b) (Alós-Ferrer and Ritzberger (2013, Section 3)).

Finally, let $\mathcal{Z}$ be the collection of maximal chains in $(T,\preccurlyeq)$, and call $Z \in \mathcal{Z}$ a *play*. Plays can be either finite or infinite. Accordingly, $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}}{\cup}\mathcal{Z}_{\mathsf{inft}}$, where

$$\mathcal{Z}_{\mathsf{ft}} := \{ \text{ finite maximal chains in } (T,\preccurlyeq) \} \text{ and} \tag{6a}$$

$$\mathcal{Z}_{\mathsf{inft}} := \{ \text{ infinite maximal chains in } (T,\preccurlyeq) \}\ . \tag{6b}$$

For example, a game with a finite number of nodes has no infinite plays. In contrast, an infinitely repeated stage game has no finite plays. In-between, an infinite centipede game has some some finite plays and some infinite plays (Section 3.1's Π^{ce} is a preform for this well-known game).

Part (a) of the following proposition shows that each finite play can be uniquely associated with a non-decision node. It does so by means of the maximization operator for $\preccurlyeq$. Meanwhile, part (b) shows that each infinite play can be uniquely associated with an infinite sequence of nodes (there is no single node associated with an infinite play). For this, define the function E from $\mathcal{Z}_{\mathsf{inft}}$ into $T^{\mathbb{N}_1}$ by

$$E(Z) := (t^v)_{v\geq 1} \ , \tag{7}$$

where each t^v is the unique element t of Z for which $k(t) = v$. Call E the *enumeration* operator.

Proposition 2.1. *Suppose* $(T, C, \otimes)$ *satisfies (1a)–(1b) and derive its* F, t^o, p, k, $\prec$, $\preccurlyeq$, $\mathcal{Z}_{\mathsf{ft}}$, $\mathcal{Z}_{\mathsf{inft}}$, *and* E. *Then the following hold.*

(a) $\mathcal{Z}_{\mathsf{ft}} \ni Z \mapsto \max Z$ *is a bijection onto* $T \smallsetminus F^{-1}(C)$. *Its inverse is* $\{p^m(t)|k(t){\geq}m{\geq}1\}{\cup}\{t\} \hookleftarrow t \in T \smallsetminus F^{-1}(C)$.

(b) E *is a well-defined bijection from* $\mathcal{Z}_{\mathsf{inft}}$ *onto*

$$\mathcal{Y} := \{ \ (t^v)_{v\geq 1} \mid t^o{=}p(t^1) \text{ and } (\forall v{\geq}1)\, t^v{=}p(t^{v+1}) \ \} \ .$$

Its inverse is $\{t^o\}{\cup}\{t^v|v{\geq}1\} \hookleftarrow (t^v)_{v\geq 1} \in \mathcal{Y}$. *(Proof A.3.)*

2.3. Choices

Section 2.1 called $\mathcal{H} = \{F^{-1}(c)|c\}$ the collection of "information sets".[6] Here I provide justification for that terminology. In the standard literature, [a] the collection of information sets partitions the set of decision nodes, and [b] two nodes in the same information set share the same set of feasible options. Feature [a] is assured by (1c) itself since $F^{-1}(C)$ is the set of decision nodes. Feature [b] is assured by (8a) of Proposition 2.2 below.

Further, (8b) of Proposition 2.2 shows that different information sets have different choices. Although this imposes a loss of generality in the mathematical sense, it does not impose a loss of generality in the modelling sense because one can always introduce more choices until each information set has its own choices.

Incidentally, (8b) also concerns my decision to use the term "choice". In the literature, there is a correlation between [1] assuming something like (8b) and [2] using the term "choice" rather than another term

[6] This implicit specification of information sets mimics a similar construction by Ritzberger (2002, page 97).

such as "action". For example, both [1] and [2] are done by von Neumann and Morgenstern (1944, Sections 9 and 10) and Ritzberger (2002, Section 3.2). In contrast, neither [1] nor [2] is done by Osborne and Rubinstein (1994, Section 11.1). Accordingly, I use the term "choice" rather than another term such as "action".

Proposition 2.2. *Suppose that $(T,C,\otimes)$ satisfies (1a) and (1c). Derive F and $\mathcal{H}$. Then the following hold.*

(8a) $$(\forall t,t')\ \ [(\exists H)\{t,t'\}\subseteq H]\ \Rightarrow\ F(t)=F(t') \text{ and}$$

(8b) $$(\forall t,t')\ \ [(\nexists H)\{t,t'\}\subseteq H]\ \Rightarrow\ F(t)\cap F(t')=\varnothing\ .$$

(Proof A.4.)

Finally, define

(9) $$q := \{(t^\sharp,c)|(\exists t)(t,c,t^\sharp)\in\otimes\}\ .$$

Since $\otimes$ is a bijection onto $T\smallsetminus\{t^o\}$ by (1a), q is a function from $T\smallsetminus\{t^o\}$. Accordingly, call q the *previous-choice* function,[7] and call $q(t^\sharp)$ the choice *previous* to $t^\sharp$.

The definition of q at (9) closely resembles the definition of p at (1b). This resemblance is not coincidental: Lemma A.5(b) shows that p is the first component of $\otimes^{-1}$, and that q is the second component of $\otimes^{-1}$. In other words, $(p,q)=\otimes^{-1}$. This identity is useful in proofs.

In summary, many entities can be derived from a preform $\Pi = (T,C,\otimes)$. Equations (1) and (9) derive F, t^o, p, $\mathcal{H}$, and q. The set of decision nodes is $F^{-1}(C)$. Equations (3)–(7) derive k, $\prec$, $\preccurlyeq$, $\mathcal{Z}$, $\mathcal{Z}_{\mathsf{ft}}$, $\mathcal{Z}_{\mathsf{inft}}$, and E.

3. Morphisms

3.1. Definition

A *(preform) morphism* α is a quadruple $[\Pi,\Pi',\tau,\delta]$ such that

(10a) $$\tau{:}T\rightarrow T'\ ,$$

(10b) $$\delta{:}C\rightarrow C'\ ,\ \text{and}$$

(10c) $$\{\ (\tau(t),\delta(c),\tau(t^\sharp))\ |\ (t,c,t^\sharp)\in\otimes\ \}\subseteq\otimes'\ ,$$

where $\Pi=(T,C,\otimes)$ and $\Pi'=(T',C',\otimes')$ are preforms.

[7] This resembles the function α assumed by Mas-Colell, Whinston, and Green (1995, page 227).

Proposition 3.1. *Suppose that the preform* $\Pi = (T, C, \otimes)$ *determines* F, t^o, p, *and* q, *and that the preform* $\Pi' = (T', C', \otimes')$ *determines* F', p', *and* q'. *(a) Then, a quadruple* $[\Pi, \Pi', \tau, \delta]$ *is a morphism iff it satisfies (10a)–(10b) and*

$$(\forall (t,c) \in F)\ (\tau(t), \delta(c)) \in F' \text{ and} \tag{11a}$$

$$(\forall (t,c) \in F)\ \tau(t \otimes c) = \tau(t) \otimes' \delta(c)\ . \tag{11b}$$

(b) Also, a quadruple $[\Pi, \Pi', \tau, \delta]$ *is a morphism iff it satisfies (10a)–(10b) and*

$$(\forall t^\sharp \neq t^o)\ \tau(p(t^\sharp)) = p'(\tau(t^\sharp)) \text{ and} \tag{12a}$$

$$(\forall t^\sharp \neq t^o)\ \delta(q(t^\sharp)) = q'(\tau(t^\sharp))\ . \tag{12b}$$

(Proof B.2.)

Characterization (11) concerns each member of $\otimes$'s domain. That is, (11) concerns each feasible node-choice pair (t, c). (11a) requires that its image is feasible. (11b) requires that the image of its result is the result of its image.

Meanwhile, characterization (12) concerns each member of $\otimes$'s range. That is, (12) concerns each non-initial node $t^\sharp$. (12a) requires that the image of its predecessor is the predecessor of its image. Similarly (12b) requires that the image of its previous choice is the previous choice of its image. Incidentally, (12a) is equivalent to $t = p(t^\sharp)$ implying $\tau(t) = p'(\tau(t^\sharp))$, and (12b) is equivalent to $c = q(t^\sharp)$ implying $\delta(c) = q'(\tau(t^\sharp))$ (Lemma B.3).

If two preforms are related by a morphism, there are certain relationships between the items derived from them. Some such results appear above in (11a), (12a), and (12b). Others appear in the following proposition.

Proposition 3.2. *Suppose* $[\Pi, \Pi', \tau, \delta]$ *is a morphism, where* $\Pi = (T, C, \otimes)$ *determines* F, t^o, p, $\mathcal{H}$, k, $\prec$, $\preccurlyeq$, $\mathcal{Z}_{\mathsf{ft}}$, *and* $\mathcal{Z}_{\mathsf{inft}}$, *and where* $\Pi' = (T', C', \otimes')$ *determine* F', t'^o, p', $\mathcal{H}'$, k', $\prec'$, $\preccurlyeq'$, $\mathcal{Z}'_{\mathsf{ft}}$, *and* $\mathcal{Z}'_{\mathsf{inft}}$. *Then the following hold.*

(a) $t'^o \preccurlyeq' \tau(t^o)$.
(b) If $t \in F^{-1}(C)$, *then* $\tau(t) \in (F')^{-1}(C')$.
(c) If $m \geq 1$ *and* $t^1 = p^m(t^2)$, *then* $\tau(t^1) = (p')^m(\tau(t^2))$.
(d) $k'(\tau(t)) = k(t) + k'(\tau(t^o))$.
(e) If $t^1 \prec t^2$, *then* $\tau(t^1) \prec' \tau(t^2)$.

(f) If $t^1 \preccurlyeq t^2$*, then* $\tau(t^1) \preccurlyeq' \tau(t^2)$.
(g) If $S{\subseteq}T$ *is a chain, then* $\tau|_S$ *is injective and* $\tau(S)$ *is a chain.*[8]
(h) $(\forall Z{\in}\mathcal{Z}_{\mathsf{inft}})(\exists Z'{\in}\mathcal{Z}'_{\mathsf{inft}})\ \tau(Z) \subseteq Z'$.[8]
(i) $(\forall Z{\in}\mathcal{Z}_{\mathsf{ft}})(\exists Z'{\in}\mathcal{Z}'_{\mathsf{ft}}{\cup}\mathcal{Z}'_{\mathsf{inft}})\ \tau(Z) \subseteq Z'$.[8]
(j) $(\forall H)(\exists H')\ \tau(H) \subseteq H'$.[8] *(Proof B.4.)*

This paragraph considers some examples. Define the "a-or-b" preform Π^{ab} by

$$T^{\mathsf{ab}} = \{\mathsf{O},\mathsf{A},\mathsf{B}\}\ ,$$
$$C^{\mathsf{ab}} = \{\mathsf{a},\mathsf{b}\}\ ,\ \text{and}$$
$$\otimes^{\mathsf{ab}} = \{\ (\mathsf{O},\mathsf{a},\mathsf{A}),\ (\mathsf{O},\mathsf{b},\mathsf{B})\ \}\ .$$

Define the "centipede" preform Π^{ce} by

$$T^{\mathsf{ce}} = \{\mathsf{1},\mathsf{2},\mathsf{3},...\}{\cup}\{\bar{\mathsf{1}},\bar{\mathsf{2}},\bar{\mathsf{3}},...\}\ ,$$
$$C^{\mathsf{ce}} = \{\mathsf{1stop},\mathsf{1go},\mathsf{2stop},\mathsf{2go},...\}\ ,\ \text{and}$$
$$\otimes^{\mathsf{ce}} = \{\ (\mathsf{1},\mathsf{1stop},\bar{\mathsf{1}}),\ (\mathsf{1},\mathsf{1go},\mathsf{2}),\ (\mathsf{2},\mathsf{2stop},\bar{\mathsf{2}}),\ (\mathsf{2},\mathsf{2go},\mathsf{3}),\ ...\ \}\ .$$

There are many morphisms from Π^{ab} to Π^{ce}. One injective morphism is $[\Pi^{\mathsf{ab}},\Pi^{\mathsf{ce}},\tau,\delta]$ where

$$\tau(\mathsf{O}) = \mathsf{2}\ ,\tau(\mathsf{A}) = \bar{\mathsf{2}}\ ,\tau(\mathsf{B}) = \mathsf{3}\ ,$$
$$\delta(\mathsf{a}) = \mathsf{2stop}\ ,\ \text{and}\ \delta(\mathsf{b}) = \mathsf{2go}\ .$$

Note that $\{(\tau(t),\delta(c),\tau(t^\sharp))|(t,c,t^\sharp){\in}\otimes^{\mathsf{ab}}\} = \{(\mathsf{2},\mathsf{2stop},\bar{\mathsf{2}}),(\mathsf{2},\mathsf{2go},\mathsf{3})\}$ $\subseteq \otimes^{\mathsf{ce}}$, as required by (10c). Meanwhile, one non-injective morphism is $[\Pi^{\mathsf{ab}},\Pi^{\mathsf{ce}},\tau^*,\delta^*]$ where

$$\tau^*(\mathsf{O}) = \mathsf{2}\ ,\tau^*(\mathsf{A}) = \tau^*(\mathsf{B}) = \mathsf{3}\ ,$$
$$\text{and}\ \delta^*(\mathsf{a}) = \delta^*(\mathsf{b}) = \mathsf{2go}\ .$$

Note that $\{(\tau^*(t),\delta^*(c),\tau^*(t^\sharp))|(t,c,t^\sharp){\in}\otimes^{\mathsf{ab}}\} = \{(\mathsf{2},\mathsf{2go},\mathsf{3})\} \subseteq \otimes^{\mathsf{ce}}$, as required by (10c).

3.2. THE CATEGORY **ncPreform**

This subsection defines the category **ncPreform**, which is called the *category of node-and-choice preforms*. Let an object be a node-and-choice preform $\Pi = (T,C,\otimes)$. Let an arrow be a preform morphism

[8] In Proposition 3.2(g)–(j), and in Proposition 3.3(h)–(j), the symbol τ is overloaded. Specifically, if S is a set of nodes in Π, then $\tau(S) := \{\tau(t)|t{\in}S\}$.

$\alpha = [\Pi, \Pi', \tau, \delta]$. Let source, target, identity, and composition be

$$\alpha^{\mathsf{src}} = [\Pi, \Pi', \tau, \delta]^{\mathsf{src}} = \Pi \ ,$$
$$\alpha^{\mathsf{trg}} = [\Pi, \Pi', \tau, \delta]^{\mathsf{trg}} = \Pi' \ ,$$
$$\mathsf{id}_\Pi = [\Pi, \Pi, \mathsf{id}_T^{\mathbf{Set}}, \mathsf{id}_C^{\mathbf{Set}}] \ , \text{ and}$$
$$\alpha' \circ \alpha = [\Pi', \Pi'', \tau', \delta'] \circ [\Pi, \Pi', \tau, \delta] = [\Pi, \Pi'', \tau' \circ \tau, \delta' \circ \delta] \ ,$$

where $\mathsf{id}^{\mathbf{Set}}$ is the identity in **Set**.

Theorem 1. **ncPreform** *is a category. (Proof B.5.)*

3.3. Isomorphisms in **ncPreform**

Theorem 2. *Suppose that $\alpha = [\Pi, \Pi', \tau, \delta]$ is a morphism. Then α is an isomorphism iff τ and δ are bijections. Further, if α is an isomorphism, then $\alpha^{-1} = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]$. (Proof B.6.)*

Proposition 3.3. *Suppose $[\Pi, \Pi', \tau, \delta]$ is an isomorphism, where $\Pi = (T, C, \otimes)$ determines F, t^o, p, q, $\mathcal{H}$, k, $\prec$, $\preccurlyeq$, $\mathcal{Z}_{\mathsf{ft}}$, $\mathcal{Z}_{\mathsf{inft}}$, and E, and where $\Pi' = (T', C', \otimes')$ determines F', t'^o, p', q', $\mathcal{H}'$, k', $\prec'$, $\preccurlyeq'$, $\mathcal{Z}'_{\mathsf{ft}}$, $\mathcal{Z}'_{\mathsf{inft}}$, and E'. Then the following hold.*
(a) $(\tau, \delta, \tau)|_\otimes$ is a bijection from $\otimes$ onto $\otimes'$.
(b) $(\tau, \delta)|_F$ is a bijection from F onto F'.
(c) $\tau|_{F^{-1}(C)}$ is a bijection from $F^{-1}(C)$ onto $(F')^{-1}(C')$.
(d) $(\tau, \tau)|_p$ is a bijection from p onto p'.
(e) $(\tau, \delta)|_q$ is a bijection from q onto q'.
(f) $(\tau, \tau)|_\prec$ is a bijection from $\prec$ onto $\prec'$.
(g) $(\tau, \tau)|_\preccurlyeq$ is a bijection from $\preccurlyeq$ onto $\preccurlyeq'$.
(h) $\tau|_\mathcal{H}$ is a bijection from $\mathcal{H}$ onto $\mathcal{H}'$.[8]
(i) $\tau|_{\mathcal{Z}_{\mathsf{ft}}}$ is a bijection from $\mathcal{Z}_{\mathsf{ft}}$ onto $\mathcal{Z}'_{\mathsf{ft}}$.[8]
(j) $\tau|_{\mathcal{Z}_{\mathsf{inft}}}$ is a bijection from $\mathcal{Z}_{\mathsf{inft}}$ onto $\mathcal{Z}'_{\mathsf{inft}}$.[8]
(k) $\tau(t^o) = t'^o$.
(l) $k'(\tau(t)) = k(t)$.
(m) $(\forall Z \in \mathcal{Z}_{\mathsf{inft}})(\forall v \geq 1)\ \tau(E[Z]^v) = E'[\tau(Z)]^v$. (Proof B.9.)

Some pairs of preforms are uniquely isomorphic in the sense that there is exactly one pair of isomorphisms between them. For example, define the "Roman centipede" preform Π^{R} by

$$T^{\mathsf{R}} = \{\mathsf{i}, \mathsf{ii}, \mathsf{iii}, ...\} \cup \{\bar{\mathsf{i}}, \bar{\mathsf{ii}}, \bar{\mathsf{iii}}, ...\} \ ,$$

$$C^{\mathsf{R}} = \{\mathsf{i{:}finio}, \mathsf{i{:}procedo}, \mathsf{ii{:}finio}, \mathsf{ii{:}procedo}, ...\} \ , \text{ and}$$

$$\otimes^{\mathsf{R}} = \{ \ (\mathsf{i}, \mathsf{i{:}finio}, \bar{\mathsf{i}}), \ (\mathsf{i}, \mathsf{i{:}procedo}, \mathsf{ii}), \ (\mathsf{ii}, \mathsf{ii{:}finio}, \bar{\mathsf{ii}}), \ (\mathsf{ii}, \mathsf{ii{:}procedo}, \mathsf{iii}), \ ... \ \} \ .$$

It can be shown that this preform is uniquely isomorphic to the preform Π^{ce} defined in Section 3.1. Essentially, there are exactly two nodes at each nonzero stage, and exactly one of these two is a decision node. Hence parts (c) and (l) of Proposition 3.3 determine τ. Then τ and (10c) determine δ.

In contrast, other pairs of preforms are isomorphic but not uniquely isomorphic. For example, define the "x-or-y" preform Π^{xy} by

$$T^{\mathsf{xy}} = \{\mathsf{O}, \mathsf{X}, \mathsf{Y}\} \ ,$$

$$C^{\mathsf{xy}} = \{\mathsf{x}, \mathsf{y}\} \ , \text{ and}$$

$$\otimes^{\mathsf{xy}} = \{ \ (\mathsf{O}, \mathsf{x}, \mathsf{X}), \ (\mathsf{O}, \mathsf{y}, \mathsf{Y}) \ \} \ .$$

There are two distinct isomorphisms from Π^{xy} to the preform Π^{ab} defined in Section 3.1. One is $[\Pi^{\mathsf{xy}}, \Pi^{\mathsf{ab}}, \tau, \delta]$ in which $\tau = \{(\mathsf{X}, \mathsf{A}), (\mathsf{Y}, \mathsf{B})\}$ and $\delta = \{(\mathsf{x}, \mathsf{a}), (\mathsf{y}, \mathsf{b})\}$. Another is $[\Pi^{\mathsf{xy}}, \Pi^{\mathsf{ab}}, \tau^*, \delta^*]$ in which $\tau^* = \{(\mathsf{X}, \mathsf{B}), (\mathsf{Y}, \mathsf{A})\}$ and $\delta = \{(\mathsf{x}, \mathsf{b}), (\mathsf{y}, \mathsf{a})\}$.

Appendix **A.** For Preforms

Lemma A.1. *Suppose that $(T, C, \otimes)$ satisfies (1a) and derive its F and t^o. (a) Then, $F^{-1}(C) = p(T \smallsetminus \{t^o\})$, where p is defined as in (1b). (b) Also, $F^{-1}(C) = \{t^1 | (\exists t^2) t^1 \prec t^2\}$, where $\prec$ is defined by (4).*

Proof. (a). I argue

$$\begin{aligned} F^{-1}(C) &= \{ \ t \mid (\exists c) \ (t, c) \in F \ \} \\ &= \{ \ t \mid (\exists c)(\exists t^\sharp) \ (t, c, t^\sharp) \in \otimes \ \} \\ &= \{ \ t \mid (\exists t^\sharp) \ t = p(t^\sharp) \ \} \\ &= p(T \smallsetminus \{t^o\}) \ , \end{aligned}$$

where the first equality is a rearrangement, the second follows from (1a), the third follows from the definition of p, and the fourth holds because the domain of p is $T \smallsetminus \{t^o\}$.

(b). By part (a), it suffices to prove

$$p(T \smallsetminus \{t^o\}) = \{ \ t^1 \mid (\exists t^2) \ t^1 \prec t^2 \ \} \ .$$

To prove the $\subseteq$ direction, suppose there exists $t^\sharp$ such that $t = p(t^\sharp)$. Then $t \prec t^\sharp$ by the definition of $\prec$. Conversely, suppose $t^1 \prec t^2$. Then by the definition of $\prec$, there exists $m{\geq}1$ such that $t^1 = p^m(t^2)$. If m equals 1, $t^1 = p(t^2)$ (and $t^2 \in T{\smallsetminus}\{t^o\}$ because this set is the domain of p). Otherwise, $t^1 = p(p^{m-1}(t^2))$ (and $p^{m-1}(t^2) \in T{\smallsetminus}\{t^o\}$ because this set is the domain of p). □

Lemma A.2. *Suppose (T, p) is a tree oriented toward t^o, and derive k, $\prec$, $\preccurlyeq$, $\mathcal{Z}_{\mathsf{ft}}$, and $\mathcal{Z}_{\mathsf{inft}}$ by (3)–(6). Then the following hold.*

(a) $t^1 \prec t^2$ iff both $k(t^1) < k(t^2)$ and $t^1 = p^{k(t^2)-k(t^1)}(t^2)$.

(b) $\prec$ is the asymmetric part of $\prec$.

(c) $t^1 \preccurlyeq t^2$ iff both $k(t^1) \leq k(t^2)$ and $t^1 = p^{k(t^2)-k(t^1)}(t^2)$, where p^0 is the identity function.

(d) $(T, \preccurlyeq)$ is a partially ordered set.

(e) If $S{\subseteq}T$ is a chain, $S{\cup}\{p^m(t)|t{\in}S, k(t){\geq}m{\geq}1\}$ is a chain.

(f) If $S{\subseteq}T$ is an infinite chain, $S{\cup}\{p^m(t)|t{\in}S, k(t){\geq}m{\geq}1\} \in \mathcal{Z}_{\mathsf{inft}}$.

(g) If $S{\subseteq}T$ is a chain, there exists $Z \in \mathcal{Z}_{\mathsf{ft}}{\cup}\mathcal{Z}_{\mathsf{inft}}$ such that $S \subseteq Z$.

(h) If $t \in Z \in \mathcal{Z}_{\mathsf{ft}}{\cup}\mathcal{Z}_{\mathsf{inft}}$ and $k(t){\geq}m{\geq}1$, then $p^m(t) \in Z$.

Proof. (a). The reverse direction follows immediately from the definition of $\prec$. To see the forward direction, suppose $t^1 \prec t^2$. Then by the definition of $\prec$, there exists an $m \geq 1$ such that $t^1 = p^m(t^2)$. Meanwhile by the definition of $k(t^1)$, I have $t^o = p^{k(t^1)}(t^1)$. Combining these two yields $t^o = p^{k(t^1)}(t^1) = p^{k(t^1)}(p^m(t^2)) = p^{k(t^1)+m}(t^2)$. Thus $k(t^2) = k(t^1){+}m$ by the definition of $k(t^2)$. So $m = k(t^2){-}k(t^1)$. This and the definition of m imply both $k(t^2) > k(t^1)$ and $t^1 = p^{k(t^2)-k(t^1)}(t^2)$.

(b). By the definition of $\preccurlyeq$, it suffices to prove that $\prec$ is asymmetric. This relation is asymmetric because if both $t^1 \prec t^2$ and $t^2 \prec t^1$ held, part (a) would imply both $k(t^1) < k(t^2)$ and $k(t^2) < k(t^1)$.

(c). By using the definition of $\preccurlyeq$ for the first equivalence, and by using part (a) for the second equivalence,

$$\begin{aligned}
& t^1{\preccurlyeq}t^2 \\
\Leftrightarrow\ & t^1{\prec}t^2 \text{ or } t^1{=}t^2 \\
\Leftrightarrow\ & [\ k(t^1){<}k(t^2) \text{ and } t^1{=}p^{k(t^2)-k(t^1)}(t^2)\ ] \text{ or} \\
& [\ k(t^1){=}k(t^2) \text{ and } t^1{=}p^{k(t^2)-k(t^1)}(t^2)\ ] \\
\Leftrightarrow\ & k(t^1){\leq}k(t^2) \text{ and } t^1{=}p^{k(t^2)-k(t^1)}(t^2)\ .
\end{aligned}$$

(d). Reflexivity holds by the definition of $\preccurlyeq$. Transitivity holds by [1] the definition of $\preccurlyeq$ and [2] the transitivity of $\prec$, which follows immediately from its definition. To show antisymmetry, suppose $t^1 \preccurlyeq t^2$ and $t^2 \preccurlyeq t^1$. Then by two applications of part (c), $k(t^1) = k(t^2)$. Thus by $t^1 \preccurlyeq t^2$ and part (c) again, $t^1 = p^0(t^2) = t^2$.

(e). Let p^0 be the identity function, so that

$$S\cup\{p^m(t)|t\in S, k(t)\geq m\geq 1\} = \{p^m(t)|t\in S, k(t)\geq m\geq 0\} \ .$$

Then consider $p^{m^1}(t^1)$ and $p^{m^2}(t^2)$ such that $\{t^1, t^2\}\in S$, $k(t^1)\geq m^1\geq 0$, and $k(t^2)\geq m^2\geq 0$. Since S is a chain, assume without loss of generality that $t^1 \preccurlyeq t^2$. Thus by part (c), there is an $m\geq 0$ such that $t^1 = p^m(t^2)$. If $m^1{+}m > m^2$, $p^{m^1}(t^1) = p^{m^1+m}(t^2) \prec p^{m^2}(t^2)$. If $m^1{+}m = m^2$, $p^{m^1}(t^1) = p^{m^1+m}(t^2) = p^{m^2}(t^2)$. If $m^1{+}m < m^2$, $p^{m^1}(t^1) = p^{m^1+m}(t^2) \succ p^{m^2}(t^2)$.

(f). Suppose S is an infinite chain. Since S is a chain and since $\min S$ exists, I may number the elements of S so that $\min S = t^1 \prec t^2 \prec t^3$ ... . Thus by part (a), $(\forall n\geq 1)\ k(t^n) < k(t^{n+1})$. Hence $(\forall n\geq 1)\ k(t^n) \geq n{-}1$.

Now consider $\bar{S} := S\cup\{p^m(t)|t\in S, k(t)\geq m\geq 1\}$. By part (e), $\bar{S}$ is a chain. Further, it is infinite because S is infinite. Thus it remains to be shown that $\bar{S}$ is maximal. Accordingly, suppose that it were not maximal. Then there would be some $t' \notin \bar{S}$ such that $\bar{S}\cup\{t'\}$ is a chain.

This paragraph shows that $(\forall n\geq 1)\ k(t') \geq n$. Take any $n\geq 1$. Since $t' \notin \bar{S}$, and since t^n and all its predecessors are in $\bar{S}$, it must be that $t' \succ t^n$. Thus by part (a), $k(t') > k(t^n)$. Thus, since $k(t^n) \geq n{-}1$ by the second-previous paragraph, $k(t') \geq n$.

By the previous paragraph, $k(t') \notin \mathbb{N}_0$. This contradicts the definition of k.

(g). Suppose S is a chain. On the one hand, suppose S is infinite. Then part (f) shows that it is a subset of a member of $\mathcal{Z}_{\mathsf{inft}}$. On the other hand, suppose S is finite. Then $\max S$ exists, and two cases arise. These cases are defined in the first sentences of the next two paragraphs.

[1] Suppose that [a] $\max S$ does not have a successor or [b] $\max S$ has a successor that does not have a successor. In either [a] or [b], let t^* denote the node without a successor. Then $S\cup\{t^*\}$ is a chain. Thus by part (e), $\bar{S} = (S\cup\{t^*\})\cup\{p^m(t)|t\in S\cup\{t^*\}, k(t)\geq m\geq 1\}$ is a chain. If $\bar{S}$ were not maximal, there would be some $t' \notin \bar{S}$ such that $\bar{S}\cup\{t'\}$ is

a chain. Since $\bar{S}$ contains all the predecessors of t^*, it must be that $t' \succ t^*$. But this contradicts the assumption that t^* does not have a successor.

[2] Suppose that $\max S$ has a successor and that every successor of $\max S$ has a successor. Then define S^1 by $S^1 = S \cup \{t^1\}$ where t^1 is some successor of $\max S$. Then, for every $n \geq 2$, define $S^n = S^{n-1} \cup \{t^2\}$ where t^n is some successor of t^{n-1}. Then $\cup_{n \geq 1} S^n$ is an infinite chain. Thus part (f) shows that it is a subset of a member of $\mathcal{Z}_{\mathsf{inft}}$.

(h). Suppose $t \in Z \in \mathcal{Z}_{\mathsf{ft}} \cup \mathcal{Z}_{\mathsf{inft}}$ and $k(t) \geq m \geq 1$. I argue

$$p^m(t) \in Z \cup \{p^{m'}(t') | t' \in Z, k(t') \geq m' \geq 1\} \subseteq Z \ .$$

The set membership holds because $t \in Z$ and $k(t) \geq m \geq 1$. The set inclusion holds because [1] $Z \cup \{p^{m'}(t') | t' \in Z, k(t') \geq m' \geq 1\}$ is a chain by part (e) and [2] Z is maximal by $Z \in \mathcal{Z}_{\mathsf{ft}} \cup \mathcal{Z}_{\mathsf{inft}}$. □

Proof A.3 (for Proposition 2.1)**.** *(a).* I must show that

$$\mathcal{Z}_{\mathsf{ft}} \ni Z \mapsto \max Z \tag{13}$$

is a bijection onto $T \smallsetminus F^{-1}(C)$, and that its inverse is

$$\{p^m(t) | k(t) \geq m \geq 1\} \cup \{t\} \leftarrow\!\shortmid\, t \in T \smallsetminus F^{-1}(C) \ . \tag{14}$$

These results follow from the next two paragraphs.

This paragraph argues that the function (13) followed by the function (14) is the identity function on $\mathcal{Z}_{\mathsf{ft}}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{ft}}$. The remainder of this paragraph argues

$$\begin{gathered} Z \mapsto \max Z \mapsto \\ \{\ p^m(\max Z) \mid k(\max Z) \geq m \geq 1\ \} \cup \{\ \max Z\ \} = Z\ , \end{gathered} \tag{15}$$

where the two arrows apply the functions (13) and (14), respectively. By inspection, the first arrow applies the function (13). Before applying the function (14), it must be shown that $\max Z$ exists and is an element of $T \smallsetminus F^{-1}(C)$. First, $\max Z$ exists and is an element of T because $Z \subseteq T$ is a finite chain. Second, $\max Z$ is not an element of $F^{-1}(C)$, for if it were, [1] it would have a successor by Lemma A.1(b), thus [2] Z would not be a maximal chain, and thus [3] $Z \notin \mathcal{Z}_{\mathsf{ft}}$ in contradiction to the definition of Z. Accordingly, the second arrow in (15) applies the function (14) at $t = \max Z$. To continue, the $\subseteq$ direction of the equality in (15) holds by Lemma A.2(h) applied at $t = \max Z$. To see the $\supseteq$ direction, take any $t \in Z$. Because Z is a chain that contains $\max Z$, either $t \preccurlyeq \max Z$ or $\max Z \prec t$. The former implies that t is in the left-hand side. The latter contradicts the definition of the maximum operator.

This paragraph argues that the function (14) followed by the function (13) is the identity function on $T \smallsetminus F^{-1}(C)$. Accordingly, take any $t \in T \smallsetminus F^{-1}(C)$. The remainder of this paragraph argues

$$\begin{aligned} t &\mapsto \{p^m(t)|k(t){\geq}m{\geq}1\}{\cup}\{t\} \mapsto \\ &\max\{p^m(t)|k(t){\geq}m{\geq}1\}{\cup}\{t\} = t \ , \end{aligned} \tag{16}$$

where the two arrows apply the function (14) and (13), respectively. By inspection, the first arrow applies the function (14). Before applying the function (13), it must be shown that $S := \{p^m(t)|k(t){\geq}m{\geq}1\}{\cup}\{t\}$ is an element of $\mathcal{Z}_{\mathsf{ft}}$. Since S is a finite chain by inspection, I only need to show that S is maximal. Accordingly, suppose there were a $t' \notin S$ such that $S{\cup}\{t'\}$ was a chain. Because $t \in S$ and $S{\cup}\{t'\}$ is a chain, either $t' \preccurlyeq t$ or $t \prec t'$. The first case is impossible for it would imply that $t' \in S$, in contradiction to the definition of t'. The second case would imply [1] that t has a successor, and thus [2] that $t \in F^{-1}(C)$ by Lemma A.1(b). This would contradict the definition of t. Accordingly, the second arrow in (16) applies the function (13) at $Z = S$. The equality is immediate.

(b). This paragraph shows that E is a well-defined function from $\mathcal{Z}_{\mathsf{inft}}$ into $T^{\mathbb{N}_1}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. It must be shown that

$$(\forall v{\geq}1)(\exists! t{\in}Z)\ k(t) = v \ .$$

Take any $v \geq 1$. First, consider uniqueness. It must be shown that there are not two nodes in Z at stage v. This holds because distinct nodes in a chain have different stages by Lemma A.2(a). Second, consider existence. Let $S := \{t'{\in}Z|k(t'){\leq}v\}$. Since distinct nodes in a chain have different stages by Lemma A.2(a), S is finite. Thus, since Z is infinite, there is some $t^* \in Z$ such that $k(t^*) > v$. Let $t = p^{k(t^*)-v}(t^*)$. Note $t \in Z$ by Lemma A.2(h) at its t equal to t^* and its m equal to $k(t^*){-}v$. Further note that

$$t^o = p^{k(t^*)}(t^*) = p^v(p^{k(t^*)-v}(t^*)) = p^v(t) \ ,$$

where the first equality holds by the definition of $k(t^*)$, the second is a rearrangement, and the thirds holds by the definition of t. Thus $k(t) = v$ by the definition of $k(t)$.

This paragraph shows that E maps from $\mathcal{Z}_{\mathsf{inft}}$ into $\mathcal{Y} \subseteq T^{\mathbb{N}_1}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. By the previous paragraph, I may let $E(Z) = (t^v)_{v\geq1}$. It must be shown that $t^o = p(t^1)$ and that $(\forall v{\geq}1)\ t^v = p(t^{v+1})$. Since $k(t^1) = 1$ by the definition of E, $p(t^1) = t^o$ by the definition of k. Next take any $v \geq 1$. By the definition of E, [1] $\{t^v, t^{v+1}\} \subseteq Z$, [2] $k(t^v) = v$, and [3] $k(t^{v+1}) = v{+}1$. By [1], $t^v \prec t^{v+1}$ or $t^{v+1} \preccurlyeq t^v$. Thus $t^v \prec t^{v+1}$ because the alternative is impossible by [2], [3], and Lemma A.2(c). Finally, $t^v \prec t^{v+1}$ implies $t^v = p(t^{v+1})$ by [2], [3], and Lemma A.2(a).

The next two paragraphs prove that E is a bijection from $\mathcal{Z}_{\mathsf{inft}}$ onto $\mathcal{Y}$, and that its inverse is

$$\{t^o\}\cup\{t^v|v{\geq}1\} \hookleftarrow (t^v)_{v\geq1} \in \mathcal{Y} \ . \tag{17}$$

This paragraph argues that E followed by the function (17) is the identity function on $\mathcal{Z}_{\mathsf{inft}}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. I argue

$$Z \mapsto E(Z) \mapsto$$
$$\{t^o\}\cup\{E(Z)^v|v{\geq}1\} = Z \ ,$$

where the arrows apply the functions E and (17), respectively. The first arrow applies E by inspection. The second arrow applies (17) because $E(Z) \in \mathcal{Y}$ by the second-previous paragraph. To see the $\subseteq$ direction of the equality, take any $t \in \{t^o\}\cup\{E(Z)^v|v{\geq}1\}$. If $t = t^o$, then $t \in Z$ because t^o belongs to every maximal chain and Z is a maximal chain. If $t = E(Z)^v$ from some $v \geq 1$, then $t \in Z$ by the definition of E. To see the $\supseteq$ direction of the equality, take any $t \in Z$. If $k(t) = 0$, then $t = t^o$. If $k(t) \geq 1$, then $t = E(Z)^{k(t)}$ by the definition of E.

This paragraph argues that the function (17) followed by E is the identity function on $\mathcal{Y}$. Accordingly, take any $(t^v)_{v\geq1} \in \mathcal{Y}$. I argue

$$(t^v)_{v\geq1} \mapsto \{t^o\}\cup\{t^v|v{\geq}1\} \mapsto$$
$$E(\ \{t^o\}\cup\{t^v|v{\geq}1\}\ ) = (t^v)_{v\geq1} \ ,$$

where the arrows apply the functions (17) and E, respectively. The first arrow applies (17) by inspection. Before applying E, it must be shown that $S := \{t^o\}\cup\{t^v|v{\geq}1\}$ belongs to $\mathcal{Z}_{\mathsf{inft}}$. In other words, it must be shown that S is an infinite maximal chain. The definitions of $(t^v)_{v\geq1}$ and $\mathcal{Y}$ assure that S is a chain and that S contains a node of every stage. This easily implies that S is infinite. It also implies that S is maximal because distinct nodes in a chain have different stages by

Lemma A.2(a). Hence S belongs to $\mathcal{Z}_{\mathsf{inft}}$ and the second arrow applies E. The equality follows from the fact that $(\forall v{\geq}1)\ k(t^v) = v$ by the definitions of $(t^v)_{v\geq 1}$ and $\mathcal{Y}$. □

Proof A.4 (for Proposition 2.2)**.** To show the contrapositive of (8a), suppose $F(t) \neq F(t')$. Without loss of generality, suppose $\hat{c} \in F(t)$ but $\hat{c} \notin F(t')$. Let $\hat{H} = F^{-1}(\hat{c})$ and note that $t \in \hat{H}$ but $t' \notin \hat{H}$. Thus, since $\mathcal{H} = \{F^{-1}(c)|c\}$ is a partition by (1c), there cannot be an H that contains both t and t'.

To show the contrapositive of (8b), suppose that $F(t){\cap}F(t') \neq \varnothing$. Then there is c such that $c \in F(t)$ and $c \in F(t')$. Hence both t and t' belong to $H := F^{-1}(c)$. □

Lemma A.5. *Suppose $(T, C, \otimes)$ satisfies (1a), derive p by (1b), and derive q by (9). Then the following hold.*
(a) $\otimes = \{\ (p(t^\sharp), q(t^\sharp), t^\sharp) \mid t^\sharp{\neq}t^o\ \}$.
(b) $\otimes^{-1} = (p, q)$.

Proof. (a) To show the $\subseteq$ direction, take any $(t, c, t^\sharp) \in \otimes$. Then [1] $t^\sharp \neq t^o$ by (1a), [2] $t = p(t^\sharp)$ by the definition of p, and [3] $c = q(t^\sharp)$ by the definition of q. Conclusions [2] and [3] imply $(t, c, t^\sharp) = (p(t^\sharp), q(t^\sharp), t^\sharp)$. Thus conclusion [1] implies that $(t, c, t^\sharp)$ belongs to $\{\ (p(t^\sharp), q(t^\sharp), t^\sharp)) \mid t^\sharp{\neq}t^o\ \}$.

To show the $\supseteq$ direction, take any $t^\sharp \neq t^o$. Then by (1a) there exists (t, c) such that $(t, c, t^\sharp) \in \otimes$. By the definition of p, $t = p(t^\sharp)$. By the definition of q, $c = q(t^\sharp)$. Therefore by the last three sentences, $(p(t), q(t), t^\sharp) \in \otimes$.

(b). Part (a) suffices because (1a) assumes that $\otimes$ is a bijection when viewed as a function from the first two components of its constituent triples to the third component of its constituent triples. □

Appendix B. For Morphisms

Lemma B.1. *Suppose that the preform $\Pi = (T, C, \otimes)$ determines F, t, p, and q, and that the preform $\Pi' = (T', C', \otimes')$ determines F', t'^o, p', and q'. Further suppose that $\tau{:}T{\to}T'$ and $\delta{:}C{\to}C'$. Then the following three conditions are equivalent.*

$$(a) \qquad \{\ (\tau(t), \delta(c), \tau(t^\sharp)) \mid (t, c, t^\sharp){\in}\otimes\ \} \subseteq \otimes'\ .$$

$$(b) \qquad \begin{gathered} (\forall (t,c){\in}F)\ (\tau(t),\delta(c)) \in F' \textit{ and} \\ (\forall (t,c){\in}F)\ \tau(t{\otimes}c) = \tau(t){\otimes}'\delta(c)\ . \end{gathered}$$

$$(c) \qquad \begin{gathered} (\forall t^\sharp{\neq}t^o)\ \tau(p(t^\sharp)) = p'(\tau(t^\sharp)) \textit{ and} \\ (\forall t^\sharp{\neq}t^o)\ \delta(q(t^\sharp)) = q'(\tau(t^\sharp))\ . \end{gathered}$$

Proof. (a)⇒(b). Assume (a). To show the first half of (b), I argue

$$\begin{aligned} F' &= \{\ (t',c') \mid (\exists t'^\sharp)\ (t',c',t'^\sharp){\in}{\otimes}'\ \} \\ &\supseteq \{\ (t',c') \mid (\exists t'^\sharp)(\exists (t,c,t^\sharp){\in}{\otimes})\ (t',c',t'^\sharp){=}(\tau(t),\delta(c),\tau(t^\sharp))\ \} \\ &= \{\ (t',c') \mid (\exists (t,c,t^\sharp){\in}{\otimes})\ (t',c'){=}(\tau(t),\delta(c))\ \} \\ &= \{\ (\tau(t),\delta(c)) \mid (\exists t^\sharp)\ (t,c,t^\sharp){\in}{\otimes}\ \} \\ &= \{\ (\tau(t),\delta(c)) \mid (t,c){\in}F\ \}\ . \end{aligned}$$

The first equality holds by the definition of F', and the set inclusion holds by (a). The second and third equalities are rearrangements, and the fourth holds by the definition of F. To see the second half of (b), take any $(t,c) \in F$. Then $(t,c,t{\otimes}c) \in \otimes$. Thus by (a), $(\tau(t),\delta(c),\tau(t{\otimes}c)) \in \otimes'$. Thus $\tau(t){\otimes}'\delta(c) = \tau(t{\otimes}c)$.

(a)⇐(b). Assume (b). Take any $(t,c,t^\sharp) \in \otimes$. Then $(t,c) \in F$ by the definition of F. Thus by (b), $\tau(t){\otimes}'\delta(c) = \tau(t{\otimes}c)$. Thus since $t{\otimes}c = t^\sharp$ by the definition of $(t,c,t^\sharp)$, I have $\tau(t){\otimes}'\delta(c) = \tau(t^\sharp)$. Thus $(\tau(t),\delta(c),\tau(t^\sharp)) \in \otimes'$.

(a)⇒(c). Assume (a). Take any $t^\sharp \neq t^o$. Then by Lemma A.5(a), $(p(t^\sharp),q(t^\sharp),t^\sharp) \in \otimes$. Thus by (a),

$$(\ \tau(p(t^\sharp)),\ \delta(q(t^\sharp)),\ \tau(t^\sharp)\ ) \in \otimes'\ .$$

This implies $\tau(p(t^\sharp)) = p'(\tau(t^\sharp))$ by the definition of p'. Further, it implies $\delta(q(t^\sharp)) = q'(\tau(t^\sharp))$ by the definition of q'.

(a)⇐(c). Assume (c). To begin, I argue

$$(\forall t{\neq}t^o)\ \tau(t) \neq t'^o\ . \tag{18}$$

Take any $t \neq t^o$. By the first half of (c), $\tau(t)$ is in the domain of p'. Thus, since the domain of p' is $T'{\smallsetminus}\{t'^o\}$, $\tau(t) \neq t'^o$. Then, I argue

$$\begin{aligned} \otimes' &= \{\ (\ p'(t'^\sharp),\ q'(t'^\sharp),\ t'^\sharp\ ) \mid t'^\sharp{\neq}t'^o\ \} \\ &\supseteq \{\ (\ p'(\tau(t^\sharp)),\ q'(\tau(t^\sharp)),\ \tau(t^\sharp)\ ) \mid \tau(t^\sharp){\neq}t'^o\ \} \\ &\supseteq \{\ (\ p'(\tau(t^\sharp)),\ q'(\tau(t^\sharp)),\ \tau(t^\sharp)\ ) \mid t^\sharp{\neq}t^o\ \} \end{aligned}$$

$$= \{ \ (\ \tau(p(t^\sharp)), \ \tau(q(t^\sharp)), \ \tau(t^\sharp) \ ) \mid t^\sharp{\neq}t^o \ \}$$
$$= \{ \ (\tau(t), \delta(c), \tau(t^\sharp)) \mid t{=}p(t^\sharp), \ c{=}q(t^\sharp), \ t^\sharp{\neq}t^o \ \}$$
$$= \{ \ (\tau(t), \delta(c), \tau(t^\sharp)) \mid (t, c, t^\sharp){\in}\otimes \ \}$$

The first equality holds by Lemma A.5(a) for Π'. The first set inclusion holds by the assumption that $\tau{:}T{\to}T'$. The second set inclusion holds by (18). The second equality holds by both halves of (c). The third equality is a rearrangement. The fourth equality holds by Lemma A.5(a) for Π. □

Proof B.2 (for Proposition 3.1)**.** I argue

$$\begin{aligned} & [\Pi, \Pi', \tau, \delta] \text{ is a morphism} \\ \Leftrightarrow\ & [\Pi, \Pi', \tau, \delta] \text{ satisfies (10a)–(10b) and (10c)} \\ \Leftrightarrow\ & [\Pi, \Pi', \tau, \delta] \text{ satisfies (10a)–(10b) and (11)} \\ \Leftrightarrow\ & [\Pi, \Pi', \tau, \delta] \text{ satisfies (10a)–(10b) and (12)} \ . \end{aligned}$$

The first equivalence is the definition of a morphism. The next two equivalences follow from Lemma B.1. □

Lemma B.3. *Suppose the preform $\Pi = (T, C, \otimes)$ determines p and q, and the preform $\Pi' = (T', C', \otimes')$ determines p' and q'. Further suppose $\tau{:}T{\to}T'$ and $\delta{:}C{\to}C'$. (a) Then, (12a) is equivalent to*

$$\{ \ (\tau(t^\sharp), \tau(t)) \mid (t^\sharp, t){\in}p \ \} \subseteq p' \ .$$

(b) Also, (12b) is equivalent to

$$\{ \ (\tau(t^\sharp), \delta(c) \mid (t^\sharp, c){\in}q \ \} \subseteq q' \ .$$

Proof. (a). I argue

$$\begin{aligned} & (\forall t^\sharp{\neq}t^o) \ \tau(p(t^\sharp)) = p'(\tau(t^\sharp)) \\ \Leftrightarrow\ & \{ \ (\tau(t^\sharp), \tau(p(t^\sharp)) \mid t^\sharp{\neq}t^o \ \} \subseteq p' \\ \Leftrightarrow\ & \{ \ (\tau(t^\sharp), \tau(t)) \mid t{=}p(t^\sharp), t^\sharp{\neq}t^o \ \} \subseteq p' \\ \Leftrightarrow\ & \{ \ (\tau(t^\sharp), \tau(t)) \mid (t^\sharp, t){\in}p \ \} \subseteq p' \ . \end{aligned}$$

The first two equivalences are rearrangements. The last holds because the domain of p is $T{\smallsetminus}\{t^o\}$.

(b). I argue

$$\begin{aligned} & (\forall t^\sharp{\neq}t^o) \ \delta(q(t^\sharp)) = q'(\tau(t^\sharp)) \\ \Leftrightarrow\ & \{ \ (\tau(t^\sharp), \delta(q(t^\sharp)) \mid t^\sharp{\neq}t^o \ \} \subseteq q' \end{aligned}$$

$$\Leftrightarrow \{\ (\tau(t^\sharp), \delta(c)) \mid c{=}q(t^\sharp), t^\sharp{\neq}t^o\ \} \subseteq q'$$
$$\Leftrightarrow \{\ (\tau(t^\sharp), \delta(c)) \mid (t^\sharp, c){\in}q\ \} \subseteq q'\ .$$

The first two equivalences are rearrangements. The last holds because the domain of q is $T{\smallsetminus}\{t^o\}$. □

Proof B.4 (for Proposition 3.2)**.** *(a).* This is trivial. It holds because $\tau(t^o) \in T'$ and because $(\forall t')\ t'^o \preccurlyeq' t'$.

(b). Suppose $t \in F^{-1}(C)$. Then by Lemma A.1(a) for Π, there exists a $t^\sharp$ such that $t = p(t^\sharp)$. Thus by Proposition 3.1(b) and Lemma B.3(a), $\tau(t) = p'(\tau(t^\sharp))$. Thus by Lemma A.1(a) for Π', $\tau(t) \in (F')^{-1}(C')$.

(c). Suppose $m{\geq}1$ and $t^1 = p^m(t^2)$.

This paragraph shows by induction on $i{\geq}1$ that

$$(\forall m{\geq}i{\geq}1)\ \tau(p^i(t^2)) = (p')^i(\tau(t^2))\ . \tag{20}$$

The initial step (i=1) holds by (12a) of Proposition 3.1, applied at $t^\sharp = t^2$ (note $t^2{\neq}t^o$ because $p^m(t^2)$ exists and $m{\geq}1$). To show the inductive step ($m{\geq}i{>}1$), I argue

$$\begin{aligned}\tau \circ p^i(t^2) &= \tau \circ p \circ p^{i-1}(t^2)\\ &= p' \circ \tau \circ p^{i-1}(t^2)\\ &= p' \circ (p')^{i-1} \circ \tau(t^2)\\ &= (p')^i \circ \tau(t^2)\ .\end{aligned}$$

The first equality is a rearrangement. The second equation holds by (12a) of Proposition 3.1, applied at $t^\sharp = p^{i-1}(t^2)$ (note $p^{i-1}(t^2){\neq}t^o$ because $p^m(t^2)$ exists and $m{\geq}i$). The third equation holds by the inductive hypothesis, and the fourth is a rearrangement.

Finally, I argue

$$\tau(t^1) = \tau(p^m(t^2)) = (p')^m(\tau(t^2))\ .$$

The first equality holds by the assumption $t^1 = p^m(t^2)$, the second holds by (20) at $i{=}m$.

(d). By the definition of $k'(\tau(t))$, it suffices to show

$$\begin{aligned}t'^o &= (p')^{k'(\tau(t^o))}[\tau(t^o)]\\ &= (p')^{k'(\tau(t^o))}[(p')^{k(t)}(\tau(t))]\\ &= (p')^{k(t)+k'(\tau(t^o))}(\tau(t))\ .\end{aligned}$$

The first equality follows from the definition of $k'(\tau(t^o))$. To see the second equality, note $t^o = p^{k(t)}(t)$ by the definition of $k(t)$. Hence $\tau(t^o) = (p')^{k(t)}(\tau(t))$ by part (c). The final equality is a rearrangement.

(e). Suppose $t^1 \prec t^2$. Then by the definition of $\prec$, there exists $m{\geq}1$ such that $t^1 = p^m(t^2)$. Thus by part (c), $\tau(t^1) = (p')^m(\tau(t^2))$. Thus by the definition of $\prec'$, $\tau(t^1) \prec' \tau(t^2)$.

(f). Suppose $t^1 \preccurlyeq t^2$. Then by the definition of $\preccurlyeq$, either $t^1 = t^2$ or $t^1 \prec t^2$. In the case of equality, $\tau(t^1) = \tau(t^2)$. In the case of precedence, part (e) implies $\tau(t^1) \prec \tau(t^2)$. Thus in either case, $\tau(t^1) \preccurlyeq \tau(t^2)$.

(g). Suppose $S \subseteq T$ is a chain.

To show that $\tau|_S$ is injective, suppose t^1 and t^2 are distinct members of S. Since S is a chain, $t^1 \prec t^2$ without loss of generality. Hence $\tau(t^1) \prec' \tau(t^2)$ by part (e). Hence $\tau(t^1)$ and $\tau(t^2)$ are distinct.

To show that $\tau(S)$ is a chain, take any distinct t'^1 and t'^2 in $\tau(S)$. Since both are in $\tau(S)$, there exist distinct t^1 and t^2 in S such that $\tau(t^1) = t'^1$ and $\tau(t^2) = t'^2$. Thus since S is a chain, $t^1 \prec t^2$ without loss of generality. Hence $\tau(t^1) \prec' \tau(t^2)$ by part (e). Hence $t'^1 \prec' t'^2$ by the definition of t^1 and t^2.

(h). Take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. Since Z is an infinite chain in T, part (g) implies that $\tau(Z)$ is an infinite chain in T'. Thus by Lemma A.2(f) applied to (T',p') at $S' = \tau(Z)$, there exists $Z' \in \mathcal{Z}'_{\mathsf{inft}}$ such that $\tau(Z) \subseteq Z'$.

(i). Take any $Z \in \mathcal{Z}_{\mathsf{ft}}$. Since Z is a chain in T, part (g) implies that $\tau(Z)$ is a chain in T'. Thus by Lemma A.2(g) applied to (T',p') at $S' = \tau(Z)$, there exists $Z' \in \mathcal{Z}'_{\mathsf{ft}} \cup \mathcal{Z}'_{\mathsf{inft}}$ such that $\tau(Z) \subseteq Z'$.

(j). Take any H. By (1c) for Π, there exists c such that $H = F^{-1}(c)$. Let $H' = (F')^{-1}(\delta(c))$. Note $H' \in \mathcal{H}'$ by (1c) for Π'. Thus it suffices to argue

$$
\begin{aligned}
\tau(H) &= \{\ \tau(t) \mid t{\in}H\ \} \\
&= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } t{\in}H\ \} \\
&= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } t{\in}F^{-1}(c)\ \} \\
&= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (t,c){\in}F\ \} \\
&\subseteq \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (\tau(t),\delta(c)){\in}F'\ \} \\
&= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (t',\delta(c)){\in}F'\ \} \\
&\subseteq \{\ t' \mid (t',\delta(c)){\in}F'\ \}
\end{aligned}
$$

$$= (F')^{-1}(\delta(c))$$
$$= H' \ .$$

The first and second equalities are rearrangements, the third follows from the definition of c, and the fourth is a rearrangement. The first inclusion follows from (11a) of Proposition 3.1(a). The fifth equality is a rearrangement. The second inclusion follows from $\tau(T) \subseteq T'$, which follows from (10a). The sixth equality is a rearrangement, and the final equality follows from the definition of H'. □

Proof B.5 (for Theorem 1)**.** This paragraph notes that, for every preform $\Pi = (T, C, \otimes)$, the quadruple $[\Pi, \Pi, \mathsf{id}_T^{\mathbf{Set}}, \mathsf{id}_C^{\mathbf{Set}}]$ is a morphism. By inspection $\mathsf{id}_T^{\mathbf{Set}}$ satisfies (10a) and $\mathsf{id}_C^{\mathbf{Set}}$ satisfies (10b). Further, (10c) holds with equality.

This paragraph shows that, if $\alpha = [\Pi, \Pi', \tau, \delta]$ and $\alpha' = [\Pi', \Pi'', \tau', \delta']$ are morphisms, then $[\Pi, \Pi'', \tau' \circ \tau, \delta' \circ \delta]$ is a morphism. Accordingly, take any such α and α'. Let $\Pi = (T, C, \otimes)$, $\Pi' = (T', C', \otimes')$, and $\Pi'' = (T'', C'', \otimes'')$. Note that $\tau{:}T \to T'$ by (10a) for α, and that $\tau'{:}T' \to T''$ by (10a) for α'. Hence $\tau' \circ \tau{:}T \to T''$, which is (10a) for $\alpha' \circ \alpha$. A parallel argument shows $\delta' \circ \delta{:}C \to C''$, which is (10b) for $\alpha' \circ \alpha$. Finally, to show that (10c) holds for $\alpha' \circ \alpha$, I argue

$$\begin{aligned}
&\{\ (\tau' \circ \tau(t), \delta' \circ \delta(c), \tau' \circ \tau(t^\sharp))\ |\ (t, c, t^\sharp) \in \otimes\ \} \\
= &\{\ (\tau'(t'), \delta'(c'), \tau'(t'^\sharp))\ |\ (t', c', t'^\sharp) \in \{(\tau(t), \delta(c), \tau(t^\sharp)) | (t, c, t^\sharp) \in \otimes\}\ \} \\
\subseteq &\{\ (\tau'(t'), \delta'(c'), \tau'(t'^\sharp))\ |\ (t', c', t'^\sharp) \in \otimes'\ \} \\
\subseteq &\otimes'' \ .
\end{aligned}$$

The equality is a rearrangement. The first inclusion holds by (10c) for α, and the second inclusion holds by (10c) for α'.

The first paragraph of this proof shows that the identity arrow id_Π is well-defined for any preform Π. The second paragraph shows that the composition $\alpha' \circ \alpha$ is well-defined for any arrows α and α'. The unit and associative laws are immediate. Thus **ncPreform** is a category (e.g. Awodey (2010, Section 1.3)). □

Proof B.6 (for Theorem 2)**.** Throughout this proof, assume that $\alpha = [\Pi, \Pi', \tau, \delta]$ is a morphism, where $\Pi = (T, C, \otimes)$ and where $\Pi' = (T', C', \otimes')$.

In this paragraph, suppose that $\alpha = [\Pi, \Pi', \tau, \delta]$ is an isomorphism, and let $\alpha^{-1} = [\Pi^*, \Pi^{**}, \tau^*, \delta^*]$ be its inverse. Then

$$[\Pi^*, \Pi^{**}, \tau^*, \delta^*]\circ[\Pi, \Pi', \tau, \delta] = \mathsf{id}_\Pi = [\Pi, \Pi, \mathsf{id}^{\mathbf{Set}}_T, \mathsf{id}^{\mathbf{Set}}_C] \text{ and} \tag{21a}$$

$$[\Pi, \Pi', \tau, \delta]\circ[\Pi^*, \Pi^{**}, \tau^*, \delta^*] = \mathsf{id}_{\Pi'} = [\Pi', \Pi', \mathsf{id}^{\mathbf{Set}}_{T'}, \mathsf{id}^{\mathbf{Set}}_{C'}] \ , \tag{21b}$$

where the first equality in both lines follows from the definition of α^{-1}, and the second equality in both lines follows from the definition of id. The well definition of $\circ$ in (21a) implies

$$\Pi^* = \Pi' \ . \tag{22}$$

The well definition of $\circ$ in (21b) implies

$$\Pi^{**} = \Pi \ . \tag{23}$$

The third component of (21a) implies that $\tau^*\circ\tau = \mathsf{id}^{\mathbf{Set}}_T$. The third component of (21b) implies that $\tau\circ\tau^* = \mathsf{id}^{\mathbf{Set}}_{T'}$. The last two sentences imply that τ is a bijection from T onto T' and that

$$\tau^* = \tau^{-1} \ . \tag{24}$$

Similarly, the fourth components of (21a) and (21b) imply that δ is a bijection from C onto C' and that

$$\delta^* = \delta^{-1} \ . \tag{25}$$

The previous two sentences have shown that τ and δ are bijections. Further,

$$\alpha^{-1} = [\Pi^*, \Pi^{**}, \tau^*, \delta^*] = [\Pi', \Pi, \tau^{-1}, \delta^{-1}] \ ,$$

where the first equality follows from the definition of α^{-1} in the first sentence of this paragraph, and where the second equality follows from (22)–(25).

It remains to prove the reverse direction of the theorem's second sentence. Accordingly, suppose that τ and δ are bijections. Define $\alpha^* = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]$. Then

$$\alpha^*\circ\alpha = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]\circ[\Pi, \Pi', \tau, \delta] = [\Pi, \Pi, \mathsf{id}^{\mathbf{Set}}_T, \mathsf{id}^{\mathbf{Set}}_C] = \mathsf{id}_\Pi \text{ and}$$

$$\alpha\circ\alpha^* = [\Pi, \Pi', \tau, \delta]\circ[\Pi', \Pi, \tau^{-1}, \delta^{-1}] = [\Pi', \Pi', \mathsf{id}^{\mathbf{Set}}_{T'}, \mathsf{id}^{\mathbf{Set}}_{C'}] = \mathsf{id}_{\Pi'} \ .$$

Thus α is an isomorphism. □

Lemma B.7. *Suppose $[\Pi, \Pi', \tau, \delta]$ is an isomorphism, where $\Pi = (T, C, \otimes)$ determines F, p, q, $\prec$, and $\preccurlyeq$, and where $\Pi' = (T', C', \otimes')$ determines F', p', q', $\prec'$, and $\preccurlyeq'$. Then the following hold.*

(a) $(\tau, \delta, \tau)|_{\otimes}$ *is a bijection from* $\otimes$ *onto* $\otimes'$.
(b) $(\tau, \delta)|_F$ *is a bijection from* F *onto* F'.
(c) $\tau|_{F^{-1}(C)}$ *is a bijection from* $F^{-1}(C)$ *onto* $(F')^{-1}(C)$.
(d) $(\tau, \tau)|_p$ *is a bijection from* p *onto* p'.
(e) $(\tau, \delta)|_q$ *is a bijection from* q *onto* q'.
(f) $(\tau, \tau)|_{\prec}$ *is a bijection from* $\prec$ *onto* $\prec'$.
(g) $(\tau, \tau)|_{\preccurlyeq}$ *is a bijection from* $\preccurlyeq$ *onto* $\preccurlyeq'$.

Proof. Theorem 2 implies

$$\tau \text{ is a bijection from } T \text{ onto } T' \ , \tag{26a}$$

$$\delta \text{ is a bijection from } C \text{ onto } C' \ , \text{ and} \tag{26b}$$

$$\alpha^{-1} = [\Pi', \Pi, \tau^{-1}, \delta^{-1}] \ . \tag{26c}$$

(a). By (10c) for α, $(\tau, \delta, \tau)|_{\otimes}$ is a well-defined function from $\otimes$ into $\otimes'$. It is injective by (26a)–(26b). To show it is surjective, take any $(t', c', t'^{\sharp}) \in \otimes'$. By (26c), and by (10c) for α^{-1},

$$(\tau^{-1}(t'), \delta^{-1}(c'), \tau^{-1}(t'^{\sharp})) \in \otimes \ .$$

Thus $(\tau, \delta, \tau)(\tau^{-1}(t'), \delta^{-1}(c'), \tau^{-1}(t'^{\sharp})) = (t', c', t'^{\sharp})$ is in the range of $(\tau, \delta, \tau)|_{\otimes}$.

(b). By (11a) of Proposition 3.1(a) for α, $(\tau, \delta)|_F$ is a well-defined function from F into F'. It is injective by (26a)–(26b). To show it is surjective, take any $(t', c') \in F'$. By (26c), and by (11a) of Proposition 3.1(a) for α^{-1},

$$(\tau^{-1}(t'), \delta^{-1}(c')) \in F \ .$$

Thus $(\tau, \delta)(\tau^{-1}(t'), \delta^{-1}(c')) = (t', c')$ is in the range of $(\tau, \delta)|_F$.

(c). Proposition 3.2(b) for α implies $\tau|_{F^{-1}(C)}$ is a well-defined function from $F^{-1}(C)$ into $(F')^{-1}(C')$. It is injective by (26a). To show it is surjective, take any $t' \in (F')^{-1}(C')$. Proposition 3.2(b) for α^{-1} implies $\tau^{-1}(t') \in F^{-1}(C)$. Thus $\tau(\tau^{-1}(t')) = t'$ is in the range of $\tau|_{F^{-1}(C)}$.

(d). By Proposition 3.1(b) for α, (12a) holds. Thus by Lemma B.3(a) for α, $(\tau, \tau)|_p$ is a well-defined function from p into p'. It is injective by (26a). To show it is surjective, take any $(t'^{\sharp}, t') \in p'$. By (26c), and by Proposition 3.1(b) for α^{-1}, I have (12a) for α^{-1}. Thus by Lemma B.3(a) for α^{-1}, $(\tau^{-1}, \tau^{-1})|_{p'}$ is a well-defined function from p'

into p.[9] Applying this at the $(t'^\sharp, t')$ defined three sentences ago yields

$$(\tau^{-1}(t'^\sharp), \tau^{-1}(t')) \in p\ .$$

Thus $(\tau,\tau)(\tau^{-1}(t'^\sharp), \tau^{-1}(t')) = (t'^\sharp, t')$ is in the range of $(\tau,\tau)|_p$.

(e). By Proposition 3.1(b) for α, (12b) holds. Thus by Lemma B.3(b) for α, $(\tau,\delta)|_q$ is a well-defined function from q into q'. It is injective by (26a)–(26b). To show it is surjective, take any $(t'^\sharp, c') \in q'$. By (26c), and by Proposition 3.1(b) for α^{-1}, I have (12b) for α^{-1}. Thus by Lemma B.3(b) for α^{-1}, $(\tau^{-1},\delta^{-1})|_{q'}$ is a well-defined function from q' into q.[10] Applying this at the $(t'^\sharp, c')$ defined three sentences ago yields

$$(\tau^{-1}(t'^\sharp), \delta^{-1}(c')) \in q\ .$$

Thus $(\tau,\delta)(\tau^{-1}(t'^\sharp), \delta^{-1}(c')) = (t'^\sharp, c')$ is in the range of $(\tau,\delta)|_q$.

(f). Proposition 3.2(e) implies that $(\tau,\tau)|_\prec$ is a well-defined function from $\prec$ into $\prec'$. It is injective by (26a). To show it is surjective, take any $(t'^1, t'^2) \in {\prec'}$. By (26c), and by Proposition 3.2(e) for α^{-1},

$$(\tau^{-1}(t'^1), \tau^{-1}(t'^2)) \in {\prec}\ .$$

Thus $(\tau,\tau)(\tau^{-1}(t'^1), \tau^{-1}(t'^2)) = (t'^1, t'^2)$ is in the range of $(\tau,\tau)|_\prec$.

(g). This proof is similar to that of the previous part. Merely replace $\prec$ with $\preccurlyeq$, and replace Proposition 3.2(e) with Proposition 3.2(f). □

Lemma B.8. *Suppose $[\mathit{\Pi}, \mathit{\Pi}', \tau, \delta]$ is an isomorphism, where $\mathit{\Pi} = (T, C, \otimes)$ determines $\mathcal{H}$ and $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}} \cup \mathcal{Z}_{\mathsf{inft}}$, and where $\mathit{\Pi}' = (T, C, \otimes)$ determines $\mathcal{H}'$ and $\mathcal{Z}' = \mathcal{Z}'_{\mathsf{ft}} \cup \mathcal{Z}'_{\mathsf{inft}}$. Then the following hold.*
(a) $(\forall H \in \mathcal{H})\ \tau(H) \in \mathcal{H}'$.
(b) $(\forall Z \in \mathcal{Z})\ \tau(Z) \in \mathcal{Z}'$.

Proof. By Theorem 2, τ is a bijection from T onto T', δ is a bijection from C onto C', and $\alpha^{-1} = [\mathit{\Pi}', \mathit{\Pi}, \tau^{-1}, \delta^{-1}]$. These facts will sometimes be used implicitly. Also, let $\mathit{\Pi}$ determine F, and $\mathit{\Pi}'$ determine F'.

[9] The previous two sentences resemble the paragraph's first two sentences. To be explicit, (12a) for α^{-1} is $(\forall t'^\sharp_* {\neq} t'^o)\ \tau^{-1}(p'(t'^\sharp_*)) = p(\tau^{-1}(t'^\sharp_*))$. Lemma B.3(a) for α^{-1} shows this is equivalent to $\{(\tau^{-1}(t'^\sharp_*), \tau^{-1}(t'_*)) | (t'^\sharp_*, t'_*) {\in} p'\} \subseteq p$.

[10] The previous two sentences resemble the paragraph's first two sentences. To be explicit, (12b) for α^{-1} is $(\forall t'^\sharp_* {\neq} t'^o)\ \delta^{-1}(q'(t'^\sharp_*)) = q(\tau^{-1}(t'^\sharp_*))$. Lemma B.3(b) for α^{-1} shows this is equivalent to $\{(\tau^{-1}(t'^\sharp_*), \delta^{-1}(c'_*)) | (t'^\sharp_*, c'_*) {\in} q'\} \subseteq q$.

(a). Take any H. By the definition of $\mathcal{H}$, there exists c such that $H = F^{-1}(c)$. Note that

$$
\begin{aligned}
(27) \qquad H &= F^{-1}(c) \\
&= \{\ t \mid (t,c){\in}F\ \} \\
&= \{\ t \mid (\exists(t',c'){\in}F')\ t{=}\tau^{-1}(t'), c{=}\delta^{-1}(c')\ \} \\
&= \{\ t \mid (\exists t')\ (t',\delta(c)){\in}F', t{=}\tau^{-1}(t')\ \} \\
&= \{\ \tau^{-1}(t') \mid (t',\delta(c)){\in}F'\ \} \\
&= \{\ \tau^{-1}(t') \mid t' \in (F')^{-1}(\delta(c))\ \} \\
&= \tau^{-1}(\ (F')^{-1}(\delta(c))\ )\ ,
\end{aligned}
$$

where the first equation holds by the definition of c, the third equation holds by Lemma B.7(b), and the remaining equations are rearrangements. Because τ is a bijection, (27) implies $\tau(H) = (F')^{-1}(\delta(c))$. Thus $\tau(H) \in \mathcal{H}'$ by the definition of $\mathcal{H}'$.

(b). Take any Z. Then by Proposition 3.2(g) applied to α at $S = Z$, $\tau(Z)$ is a chain. Hence it remains to be shown that $\tau(Z)$ is maximal. Suppose not. Then there is $t' \notin \tau(Z)$ such that $\tau(Z){\cup}\{t'\}$ is a chain. By Proposition 3.2(g) applied to α^{-1} at $S' = \tau(Z){\cup}\{t'\}$, $\tau^{-1}(\tau(Z){\cup}\{t'\}) = Z{\cup}\{\tau^{-1}(t')\}$ is a chain. Note $\tau^{-1}(t') \notin Z$ because τ is a bijection and because $t' \notin \tau(Z)$. This contradicts the maximality of Z. □

Proof B.9 (for Proposition 3.3)**.** Parts (a) and (g) follow from Lemma B.7. Before proving the remainder, note Theorem 2 implies that τ is a bijection from T onto T', that δ is a bijection from C onto C', and that $\alpha^{-1} = [\mathit{\Pi}', \mathit{\Pi}, \tau^{-1}, \delta^{-1}]$. These facts will sometimes be used implicitly.

(h). Lemma B.8(a) implies that $\tau|_{\mathcal{H}}$ is a well-defined function from $\mathcal{H}$ into $\mathcal{H}'$. It is injective because τ is injective. To show that it is surjective, take any $H' \in \mathcal{H}'$. By Lemma B.8(a) applied to α^{-1}, $\tau^{-1}(H') \in \mathcal{H}$. Thus $\tau(\tau^{-1}(H')) = H'$ is in the range of $\tau|_{\mathcal{H}}$.

(i)–(j). Let $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}}{\cup}\mathcal{Z}_{\mathsf{inft}}$ and $\mathcal{Z}' = \mathcal{Z}'_{\mathsf{ft}}{\cup}\mathcal{Z}'_{\mathsf{inft}}$. Since τ is a bijection, the cardinality of S equals the cardinality of $\tau(S)$ for any set $S \subseteq T$. Thus it suffices for both parts (i) and (j) to show that $\tau|_{\mathcal{Z}}$ is a bijection from $\mathcal{Z}$ onto $\mathcal{Z}'$.

Lemma B.8(b) implies that $\tau|_{\mathcal{Z}}$ is a well-defined function from $\mathcal{Z}$ into $\mathcal{Z}'$. It is injective because τ is injective. To show that it is surjective, take any $Z' \in \mathcal{Z}'$. By Lemma B.8(b) applied to α^{-1}, $\tau^{-1}(Z') \in \mathcal{Z}$. Thus $\tau(\tau^{-1}(Z')) = Z'$ is in the range of $\tau|_{\mathcal{Z}}$.

(k). Since t^o weakly precedes all nodes in T, $t^o \preccurlyeq \tau^{-1}(t'^o)$. Thus by part (g), $\tau(t^o) \preccurlyeq' t'^o$. Meanwhile, since t'^o weakly precedes all nodes in T', $t'^o \preccurlyeq' \tau(t^o)$. The last two sentences imply $\tau(t^o) = t'^o$ because $\preccurlyeq'$ is antisymmetric (Lemma A.2(d) for (T', p')).

(l). By part (k) and by the definition of $k'(t'^o)$, $k'(\tau(t^o)) = k'(t'^o) = 0$. Thus by Proposition 3.2(d), $k'(\tau(t)) = k(t) + k'(\tau(t^o)) = k(t)$.

(m). Take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. The expression $E'[\tau(Z)]$ is well-defined because $\tau(Z) \in \mathcal{Z}'_{\mathsf{inft}}$ by part (j). Now take any $v{\geq}1$. By the definition of E, $E[Z]^v$ is a stage-v member of Z. Thus by part (l), $\tau(E[Z]^v)$ is a stage-v member of $\tau(Z)$. Thus by the definition of E', $\tau(E[Z]^v)$ equals $E'[\tau(Z)]^v$. □

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