Addendum to "The Category of Node-and-Choice Preforms for Extensive-Form Games"

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The last paragraph of Streufert (2016, Proof B.6) neglected to prove that $[\mathit{\Pi}',\mathit{\Pi},\tau^{-1},\delta^{-1}]$ is a morphism. The following lemma fills that gap.

Lemma. *Suppose $[\mathit{\Pi},\mathit{\Pi}',\tau,\delta]$ is a morphism, where $\mathit{\Pi}=(T,C,\otimes)$ and $\mathit{\Pi}'=(T',C',\otimes')$. Further suppose τ and δ are bijections. Then $[\mathit{\Pi}',\mathit{\Pi},\tau^{-1},\delta^{-1}]$ is a morphism.*

Proof. It must be shown that

$$\tau^{-1}{:}T'{\to}T\ ,$$

$$\delta^{-1}{:}C'{\to}C\ ,\ \text{and}$$

$$\{\ (\tau^{-1}(t'),\delta^{-1}(c'),\tau^{-1}(t'^{\sharp}))\mid(t',c',t'^{\sharp}){\in}\otimes'\ \}\subseteq\otimes\ .$$

The first two are immediate, and the remainder of this proof shows the third. Accordingly, assume $(t',c',t'^{\sharp})\in\otimes'$. For notational ease, define $t=\tau^{-1}(t')$ and $t^{\sharp}=\tau^{-1}(t'^{\sharp})$.

This paragraph shows that $t^{\sharp}\neq t^{o}$, where t^{o} is the root node of $\mathit{\Pi}$. Suppose $t^{\sharp}$ were t^{o}. Then $t^{\sharp}\preccurlyeq t$ because t^{o} weakly precedes every node. Thus by Streufert (2016, Proposition 3.2(f)), $\tau(t^{\sharp})\preccurlyeq'\tau(t)$. Thus by the definitions of t and $t^{\sharp}$, $t'^{\sharp}\preccurlyeq' t'$. This contradicts $t'\prec' t'^{\sharp}$, which follows from the assumption that $t'{\otimes}'c'=t'^{\sharp}$.

Since the range of $\otimes$ is $T{\setminus}\{t^{o}\}$, and since $t^{\sharp}\neq t^{o}$ by the previous paragraph, Streufert (2016, Lemma A.5(b)) implies that

$$p(t^{\sharp})\otimes q(t^{\sharp})=t^{\sharp}\ . \tag{1}$$

Note that

$$\tau{\circ}p(t^{\sharp})\otimes'\delta{\circ}q(t^{\sharp})=\tau(t^{\sharp})=t'^{\sharp}\ ,$$

where the first equality holds by (1) and Streufert (2016, (10c)), and the second equality holds by the definition of $t^{\sharp}$. Because of the previous equality, because $t'{\otimes}'c'=t'^{\sharp}$ by assumption, and because $\otimes'$ is a bijection by Streufert (2016, (1a)),

$$\tau{\circ}p(t^{\sharp})=t'\ \text{ and }\ \delta{\circ}q(t^{\sharp})=c'\ .$$

Hence

$$p(t^\sharp) = \tau^{-1}(t') \text{ and } q(t^\sharp) = \delta^{-1}(c') \ .$$

Now take (1) and replace its three terms by means of [a] the two equalities in the last sentence and [b] the definition of $t^\sharp$. The result is

$$\tau^{-1}(t') \otimes \delta^{-1}(c') = \tau^{-1}(t'^\sharp) \ .$$

□

References

Streufert, P. A. (2016): "The Category of Node-and-Choice Preforms for Extensive-Form Games," Western University, Department of Economics Research Report Series 2016-2, August, 28 pages.