Minting and Imitating Commodity Money

Vadim Iaralov

Introduction

This paper was inspired by stories of "bad" continental imitations of English sterlings in 13-14th centuries – crockards and pollards. The counterfeiters (often foreign princes) clipped current issues or imitated them, so that the coins were worse but not far worse than the current issue.[1] Inherent variation of coin quality, since minting was a manual process, allowed these coins to circulate and be accepted. The proportion of identified tampered coins to the total may have actually been small at times – 3% in 1290.[2] Nevertheless, the rulers were very much concerned about the counterfeiters and consequently passed grave laws, such as death penalties to deter them. But perhaps it was not so much an issue of honour and morality but rather that the rulers were concerned about their mints' profits. This paper will develop a model to study how the existence of counterfeiters may be harmful to the rulers' coffers and how various macroeconomic variables may be affected. These crockards and pollards were effectively all removed from circulation by arbitrage (esp. by Italian bankers) when their exchange rate was set at ½, below their intrinsic value in recoinage of 1300.[3] This is disappointing because few of these imitations survived to the present day, so it is harder to determine the finess and weight accurately and precisely than otherwise would have been possible.

Literature survey

Commodity money models are usually specified by a search (random-matching) model or general equilibrium framework. Sargent and Velde (1999) use the general equilibrium approach. Their dynamic model has a representative household, the firm, the government and international trade. The main feature of their model is to have two types of coins: pennies and dollars. These coins have different varying intrinsic values (finess) set by the government, different fixed face values and different given production costs. Each coin has a different minting and melting point. A lot of complexity is generated by preventing arbitrage and different equilibria are possible. The functional forms are kept general and a specific functional form ln(.) for utility appears only briefly in an example. The model is set-up based on stylized historical facts of the bimetallic system and analyzes how growth of endowment may lead to shortages of small coins. It suggests that the shortage may be alleviated through reinforcing larger coins or debasing smaller coins to "realign" the intervals for minting-melting. It is then compared to some historical events, notably the Affair of the Quattrini when the small coins were debased to deflect the flow of Pisan small coins. In this model, fluctuating national income often forces shortages of small coins, which are relatively expensive to produce. These shortages were typically eased by debasements until the minting stops as the exchange rate of petty coins declines to the commodity value and the shortages come back, so debasements have to continue. Counterfeiting is mentioned only briefly (not modeled) when it restrains what gross seignorage governments may charge - a

[1] However, some petty money was counterfeited with no silver content during 1460's. Spufford (1988), p.361
[2] Mayhew (1979), p.130
[3] Ibid, p.137

positive seignorage only if it has lower costs than its competitors. Also, it is said that law enforcement (death penalty) increases expected costs for counterfeiting coins, allowing for higher seignorage. While there is no explicit calibration, the way the unknowns (such as mint equivalents) were defined based on real historical facts does allow room for some empirical monetary policy.

Nosal and Wallace (2004) highlight that is very important to model counterfeiting as a negative activity with distortions and costs. They criticize some previous models, where counterfeiting may be even welfare-improving if it reduced money shortages. Their agents cannot commit and information on their past trades is unavailable. They use modular arithmetic to prevent double coincidence of wants, so money is used. A certain fraction of agents has some kind of fiat money (genuine or counterfeit) and they buy goods from sellers who do not have money. They do not provide functional forms at all, so calibrating their model to the real world is impossible. In particular, the detection signal is exogenously given and it is unspecified what functional form it could have. Their conclusion is very surprising – counterfeiting does not exist in equilibrium. Since the reality is that counterfeiting existed for centuries, this model cannot be satisfactory.

A deterministic dynamic matching model with fiat money is used by Monnet (2005). This model is more complicated than Nosal and Wallace (2004) and counterfeiting does occur if it is not prohibitively expensive. Impact of a bank providing standing facility to prevent counterfeiting when shortages of money arise is considered. Also, enforcement (detection and confiscation) is performed by banks with exogenous unspecified probability. This makes owning and accepting counterfeit bills less pleasant. If the counterfeiting activity is cheap and unenforced, then it tends to create inflation that would offset additional counterfeiting in equilibrium out by decreasing marginal benefit of imitating real bank notes with the marginal cost being fixed. Finally, private histories are taken into account. This model is more reasonable than Nosal and Wallace (2004) because counterfeiting may persist in equilibrium even if there is some law enforcement – it matches the real world better.

There doesn't seem to be much on models of commodity money with counterfeiting. There are commodity-money models without counterfeiting and there are fiat money models with counterfeiting. One important difference is that commodity money can store value while hoarded (not modeled in this paper, either) and another is that counterfeits can be made of different quality at a trade-off of how much precious metal (silver) is to include. Counterfeiting commodity money is certainly costly if there is any kind of reasonable detection mechanism because some real precious metal needs to be used.

Description of the model

This paper studies presence of counterfeiting in a model economy with metal coins that have the same face values. The model develops a partial equilibrium of a closed economy with only the money market clearing because consumers are unspecified, their real demand for money is treated exogenously and the goods market (e.g. trading silver) is unspecified. Specifying consumer's problem and market for silver would be the natural extension. The money market (Appendix VII) is simpler than in Sussman and Zeira (2003)[4] because reminting/reinforcement is not included. These coins are accepted by-tale if certified (possibly, incorrectly) that their intrinsic value equals the standard finess. Only the silver bullion is used to mint money at the royal mint. Hence, supply is infinitely elastic

[4] p.21, Fig. 6

at P, the mint price of silver[5]. Money demand is taken from Williamson (2002)[6]: $M^d = Pl$, here l is exogenous real demand for money. $Pl = M^d = M^s = q_m + q_c$ or $q_m = Pl - q_c$. It is assumed that the counterfeiter releases his counterfeited money into the economy first and the mint satisfies the excess money demand afterwards.

Representative counterfeiter buys silver at market price P, mints coins that have less silver than royal coins and goes to the royal mint to certify them as legitimate, perhaps worn coins[7]. If the coins are certified to be similar to the mint's coins (deemed plausibly authentic), they may now be exchanged 1:1 for the current royal coins. If the coins are not certified (deemed forged), then they are discarded.[8] The verification process is costly: fees and possible criminal penalties. If the counterfeiter comes to the mint rarely (q_c is small relative to Pl), then the costs are small because he is assumed to be honest. But if the mint suspect the counterfeiter is responsible for forging coins, the cost is infinite – he is hanged ($q_c = Pl$, $q_m = 0$, so all coins in the economy are forged).

In a modification of the cournot duopoly model, mint takes q_c to be exogenous.[9] The mint is willing to make as many coins as needed to satisfy the excess money demand and takes a seignorage charge, s. The silver content (finess) in royal mint's coins, f_m, is publicly known. There is a bulk reputation adjustment (Appendix III) on coin's official finess that enters the mint's profit. This adjustment is maximized at some $0<f_m\leq 1$, is negative infinity at $f_m=0$ (expectations of currency substitution or reinforcement making l=0 should prevent that), zero at $f_m=1$ and negative for $f_m>1$ (ad-hoc feasibility penalty to prevent mints profiting from impossible finess).

Hypothetical scenario (not equilibrium outcomes but a simple example of how the model works): King's mint takes 1000 grams of pure silver and gives back 1800 1g coins with 0.50g of silver each and 0.50g of base metal (copper).The king keeps 200 coins at 10% seignorage rate, assuming brassage and cost of copper is negligible. If each 1g coin has \$1 face value, then 1kg of silver has a mint price of \$1800. 1g of silver has mint price (\$) of $1.8 = \frac{1-0.1}{0.5}$ or $P = \frac{1-s}{f_m}$

A counterfeiter wants to fake 10 royal 1g coins of $f_m = 0.5$ by making 10 1g coins of $f_c = 0.4$. He needs 0.4(10) g = 4g of silver, which costs 4(\$1.8)=\$7.2. He certifies 10 imitated coins at the mint to trade for royal coins 1:1, who confiscates them with a certain probability. The probability θ_Δ depends on the difference between f_c and f_m.$\Delta=f_m-f_c$. If $f_c=0$ ($\Delta=f_m$), then the imitated coin is pure copper, has no similarities with the royal currency, and is always confiscated ($\theta=1$). It is assumed that a coin that is half as good as the royal coin is confiscated half the time[10] ($\Delta=0.5f_m$ implies $\theta=0.5$). Finess testing technology is proxied by parameter k, so that for k=3 is the best technology[11] whenever $\Delta<0.5f_m$, with technology getting worse for these small deltas as k increases - this will turn out to be

[5] Q is used in Sussman (1993)

[6] p.307, Williamson (2002)

[7] The weight is constant but finess decreased, which should be equivalent to weight decreasing at constant finess in the real world. For simplicity, weight of coins is constant. Also, N*V=1 for simplicity as defined in Sussman (1993)

[8] An extension of this model could have the confiscated coins being added to the mint's profit.

[9] The results would be different if the mint assumed qc to be a varying function of s and fm in a Stackelberg-like equilibrium that is more difficult to solve.

[10] This assumption is, unfortunately, not based on microeconomic principles of technology or expected distributions of depreciated coins. It affects the shape of the detection function.

[11] The functional form of θ based on k^{th} discrete roots does not allow for further improvements of a non-perfect technology at k=3. By modeling the sigmoid detection curve on the logistic function, it is possible to make equivalent of k's continuous. See Appendix IV.

the only subset of Δ in equilibrium, so high k can be interpreted to mean poor technology (k has the opposite interpretation for $\Delta>0.5f_m$). Appendix I justifies that θ_Δ as defined below exhibits these relationships. The counterfeiter estimates $\theta = 0.0783$ and relative counterfeiting penalty C=20, so his expected profit is (1-0.0783)($10)-$7.2+$20ln(170/180)= $0.87, if the Mint makes 170 more coins. Because he expects positive profits, the counterfeiter will make these 10 coins. Clearly, counterfeiting is a costly activity since in this case only $0.87 is made on imitating $10.

Defining and solving for the equilibrium

Static duopoly equilibrium given parameters {l, R, C, k, ψ}[12]: sets of "prices" $\{f_c^*, f_m^*, s^*, P^*\}$ and allocations $\{q_c^*, q_m^*\}$ such that

1. Given $\{q_c^*\}$, Mint chooses $\{s^*, f_m^*\}$ to maximize Π_m.
$$\Pi_m = s(Pl - q_c) + R\left(\psi - (\psi-1)f_m - \frac{1}{f_m^{\ 2}}\right) \quad \text{, s.t.} \quad P = \frac{1-s}{f_m}$$
2. Given $\{s^*, f_m^*\}$, Counterfeiter chooses $\{q_c^*, f_c^*\}$ to maximize Π_c.
$$\Pi_c = (1-\theta_\Delta)q_c - Pq_c f_c + C\ln\left(\frac{q_m}{Pl}\right),$$
where: $\theta_\Delta = \frac{1}{2} + \frac{1}{2}\left(\frac{2\Delta}{f_m} - 1\right)^{\frac{1}{k}}$, $\Delta = f_m - f_c$ and $q_m = Pl - q_c$
Or equivalently[13],
$$\max\ \Pi_c = \left(\frac{1}{2} + \frac{1}{2}\left(2F-1\right)^{\frac{1}{k}}\right)q_c - (1-s)q_c F + C\ln\left(\frac{Pl - q_c}{Pl}\right)$$
by choosing $\{q_c^*, F^*\}$, where $F = \frac{f_c}{f_m}$, so $f_c^* = F^* \cdot f_m^*$
3. Money market clears:
   (a) $M^d = Pl = q_m + q_c = M^s$ (already substituted above to eliminate q_m).
   (b) $P = \frac{(1-s)}{f_m}$

First order conditions[14] form a non-linear system of equations that implicitly defines endogenous equilibrium "prices" and quantities as functions of the exogenous parameters.

$$\frac{\partial \Pi_m}{\partial f_m} = 0; \frac{\partial}{\partial f_m}\left(s(Pl - q_c) + R\left(\psi - (\psi-1)f_m - \frac{1}{f_m^{\ 2}}\right)\right) = 0$$

[12] Appendices I, II, III, V analyze implications of exogenous parameters.
[13] See Lemma 2 of Proposition 1 below.
[14] First derivatives of the payoff functions w.r.t. the relevant choice varibles are zero

$$-\frac{s(1-s)l}{f_m^{\ 2}}+R\left(-(\psi-1)-\frac{2}{f_m^{\ 3}}\right)=0 \tag{1}$$

$f_m(R=20, l=100, s=0.3749, \psi=3)=0.8076$

$$\frac{\partial \Pi_{\mathrm{m}}}{\partial s}=0;\frac{\partial}{\partial s}\left(s(Pl-q_c)+R\left(\psi-(\psi-1)f_m-\frac{1}{f_m^{\ 2}}\right)\right)=0$$

$$\frac{(1-s)l}{f_m}-q_c\frac{sl}{f_m}=0 \tag{2}$$

$$\frac{\partial \Pi_{\mathrm{c}}}{\partial q_c}=0;\frac{\partial}{\partial q_c}\left(\left(\frac{1}{2}+\frac{1}{2}\left(2F-1\right)^{\frac{1}{k}}\right)q_c-(1-s)q_cF+C\ln\left(\frac{Pl-q_c}{Pl}\right)\right)=0$$

$$\frac{1}{2}+\frac{1}{2}\left(2F-1\right)^{\frac{1}{k}}-(1-s)F-\frac{C}{\left(\frac{(1-s)l}{f_m}-q_c\right)}=0 \tag{3}$$

$$\frac{\partial \Pi_{\mathrm{c}}}{\partial F}=0;\frac{\partial}{\partial F}\left(\left(\frac{1}{2}+\frac{1}{2}\left(2F-1\right)^{\frac{1}{k}}\right)q_c-(1-s)q_cF+C\ln\left(\frac{Pl-q_c}{Pl}\right)\right)=0$$

$$-(1-s)q_c+\frac{\left(2F-1\right)^{\frac{1}{k}}q_c}{k\left(2F-1\right)}=0 \tag{4}$$

$F=\frac{1}{2}e^{\left(\frac{-k\ln(k-ks)}{k-1}\right)}+\frac{1}{2}$ $\qquad F(k=3, s=0.3749)=0.6947$

This only gave the root above 0.5, but there are two roots:

$|2F-1|=(k-ks)^{\left(\frac{k}{1-k}\right)}$ $\quad F_1=\frac{1}{2}+\frac{1}{2}(k-ks)^{\left(\frac{k}{1-k}\right)}$ $\quad F_2=\frac{1}{2}-\frac{1}{2}(k-ks)^{\left(\frac{k}{1-k}\right)}$

$F_1(k=3, s=0.3749)=0.6947$

$F_2(k=3, s=0.3749)=0.3053$

In Appendix IV, it is shown that second order conditions imply that F>0.5 for a maximum profit, so $F=F_1$ is used below throughout.

Solution further can proceed in two different ways:

Method (I): $s=\frac{l-q_cf_m}{2l}$ and equation (1) can be solved for $f_m'(q_c, l, R, \psi)$ by cubic formula and substituted back to get $s'(q_c, l, R, \psi)$.

$q_c = Pl - \frac{2C}{s-(1-s)(k-ks)^{\left(\frac{k}{1-k}\right)} + (k-ks)^{\left(\frac{k}{1-k}\right)}}$ gives $q_c'(s, f_m, C, k)$ from equation (3) and $F=F_1$. Then fixed-point convergence algorithm to steady-state will give equilibrium values (not functions themselves):

$f_m^*(l=l_0, R=R_0, k=k_0, \psi=\psi_0)$, $s^*(l=l_0, R=R_0, k=k_0, \psi=\psi_0)$ and

$q_c^*(l=l_0, R=R_0, k=k_0, \psi=\psi_0)$:

Initial condition, for t = 0: $q_c'(t)=0$,

Laws of motion, for t ≥1:

$s(t) = s'(q_c(t-1), l=l_0, R=R_0, k=k_0, \psi=\psi_0)$,

$f_m(t) = f_m'(q_c(t-1), l=l_0, R=R_0, k=k_0, \psi=\psi_0)$,

$q_c(t) = q_c'(s(t), f_m(t), C=C_0, k=k_0)$.

This approach has the following economic intuition: the mint still takes q_c exogenously but because of asymmetric information in period t, it does not know number of newly counterfeited coins and it treats q_c as a martingale, $E(q_c(t)\,|\,Information_t) = q_c(t-1)$. The best estimate of today's quantity of counterfeit coins is yesterday's quantity. This is better known as "adaptive expectations." It is not a novel result that adaptive expectations converge to Nash equilibrium for a cournout oligopoly.

Presumably, the survey of money flows can be done only at the end of the period as counterfeiters would act in secrecy. On the other hand, the mint's seignorage and finess at time t are public, so the counterfeiter knows them exactly.

$\lim_{t\to\infty} q_c(t) = q_c(l,R,C,k,\psi)$, $\lim_{t\to\infty} f_m(t) = f_m(l,R,C,k,\psi)$, $\lim_{t\to\infty} s(t) = s(l,R,C,k,\psi)$

This can be solved in Excel because t=10 is typically sufficient for 5 decimals.

Method (II): The same three equations as in Method (I) can be solved simultaneously in Maple 9.5 for a given $k=k_0$ to yield steady-state functions of (l, R, C, $k=k_0$, ψ) expressed implicitly as roots to unwieldy 15^{th}+ (depending on k_0) order polynomials. This only makes sense for interior solutions because there is no check to prevent negative q_c or $f_m>1$ that occur in certain equilibria. The rectangular domains of 3-D graphs in the Appendices were chosen to be mostly interior by checking boundary corners via solution method (I).

Both (I) and (II) use floating-point calculations to give numbers (not functions) as final answers. Individual experiments verify that (I) converges very rapidly to (II), indicating that the equilibrium is very stable.

Proposition 1:

$$\lim_{q_c\to 0}\left(\frac{\Pi_c^{\max}+\Pi_c^{\min}}{2q_c}\right) = \frac{1}{2}s - \frac{C}{Pl},$$

where $\Pi_c^{\max}$ is the indirect objective function to maximizing counterfeiter's profit, given any $q_c > 0$, by choosing $F=F_1$ (>0.5) and $\Pi_c^{\min}$ is the indirect objective function to minimizing counterfeiter's profit by choosing $F=F_2$ (<0.5) – see Appendix IV for second order conditions.

Proposition 1 relates per unit "average" (non-optimal) per unit profit of counterfeiter to seignorage and relative counterfeiting penalty at entry into the business.

It could be shown that $\Pi_c^{\min}(k=3) < \frac{q_c}{2}\left(s - \frac{2}{\sqrt{27s}}\right)$ since $C\ln\left(\frac{Pl - q_c}{Pl}\right) < 0$. Since s*≤0.5 from

$s = \frac{l - q_c f_m}{2l}$ and $\Pi_c^{\min}(k=3, 0 \le s \le 0.5) \le \Pi_c^{\min}(k=3, s=0.5) = q_c\left(0.5 - \sqrt{\frac{64}{27}}\right) < 0$

It should also be the case that $\Pi_c^{\min}(k \ge 3) \le \Pi_c^{\min}(k=3)$, so $\Pi_c^{\min} < 0$ for any equilibrium, implying

Proposition 2:[15]

$$s - \frac{2C}{Pl} \le \lim_{q_c \to 0}\left(\frac{\Pi_c^{\max}}{q_c}\right)$$

This bound (if it is positive) is a sufficient condition for positive production of counterfeited coins. Unfortunately, this is a very weak bound and not very useful practically. Four examples where it is verified:

a) {l=65, C=23, R=29, k=5, ψ=4} implies{q_c= 0.01794,Π_c^{max}/q_c=0.00013,s–2C/(Pl)= -0.63568}

b) {l=60,C=16,R=13,k=3,ψ=2} implies{q_c= 0.62979, Π_c^{max}/q_c=0.00522,s–2C/(Pl)= -0.52088}

c) {l=72.75,C=20,R=60,k=3,ψ=3} implies{q_c=0.00762,Π_c^{max}/q_c=0.00005,s–2C/(Pl)= -0.54410}

d) {l=172,C=21,R=30,k=3,ψ=3} implies{q_c=95.80425, Π_c^{max}/q_c=0.14865,s–2C/(Pl)= 0.00041}

In (a)-(c), the bound is negative, so it is not very useful. But whenever the bound is positive, like in (d), it forces significant entry in equilibrium because there is profit to be made, so it is not certain how to measure the limit at entry in this case.

[15] Steps supposedly leading to Proposition 2 should be proven, ideally.

<u>Further areas of research; fitting data</u>

1. *Current model*

(i) Interactions between counterfeiter's entry and macroeconomic variables: $\{P^*, s^*, \Pi_m + \Pi_c\}$.

<u>Hypothesis</u>: these never increase after period t=1 (inductive evidence from dynamic convergence), while f_m never decreases, e.g. it increases if $\{l=270, C=15, R=37, k=3, \psi=2\}$ from 0.80759 to 0.98074.

(ii) Analyze comparative statics (3-D graphs in Appendix V).

(iii) Tighten bound in Proposition 2 by knowing certain restrictions on given parameters.

(iv) Make dynamics more useful by changing parameters midway.

(v) Specify the boundary solutions (at least one of $f_m=1$ or $q_c=0$ is true) more explicitly as functions of parameters then adjust Maple's direct solutions (method 2) to become piecewise at the boundaries to make comparative-statics plots more reliable.

2. *Model extensions*

(i) Introduce consumers, silver goods market, endogenize l (real money demand).

(ii) Introduce petty money with a smaller face value, so the problem of small change/counterfeiting small change can be dealt with.

(iii) Use other detection functions instead of $\theta(\Delta)$ – see Appendix VI.

3. *Empirical work*

(i) Estimate plausible parameters to roughly match mint data from debasement and non-debasement periods from Grenoble archive data in Sussman (1993). Data on s^*, P^*, f_m^* is likely to be available but the most coveted data is q_c^* and f_c that may be harder to get.

(ii) Look at different episodes of counterfeited commodity money in history and test the model.

(iii) If there is a lot of data and it is somewhat reliable then sophisticated econometrics tools can be used to estimate parameters via mirror image of equilibrium solving above: treat endogenous variables as exogenous (provided by data) and solve for exogenous parameters, especially estimate confidence intervals.

Mayhew (1977) writes:

> "Firstly, the majority (71 out of 90) of continental sterlings were of sterling finess – i.e. about 92.5% silver. Secondly, some sterlings were struck at a lower finess – sometimes about ¾, or 2/3 or 1/2. Certain princes, particularly later in the 14th century struck only debased sterlings. However, a number of princes struck sterlings of varying finess. Consequently, my third conclusion is that we do not have anything like enough analyses to provide a full picture."[16]

He also notes, "…crockards and pollards were struck about 10% lighter than English sterlings and this is confirmed by an exchange rate of 22s pollards for 20s sterlings."[17] However, inherent variation in English coins made continental imitations "fit well" among English coins. Some model's predictions: $\{l=100, C=27, R=19, k=3, \psi=2\}$ implies $\{s=0.48674, f_m=0.92301, f_c=0.70305, q_c=2.87219, q_m=52.73434\}$. Here the finess and small amount of counterfeit coins is reasonable but seignorage seems too high. Whereas, $\{l=150, C=10, R=500, k=3, \psi=3\}$ implies $\{s=0.20692, f_m=0.99180, f_c=0.63102, q_c=88.65369, q_m=31.29303\}$. Here seignorage and finess are somewhat reasonable; while it seems unlikely that ¾ of all coins were counterfeited.

[16] Mayhew (1977), p.128

[17] Ibid, p.129

Conclusion

The focus was on the counterfeiter-mint duopoly. The model of partial equilibrium was developed to clear the money market. The mint chooses seignorage and finess of royal coins, taking quantity of counterfeits exogenously. The counterfeiter chooses finess of counterfeited coins and their quantity, taking seignorage and finess of royal coins exogenously. This model has five parameters $\{l, C, R, k, \psi\}$ that have different comparative statics on the optimal choices of macroeconomic variables and should allow sufficient flexibility to calibrate the model to get meaningful results. Specific functional forms were assumed to achieve numerical solutions and parameters were used to create some flexibility in the functions. The subjective costs/adjustments are interesting because they capture relatively how much non-monetary benefits or costs are created in the relevant payoff function. These non-monetary components were necessary and sufficient (most of the time) for second order conditions to hold. The model was solved in two ways, numerically and by computer. The steady-state convergence approach may yield some dynamic extensions with economic meaning.

Clear results of the model are $s \leq 0.5$ and $F>0.5$ in equilibrium. This means that superficially silver-plated copper coins would not be optimal but rather coins that are not altogether different from the mint (partially depending on the technology available). A strange result was proven that related per unit "average" (non-optimal) per unit profit of counterfeiter to seignorage and relative counterfeiting cost at entry into the counterfeiting business. This implied a weak sufficient condition for entry. The model should be extended to include welfare analysis by introducing consumers and the goods market. It may also be helpful to clarify requirements for an interior solution (sometimes only complex solutions exist). There is a hypothesis that $\{P^*, s^*, \Pi_m + \Pi_c\}$ cannot increase and f_m cannot increase from a counterfeiter's entry. These results should ideally be proved or disproved formally. This model in applied economics should be fairly suited to calibration from the real-world data, at least compared to other commodity-money models that are too general in their functional forms.

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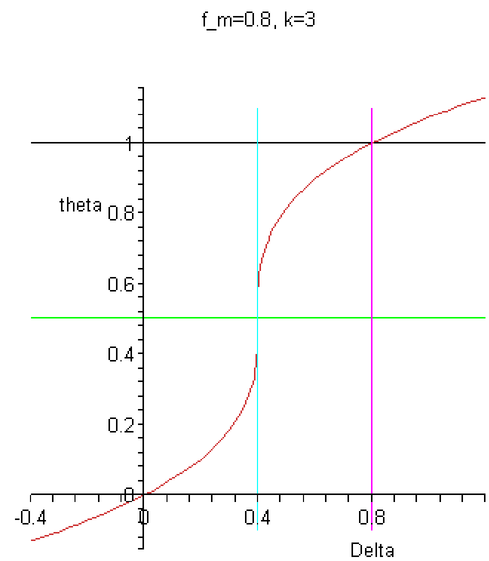
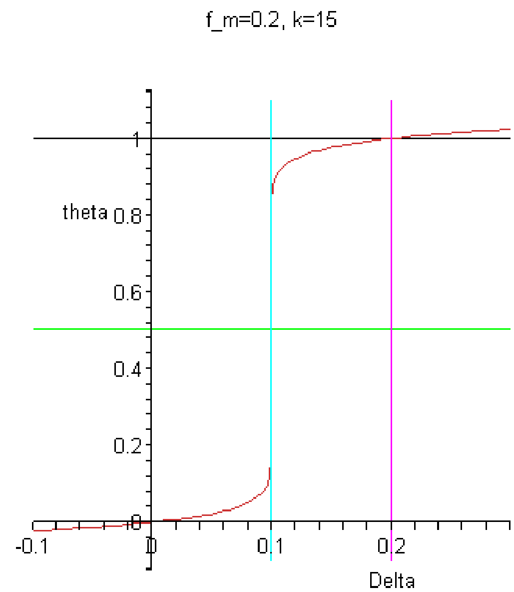
Appendix I: How k and f_m affect $\theta(\Delta)$

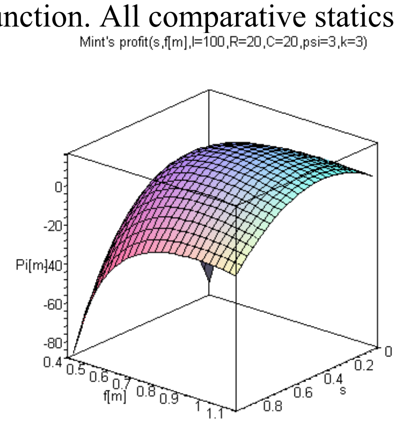




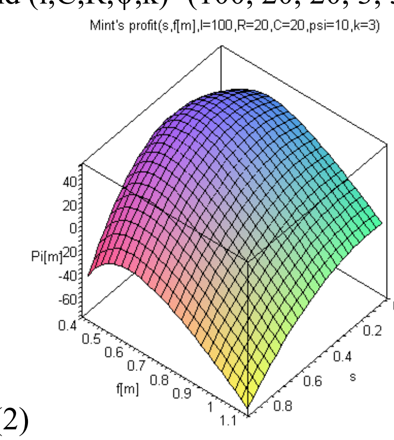
Detection technology is worse at equilibrium when k=15 because when $\Delta\approx 0$, θ is smaller that is there is less chance of detection. Because k^{th} root functions are used to model technology, k=3 is the best possible – smallest odd k that forms a sigmoid.

Appendix II: How k and ψ affect equilibrium Π_c and Π_m

1) Mint's profit function at equilibrium counterfeiter's best response q_c*=30.98, explicit parameters {l, R, ψ}={100,20,3} and implicit parameters affecting q_c*, {C, k}={20,3}. Mint's profit function is maximized uniquely at {s*, f_m*}={0.3749,0.8076} with maximum of {14.435}. The maximum {s*, f_m*} as a function of {l, C, R, k, ψ} forms Mint's equilibrium best response function. $\Pi_m\left(s^*(l,R,C,k,\psi), f_m^*(l,R,C,k,\psi), q_c^*(l,R,C,k,\psi)\right)=\Pi_m^*(l,R,C,k,\psi)$ is the indirect objective function. All comparative statics (3D plots) are around (l,C,R,ψ,k)=(100, 20, 20, 3, 3).



(1)



(2)

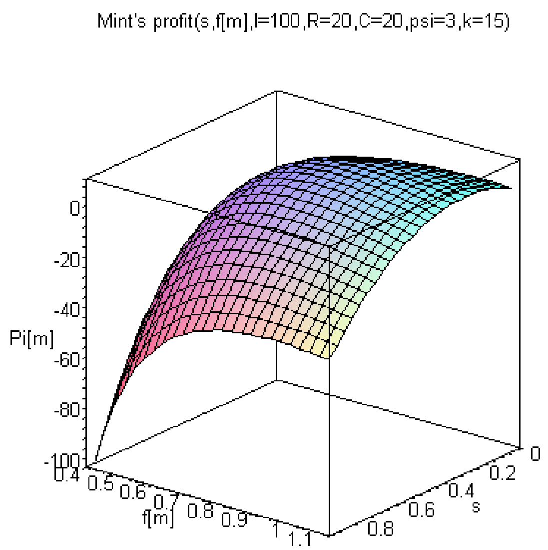
$$\Pi_m(q_c=30.979)=s\left(\frac{100(1-s)}{f_m}-30.979\right)+60-40f_m-\frac{20}{f_m^{\ 2}},$$ for graph(1)

2) Mint's profit function at equilibrium counterfeiter's best response q_c*=77.1596, explicit parameters {l, R, ψ}={100,20,10} and implicit parameters affecting q_c*, {C, k}={20,3}. Mint's

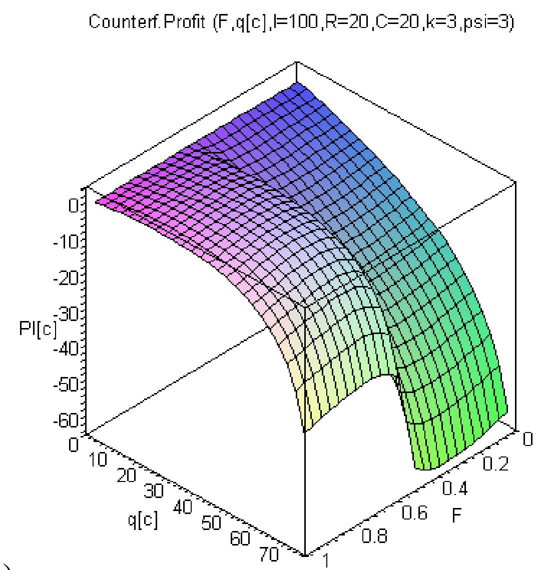
profit function is maximized uniquely at $\{s^*, f_m^*\}=\{0.3027, 0.8262\}$ with maximum of $\{49.971\}$. Another point on the Mint's equilibrium best response function (ψ increased), Π_m increased.

3) Mint's profit function at equilibrium counterfeiter's best response $q_c^*=47.7658$, explicit parameters $\{l, R, \psi\}=\{100,20,3\}$ and implicit parameters affecting q_c^*, $\{C, k\}=\{20,15\}$. Mint's profit function is maximized uniquely at $\{s^*, f_m^*\}=\{0.2905, 0.54293\}$ with maximum of $\{8.741\}$. Another point on the Mint's equilibrium best response function (k increased), Π_m decreased.

$\Pi_c(s=0.37489, f_m=0.80765) = -0.62510 q_c F + q_c\left(\frac{1}{2}+\frac{1}{2}signum\left(F-\frac{1}{2}\right)|2F-1|^{1/3}\right)$, for graph(4)



(3)

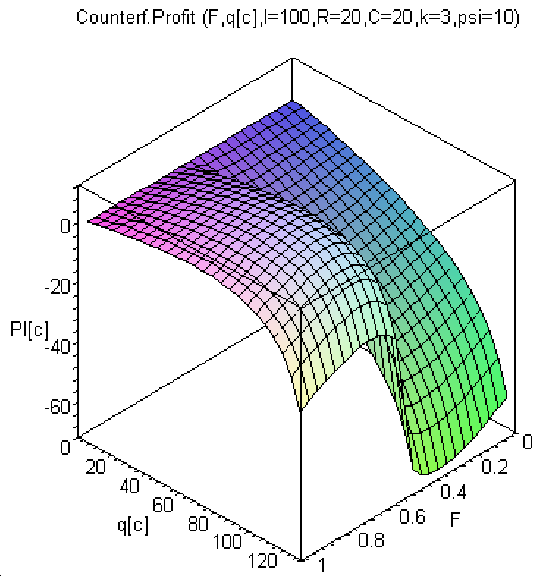


(4)

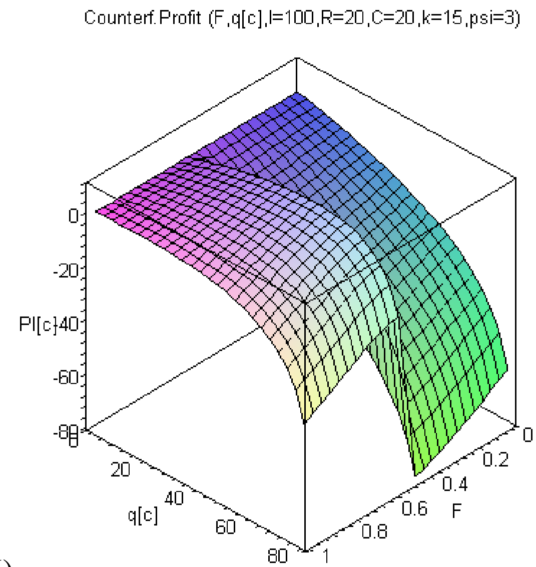
4) Counterfeiter's profit function at equilibrium mint's best response $\{s^*, f_m^*\}=\{0.3749, 0.8076\}$, explicit parameters $\{l,C,k\}=\{100,20,3\}$ and implicit parameters affecting $\{s^*, f_m^*\}$, $\{R, \psi\}=\{20, 3\}$. Counterfeiter's profit function is maximized uniquely at $\{q_c^*, F^*\}=\{30.979, 0.6947\}$ with maximum of $\{3.123\}$.A point on the Counterfeiter's equilibrium best response function.

$\Pi_c\left(s^*(l,R,C,k,\psi), f_m^*(l,R,C,k,\psi), F(l,R,C,k,\psi), q_c^*(l,R,C,k,\psi)\right) = \Pi_m^*(l,R,C,k,\psi)$

is the indirect objective function.



(5)



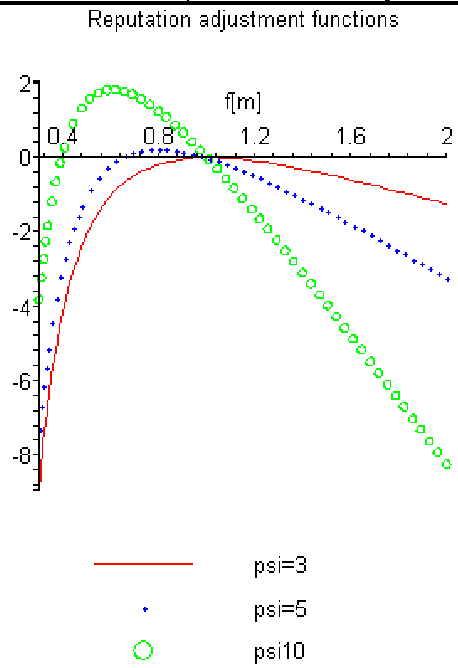
(6)

5) Counterfeiter's profit function at equilibrium mint's best response {s*, f_m*}={0.2905, 0.5429}, explicit parameters {l, C, k}={100,20,3} and implicit parameters affecting {s*, f_m*}, {R, ψ}={20, 10}. Counterfeiter's profit function is maximized uniquely at {q_c*, F*} ={77.1596,0.6610} with maximum of {10.983}. Another point on the Counterfeiter's equilibrium best response function. Increase in ψ stretched the surface along q_c axis.

6) Counterfeiter's profit function at equilibrium mint's best response {s*, f_m*}={0.3027, 0.8262}, explicit parameters {l, C, k}={100,20,15} and implicit parameters affecting {s*, f_m*}, {R, ψ}={20, 3}.Counterfeiter's profit function is maximized uniquely at {q_c*,F*}={47.766,0.5404} with maximum of {9.386}. Another point on the Counterfeiter's equilibrium best response function. Increase in k stretched the surface along $\square_c$ axis.

Graphs (4)-(6) show that the choice between F_1>0.5 and F_2<0.5 for counterfeiting is stark.
Second order conditions appear to be valid graphically for all graphs.

Appendix III: How ψ affects the reputation adjustment function



“Reputation adjustment” punishes the mint for choosing $f_m>1$ or $f_m=0$ to force second order conditions on profit maximization when choosing mint's finess. For all ψ, reputation adjustment is zero if $f_m=1$, so Mint's profits at $f_m = 1$ are exactly $s(Pl-q_c)$ If ψ >3, then at certain "natural" finess < 1 where “reputation adjustment” is maximum, the Mint also gains a small bonus of R*Reputation(f_m, ψ), for example, because coins' alloy is more durable. This "natural" finess weakly anchors Mint's optimal finess. Mints with low ψ favour higher finess, ceteris paribus.

Appendix IV: Second-order conditions

> Hm:=hessian(eval(profit_m,P=p),[s,f[m]]);

$$Hm := \begin{bmatrix} -\frac{2l}{f_m} & -\frac{(1-s)l}{f_m^2}+\frac{sl}{f_m^2} \\ -\frac{(1-s)l}{f_m^2}+\frac{sl}{f_m^2} & \frac{2s(1-s)l}{f_m^3}-\frac{6R}{f_m^4} \end{bmatrix}$$

> is(Hm[1,1]<0) assuming l>0, f[m]>0; *true*

is(Hm[2,2]<0) assuming additionally, s=0.374898215,l=100,R=20,f[m]=0.807646097; *true*

Hm[2,2]<0;

$$\frac{2s(1-s)l}{f_m^3}-\frac{6R}{f_m^4}<0, \qquad 2s(1-s)f_m-\frac{6R}{l}<0, \qquad s(1-s)f_m<\frac{3R}{l}$$

Since $s(1-s)f_m \le (0.5)(0.5)(1)=0.25$, therefore $\frac{\partial^2\Pi_m}{\partial f_m^{\ 2}} < 0$ whenever $0.25 < \frac{3R}{l}$ and the condition is weaker for general $s<0.5, f_m<1$.

E.g., here we have 0.374898215*(1-0.374898215)*(0.807646097)=0.1892714941 < 0.25.
> det(Hm)>0;

$$0 < \frac{l(12R - lf_m)}{f_m^5}, \qquad 0 < 12R - lf_m, \qquad f_m < \frac{12R}{l}$$

This reduces to $0.25 < \frac{3R}{l}$, once again.

> Hc:=hessian(profit_c,[q[c],F]);

$$Hc := \begin{bmatrix} -\dfrac{C}{\left(-q_c - \dfrac{(-1+s)\,l}{f_m}\right)^2} & -1+s+\dfrac{(2F-1)^{\left(\frac{1}{k}\right)}}{k\,(2F-1)} \\ -1+s+\dfrac{(2F-1)^{\left(\frac{1}{k}\right)}}{k\,(2F-1)} & \dfrac{2\,(2F-1)^{\left(\frac{1}{k}\right)}\,q_c}{k^2\,(2F-1)^2} - \dfrac{2\,(2F-1)^{\left(\frac{1}{k}\right)}\,q_c}{k\,(2F-1)^2} \end{bmatrix}$$

> diff(profit_c,q[c]$2)<0;

$$\frac{-C}{\left(\frac{(1-s)l}{f_m} - q_c\right)} < 0$$

is(%) assuming q[c]<(1-s)*l/f[m], C>0; *true*
> Diff(PI[c],F$2)=factor(Hc[2,2]);

$$\frac{\partial^2\Pi_c}{\partial F^2} = -\frac{2\left(2F-1\right)^{\frac{1}{k}} q_c(k-1)}{k^2\left(2F-1\right)^2}$$

For odd k: $k \ge 3$, $(2F-1)^{1/k} < 0$ if $F<0.5$, making numerator negative ($q_c>0$). Since denominator is positive, the whole fraction is positive for $F<0.5$. Hence, $F>0.5$ is necessary for

$$\frac{\partial^2\Pi_c}{\partial F^2} < 0$$

> is(Hc[1,1]<0) assuming q[c]<(1-s)*l/f[m],C>0; *true*
is(Hc[2,2]<0) assuming F=0.7,q[c]>0,k>=3; *true*

is(det(Hc)>0) assuming additionally, k>=3,F>0,F<1,f[m]=0.807646097,l=100,C=20,q[c]<77,q[c]>0,s>=0,s<=0.3781125; *true*

is(det(Hc)>0) assuming additionally, k>=3,F>0,F<1,f[m]=0.807646097,l=100,C=20,q[c]<77,q[c]>0,s>=0,s<=0.3781126; *FAIL*

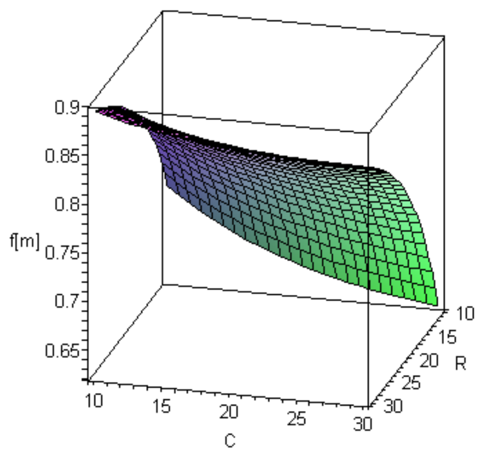
is(1=0); *false*

For some reason Maple 9.5 could not properly evaluate the fourth line (neither true nor false).

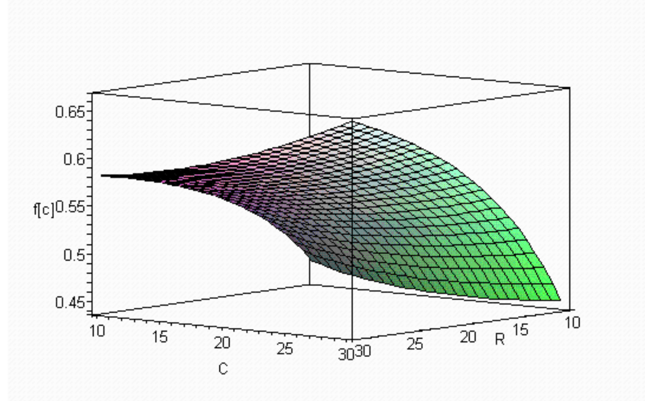
Second order condition for maximum, $\frac{\partial^2 \Pi_c}{\partial q_c^{\,2}} < 0$ always holds for the domain of interest. $\frac{\partial^2 \Pi_c}{\partial F^2} < 0$ as well . Whether the determinant of the Hessian matrix is positive could depend on the parameters but it is positive for this equilibrium s*=0.374898215<0.3781125 (third line shows this is true). Hence, there is maximum Π_c whenever first order conditions are satisfied.

Appendix V: 3-D comparative statics

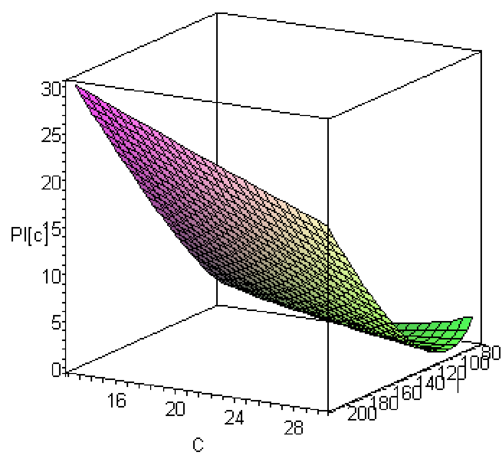
Mint's finess(I=100,C,R,psi=3,k=3)



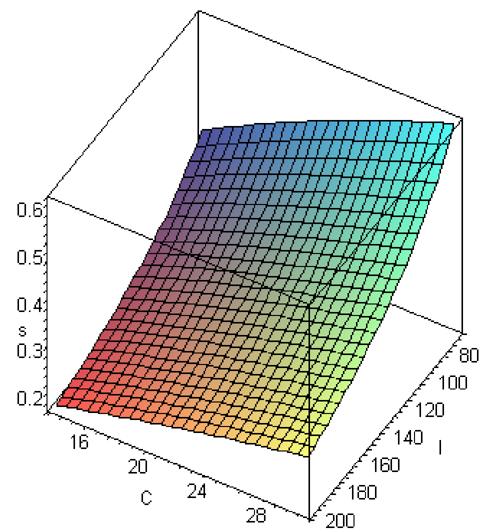
Counterfeit finesse(I=100,C,R,psi=3,k=3)



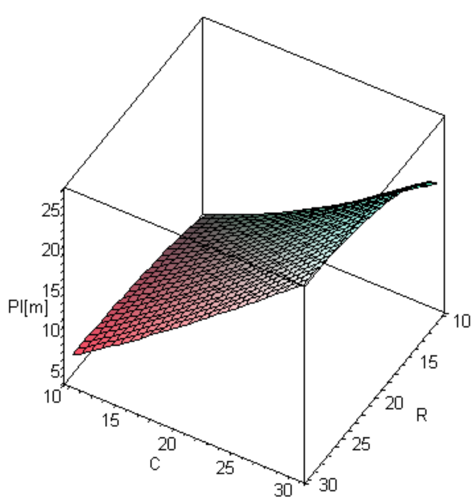
Counterfeit profit(I,C,R=20,psi=3,k=3)



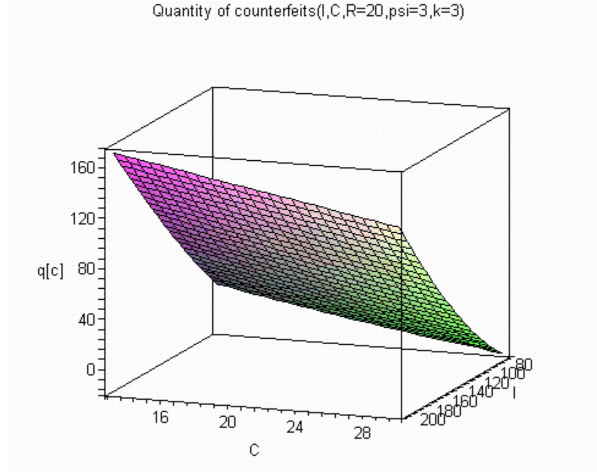
Seignorage(I,C,R=20,psi=3,k=3)



Mint's profit(I=100,C,R,psi=3,k=3)



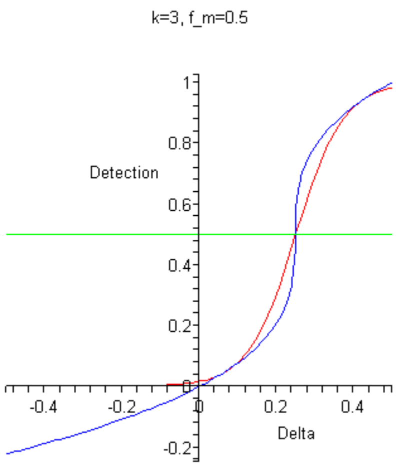
Quantity of counterfeits(I,C,R=20,psi=3,k=3)

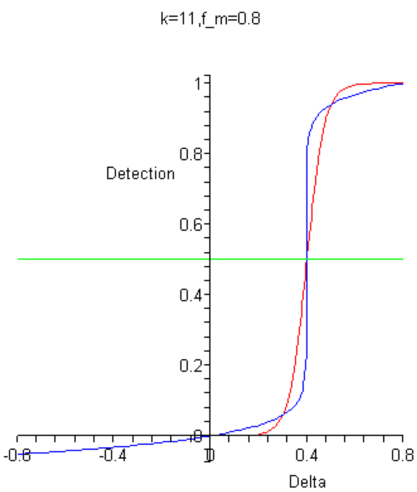


Appendix VI: Other sigmoid functions

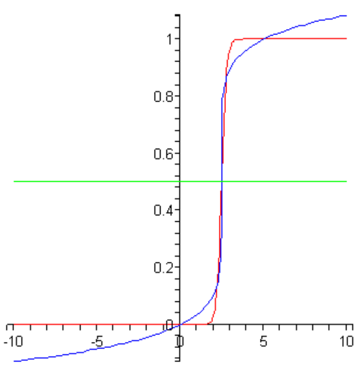
k:=3:f_m:=0.5:plot([1/(1+exp(-(2*k/f_m*(x-f_m/2)))),(1+(2*x/f_m-1)^(1/k))/2,1/2],x=-f_m..f_m,title="k=11,f_m=0.8", color = [red, blue, green], labels=[Delta,Detection]);

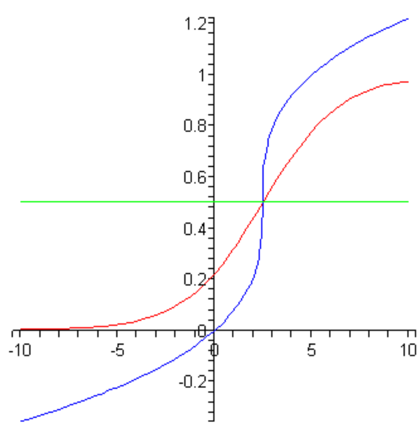
k:=11:f_m:=0.8:plot([1/(1+exp(-(2*k/f_m*(x-f_m/2)))),(1+(2*x/f_m-1)^(1/k))/2,1/2],x=-f_m..f_m,title="k=11,f_m=0.8", color = [red, blue, green], labels=[Delta,Detection]);



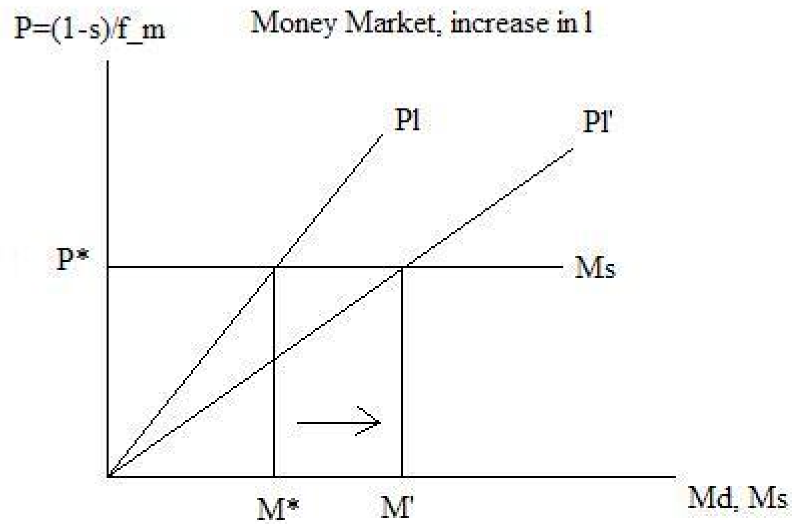


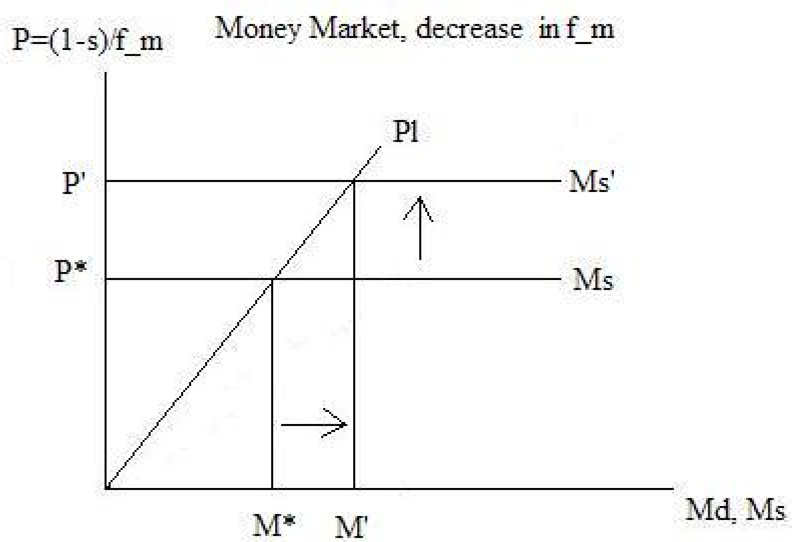
smartplot(1/(1+exp(-(7*(x-2.5)))),(1+(2*x/5-1)^(1/7))/2,1/2);
smartplot(1/(1+exp(-(0.5*(x-2.5)))),(1+(2*x/5-1)^(1/3))/2,1/2);





Appendix VII: Money market





Appendix VIII: Proof of Proposition 1

$$\lim_{q_c \to 0}\left(\frac{\Pi_c^{max}+\Pi_c^{min}}{2q_c}\right)=\frac{1}{2}s-\frac{C}{Pl}$$

Proof of Proposition 1:[18]

Lemma 1: $F_1 = 1 - F_2$.

Proof: from first order conditions,

$$\frac{\partial}{\partial F}\left(\left(\frac{1}{2}+\frac{1}{2}(2F-1)^{\frac{1}{k}}\right)q_c-(1-s)q_cF+C\ln\left(\frac{Pl-q_c}{Pl}\right)\right)=0$$

(assuming k is odd, so the k^th root of a negative number is defined)

$-(0.5)(1/k)(1-2F)^{(1-k)/k}(-2)q_c-(1-s)\,q_c = 0$

Since $q_c>0$, $(1/k)(1\text{-}2F)^{(1-k)/k} = (1-s)$

$(1-2F)^{(1-k)/k} = k(1-s)$

$|1-2F|=|2F-1|=(k(1-s))^{k/(1-k)}$ (because 1–k is even, there are two solutions)

If $2F_1-1>0$ (i.e. $F_1>0.5$), $2F_1-1 = (k(1-s))^{k/(1-k)}$

$F_1 = 0.5(1+(k(1-s))^{k/(1-k)})$

If $2F_2-1<0$ (i.e. $F_2<0.5$), $2F_1-1 = -(k(1-s))^{k/(1-k)}$

$F_2 = 0.5(1-(k(1-s))^{k/(1-k)})$

$LS = F_1 = 0.5(1+(k(1-s))^{k/(1-k)})$

$RS = 1 - F_2 = 1 - 0.5(1-(k(1-s))^{k/(1-k)}) = 0.5+0.5(k(1-s))^{k/(1-k)} = 0.5(1+(k(1-s))^{k/(1-k)})$

Therefore, LS=RS, and Lemma 1 is proven.

Lemma 2:[19] For all $0\leq F\leq 1$, $\theta(F)+\theta(1\text{-}F) = 1$, where

$$\theta_\Delta=\frac{1}{2}+\frac{1}{2}\left(\frac{2\Delta}{f_m}-1\right)^{\frac{1}{k}}.\ \Delta = f_m - f_c. \qquad F=\frac{f_c}{f_m}$$

Proof: $\theta_\Delta=\frac{1}{2}+\frac{1}{2}\left(\frac{2(f_m-f_c)}{f_m}-1\right)^{\frac{1}{k}}.$ $\qquad \theta_\Delta=\frac{1}{2}+\frac{1}{2}\left(1-\frac{2f_c}{f_m}\right)^{\frac{1}{k}}.$ $\theta_F=\frac{1}{2}+\frac{1}{2}(1-2F)^{1/k}$

Detection function (theta) becomes univariate in F, relative proportion of counterfeiter's finess to the mint's finess.

$\theta(F) + \theta(1-F) = 0.5+0.5(1-2F)^{1/k}+0.5+0.5(1-2(1-F))^{1/k} = 1+0.5(1-2F)^{1/k} + 0.5(-1+2F)^{1/k}$

$\theta(F) + \theta(1-F) = 1+0.5((1-2F)^{1/k} +(-1)(1-2F)^{1/k})$ for odd k

$\theta(F) + \theta(1-F)=1+0.5((1-2F)^{1/k} - (1-2F)^{1/k})=1+0.5(0)=1$. Lemma 2 is proven.

Lemma 3: For all $0\leq F\leq 1$,

$$\Pi_c(F)+\Pi_c(1-F)=sq_c+2C\ln\left(\frac{Pl-q_c}{Pl}\right)$$

Proof:

[18] Any self-respecting economics paper needs to prove an obscure Proposition 1 to get published.

[19] The underlying shape of the detection function is probably necessary where we assumed $\theta(\Delta=0.5f_m)=0.5$, a skewed detection function need not satisfy Lemma 2

$$\Pi_c(F) = \left(1-\theta(F)\right) q_c - Pq_c f_c + C\ln\left(\frac{Pl-q_c}{Pl}\right), \Pi_c(F) = \left(1-\theta(F)\right) q_c - \frac{(1-s)q_c f_c}{f_m} + C\ln\left(\frac{Pl-q_c}{Pl}\right)$$

$$\Pi_c(F) = \left(1-\theta(F)\right) q_c - (1-s)q_c F + C\ln\left(\frac{Pl-q_c}{Pl}\right)$$

$$\Pi_c(1-F) = \left(1-\theta(1-F)\right) q_c - (1-s)q_c(1-F) + C\ln\left(\frac{Pl-q_c}{Pl}\right)$$

$$\Pi_c(F) + \Pi_c(1-F) = q_c\left(2-\theta(1-F)-\theta(F)\right) - (1-s)q_c + 2C\ln\left(\frac{Pl-q_c}{Pl}\right)$$

By Lemma 2: $\Pi_c(F) + \Pi_c(1-F) = q_c - (1-s)q_c + 2C\ln\left(\frac{Pl-q_c}{Pl}\right)$

$\Pi_c(F) + \Pi_c(1-F) = q_c s + 2C\ln\left(\frac{Pl-q_c}{Pl}\right)$, Lemma 3 is proven.

Lemma 4:

$$\lim_{q_c\to 0}\left(\frac{\Pi_c(F)+\Pi_c(1-F)}{2q_c}\right) = \frac{1}{2}s - \frac{C}{Pl}$$

Proof:

From Lemma 3, $\dfrac{\Pi_c(F)+\Pi_c(1-F)}{2q_c} = \dfrac{1}{2}s + \dfrac{C\ln\left(\frac{Pl-q_c}{Pl}\right)}{q_c}$

$$\lim_{q_c\to 0}\left(\frac{\Pi_c(F)+\Pi_c(1-F)}{2q_c}\right) = \frac{1}{2}s + \lim_{q_c\to 0}\left(\frac{C\ln\left(\frac{Pl-q_c}{Pl}\right)}{q_c}\right)$$

RS = $\dfrac{1}{2}s + \dfrac{\lim_{q_c\to 0}\left(C\ln\left(\frac{Pl-q_c}{Pl}\right)\right)}{\lim_{q_c\to 0} q_c}$, 0/0 indeterminate form. Applying L'Hopital's Rule,

RS=$\dfrac{1}{2}s + \dfrac{\lim_{q_c\to 0}\left(\frac{\partial}{\partial q_c}C\ln\left(\frac{Pl-q_c}{Pl}\right)\right)}{\lim_{q_c\to 0}\frac{d}{dq_c}q_c} = \dfrac{1}{2}s + \dfrac{\lim_{q_c\to 0}\frac{-C}{Pl-q_c}}{\lim_{q_c\to 0}1} = \dfrac{1}{2}s - \dfrac{C}{Pl}$. Lemma 4 is proven.

Lemma 1, 4 and second-order conditions imply Proposition 1 is proven.