Strategic Partnership Formation

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Abstract

In this paper we define and analyze the partnership formation problem with equal sharing of output. This problem extends the partnership problem with heterogeneous ability by adding a stage in which agents strategically form partnerships. We investigate the problem within a matching framework, using stability as an equilibrium notion. Particular attention is paid to a specific type of matching referred to as consecutive-ability matching. A stable matching algorithm is defined for the partnership formation problem. Generalizing two results from Sherstyuk (1998), we give conditions for the efficiency and stability of the consecutive-ability matching when agents strategically choose effort. We prove a stronger theorem, which directly characterizes supermodular utility functions for which consecutive-ability matching is stable. Under incomplete information, we show how the unique stability of consecutive-ability matching is violated.

1. Introduction

In many economic settings, individuals cooperate to achieve some common goal. Two authors co-writing an academic paper or a team of attorneys working together on a lawsuit are two of the many real life examples of such settings. Agents in a team strategically choose which effort exertion level they wish to contribute to a joint production process in order to achieve some share of output according to some predetermined sharing rule. The decision making procedure and the strategic interaction of effort choices within a team setting are referred to as the partnership problem. In a typical partnership problem, agents simultaneously choose costly effort which maximizes their welfare given the effort exertion levels of their partners, and the underlying sharing rule.

Many interesting results in partnership problems stem from the existence of strategic complementarity or strategic substitutability in effort choice. The former notion captures situations in which higher effort exerted by one partner increases the marginal productivity of the other partners. The latter notion captures situations in which higher effort exerted by one partner decreases the marginal productivity of the other partners.[1] If there exists strategic interdependence (either

[1] See Bulow, Geanakoplos, and Klemperer (1985) for rigorous definitions of these notions and some economic applications.

substitutability or complementarity) in terms of effort, then the ability of each partner may also affect the productivity of the team – by contributing more effort, a more capable partner may enhance not only his/her own productivity, but also the rest of the team's productivity.

This paper will investigate the formation of partnerships which exhibit strategic complementarity or substitutability, building on the aforementioned partnership problem. We consider a pool of individuals who need to be assigned into partnerships or need to form them via some decentralized matching mechanism. Our problem, therefore, consists of two stages: a partnership formation stage followed by strategic choice of effort within the formed partnerships. What would be the matching outcome when the abilities of those individuals differ and they are free to choose their partners? Furthermore, is this decentralized outcome socially efficient as measured by various criteria (for example, aggregate output of all the partnerships or aggregate welfare)?

As a motivational example, consider an intercollegiate tennis competition. When tennis teams from two universities compete, each will send teams to play a certain number of doubles matches. When the players designated to play doubles matches freely form teams among themselves, it appears that players usually choose to play with other players of similar ability.[2] Our investigation might help explain why the composition of doubles teams has this feature and whether teams formed in this way maximize the success of the entire team as measured by total matches won.

2. Literature Review

Partnership problems have been widely studied in economics. It is well known that inefficiency exists in a partnership with strategic complementarity.[3] In such partnerships, individuals do not fully internalize the marginal benefit of the effort they contribute to the team and thus under-provide effort. Much of the literature on partnerships deals with various mechanisms or conditions that attempt to alleviate this inefficiency.

There exists research which focuses on inefficiencies due to the structure of the partnership. Farrell and Scotchmer (1988) were first to point out that with heterogeneous agents and an equal sharing rule (that is, each partner receives an equal share of output),[4] teams formed freely by the agents may be inefficiently small. The reason is that when economies of scale and strategic complementarity

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[2] This observation was made by a friend of the authors who had substantial experience in competitive tennis.
[3] This was first shown by Holmstrom (1982).
[4] Farrell and Scotchmer (1988) embed the equal output sharing rule in the definition of a partnership as they define a partnership to be a "coalition that divides its output equally". In contrast, we use the term 'partnership' regardless of the underlying output sharing rule.

are present, it is socially efficient to form large teams; however, due to the equal sharing rule, the more capable members do not want to subsidize less able members merely for their participation. Thus, the team size is too small in equilibrium.

Sherstyuk (1998) extends the research of Farrell and Scotchmer (1988) by focusing on the inefficiency due to the composition of the partnerships instead of on the inefficiency due to their size. In a framework where N individuals are ranked by their ability, and a K-member team output function is increasing in each member's ability, Sherstyuk shows that the stable matches – those arising in a decentralized manner – are consecutive. That is, not only are the ability ranks of the members in a given team consecutive, but also the teams are ranked consecutively according to their most and least capable members. Furthermore, the efficiency of this stable outcome depends on the features of the production technology. More specifically, if the production function exhibits strategic complementarity, then the stable outcome is efficient, but the same does not necessarily hold if the production function exhibits strategic substitutability.[5]

We would like to continue the research on partnership formation. First we generalize the aforementioned result from Sherstyuk (1998) by allowing agents to choose effort after partnerships are formed, whereas in Sherstyuk, output is predetermined by the ability level of each partner. In other words, we are considering a two staged game, where in the first stage, heterogeneous agents form partnerships and in the second stage the standard partnership game is played under complete information.

Following the analysis of the complete information case, we shall investigate this problem under an incomplete information setting. We would like to investigate the effects of incomplete information on efficiency and stability of the matching outcome. We feel that this is a relevant avenue for research because in many cases, ability of agents is not perfectly observed by others. Consider again the tennis team example. When choosing a doubles partner, the senior players may not know the abilities of the junior players with certainty; however, they may have some idea about junior players' ability by the very fact that they are selected for the team. As in Farrell and Scotchmer (1988), we will assume the equal sharing constraint throughout the paper. Frequently, partnerships adhere to equal sharing of profits regardless of the composition of partners' contribution. Co-authors of an economics paper usually share the credit equally and partners in a law firm often share profits equally. There are cases where equal sharing does not

[5] Strategic complementarity (substitutability) in a partnership game is usually represented by a production function which is supermodular (submodular) in effort levels. A twice differentiable production function is said to be supermodular (submodular) if it has positive (negative) cross partial derivatives in all of its arguments.

apply. For example, investment banks often distribute bonuses to the proprietary traders according to their contribution to the firm's profit.

3. The Model

Consider a group of $n \in \{4, 5, \ldots\}$ agents who need to form teams of size $k \in \{m \in \mathbb{N} : m \text{ divides } n\}$. Agents differ in their ability level, which can be thought of as their innate productivity, denoted by θ_i for every agent $i \in I = \{1, \ldots, n\}$. Assume that $\theta = (\theta_1, \ldots, \theta_n)$ is common knowledge where $\theta_1 > \theta_2 > \ldots > \theta_n$. In a given partnership, each agent i exerts effort x_i. Thus, in the case of $k = 2$, agent i's utility is given by $u_i(x_i, x_j, \theta_i, \theta_j)$ where $i \neq j \in I$. We define a matching to be an arbitrary assignment of the n agents into teams of k. We characterize matching outcomes by notions of stability and efficiency. First, we define stable matching outcomes, and a particular class of matching outcomes in which we are interested – the class of consecutive-ability matchings. Second, we define the matching algorithm which describes the dynamics to equilibrium. We then proceed to analyze the efficiency of consecutive-ability matchings. While some notions apply for the more general case, attention is mostly restricted to the case of $k = 2$ for concreteness. We leave it for future research to build on our paper in order to investigate the case of $k > 2$.

3.1 Stable decentralized matching outcomes

Definition 1. A matching outcome (or simply, a matching) is a function $\mu : I \to I$ such that $\forall i, j \in I$ the following hold:

1. $\mu(i) \neq i$
2. $\mu(i) = j$ if and only if $\mu(j) = i$.[6]

Definition 2. A stable matching outcome is one such that there exist no two agents who would rather be matched with each other but are matched differently under this outcome. Formally, there exists no $\{a, b, c, d\} \subseteq I$ such that $\mu(a) = c$ and $\mu(b) = d$ for which $u_a(x_a, x_b, \theta_a, \theta_b) \geq u_a(x_a, x_c, \theta_a, \theta_c)$ and $u_b(x_b, x_a, \theta_b, \theta_a) \geq u_b(x_b, x_d, \theta_b, \theta_d)$.

We are interested in the stability of matchings since it defines an equilibrium notion in the sense that it is a reasonable outcome when agents form teams according to a decentralized mechanism and act as utility maximizers.

Definition 3. A consecutive-ability matching is $\mu(i) = i + 1 \; \forall i \in \{1, 3, 5, \ldots, n - 1\}$.[7]

[6] Notice that together these two conditions imply that every agent finds a match.

As mentioned above, Sherstyuk gives the following result:[8]

Proposition 4. ***(Sherstyuk, 1998)*** *In a team formation setting where agents' input into production is predetermined by their ability (that is, agents do not choose effort level after partnerships are formed), the unique stable matching outcome is consecutive-ability matching, for both supermodular and submodular production technologies.*

We would like to show that this result holds also in a team formation setting where agents strategically choose effort. We show the result indeed holds in two steps: first, we give a condition under which all agents have an identical preference ranking, and second, we show that consecutive matching follows from the fact that agents have identical preference rankings.

Let us describe a particular decentralized algorithm for achieving a stable matching in the partnership formation problem. The algorithm we introduce is a variation of the deferred acceptance algorithm described in Gale and Shapley (1962). It is a variation since, first, in our algorithm acceptance is not deferred – agents cannot hold proposals; second, our market is slightly different than the one analyzed in Gale and Shapley (1962). Namely, in their model there are two distinct groups (for example, men versus women, or students versus colleges), while in our team formation problem, agents may team up with any member of the other $n - 1$ agents in the market, so the marriage analogy is not quite appropriate in our case.[9]

In the algorithm, at each stage agents propose partnership to their most preferred agent, and if it is the case that two partners have proposed to each other, then they are publicly declared as partners. Once a pair of agents has been declared, both agents are removed from consideration of the remaining agents in the next step, that is, they are removed from remaining agents' preference ranking. We argue that since every agent proposes to his most preferred choice given previous rejections, the outcome is stable. To see this point explicitly, consider the first round of the algorithm. Since agents have proposed to their most preferred agent, agents who form partnerships in this stage do not wish to deviate from this partnership. Now consider the second round. Agents who have teamed up in the

[7] Notice that the case where $|I| = n$ is odd is not allowed since we assume that $k = 2$ divides n. From here on, we do not consider situations in which with certainty some agents remain without partners since all of these results follow trivially from the present case and do not provide deeper insight. For example, if n was odd, consecutive-ability matching will be dictate the partnerships $(1, 2); (3, 4); \ldots ; (n - 2, n - 1); (n)$ or $(1); (2, 3); \ldots ; (n - 1, n)$.

[8] This proposition can be found in Sherstyuk, 198, 336.

[9] The notion of consecutive-ability matching in a team formation problem is analogous to perfect positive associative matching, as defined in Becker (1973) in markets like the marriage market and the student-college admission market. Therefore, while it resembles consecutive-matching, we avoid using the term assortative matching for the team formation problem.

first round are removed from consideration of the remaining agents. Therefore, agents propose to their most preferred agents excluding those who have teamed up in the first round, and thus would not like to deviate from this outcome. This line of reasoning can be inductively applied at every stage. Hence, the matching outcome resulting from the algorithm given above is stable.

We will consider a utility function which captures strategic complementarity or substitutability. Results regarding stability and efficiency will be shown for this utility function. Moreover, we provide a more general result with respect to the stability of consecutive ability-matching for a far broader class of utility functions. Let us start with a theorem which will be used in the stability analysis. Denoting agent i's equilibrium effort level when matched with j by $x_i^*\ (\theta_i,\ \theta_j)$, we have the following theorem:

Theorem 5. *If* $\dfrac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j} > 0\ \ \forall i, j \in I$ *or* $\dfrac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j} < 0\ \ \forall i, j \in I$ *then the unique stable matching outcome is consecutive.*

Proof. Proceed by executing the algorithm described above. First, suppose that

$$\frac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j} > 0\ \ \forall i \in I.$$

Since $\theta_1 > ... > \theta_n$) by assumption, $\dfrac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j} > 0\ \ \forall i \in I$ implies $(i, 1) \succ (i, 2) \succ . . . \succ (i, n)\ \forall i \in I$[10] where the notation $(\alpha, \beta) \succ (\alpha, \gamma)$ means that agent α strictly prefers matching with β to matching with γ. In the first step, all agents except agent 1 propose partnership to agent 1 as he is at the top of all agents' preference rankings, while agent 1 proposes to agent 2. Agent 1 rejects proposals from all agents $i \in \{3, . . . , n\}$ (because agent 2 is at the top of his ranking) and agents 1 and 2 are declared partners; thus, $\mu(1) = 2$. In the second stage, we have $(i, 3) \succ (i, 4) \succ . . . \succ (i, n)\ \forall i \in \{3, . . . , n\}$. All remaining agents except agent 3 propose to agent 3, while agent 3 proposes to agent 4. Agent 3 rejects proposals from all agents $i \in \{5, . . . , n\}$ since he strictly prefers agent 4; thus, $\mu(3) = 4$. Generally, in stage m (where $m \le \frac{n}{2}$ if m even, $m \le \frac{n}{2} - 1$ if m odd), all agents propose to agent $2m - 1$. Agent $2m - 1$ proposes only to agent $2m$. Agent $2m - 1$ rejects proposals from all agents $i \in \{2m + 1, . . . , n\}$, which gives

[10] Not including elements objects such as (i, i) for some $i \in I$ since they are meaningless by construction of μ – agents cannot match with themselves.

$\mu(2m-1)=2m$ for any such m. Therefore, the stable matching outcome resulting from applying the above algorithm is consecutive.

To show uniqueness, suppose that there exists another stable matching $\phi: I \mapsto I$. Since $(i, 1) \succ (i, 2) \succ \ldots \succ (i, n)\ \forall i \in I$ implies, in particular, that $(1, 2) \succ (1, i)$ and $(2, 1) \succ (2, i)\ \forall i = 3, \ldots, n$. Therefore, since ϕ is stable, $\phi(1) = 2$. Similarly, and since agents 3 and 4 know that $\phi(1) = 2$, we have $\phi(3) = 4$. Generally, letting $m \in I$ be odd, agents m and $m+1$ cannot match with any agents $1, \ldots, m-1$ since $\phi(i) = i + 1\ \forall i = 1, \ldots, m-1$. Therefore, since $(m, m+1) \succ (m, j)$ and $(m+1, m) \succ (m+1, j)\ \forall j = m+2, \ldots, n$ we have that $\phi(m) = m+1$. It follows that $\phi(i) = i+1\ \forall i = 1, 3, 5, \ldots, n-1$. Hence $\phi = \mu$; that is, ϕ is consecutive.

Now suppose $\dfrac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j} > 0\ \ \forall i \in I.$ Similarly to the former case, it follows that $(i, 1) \prec (i, 2) \prec (i, n)\ \forall i \in I$ and the algorithm proceeds in reverse analogy to the case of $\dfrac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j} < 0\ \ \forall i \in I.$ In the first step, all agents except agent n propose partnership to agent n, while agent n proposes to agent n - 1. Agent n rejects proposals from all agents $i \in \{1, 2, \ldots, n-2\}$ and we have $\mu(n-1) = n$. In step number m (where $m \le \frac{n}{2}$ if m even, $m \le \frac{n}{2} - 1$ if m odd), all remaining agents propose to agent $n - 2m + 2$ while he proposes only to $n - 2m + 1$. Therefore, $\mu(n - 2m + 1) = n - 2m + 2$. It follows that the stable matching achieved is consecutive-ability matching. The uniqueness argument is trivially similar to the previous one, so we omit it.

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Now let us consider a specific utility function which is typical in a partnership problem. We proceed to show that Theorem 5 indeed holds for this utility function. In particular, let

$$u_i(x_i, x_j, \theta_i, \theta_j) = \frac{1}{2}(\theta_i x_i + \theta_j x_j + c x_i x_j) - x_i^2$$

where x_i is effort exerted by player i, c is a constant of strategic complementarity or substitutability (assumed to be the same for any match). Notice that $\dfrac{\partial u_i(x_i, x_j, \theta_i, \theta_j)}{\partial x_i \partial x_j} = \dfrac{c}{2}$, which implies that c determines whether the utility function is supermodular or submodular in effort. The second order condition for maximization of this utility holds; namely, $\frac{\partial^2 u_i(x_i, x_j, \theta_i, \theta_j)}{\partial x_i^2} = -2 < 0,$ so we are justified in using first order condition to compute equilibrium. In order to guarantee first order solution we have to bound several variables, namely, let $c \in [-1, 1]$, θ_i, $\theta_j \in (0, 1]$. Denote player i's best response function by $B\,R_i$. Partners best respond according to their first order conditions:

$$B\,R_i = \frac{\theta_i}{4} + \frac{cx_j}{2}$$

$$B\,R_j = \frac{\theta_j}{4} + \frac{cx_i}{2}.$$

Solving this system simultaneously, equilibrium effort exertion levels are given by

$$x_i^*(\theta_i, \theta_j) = \frac{1}{4}\left(\frac{\theta_i + \frac{c}{2}\theta_j}{4 - c^2}\right)$$

$$x_j^*(\theta_i, \theta_j) = \frac{1}{4}\left(\frac{\theta_j + \frac{c}{2}\theta_i}{4 - c^2}\right).$$

Now, consider the utility for agent i from forming a pair with agent j given these equilibrium effort exertions. Substituting the equilibrium strategy of agents i and j into i's utility function gives

$$u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j) = \frac{1}{128}\left(\frac{14\theta_i^2 + 15\theta_i\theta_j c + 16\theta_j^2}{c^2 - 4}\right)$$

Differentiating with respect to the partner's ability gives

$$\frac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j} = \frac{1}{128}\left(\frac{15\theta_i c + 32\theta_j}{c^2 - 4}\right)$$

for all agents. Indeed, given that c is constant across individuals, $\frac{\partial u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j}$ will have the same sign for all individuals. That is, we have

$$\frac{\partial u_i(x_i^*(\cdot),x_j^*(\cdot),\theta_i,\theta_j)}{\partial\theta_j} > 0\ \forall i \quad \text{or} \quad \frac{\partial u_i(x_i^*(\cdot),x_j^*(\cdot),\theta_i,\theta_j)}{\partial\theta_j} < 0\ \forall i \tag{3.1}$$

We claim that it is because of this condition that consecutive matching will be achieved in equilibrium since under this situation all agents have the same preference ranking over the $n - 1$ possible partners.

While Theorem 5 helps characterize utility functions for which we should expect consecutive-ability matchings to be stable, it requires solving for equilibrium effort exertions. It will be more useful to have a condition on the original utility function, rather than on the indirect utility function which has equilibrium effort levels substituted into it. We will be using some results from Milgrom and Roberts (1990) to prove conditions on the utility function such that $\frac{\partial u_i(x_i^*,x_j^*,\theta_i,\theta_j)}{\partial\theta_j} > 0$ holds (which implies consecutive matching by Theorem 5). That is, we will give general results only for the case of supermodularity, while our condition for the case of submodularity (given in Theorem 5) remains slightly less useful since one needs to solve for equilibrium strategies.

Let us state the relevant results from Milgrom and Roberts (1990),[11] which will in turn be used in the proof of our next theorem. Our next theorem, Theorem 8, will specify general conditions for stability of the consecutive ability matching with supermodular utility functions.

Proposition 6. ***(Milgrom and Roberts, 1990)*** *Suppose that in a standard partnership problem, the utility function of each partner is supermodular in effort. Then for each player i, there exist maximum and minimum effort level (denote* $\underline{x}_i$ *and* $\overline{x}_i$ *respectively) that survive iterative elimination of strictly dominated strategies. Moreover,* $\underline{x}_i$ *and* $\overline{x}_i$ *are pure strategy Nash equilibria.*

Proposition 7. ***(Milgrom and Roberts, 1990)*** *Suppose that in addition to* $u_i(x_i, x_j, \theta_i, \theta_j)$ *being supermodular in effort, we also have* $\frac{\partial^2 u_i}{\partial x_i \partial\theta_i} \geq 0 \forall i \in I$. *Then* $\overline{x}_i(\theta_i,\theta_j)$ *and* $\underline{x}_i(\theta_i,\theta_j)$ *are increasing functions of* θ_i.

[11] The original theorems can be found in Milgrom and Roberts (1990: 1255, 1277). Propositions 6 and 7 in this paper correspond to Theorems 5 and 6 in Milgrom and Roberts (1990), respectively. We state here versions which are relevant to our problem.

Now we can state our generalized stability result for the case of supermodular utilities.

Theorem 8. Suppose that in a two-agent partnership problem, where the utility functions $u_i(x_i, x_j, \theta_i, \theta_j)$ and $u_i(x_i, x_j, \theta_i, \theta_j)$ are supermodular in effort, there exists an unique Nash equilibrium. Then,

$$\frac{\partial^2 u_i}{\partial x_i \partial \theta_i} \geq 0,\ \frac{\partial u_i}{\partial \theta_j} > 0,\ \frac{\partial u_i}{\partial x_j} \geq 0\ \forall i \in I,\ \forall \theta_i, \theta_j \Rightarrow \frac{\partial u_i(x_i^*, x_j^*, \theta_i, \theta_j)}{\partial \theta_j} > 0\ \forall i \in I.$$

Proof. By assumption, there exists a unique Nash equilibrium. Therefore $u_i(x_i^*, x_j^*, \theta_i, \theta_j)$ is well defined and, furthermore, by the chain rule, $\frac{\partial u_i(x_i^*, x_j^*, \theta_i, \theta_j)}{\partial \theta_j} = \frac{\partial u_i}{\partial \overline{x}_i} \cdot \frac{\partial \overline{x}_i}{\partial \theta_j} + \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial x_j}{\partial \theta_j} + \frac{\partial u_i}{\partial \theta_i} \cdot \frac{\partial \theta_i}{\partial \theta_j} + \frac{\partial u_i}{\partial \theta_j} \cdot \frac{\partial \theta_j}{\partial \theta_j}$. Now, by the envelope theorem, $\frac{\partial u_i}{\partial x_i^*} = 0$ because x^*, by definition, maximizes u_i. Further, $\frac{\partial \theta_i}{\partial \theta_j} = 0$ and $\frac{\partial \theta_j}{\partial \theta_j} = 1$ because θ_i and θ_j are exogenous. Therefore, the equation reduces to $\frac{\partial u_i(x_i^*, x_j^*, \theta_i, \theta_j)}{\partial \theta_j} = \frac{\partial u_i}{\partial x_j^*} \cdot \frac{\partial x_j^*}{\partial \theta_j} + \frac{\partial u_i}{\partial \theta_j}$. Since any Nash equilibrium survives iterated elimination of strictly dominated strategies, Proposition 6 (in particular, that $\underline{x}_j$ and $\overline{x}_j$ are pure strategy Nash equilibria) implies that $\underline{x}_j(\theta_i, \theta_j) = \overline{x}_j(\theta_i, \theta_j) = x_j^*(\theta_i, \theta_j)$. By Proposition 7, $\frac{\partial x_j^*}{\partial \theta_j} \geq 0$ and by assumption we have $\frac{\partial u_i}{\partial \theta_j} > 0$ and $\frac{\partial u_i}{\partial x_j} \geq 0$. Thus $\frac{\partial u_i}{\partial x_j^*} \cdot \frac{\partial x_j^*}{\partial \theta_j} + \frac{\partial u_i}{\partial \theta_j} = \frac{\partial u_i(x_i^*, x_j^*, \theta_i, \theta_j)}{\partial \theta_j} > 0 \forall i \in I.$

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3.2 Efficiency

Now we consider the efficiency of consecutive matching outcomes. While various notions of efficiency can be defined, leading to different results, here we regard matching outcomes which maximize the sum of agents' utilities as efficient. Analogous to results regarding efficiency in Sherstyuk (1998), we show that if every agent's utility in equilibrium is supermodular (submodular) in abilities (i.e., $\frac{\partial^2 u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial \theta_j \partial \theta_j} \geq (\leq)\ 0\ \forall i \neq j \in I$), then the consecutive-matching equilibrium is efficient (inefficient). Notice that with the specific utility function that we assumed previously, we have

$$\frac{\partial^2 u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial\theta_i \partial\theta_j} = \frac{15}{128}\left(\frac{c}{4-c^2}\right) \forall i \in I$$

In the case of strategic complementarity in effort, that is, $c > 0$, we indeed have supermodularity in ability, since $\frac{\partial^2 u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial\theta_i \partial\theta_j} = \frac{15}{128}\left(\frac{c}{4-c^2}\right) > 0 \forall i.$ Therefore, by Corollary 2 in Sherstyuk (1998, 339) we have that the consecutive matching outcome is efficient. That is, when $c > 0$ the social planner cannot improve the stable outcome if he/she seeks to maximize the sum of agents' utilities. In the case of strategic substitutability in effort, that is, $c < 0$, we have $\frac{\partial^2 u_i(x_i^*(\cdot), x_j^*(\cdot), \theta_i, \theta_j)}{\partial\theta_i \partial\theta_j} = \frac{15}{128}\left(\frac{c}{4-c^2}\right) < 0.$ Again, invoking the same corollary from Sherstyuk (1998), we have that the consecutive matching outcome is inefficient. Unlike our results regarding the stability of matching outcomes, we do not have a general characterization of utility functions that satisfy the efficiency condition given above.

4. Incomplete Information

Incomplete information with respect to agents' abilities can be introduced into the model in a variety of ways. It is possible to adopt many different approaches on the characterization of the information structure of the problem. The most general characterization involves defining what each agent knows about every other agent's ability level. Until now, we assumed that each agent knows every agent's ability (that is, complete information on abilities), and we want to alter this assumption. Deriving results under the most general characterization of the information structure, however, is a very difficult task. Therefore, we will restrict our attention to an information structure that is less general yet still reasonable. We will restrict our analysis to a partition of the set of agents I into two subsets: the first consists of agents whose ability is publicly known to all agents (public figures), and the second consisting of agents whose ability is not known to any other agent (private figures).[12]

[12] Note that an alternative characterization of incomplete information is that each agent knows only the type of agents who have the same type as him/her or within some interval centered around him/her. This characterization, however, is likely to result in consecutive matching as in the complete information case.

We will argue that under certain assumptions, incomplete information gives rise to stable outcomes that are not consecutive in nature. In other words, we claim that the consecutive matching outcome is no longer the unique stable outcome under incomplete information. As before, agents differ in ability, however, ability is now private information for some agents. That is, each agent in the group of n agents is characterized by two measures: his/her ability level, and whether this ability level is public information or not. Formally, each agent i can be characterized by a pair (θ_i, p) or (θ_i, s) where p stands for public and s stands for secret (private figure). We assume that $\theta_i \sim F\ \forall i \in I$ where F is some distribution on a two-element set $\{\underline{\theta}, \overline{\theta}\}$. We also assume that this distribution is common knowledge among all agents (common prior assumption), and we also assume that public figures can infer the distribution of abilities among private agents.

For the investigation of the effect of incomplete information on the matching outcome, we do not assume a specific functional form for agents' utility, but instead we make reasonable assumptions on agents' preference ranking of potential partners. Under these assumptions we consider the case of n = 4 (four agents) and we show that there exist non-consecutive stable matching outcomes.

Let us first state the assumptions we make on agent i's preferences assuming he/she is of the public type. We assume

$$\begin{aligned} &u_i(x_i^*(\theta_i,\overline{\theta}), x_j^*(\overline{\theta},\theta_i), \theta_i, \overline{\theta}) > E_{\theta_j}[u_i(x_i^*(\theta_i,\theta_j), x_j^*(\theta_j,\theta_i), \theta_i, \theta_j)] > \\ &\qquad u_i(x_i^*(\theta_i,\underline{\theta}), x_j^*(\underline{\theta},\theta_i), \theta_i, \underline{\theta}) \forall \theta_i \in \{\underline{\theta}, \overline{\theta}\} \end{aligned} \tag{4.1}$$

where the first term in the inequality is the utility from matching with a high ability public type, the second term is expected utility from matching with a stranger, and the third term is the utility from matching with a low ability public type.

Now suppose that agent i is of the private type. We assume

$$\begin{aligned} &u_i(x_i^*(\theta_i,\overline{\theta}), x_j^*(\overline{\theta},\theta_i), \theta_i, \overline{\theta}) > E_{\theta_j}[u_i(x_i^*(\theta_i,\theta_j), x_j^*(\theta_j,\theta_i), \theta_i, \theta_j)] > \\ &\qquad u_i(x_i^*(\theta_i,\underline{\theta}), x_j^*(\underline{\theta},\theta_i), \theta_i, \underline{\theta}) \forall \theta_i \in \{\underline{\theta}, \overline{\theta}\} \end{aligned} \tag{4.2}$$

where the first term is utility from matching with a public high type, the second term is utility from being matched with a stranger, and the third term is utility from matching with a public low type.[13]

[13] Notice that the equilibrium strategy profile in the case of complete information is distinct from the one in the case of asymmetric incomplete information, which is distinct from the one in the case of symmetric incomplete information.

Assuming the above two conditions hold, let us examine all possible information structures of the example of four agents with two agents having high ability level (high type) and two having low ability level (low type). First, the cases where both of the high types are known or both the low types are known are trivial since in the former case, high types will match with each other since both obtain the highest welfare possible in this way. In the latter case, high type agents would prefer working with a private ability agents over working with the public low ability type. There are four remaining possibilities. Using assumptions (4.1) and (4.2), our claim is that in each of these cases, there exists a nonconsecutive stable outcome. Notice that we present stable outcomes, but we do not show the dynamics to this outcome, since an interpretation of such a procedure is not clear – if some agents can not differentiate between two agents, there either needs to be some reason for him/her to propose partnership to one of them, or we need an algorithm which allows multiple proposals based on Bayesian maximization of expected utility.

4.1 Ability is private information for all agents

Consider the matching where each high type matches with a low type. Since everyone is a private figure, there is no incentive (ex-ante) for any player to break from the proposed matching to attempt forming a new team. Therefore this matching is stable, and is clearly not consecutive.

4.2 One of the high types is a public figure, while the remaining agents are private figures

Consider the following non-consecutive matching:

- Team 1: high type public figure with a low type private figure.
- Team 2: high type private figure with a low type private figure.

In this case, the low type private figure in the team 1 does not have an incentive to deviate since he strictly prefers to work with a high type public figure over private figures. The high type public figure in the team 1 has no incentive to deviate either since he cannot distinguish between the private figures. Therefore, there exists no coalition of two agents who would like to block this matching.

4.3 One of the low types is a public figure, while the remaining players are private figures

Consider the following non-consecutive matching:

- Team 1: low type public figure with a high type private figure.
- Team 2: low type private figure with a high type private figure.

Neither agent in team 2 has an incentive to deviate since they cannot identify what type the third private figure is, and teaming up with the low type public figure yields strictly lower utility.

4.4 One of each type is a public figure, while all remaining agents are private figures

Consider the following non-consecutive matching:

- Team 1: high type public figure with the low type private figure.
- Team 2: low type public figure with the high type private figure.

To see that this is a stable outcome, consider team 1. The high type public figure has no incentive to deviate since he does not know that the private figure in team 2 is of high type, and is therefore indifferent between the private types. Now, teaming up with the low type public figure gives strictly lower utility than with a private figure ex-ante. The low type private figure would be strictly worse off had he been matched with the low type public figure. Furthermore, on average, he prefers the high type public figure to a private figure. Hence, this matching is stable.

5. Conclusions and Future Research Objectives

When there is a pool of economic agents that need to form productive partnerships among themselves, the question of stability and efficiency of decentralized outcomes arises. Formally defining notions relevant to the partnership formation problem (that is, stable, consecutive matchings), our paper has generalized two main results from Sherstyuk (1998) by allowing agents to choose effort after forming a partnership rather than the utility of agents in a partnership being predetermined by abilities. We have assumed a typical utility function for a partnership game and have investigated the stability and efficiency of the decentralized matching outcome under complete information. In Theorem 5, we have proved, analogous to results presented in Sherstyuk (1998), that in the case of our specific utility function, consecutive matching is the unique stable outcome for both supermodular and submodular production functions. The condition we gave, however, is not very helpful when inspecting some utility function and trying to decide whether Theorem 5 holds. The main result of our paper, Theorem 8, partially remedies this inconvenience.

We have used two theorems from Milgrom and Roberts (1990) and the envelope theorem to prove that, under reasonable assumptions, the stable consecutive matching result can be generalized to a broad class of supermodular utility functions (i.e., Theorem 8). We leave the generalization in the case of submodular utility functions for future research. In terms of efficiency, we showed for our specific utility form that stable outcomes are efficient in the case of supermodular production functions, but not for submodular production functions. We were not able to generalize the efficiency results very far, and it remains a potential objective for further analysis.

In the case of incomplete information, we have shown that within a reasonable class of information structures, there may exist non-consecutive stable outcomes, in contrast to the complete information case. This result is trivial since it is driven simply by the lack of information on other agents' abilities. We have made some unsuccessful attempts at finding non-trivial mechanisms under which agents may communicate in order to achieve the consecutive matching outcome even when incomplete information is present. By a non-trivial mechanism, we mean that it does not necessarily implement the consecutive matching outcome, but rather it does so only under some conditions. Examples of such mechanisms in our sharing rule framework include classic signaling of ability by the private agents and screening by the public agent.

A particularly interesting mechanism is one which breaks the equal sharing constraint. If we assume effort is perfectly contractible and that agents may break equal sharing, then the public agent can try to offer a contract to the private agents which includes profit division and effort exertion levels, for which it is incentive compatible and individually rational for the private agents to state truthfully their type. It is then possible for the consecutive matching outcome to be stable under some contract. On the other hand, agents of low type can be induced, for example, by a high public agent to form a partnership by being offered to exert a low level of effort; however, such a contract is expected to give this low agent a small part of output. Indeed we leave these considerations for future research on the partnership formation problem.

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