The Effects of Credit Markets on Fertility in Closed and Open Economies

Shahar Rotberg

Abstract

This paper studies the effects of credit markets on fertility rates of poor and rich agents, using a two-period, overlapping-generation (OLG) model. Poor and rich parents choose fertility and consumption to maximize utility, which depends only on their own number of children and consumption. Initially agents are restricted from borrowing and lending. Subsequently, open and closed economies are studied where agents have access to credit markets. Results show that without credit markets, rich agents have more children than poor agents, and that with credit markets, poor agents conceive more children than rich agents only if their second period income to first period income is sufficiently larger than that of rich agents.

Acknowledgements

I owe a debt of gratitude to Professor Gregory Pavlov for his constant guidance, comments and help throughout the year. I would also like to thank Professors Terry Sicular and Christopher Robinson for their very useful insights and comments. Lastly, I would like to thank my peers Aaron Weisbrod and Jerry Lao for their very helpful comments.

1. Introduction

Historically, income and fertility have been positively correlated. However, as most research indicates, for many years now the relationship between income and fertility has been negative throughout the Western world and various developing countries (Becker 1981). For example, researchers Jones and Tertlit, who conducted empirical research on the relationship between income and fertility in the United States between the years 1826-1960, found "a strong negative relationship between income and fertility for all cohorts and estimate[d] an overall income elasticity of about -0.38 for the period" (Jones and Tertlit 2007). Another study finds that since the beginning of the 1750s, income and fertility have been negatively correlated in Europe (Coale and Watkins 1979). Similar results are found among many other countries as well. The question that Jones, Tertlit, Coale, Watkins and other researchers have tried to investigate has been a highly discussed topic in economics, as well as in sociology and demography for many years now. One main reason for the interest in that question is that if the rich have fewer children than the poor, it may mean an increased proportion of poor people in society, which can have implications on income redistribution, for example. Another reason for the continuing interest is that if income and fertility are negatively correlated, then it means that children are an inferior good, which most people find to be counterintuitive. Nonetheless, while such research investigates the existence of the phenomenon, it still does not answer the more highly debated question: Why do rich people have fewer children than poor people?

Many non-mathematical as well as mathematical theories, which will be elaborated on over the next two paragraphs, have tried to answer this question. However, it is interesting to observe that the banking system in Europe had begun expanding and resembling the modern banking system at the end of the 1690s (for example, the Bank of England was first Chartered in 1694 and the Bank of Scotland was established in 1695), providing people with easier access to credit and deposit (Hildreth 2001). Interestingly, this expansion occurred just before income and fertility had become negatively correlated. There may be numerous reasons for why income and fertility became negatively correlated since that specific time, but this paper investigates a new theory that may help to better explain the relationship between income and fertility: *What are the effects of credit markets on fertility decisions of poor people and rich people, and on the fertility differential between them?*

The non-mathematical theories started to emerge over 200 years ago. In 1798, Thomas Malthus theorized that rich people had more children because the age at marriage was lower for them than for the poor, giving them more time to reproduce (Malthus 1798). Then, in 1960 Becker introduced the theory of contraceptive knowledge, in which he argued that "contraceptive knowledge has been positively related to income", and that "when it is held constant, a positive relationship appears" (Becker 1960). Moreover, Becker stipulated that richer people "buy higher quality" children – for example, parents spend more money on "higher quality children", that is, on their education, housing, clothes, food, etc. – but do not necessarily "buy more quantity" of children. He theorizes that the increase in "quality" is large, while the increase in "quantity" is positive, but small. Following this, Jacob Mincer received empirical support to his theory about fertility being negatively correlated with the female wage rate. He found that the female's wage on the market is more strongly negatively related to family size than is family income to family size (Mincer 1963).

On the other hand, mathematical theories on the subject have been around for only around 20 to 30 years. The major difference between the theory in this paper and the other theories, which will be explained below, lies in the addition of heterogeneity in the agents' incomes. In 1988, Gary Becker and Robert Barro used an Overlapping Generation Model (OLG) with endogenous fertility in an open economy and one representative agent, who makes decisions in only one period, to find that rich agents would have more children than they would have had they been poorer (Becker and Barro, 1988). A year later they introduced a model in a closed economy, which qualitatively yielded similar conclusions (Becker and Barro 1989). In 2007, using an OLG model, Scholtz and Seshardi allow children to make transfers to their parents, and find that low-income families have more children than high-income families (Scholtz and Seshardi 2007). This basically meant that for low-income families, children act as social security.

As one can see, none of the theories that circulate in academia have been studying the effects of credit markets on fertility decisions and on the fertility differential between rich and poor. First, the model in this paper encourages the reader to think about the relationship between income and fertility from a completely new perspective, that is, to consider the role of credit markets in determining the relationship between fertility and income. Second, the model in this paper may serve as a more accurate framework for analyzing the implications that different government policies, such as social security, have on fertility rates and the fertility differential between rich and poor. Third, by studying the model in this paper, one might be better able to predict future growth trends of poor and rich populations, and possibly to understand the mechanisms that affect these trends and use them when constructing various policies.

In the next section there will be a brief discussion of an important definition, which is central to the model of this paper. Then, the economic model will be explained and analyzed, first in an open economy setup, and then in a closed economy framework.

2. Definition: Rich v. Poor

Before the economic environment is introduced, it is crucial that the reader fully understands what is conveyed by the words "Rich" and "Poor" in this paper. "Rich" and "Poor" can be relative or absolute terms, and can refer to a stock or a flow. For example, in the models by Becker and Barro from 1988 and 1989, a "Rich" or "Poor" agent is one with higher or lower stock of non-labor income, respectively. Furthermore, in the Becker and Barro models "Rich" and "Poor" are absolute terms, so an economic agent who received more non-labor income had a greater purchasing power absolutely. In contrast, a "Rich" or "Poor" agent in this paper is one with a higher or lower flow of labor income, respectively. In addition, "Rich" and "Poor" are relative terms. An economic agent can only be "Poor" if another agent is receiving a higher flow of labor income than him, thus making the other agent "Rich".

Another important point to be made is about the assumptions that are made about the underlying characteristics of "Rich" and "Poor" agents. "Rich" agents may be richer than "Poor" agents due to their higher innate ability or intelligence, for example, which make them more productive. On the other hand, being "Rich" can be a consequence of pure luck, such as living in an area experiencing an economic boom, compared to a "Poor" agent, who may be just as able, but for some reason lives in a place suffering from economic stagnation.

It is more likely that in small open economies, being "Rich" or "Poor" would be a consequence attributed mainly to ability, since mobility is usually much higher in smaller countries, (for example, Israel and Luxemburg). In contrast, it is more likely that in a closed economy "Rich" and "Poor" will be determined by both ability and luck, as those economies tend to be larger in size, and moving from one city to another may be harder (for example, USA, China and The European Union).

Lastly, since the key idea in studying the effects of a particular variable on another is to hold every other variable fixed, it is assumed in this paper that both the "Rich" and "Poor" agents have the same level of education, i.e., high school, BA, or MA and above. This assumption permits us to control for any confounding variables that might affect the relationship between income and fertility.

3. The Economic Environment

The model is a two-period, overlapping-generation (OLG), closed/open economy model. There are two agents, one poor and one rich. Both are assumed to work the same amount of time, and each one receives an income in each period. It is assumed that agents do not make labor decisions. Furthermore, a rich agent receives more income in any given period, i.e., a higher wage. The agents have a choice of how much to consume in the first and second periods of their lives, and how many children to have in the first period (An agent can only bear children in the first period of life.). For each agent, the cost of a child will be a fixed fraction of one's income. On the other hand, each child provides its parents with positive psychic utility. The utility function for both poor and rich agents is assumed to be a time separable (natural) log utility function. Although rich parents spend nominally more on each child than poor parents, it is still assumed that an agent's utility only depends on the "quantity" of children one has, and not on their "quality". Furthermore, although this is a deviation from most "standard" macro-economic models, it is assumed that the utility of an agent does not depend on the rest of one's lineage, i.e., the infinitely lived agent. The reason for the choice of such a utility function, as opposed to more "standard", infinitely lived agent utility function, is that it seems to be more realistic to assume that agents believe they cannot actually control or foresee the decisions of their children, thus they do not take them into account in their objective function. (This may be similar to the behavior of a firm that could operate as a monopoly, but believes it is operating in a competitive market.) Furthermore, children of a poor person receive income equivalent to their parents', and children of a rich person receive income equivalent to their parents'. (In a later section this assumption will be relaxed.) This assumption is made due to the strong support Solon finds for the transmission of earnings ability in many different developed countries, although in the long run it experiences a regression to the mean (Solon 2002). First, the model will be solved with the assumption that the rich and poor agents are not aware of each other and cannot borrow and lend from the rest of the world. Thus, no borrowing and lending will take place. Second, the model will be solved with the agents being aware of each other and allowed to interact on the world credit market. Therefore, borrowing and lending will, possibly, occur. Basically, the model will hold credit markets *ceteris paribus*, to see the effects it has on the optimal fertility decisions of the rich and poor agents, and on the fertility differential, that is, the difference between the fertility rates of the poor, minus the fertility rates of the rich.

4. No Trade Economies

Here it is assumed that no credit markets exist. In essence, there are two different economies. One contains only rich agents, and the other only poor agents. The economy of the rich shall henceforward be named 'Economy 1', and the economy of the poor shall be deemed 'Economy 2'. In period 0 there is 1 rich agent in 'Economy 1' and 1 poor agent in 'Economy 2'. A poor agent's income when young is 1 (Normalization for simplicity), and K when old. Similarly, a rich agent's income when young is L, and M when old. This is denoted by Poor=[1, K], rich=[L, M], with L>1,and M>K.

It is further assumed that the cost of a child as a fraction of a poor agent's income is $\frac{1}{F}$, with $F \in \Re$, and $F > 0$. In addition, the cost of a child as a fraction of a rich agent's income is lower than that of a poor agent, namely $\frac{1}{dF}$, with $d \in \Re$, and $d > 1$.

The reason this assumption is added is that in the 2009 USA Department of Agriculture Annual report on expenditures on children by families, it was found that the lowest income class spent 25% of their before-tax income on a child on average, whereas the middle and high income classes spent 16%, and 12% of their before-tax incomes on average, respectively (the estimates were for two-child families). As one can see, income and the cost of a child as a fraction of the parents' income are negatively related, which is what the assumption about d being greater than 1 is trying to capture.

4.1. 'Economy 1'

Each rich agent born in period t is faced with the following constraints in period t, and t+1: (See appendix for explanations on the notation and on how to interpret it.)

$$L \geq c_{t[t]}^{r^1} + L\, n_t^{r^1} \frac{1}{dF} - l_t^{r^1}$$

$$M \geq c_{t[t+1]}^{r^1} + r_t^{r^1} l_t^{r^1}$$

The lifetime budget constraint is

$$M + r_t^{r^1} L \geq c_{t[t+1]}^{r^1} + r_t^{r^1} c_{t[t]}^{r^1} + r_t^{r^1} L\, n_t^{r^1} \frac{1}{dF}$$

Solving for $n_t^{r^1}$ as a function of $c_{t[t]}^{r^1}, c_{t[t+1]}^{r^1}$ gives the following expression:

$$n_t^{r^1} \leq [(M + r_t^{r^1} L - c_{t[t+1]}^{r^1} - r_t^{r^1} c_{t[t]}^{r^1})dF\,] / r_t^{r^1} L$$

Each agent has a utility function given by

$$u_t^{r^1} = \log((c_{t[t]}^{r^1})(n_t^{r^1})^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{r^1})^{\psi}(n_t^{r^1})^{\frac{\beta}{2}})$$

Each agent will maximize his utility subject to his lifetime budget constraint. That means that since having more children gives the agent more positive utility, it would be wasteful for the agent not to exhaust all of the resources given to him. Thus, the weak inequality can be changed to equality to get:

$$n_t^{r^1} = [(M + r_t^{r^1} L - c_{t[t+1]}^{r^1} - r_t^{r^1} c_{t[t]}^{r^1}) dF\ ] / r_t^{r^1} L$$

Each rich agent faces the following maximization problem:

Max

$$u_t^{r^1} = \log(c_{t[t]}^{r^1}((M + r_t^{r^1} L - c_{t[t+1]}^{r^1} - r_t^{r^1} c_{t[t]}^{r^1}) Fd / r_t^{r^1} L)^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{r^1})^{\psi}((M + r_t^{r^1} L - c_{t[t+1]}^{r^1} - r_t^{r^1} c_{t[t]}^{r^1}) Fd / r_t^{r^1} L)^{\frac{\beta}{2}})$$

W.R.T. $c_{t[t]}^{r^1}, c_{t[t+1]}^{r^1}$ (Where $0 < \beta < 1$ and $0 < \psi < 1$).

Subject to $c_{t[t]}^{r^1} > 0$, $c_{t[t+1]}^{r^1} > 0$, and, $n_t^{r^1} > 0$.

These constraints are added because if any one of the arguments is 0 then the utility function is not defined, and thus cannot possibly be maximized. Also, realistically, the agent cannot consume negative amounts, so these arguments must be positive.

Taking first order conditions gives the following:

$$\frac{\partial u_t^{r^1}}{\partial c_{t[t]}^{r^1}} = 0;\ (c_{t[t+1]}^{r^1})^{\psi}(n_t^{r^1})^{\beta} + \frac{\partial u_t^{r^1}}{\partial n_t^{r^1}}(-\frac{dF}{L}) = (c_{t[t+1]}^{r^1})^{\psi}(n_t^{r^1})^{\beta} + \beta c_{t[t]}^{r^1}(c_{t[t+1]}^{r^1})^{\psi}(n_t^{r^1})^{(\beta-1)}(-\frac{dF}{L}) = 0,$$

$$\frac{\partial u_t^{r^1}}{\partial c_{t[t+1]}^{r^1}} = 0;$$

$$\psi c_{t[t]}^{r^1}(c_{t[t+1]}^{r^1})^{(\psi-1)}(n_t^{r^1})^{\beta} + \frac{\partial u_t^{r^1}}{\partial n_t^{r^1}}(-\frac{dF}{r_t^{r^1} L}) = \psi c_{t[t]}^{r^1}(c_{t[t+1]}^{r^1})^{(\psi-1)}(n_t^{r^1})^{\beta} + \beta c_{t[t]}^{r^1}(c_{t[t+1]}^{r^1})^{\psi}(n_t^{r^1})^{(\beta-1)}(-\frac{dF}{L}) = 0$$

Solving them gives:

$$c_{t[t]}^{r^1} = (M + r_t^{r^1} L) / r_t^{r^1}(\beta + \psi + 1)$$

$$c^{r1}_{t[t+1]} = \frac{(K + r^{r1}_t L)\psi}{(\beta + \psi + 1)}$$

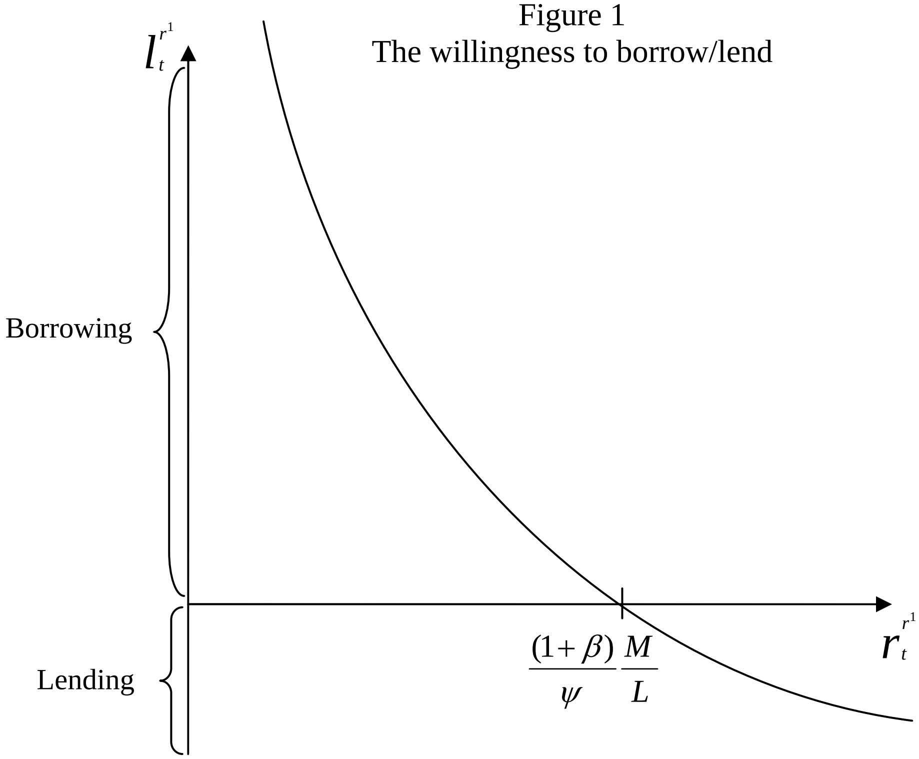
$$n^{r1}_t = [Fd\beta(M + r^{r1}_t L)]/(\beta + \psi + 1)\, r^{r1}_t L$$

These expressions maximize the utility of the rich agent because the second derivatives are negative (See appendix, Proof 1).

Plugging the expressions for $c^{r1}_{t[t]}, c^{r1}_{r[t+1]}$, and n^{r1}_t in the first period budget constraint one will get the following willingness to borrow/lend:

$$l^{r1}_t = \frac{((1+\beta)M - r^{r1}_t L\psi)}{(\beta + \psi + 1)\, r^{r1}_t} \quad \text{(See figure 1)}$$

Figure 1
The willingness to borrow/lend



In a closed economy, aggregate savings are equal to 0, i.e., $\sum_{j=1}^{N_t^{r1}} l_{t[j]}^{r1} = 0$, where subscript j denotes the individual, and N_t^{r1} denotes the total number of rich agents born in period t. However, since all rich agents are identical, then setting $l_t^{r1} = 0$ will be sufficient. Now, one can solve for the interest rate, and ultimately $c_{t[t]}^{r1}, c_{t[t+1]}^{r1}, n_t^{r1}$

(4.1.1) $r_t^{r1} = \frac{(1+\beta)}{\psi}\frac{M}{L}$

(4.1.2) $c_{t[t]}^{r1} = \frac{L}{(1+\beta)}$

(4.1.3) $c_{t[t+1]}^{r1} = M$

(4.1.4) $n_t^{r1} = \frac{\beta F}{(1+\beta)} d$

As one can observe, $r_t^{r1}, c_{t[t]}^{r1}, c_{t[t+1]}^{r1}$, and n_t^{r1} are constant across time, i.e., they do not depend on t, and in every period t, they are equal to expressions given by (4.1.1)-(4.1.4), respectively. It is interesting to note that $c_{t[t]}^{r1}$ depends on β, rather than ψ. As β increases, $c_{t[t]}^{r1}$ decreases. This is because parents with a higher β have a higher preference for the quantity of children, and so would like to bear more children, who come at the expense of concurrent consumption. It is easy to show that derivative of n_t^{r1} with respect to β is simply $\frac{dF}{(1+\beta)^2}$ >0. Thus, n_t^{r1} increases when parents prefer more children, which should not come as much of a surprise. Also, as one can observe $c_{t[t+1]}^{r1}$ depends neither on β nor ψ in this economy. The reason being that all agents are identical, and thus, everyone consumes all of their income in period t+1.

4.1.2 'Economy 2'

Each poor agent born in period t is faced with the following constraints in period t, and t+1

$$1 \geq c_{t[t]}^{p2} + n_t^{p2}\frac{1}{F} - l_t^{p2}$$

$$K \geq c_{t[t+1]}^{p^2} + r_t^{p^2} l_t^{p^2}$$

The lifetime budget constraint is given by:

$$K + r_t^{p^2} \geq c_{t[t+1]}^{p^2} + r_t^{p^2} c_{t[t]}^{p^2} + r_t^{p^2} n_t^{p^2} \frac{1}{F}$$

One can solve for $n_t^{p^2}$ as a function of $c_{t[t]}^{p^2}, c_{t[t+1]}^{p^2}$ to get:

$$n_t^{p^2} \leq [(K + r_t^{p^2} - c_{t[t+1]}^{p^2} - r_t^{p^2} c_{t[t]}^{p^2})F]/r_t^{p^2}$$

Each agent has a utility function given by

$$u_t^{p^2} = \log(c_{t[t]}^{p^2}(n_t^{p^2})^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{p^2})^{\psi}(n_t^{p^2})^{\frac{\beta}{2}})$$

Each agent will maximize his utility subject to his lifetime budget constraint. This means that since having more children gives the agent more positive utility, it would be wasteful for the agent not to exhaust all of the resources given to him. Thus, the weak inequality can be changed to an equality producing the following equation:

$$n_t^{p^2} = [(K + r_t^{p^2} - c_{t[t+1]}^{p^2} - r_t^{p^2} c_{t[t]}^{p^2})F]/r_t^{p^2}$$

Therefore, each poor agent's maximization problem is:

Max

$$u_t^{p^2} = \log(c_{t[t]}^{p^2}((K + r_t^{p^2} - c_{t[t+1]}^{p^2} - r_t^{p^2} c_{t[t]}^{p^2})F/r_t^{p^2})^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{p^2})^{\psi}((K + r_t^{p^2} - c_{t[t+1]}^{p^2} - r_t^{p^2} c_{t[t]}^{p^2})F/r_t^{p^2})^{\frac{\beta}{2}})$$

W.R.T. $c_{t[t]}^{p^2}, c_{t[t+1]}^{p^2}$

Subject to $c_{t[t]}^{p^2} > 0$, $c_{t[t+1]}^{p^2} > 0$, and, $n_t^{p^2} > 0$.

These constraints are added because if any one of the arguments is 0 then the utility function is not defined, and thus cannot possibly be maximized. Also, realistically, the agent cannot consume negative amounts, or negative amounts of children. So these arguments must be positive.

After taking first order conditions (similar to 'Economy 1'), and setting them equal to 0, the following expressions are obtained:

$$c^{p^2}_{t[t]} = (K + r^{p^2}_t)/(\beta + \psi + 1) r^{p^2}_t$$

$$c^{p^2}_{t[t+1]} = \frac{(K + r^{p^2}_t)\psi}{(\beta + \psi + 1)}$$

$$n^{p^2}_t = [F\beta(K + r^{p^2}_t)]/(\beta + \psi + 1) r^{p^2}_t$$

(These expressions maximize the utility of the poor agent by the same arguments given in 'Economy 1'.)

And the following willingness to borrow/lend:

$l^{p^2}_t = \dfrac{((1+\beta)K - r^{p^2}_t \psi)}{(\beta + \psi + 1) r^{p^2}_t}$ (Graphically it looks similar to figure 1 with the x-axis intercept at $\dfrac{(1+\beta)}{\psi}K$)

The same argument that was used in ‘Economy 1’ can be replicated here to justify using $l^{p^2}_t = 0$ as the market clearing condition. Now, one can solve for the interest rate, and ultimately for $c^{p^2}_{t[t]}, c^{p^2}_{t[t+1]}, n^{p^2}_t$

(4.2.1) $r^{p^2}_t = \dfrac{(1+\beta)}{\psi}K$

Substituting (4) back into the (1), (2), and (3) we get

(4.2.2) $c^{p^2}_{t[t]} = \dfrac{1}{(1+\beta)}$

(4.2.3) $c^{p^2}_{t[t+1]} = K$

(4.2.4) $n^{p^2}_t = \dfrac{\beta F}{(1+\beta)}$

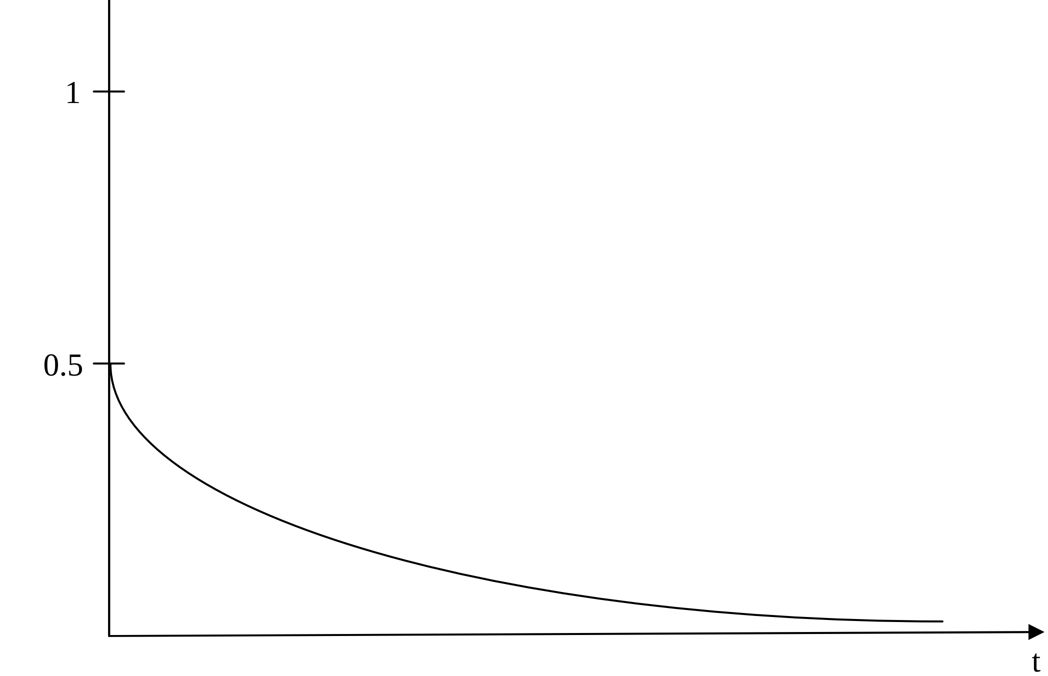
As one can observe, $r^{p^2}_t, c^{p^2}_{t[t]}, c^{p^2}_{t[t+1]}$, and $n^{p^2}_t$ are constant across time, i.e., do not depend on t, and in every period t are equal to expressions given by (4.2.1)-(4.2.4), respectively. Also, the analysis of fertility and consumptions in 'Economy 1' is applicable to 'Economy 2'.

4.1.3 'Economy 1' v. 'Economy 2'

As can be noticed right away, $n_t^{p^2} < n_t^{r^1}$ in every period t, which is due to the lower cost of children as a fraction of a rich agent's income. Let $\eta^{1,2} = n_t^{p^2} - n_t^{r^1} = \frac{\beta F}{(1+\beta)} - \frac{\beta F}{(1+\beta)} d = \frac{\beta F}{(1+\beta)}(1-d) < 0$ (since $d > 1$ by assumption), be the fertility differential between the poor and rich agents in 'Economy 1' and 'Economy 2'. In addition, it should come as no surprise that in every period t, $c_{t[t]}^{p^2} < c_{t[t]}^{r^1}$ and $c_{t[t+1]}^{p^2} < c_{t[t+1]}^{r^1}$, which is due to the combination of higher income received by the richer agent and lower real costs per child. Furthermore, it is interesting to notice that the relationship between $r_t^{p^2}$ and $r_t^{r^1}$,is indeterminate. The relationship will be determined by the ratio of an agent's second period income to first period income (which shall henceforth be referred to as 'The Ratio', or 'Ratio'. A higher 'Ratio' will also be referred to as a "steeper" profile.). The interest rate in 'Economy 1' will be higher than in 'Economy 2' only if $\frac{M}{L}$ > K, and vice versa. As will become clearer later on, 'The Ratio' will play a crucial role in determining how credit markets affect fertility decisions and the fertility differential.

Let $N_t^{r^1} = n_0^{r^1} \cdots\cdots n_{t-1}^{r^1}$ and $N_t^{p^2} = n_0^{p^2} \cdots\cdots n_{t-1}^{p^2}$ denote the total rich and poor people born in period t, respectively, and Let $a_t^{p^{1,2}} = \frac{N_t^{p^2}}{N_t^{r^1} + N_t^{p^2}}$ denote the proportion of poor agents born in period t in the total population born in period t in both 'Economy 1' and 'Economy 2'. Since $n_t^{p^2} < n_t^{r^1}$ for every period t, one can easily plot the path of $a_t^{p^{1,2}}$ (Figure 2).

Figure 2
The proportion of poor agents born in period t in the total population born in period t in both 'Economy 1' and 'Economy 2'



As shown in figure 2, $\lim_{t \to \infty} a_t^{p^{1,2}} = 0$

5. A Small Open Economy with Trade

It is worthwhile to start by analyzing the effects of credit markets on fertility rates and the fertility differential in a small open economy, since the interest rate is given, which makes the analysis somewhat less complex than that of a closed economy where the interest rate is endogenously determined.

In the small open economy, all assumptions from 'Economy 1' and 'Economy 2' are preserved but the poor and rich agents are now allowed to borrow on the world credit market at an exogenously given interest rate. This economy shall be referred to as 'Economy 3' from now onward.

5.1. 'Economy 3'

Again just as in 'Economy 1' the agents solve the same maximization problems, only this time taking the interest rate as given. The rich agent's maximization problem is:

Max

$$u_t^{r^3} = \log(c_{t[t]}^{r^3}((M + r_t^3 L - c_{t[t+1]}^{r^3} - r_t^3 c_{t[t]}^{r^3})Fd/r_t^3 L)^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{r^3})^{\psi}((M + r_t^3 L - c_{t[t+1]}^{r^3} - r_t^3 c_{t[t]}^{r^3})Fd/r_t^3 L)^{\frac{\beta}{2}})$$

W.R.T. $c_{t[t]}^{r^3}, c_{t[t+1]}^{r^3}$

Subject to $c_{t[t]}^{r^3} > 0$, $c_{t[t+1]}^{r^3} > 0$, and , $n_t^{r^3} > 0$.

The poor agent maximization problem is:

Max $u_t^{p^3} = \log(c_{t[t]}^{p^3}((K + r_t^3 - c_{t[t+1]}^{p^3} - r_t^3 c_{t[t]}^{p^3})F/r_t^3)^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{p^3})^{\psi}((K + r_t^3 - c_{t[t+1]}^{p^3} - r_t^3 c_{t[t]}^{p^3})F/r_t^3)^{\frac{\beta}{2}})$

W.R.T. $c_{t[t]}^{p^3}, c_{t[t+1]}^{p^3}$

Subject to $c_{t[t]}^{p^3} > 0$, $c_{t[t+1]}^{p^3} > 0$, and , $n_t^{p^3} > 0$.

After taking first order conditions for both agents and setting them equal to 0, the following expressions for $c_{t[t]}^{r^3}, c_{t[t+1]}^{r^3}, n_t^{r^3}$, and $c_{t[t]}^{p^3}, c_{t[t+1]}^{p^3}, n_t^{p^3}$ are obtained:

(5.1.1) $c_{t[t]}^{r^3} = (M + r_t^3 L)/(\beta + \psi + 1) r_t^3$

(5.1.2) $c_{t[t+1]}^{r^3} = \dfrac{(M + r_t^3 L)\psi}{(\beta + \psi + 1)}$

(5.1.3) $n_t^{r^3} = \dfrac{d\beta F(M + r_t^3 L)}{(\beta + \psi + 1) r_t^3 L}$

(5.1.4) $c_{t[t]}^{p^3} = (K + r_t^3)/(\beta + \psi + 1) r_t^3$

(5.1.5) $c_{t[t+1]}^{p^3} = \dfrac{(K + r_t^3)\psi}{(\beta + \psi + 1)}$

(5.1.6) $n_t^{p^3} = \dfrac{\beta F(K + r_t^3)}{(\beta + \psi + 1) r_t^3}$

In addition, the demands/supply of loans for both agents are given by:

$$l_t^{p^3} = \frac{((1 + \beta)K - \psi r_t^3)}{(\beta + \psi + 1) r_t^3}$$

$$l_t^{r^3} = \frac{((1+\beta)M - \psi r_t^3)}{(\beta+\psi+1) r_t^3}$$

Since the interest rate r_t^3 is exogenous, $c_{t[t]}^{r^3}, c_{t[t+1]}^{r^3}, n_t^{r^3}$, and $c_{t[t]}^{p^3}, c_{t[t+1]}^{p^3}, n_t^{p^3}$, are the same in every period t, and are given by the expressions in (5.1.1)-(5.1.6). It is worth noting that since r_t^3 is exogenous, a rich agent will have more children in 'Economy 3' than he did in 'Economy 1' if $r_t^3 < r_t^{r^1}$, i.e. if $\frac{(1+\beta)}{\psi}\frac{M}{L} > r_t^3$, and vice versa. Similarly, a poor agent will have more children in 'Economy 3' than he did in 'Economy 2' if $r_t^3 < r_t^{p^2}$, i.e., if $\frac{(1+\beta)}{\psi}\frac{K}{1} > r_t^3$, and vice versa. The reason being, that a lower world interest rate relative to 'Economy 1' and 'Economy 2' induces the agents to become net borrowers, giving them more resources with which to raise children. This implies that when agents have a relatively steeper profile compared to the rest the world, they will have more children if they can borrow and lend on a world credit market.

The main interest is in determining the relationship between $n_t^{r^3}$ and $n_t^{p^3}$, and ultimately to determine the path of $a_t^{p^3}$, compared to the path of $a_t^{p^{1,2}}$. Let $\eta^3 = n_t^{p^3} - n_t^{r^3} = [\beta F(K + r_t^3)]/(\beta+\psi+1) r_t^3 - [\beta F d(M + r_t^3 L)]/(\beta+\psi+1) r_t^3 L =$ $= \frac{\beta F}{(\beta+\psi+1) r_t^3}(K - \frac{M}{L}d) + \frac{\beta F}{(\beta+\psi+1)}(1-d)$, be the fertility differential between rich and poor agents in 'Economy 3'. For poor agents to have more children (5.1.7) must be satisfied.

(5.1.7) $\eta^3 = \frac{\beta F}{(\beta+\psi+1) r_t^3}(K - \frac{M}{L}d) + \frac{\beta F}{(\beta+\psi+1)}(1-d) > 0$

Since $d > 1$ by assumption, then $\frac{\beta F}{(\beta+\psi+1)}(1-d) < 0$. This means that in order for (5.1.7) to hold it is necessary that $K - \frac{M}{L}d > 0$. However, it must also be sufficiently large to offset $\frac{\beta F}{(\beta+\psi+1)}(1-d) < 0$ in order to satisfy (5.1.7). Intuitively, what this means is that in order for poor agents to bring more children than rich agents, their willingness to borrow must be sufficiently larger than that of rich agents. This happens when they have a lot of income in their second period of life relative to their first period of life. Since agents would like to smooth consumption over their lifetime, those with steeper profiles are likely to borrow more than agents with shallower profiles, dividing their higher borrowings between more consumption and more children. Thus, the relationship between $n_t^{r^3}$ and $n_t^{p^3}$ depends on 'The Ratio' of the rich agents compared to 'The Ratio' of the poor agents. Indeed, there are three cases to be analyzed.

Case 1:

If $\frac{M}{L} \geq K$ *then* $n_t^{p^3} < n_t^{r^3}$ $\forall t \in (\mathbb{N} \cup \{0\})$, $\lim_{t\to\infty} a_t^{p^3} = 0$, *and the effect on the fertility differential is ambiguous.*

In this case (5.1.7) is not satisfied at any period t, as rich agents have a steeper profile than poor agents. Thus, in each period $n_t^{p^3} < n_t^{r^3}$ leading $a_t^{p^3}$ to tend to 0, just as $a_t^{p^{1,2}}$ does. Moreover, it is interesting to study the effects of the world credit market on the fertility differential, i.e., comparing η^3 to $\eta^{1.2}$. If $r_t^{p^2} < r_t^3 < r_t^{r^1}$, then rich agents will have more children than they did in 'Economy 1', but poor agents will have fewer children than they did in 'Economy 2', increasing the fertility differential substantially, $\eta^3 < \eta^{1,2}$. However, if $r_t^3 < r_t^{p^2} < r_t^{r^1}$, then both the rich and poor agents will have more children than they would if there was no world credit market, since both become net borrowers. If $r_t^{p^2} < r_t^{r^1} < r_t^3$, then both the rich and poor agents will have fewer children than they would if there was no world credit market, as they will become net lenders, and have fewer resources to raise children. Yet, the effect of credit markets on the fertility differential is ambiguous in these two cases because a rich agent's willingness to lend may be larger or smaller than that of a poor agent depending on the world interest rate, β, ψ, K, L, and M.

Case 2:

If $\frac{M}{L} < K$ *and (5.1.7) is not satisfied at period 0 then* $n_t^{p^3} < n_t^{r^3}$ $\forall t \in (\mathbb{N} \cup \{0\})$, $\lim_{t\to\infty} a_t^{p^3} = 0$ *and the effect on the fertility differential is ambiguous.*

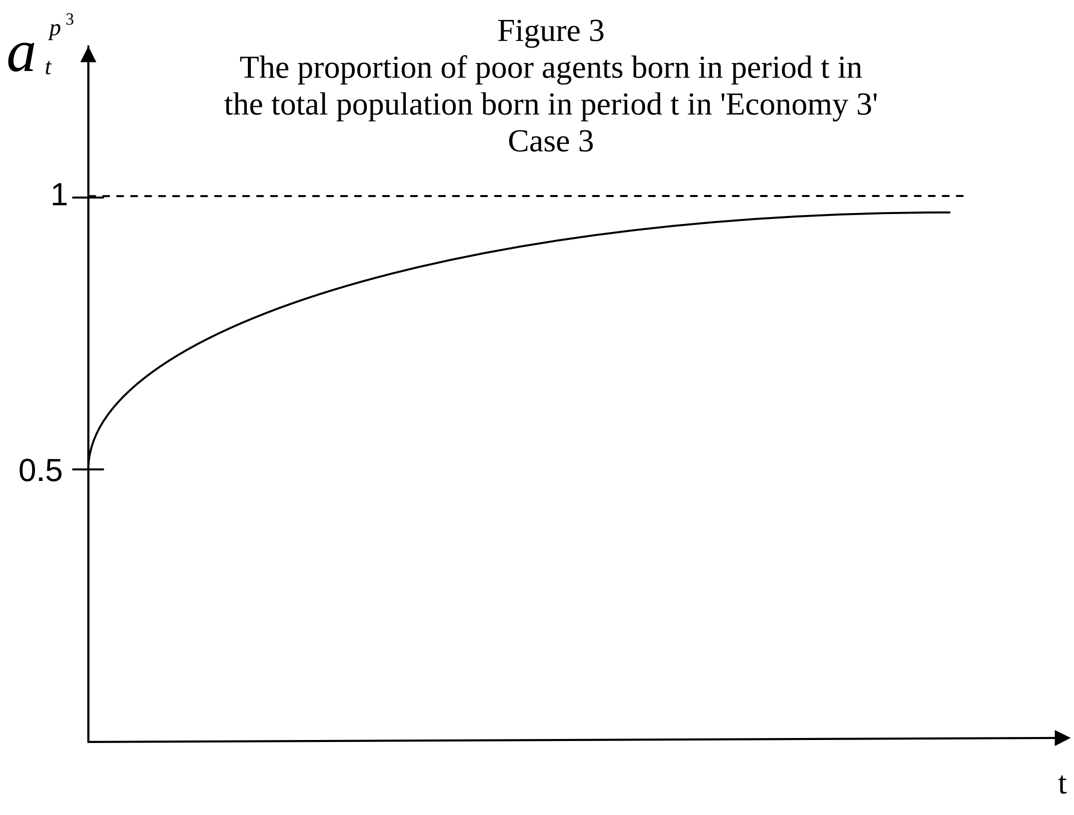
In each period $n_t^{p^3} < n_t^{r^3}$ leading $a_t^{p^3}$ to tend to 0, just as $a_t^{p^{12}}$ does (Unless (5.1.7)=0, in which case $a_t^{p^3}$=0.5 in every period t, as $n_t^{p^3} = n_t^{r^3}$ in all periods). As for the fertility differential, η^3, if $r_t^{r^1} < r_t^3 < r_t^{p^2}$, then poor agents will have more children than they did in 'Economy 2', but rich agents will have fewer children than they did in 'Economy 1', decreasing the fertility differential, $\eta^3 > \eta^{1.2}$. However, if $r_t^3 < r_t^{r^1} < r_t^{p^2}$, then both the rich and poor agents will have more children than they would have if there was no world credit market, since both become net borrowers. If $r_t^{r^1} < r_t^{p^2} < r_t^3$ then both the rich and poor agents will have fewer children than they did when there was no world credit market, as they will be net lenders and have fewer resources for the purpose of raising children. However, the effect of credit markets on the fertility differential is ambiguous in both cases because a poor agent's willingness to borrow may be higher or lower than that of a rich agent's depending on the world interest, β, ψ, K, L, and M.

Case 3:

If $\frac{M}{L} < K$ *and (5.1.7) is satisfied at period 0 then* $n_t^{p^3} > n_t^{r^3} \ \forall t \in (\mathbb{N} \cup \{0\})$, $\lim_{t\to\infty} a_t^{p^3} = 1$, *and the effect on the fertility differential is ambiguous.*

In this case, for each period t $n_t^{p^3} > n_t^{r^3}$ leading $a_t^{p^3}$ to tend to 1 (See figure 3). Note that the analysis for the fertility differential is the same as in Case 2.

Figure 3
The proportion of poor agents born in period t in the total population born in period t in 'Economy 3'
Case 3



5.2 Non-Constant World Interest Rate

If the world interest rate fluctuates across time, then if originally Case 2 or Case 3 were applicable, a varying world interest rate may induce a variation between the two cases. Thus, there may be fluctuations in fertility rates, fertility differentials, consumption behavior, and $a_t^{p^3}$. If Case 1 is applicable, then a varying world interest rate will not change the analysis.

6. A Closed Economy with Trade

In the closed economy analysis, all assumptions from 'Economy 1' and 'Economy 2' are preserved, but the rich and poor agents are allowed to trade with each other, and the interest rate is endogenously determined. This economy shall be referred to as 'Economy 4'.

6.1 'Economy 4'

Again, just as in 'Economy 1' and 'Economy 2', the agents solve the same maximization problems only with the interest rate being endogenously determined. The rich agent's maximization problem is:

Max

$$u_t^{r^4} = \log(c_{t[t]}^{r^4}((M + r_t^4 L - c_{t[t+1]}^{r^4} - r_t^4 c_{t[t]}^{r^4})Fd / r_t^4 L)^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{r^4})^{\psi}((M + r_t^4 L - c_{t[t+1]}^{r^4} - r_t^4 c_{t[t]}^{r^4})Fd / r_t^4 L)^{\frac{\beta}{2}})$$

W.R.T. $c_{t[t]}^{r^4}, c_{t[t+1]}^{r^4}$

Subject to $c_{t[t]}^{r^4} > 0$, $c_{t[t+1]}^{r^4} > 0$, and , $n_t^{r^4} > 0$.

The poor agent has the following maximization problem:

Max

$$u_t^{p^4} = \log(c_{t[t]}^{p^4}((K + r_t^{p^4} - c_{t[t+1]}^{p^4} - r_t^4 c_{t[t]}^{p^4})F / r_t^4)^{\frac{\beta}{2}}) + \log((c_{t[t+1]}^{p^4})^{\psi}((K + r_t^{p^4} - c_{t[t+1]}^{p^4} - r_t^4 c_{t[t]}^{p^4})F / r_t^4)^{\frac{\beta}{2}}$$

W.R.T. $c_{t[t]}^{p^4}, c_{t[t+1]}^{p^4}$

Subject to $c_{t[t]}^{p^4} > 0$, $c_{t[t+1]}^{p^4} > 0$, and , $n_t^{p^4} > 0$.

After taking first order conditions for both agents and setting them equal to 0, the following expressions for $c_{t[t]}^{r^4}, c_{t[t+1]}^{r^4}, n_t^{r^4}, l_t^{p^4}, c_{t[t]}^{p^4}, c_{t[t+1]}^{p^4}, n_t^{p^4}$, and $l_t^{p^4}$ are obtained:

$$c_{t[t]}^{r^4} = (M + r_t^4 L)/(\beta + \psi + 1) r_t^4$$

$$c_{t[t+1]}^{r^4} = \frac{(M + r_t^4 L)\psi}{(\beta + \psi + 1)}$$

$$n_t^{r^4} = \frac{\beta F d (M + r_t^4 L)}{(\beta + \psi + 1) r_t^4 L}$$

$$l_t^{r^4} = \frac{((1+\beta)M - \psi r_t^4 L)}{(\beta + \psi + 1) r_t^4}$$

$$c_{t[t]}^{p^4} = (K + r_t^4)/(\beta + \psi + 1) r_t^4$$

$$c_{t[t+1]}^{p^4} = \frac{(K + r_t^4)\psi}{(\beta + \psi + 1)}$$

$$n_t^{p^4} = \frac{\beta F (K + r_t^4)}{(\beta + \psi + 1) r_t^4}$$

$$l_t^{p^4} = \frac{((1+\beta)K - \psi r_t^{p^4})}{(\beta + \psi + 1) r_t^4}$$

Since it is a closed economy, aggregate savings are equal to 0, i.e., $\sum_{j=1}^{N_t^{r4}} l_{t[j]}^{r4} + \sum_{i=1}^{N_t^{p4}} l_{t[i]}^{p4} = 0$, where subscript j and i denote the individual, and $N_t^{r^4}$ and $N_t^{p^4}$ denote the total number of rich and poor agents born in period t, respectively. The aggregate demand/supply of loans is graphically similar to figure 1, with the intercept of the x-axis changing from period to period depending on the value of

$a_t^{p^4}$, i.e., the ratio of poor agents born in period t in the total population born in period t in 'Economy 4'. Since all poor agents are identical and all rich agents are identical, it is enough to solve $N_t^{r4} l_t^{r^4} + N_t^{p4} l_t^{p^4} = (1-a_t^{p4}) l_t^{r^4} + a_t^{p4} l_t^{p^4} = 0$ in order to get the interest rate r_t^4.

The expression is given by: $r_t^4 = \frac{(1+\beta)}{\psi} \frac{(a_t^{p^4} K + (1-a_t^{p^4}) M)}{(a_t^{p^4} + (1-a_t^{p^4}) L)} = \frac{(1+\beta)}{\psi} \frac{(N_t^{p^4} K + N_t^{r^4} M)}{(N_t^{p^4} + N_t^{r^4} L)}$

6.2. Existence of Steady States in 'Economy 4'

The effects of credit markets on fertility decisions will have short run and long run effects. Although the short run effects are of interest and will be analyzed, it is of particular interest to learn about the long run effects that credit markets have on 'Economy 4'. Those long run effects are the changes that will have occurred in the fertility rates, the fertility differential, the proportion of poor agents in the total population, and consumptions, once the economy has reached an equilibrium at period t'. When this happens $c_{t[t]}^{r^4}, c_{t[t+1]}^{r^4}, c_{t[t]}^{p^4}, c_{t[t+1]}^{p^4}, r_t^4, n_t^{r^4}$, and $n_t^{p^4}$ reach their steady states for any period t>=t'. Those steady states mean that from period t' onward the variables will not change with time, i.e., they will be constant for all periods t>=t'. In 'Economy 4' those steady states depend on the path of r_t^4. Once r_t^4 reaches its steady state level, so do $c_{t[t]}^{r^4}, c_{t[t+1]}^{r^4}, c_{t[t]}^{p^4}, c_{t[t+1]}^{p^4}$, and particularly, $n_t^{r^4}$ and $n_t^{p^4}$. To see how r_t^4 reaches it steady state, consider when (6.2.1) holds.

(6.2.1) $$\eta^4 = n_t^{p^4} - n_t^{r^4} = \frac{\beta F}{(\beta+\psi+1) r_t^4}\left(K - \frac{M}{L} d\right) + \frac{\beta F}{(\beta+\psi+1)}(1-d) > 0$$

There are three cases to analyze.

Case 1:

If $\frac{M}{L} \geq K$ *then* $n_t^{p^4} < n_t^{r^4}$ $\forall t \in (\mathbb{N} \cup \{0\}), \lim_{t\to\infty} a_t^{p^4} = 0$, *and the fertility differential increases.*

$\frac{\beta F}{(\beta+\psi+1)}(1-d)<0$ because $d>1$, and $K-\frac{M}{L}d<0$. Thus, $n_t^{p^4}<n_t^{r^4}$ in every period t, since $r_t^4>0$, and $F>0$. This means that $N_t^{r^4}=n_0^{r^4}\cdots n_{t-1}^{r^4}$ will grow faster than $N_t^{p^4}=n_0^{p^4}\cdots n_{t-1}^{p^4}$, making $a_t^{p^4}$ decrease in each subsequent period, tending to 0, and increasing r_t^4, until it reaches its steady state $r_t^4=\frac{(1+\beta)}{\psi}\frac{M}{L}=r_t^{r^1}$ (See figure 4). Thus, $c_{t[t]}^{r^4}, c_{t[t+1]}^{r^4}, l_t^{r^4}$, and particularly, $n_t^{r^4}$ (See figure 5) in their steady states will be the same as they would be in 'Economy 1'. Moreover, since in this case $r_t^4>r_t^{p^2}$, in every period t poor agents will be lenders. Therefore, $c_{t[t]}^{p^4}<c_{t[t]}^{p^2}, c_{t[t+1]}^{p^4}>c_{t[t+1]}^{p^2}$, and particularly, $n_t^{p^4}<n_t^{p^2}$ (See figure 6) in every period and also in their steady states. This further means that the fertility differential will increase in every period t ($\eta^4<\eta^{1,2}$) and stabilize when the interest rate reaches its steady state.

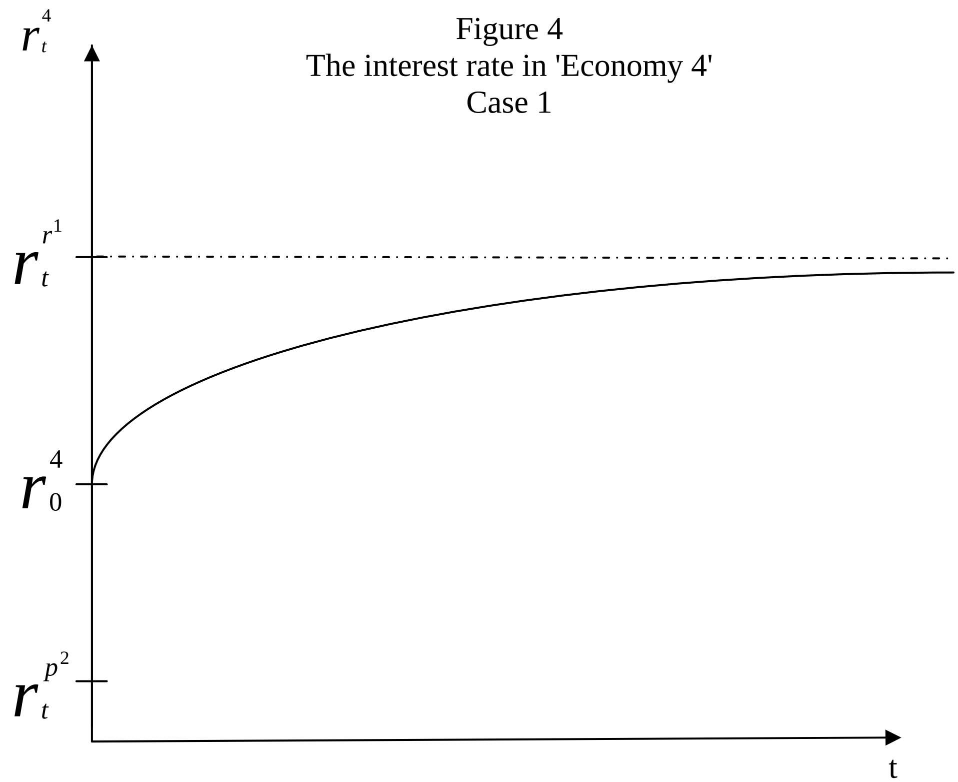


Figure 4
The interest rate in 'Economy 4'
Case 1

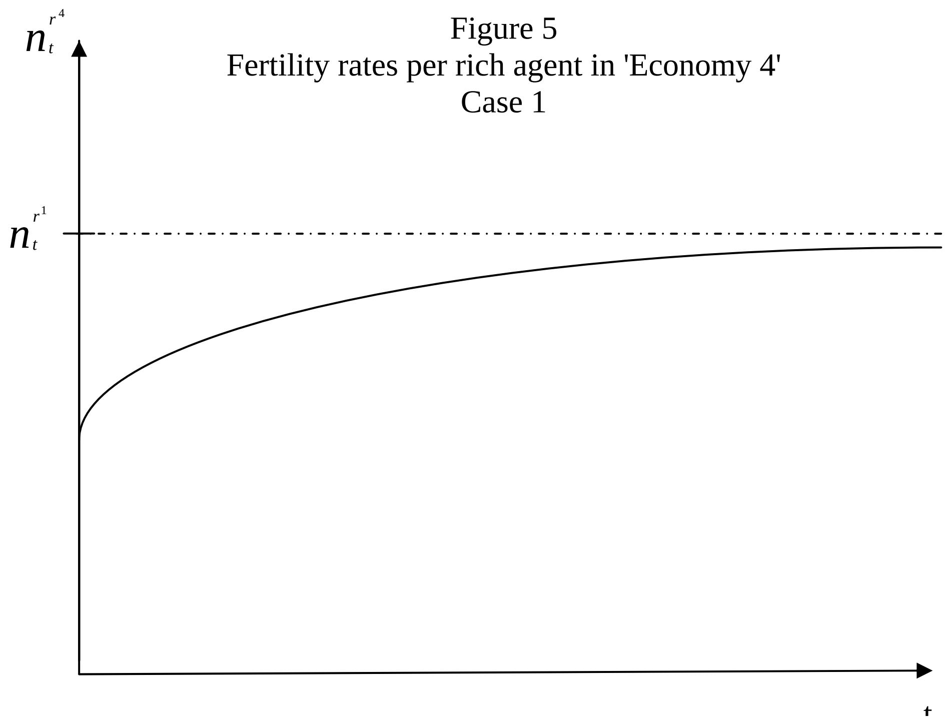


Figure 5
Fertility rates per rich agent in 'Economy 4'
Case 1

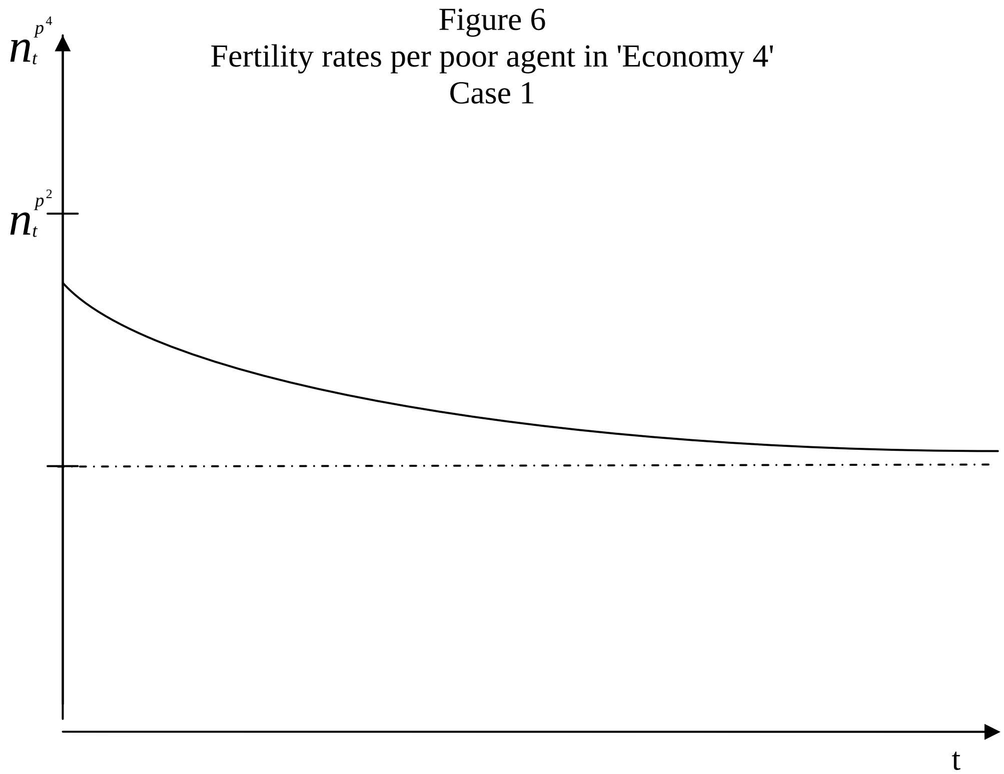


Figure 6
Fertility rates per poor agent in 'Economy 4'
Case 1

Case 2:

$\frac{M}{L} < K$ and (6.2.1) is not satisfied in period 0.

This implies that $n_0^{p^4} < n_0^{r^4}$, and so $a_1^{p^4} < a_0^{p^4}$, and $r_1^4 < r_0^4$.

Sub-Case 1:

if $K < \frac{M}{L}d$ *and (6.2.1) is not satisfied in period 0 then* $n_t^{p^4} < n_t^{r^4}$ $\forall t \in (\mathbb{N} \cup \{0\})$, $\lim_{t\to\infty} a_t^{p^4} = 0$, *and the fertility differential increases.*

By assumption $\frac{\beta F}{(\beta+\psi+1) r_0^4}(K - \frac{M}{L}d) < 0$, and as the interest rate decreases, this expression becomes more negative. One can apply this argument for subsequent periods to see that $n_t^{p^4} < n_t^{r^4}$ in every period t (See figure 8.). Thus (6.2.1) will never be satisfied. This means that $N_t^{r^4} = n_0^{r^4} \ldots n_{t-1}^{r^4}$ will grow faster than $N_t^{p^4} = n_0^{p^4} \cdots n_{t-1}^{p^4}$, making $a_t^{p^4}$ decrease in each subsequent period, tending to 0, and decreasing r_t^4, until it reaches its steady state $r_t^4 = \frac{(1+\beta)}{\psi}\frac{M}{L} = r_t^{r^1}$ (See figure 7). Thus, $c_{t[t]}^{r^4}, c_{t[t+1]}^{r^4}, l_t^{r^4}$, and more interestingly, $n_t^{r^4}$ (See figure 8) in their steady states will be the same as they would be in 'Economy 1'. Since in this case, $r_t^4 < r_t^{p^2}$ in every period t, poor agents will borrow more and so $c_{t[t]}^{p^4} > c_{t[t]}^{p^2}, c_{t[t+1]}^{p^4} < c_{t[t+1]}^{p^2}$, and particularly, $n_t^{p^4} > n_t^{p^2}$ (See figure 9) in every period t and also in their steady states. This further means that the fertility differential will be lower in the steady, i.e. $\eta^4 > \eta^{1,2}$

.

Figure 7
The interest rate in 'Economy 4'
Case 2, Sub-case 1

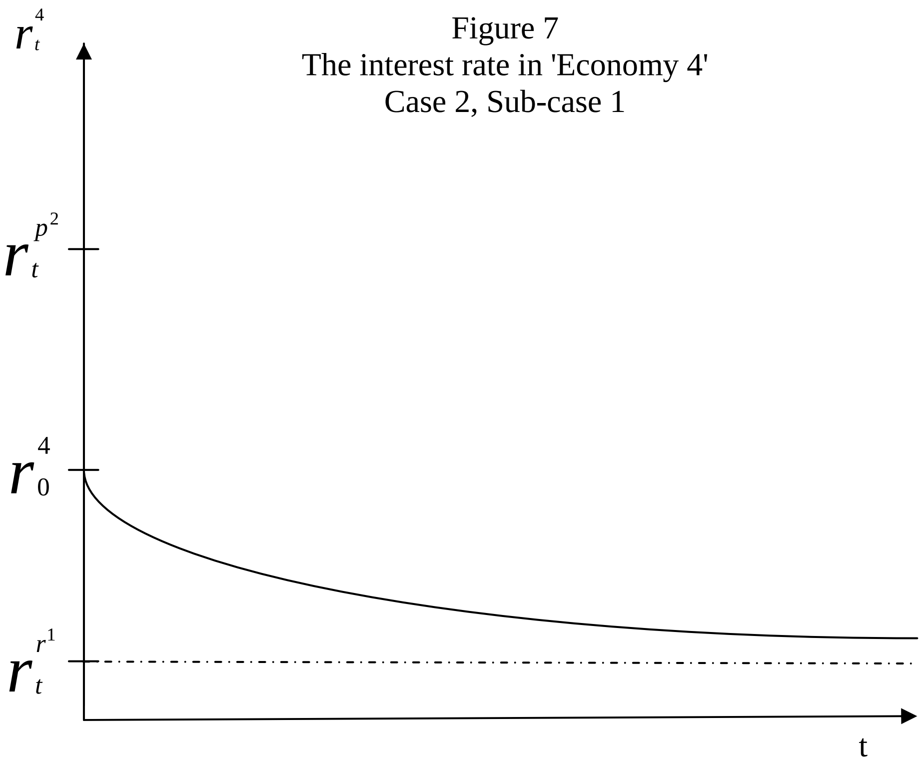


Figure 8
Fertility rates per rich agent in 'Economy 4'
Case 2, Sub-case 1

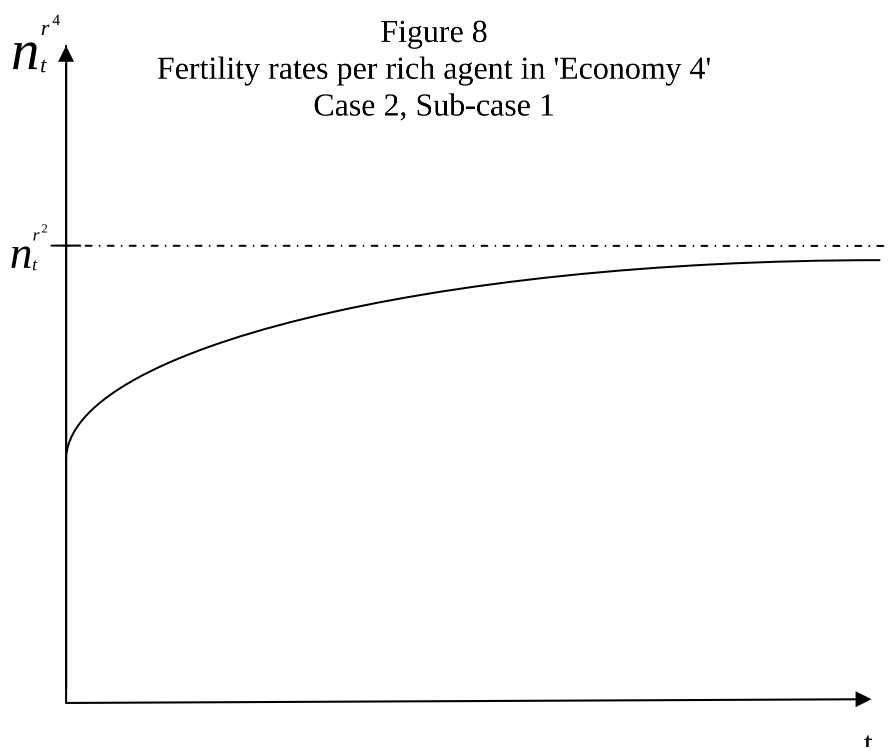
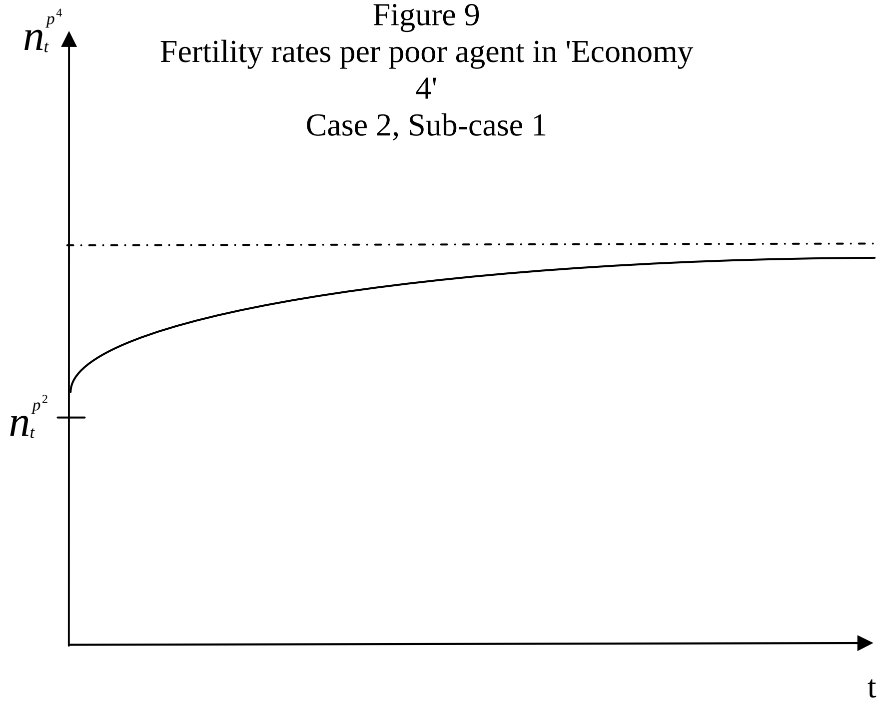


Figure 9
Fertility rates per poor agent in 'Economy 4'
Case 2, Sub-case 1



<u>Sub-Case 2</u>:

if $K > \frac{M}{L}d$ *and (6.2.1) is not satisfied in period 0 then the fertility differential decreases. Also, if* $K < \dfrac{\frac{M}{L}d\psi}{\psi + (1+\beta)(1-d)}$, *then* $n_t^{p4} < n_t^{r4}$ $\forall t \in (\mathbb{N} \cup \{0\})$ *and* $\lim_{t\to\infty} a_t^{p^4} = 0$.

Otherwise, $n_t^{p4} < n_t^{r4}$ *until some period t' where* $n_t^{p4} = n_t^{r4}$ *and*

$$\lim_{t\to\infty} a_t^{p^4} = \frac{(1+\beta)(d-1)K - (K - \frac{M}{L}d)\psi L}{K(1+\beta)(1-d)(d-1) + (K - \frac{M}{L}d)\psi(1-L)}, \text{ where } (\lim_{t\to\infty} a_t^{p^4}) \in [0, 0.5).$$

To analyze the economy in this case, one must first be aware of the fact that the economy has three different steady states for the interest rate. Two are unstable and one is stable. The interest rate that is stable is the one whose numerical value is in between the other two steady states, and the convergence to the stable steady state is from above or below, depending on its starting point, but there is no oscillation in its convergence. (See Steady States in the appendix for further explanation on stable and unstable steady states, convergence to the stable steady state, and the special cases where there are only one or two steady states.)

In 'Economy 4', one steady state is $r_t^{p^2}$, and another is $r_t^{r^1}$, where by assumption of Case 2, Sub-Case 2, $r_t^{p^2} > r_t^{r^1}$. To see this, one can refer to the previous cases, where it was shown that $r_t^{r^1}$ is a steady state. By the same reasoning, one can see that $r_t^{p^2}$ is another steady state. However, a third steady state is achieved when

$\eta^4 = n_t^{p^4} - n_t^{r^4} = \frac{\beta F}{(\beta+\psi+1) r_t^4}(K - \frac{M}{L}d) + \frac{\beta F}{(\beta+\psi+1)}(1-d) = 0$. Denote this steady state $\overline{r}^4 = \frac{K - \frac{M}{L}d}{d-1}$. Now, if $< r_t^{r^1} < \overline{r}^4 < r_t^{p^2}$ then $\overline{r}^4$ will be the stable steady state interest rate. Otherwise, $r_t^{r^1}$ will be the stable steady state interest rate. To see that $\overline{r}^4 < r_t^{p^2}$ is true, consider the following. It is straightforward to see that $r_0^4 < r_t^{p^2}$. Also, by assumption, $\frac{\beta F}{(\beta+\psi+1) r_0^4}(K - \frac{M}{L}d)$>0. Then, there are more rich agents born in the next period which puts downward pressure on the interest rate. Thus, it must be that η^4 = 0 for an interest rate lower than $r_t^{p^2}$, and so $\overline{r}^4 < r_t^{p^2}$. If $\overline{r}^4 > r_t^{r^1}$ (See figure 10), this means that the stable steady state interest rate is $\overline{r}^4$ since it is the middle value. Hence, (6.2.1) will equal to 0 at some period t=t'. This means that both the poor and the rich will bear the same number of children in this steady state. Denote this fertility rate by $\overline{n}^4$. This means that $N_t^{r^4} = n_0^{r^4} \ldots . n_{t-1}^{r^4}$ will grow faster than $N_t^{p^4} = n_0^{p^4} \cdots n_{t-1}^{p^4}$ only until period t', and so $a_t^{p^4}$ will only decrease until period t' and then remain constant and equal to $\overline{a}^{p^4}$ = $\frac{(1+\beta)(d-1)K - (K - \frac{M}{L}d)\psi L}{K(1+\beta)(1-d)(d-1) + (K - \frac{M}{L}d)\psi(1-L)}$, where $\overline{a}^{p^4}$ denotes the proportion of poor agents to the total population born in every period t>=t' (See figure 13). To see why, recall that $r_t^4 = \frac{(1+\beta)}{\psi}\frac{(N_t^{p^4}K + N_t^{r^4}M)}{(N_t^{p^4} + N_t^{r^4}L)}$. Then, $\frac{N_t^{p^4}}{N_t^{r^4}} = \frac{(1+\beta)M - \psi L r_t^4}{\psi r_t^4 - (1+\beta)K}$ and so from period t' onwards $\overline{a}^{p^4} = \frac{N_t^{p^4}}{N_t^{r^4} + N_t^{p^4}} = \frac{(1+\beta)M - \psi L \overline{r}^4}{(1+\beta)(M-K) + \psi \overline{r}^4(1-L)}$. Substituting $\overline{r}^4$ in the expression gives the expression required for $\overline{a^{p^4}}$. If, however, $\overline{r^4} < r_t^{r^1}$ then $r_t^{r^1}$ is the stable steady state interest rate, and thus $n_t^{p^4} < n_t^{r^4}$ in every period t. Therefore, $\lim_{t\to\infty} a_t^{p^4} = 0$. This means that $c_{t[t]}^{r^4}, c_{t[t+1]}^{r^4}, l_t^{r^4}$, and particularly, $n_t^{r^4}$ in their steady states will be the same as they would be in 'Economy 1'. Otherwise, if $\overline{r^4} > r_t^{r^1}$, rich agents will become lenders so $c_{t[t]}^{r^4} < c_{t[t]}^{r^1}, c_{t[t+1]}^{r^4} > c_{t[t+1]}^{r^1}$, and particularly, $n_t^{r^4} < n_t^{r^1}$ (See figure 11). Also, since the stable steady state interest rate is below $r_t^{p^2}$ in every period t, this induces poor agents to become borrowers, so $c_{t[t]}^{p^4} > c_{t[t]}^{p^2}, c_{t[t+1]}^{p^4} < c_{t[t+1]}^{p^2}$, and particularly, $n_t^{p^4} > n_t^{p^2}$ (See figure 12) in every period t and also in their steady states. This further means that the fertility differential will be lower in the steady, i.e., $\eta^4 > \eta^{1,2}$.

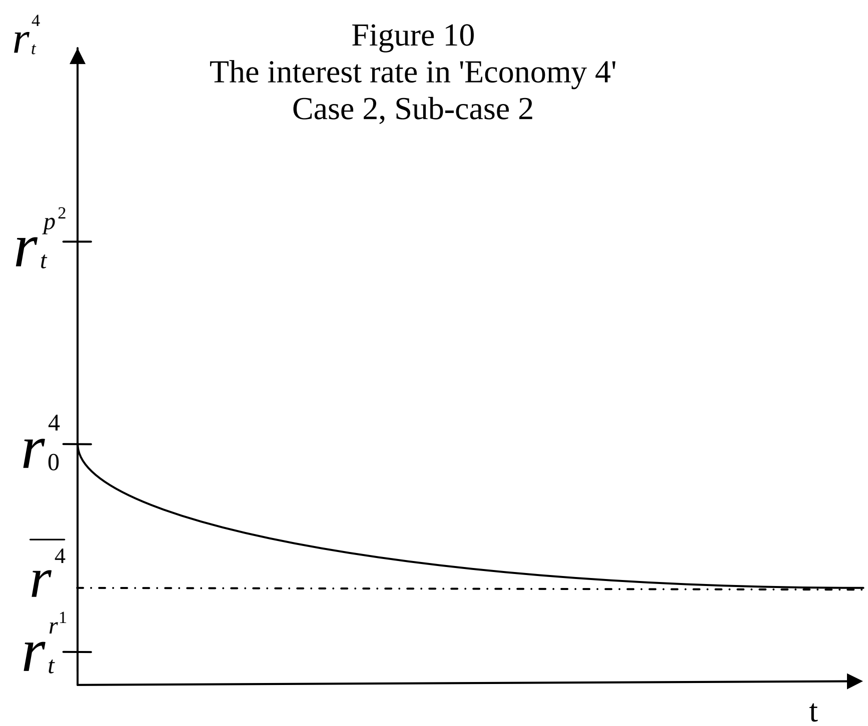


Figure 10
The interest rate in 'Economy 4'
Case 2, Sub-case 2

Figure 11
Fertility rates per rich agent in 'Economy 4'
Case 2, Sub-case 2

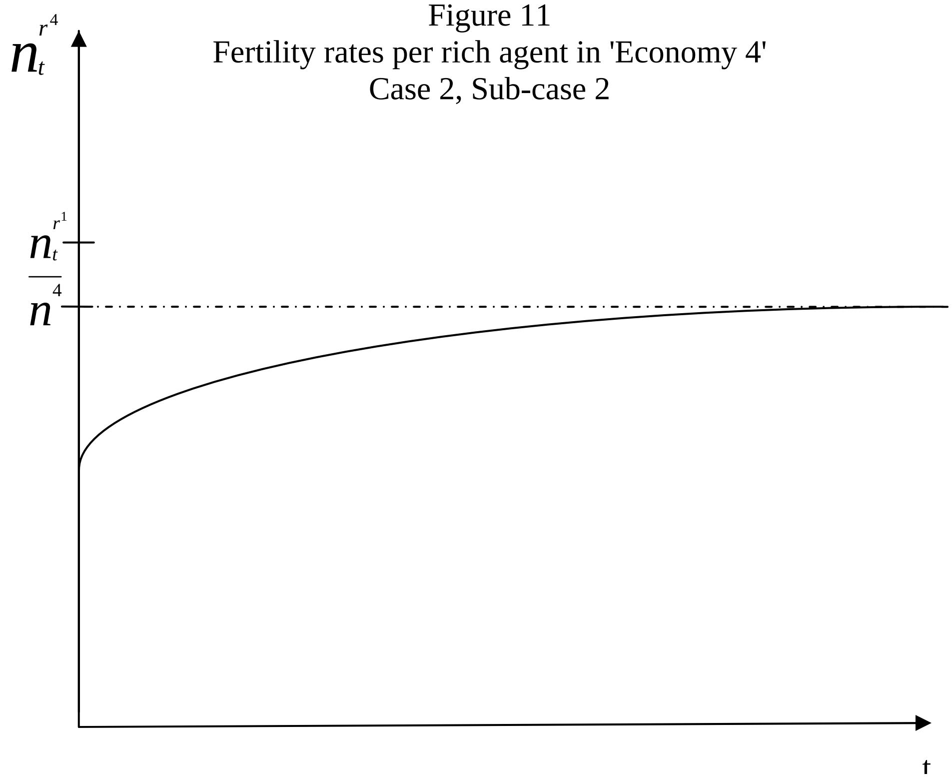
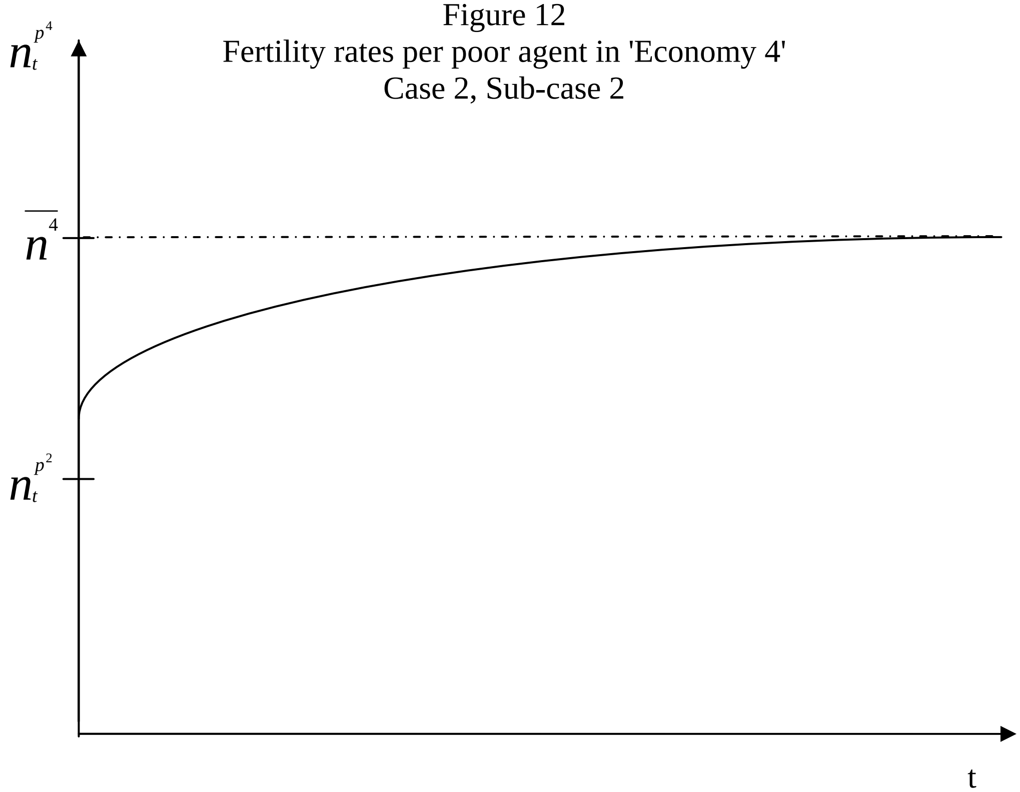


Figure 12
Fertility rates per poor agent in 'Economy 4'
Case 2, Sub-case 2



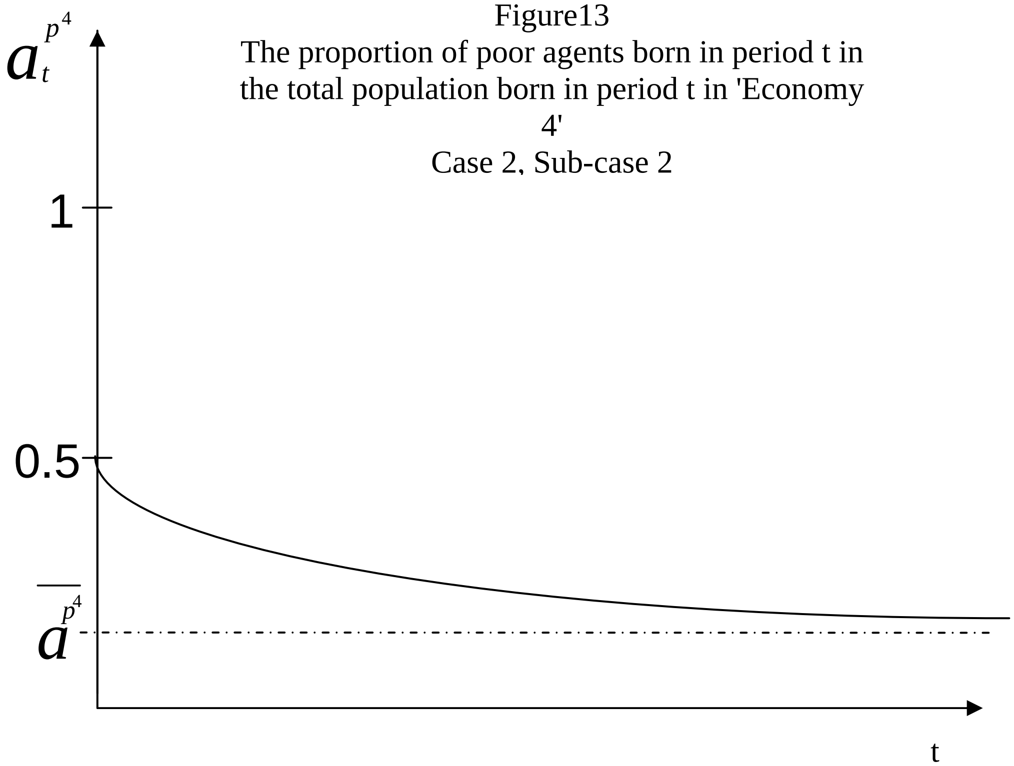


Figure13
The proportion of poor agents born in period t in the total population born in period t in 'Economy 4'
Case 2, Sub-case 2

Case 3*:*

if $K > \frac{M}{L}d$ *and (6.2.1) is satisfied in period 0 then the fertility differential decreases.*

Also, *if* $K > \dfrac{\frac{M}{L}d\psi}{\psi + (1+\beta)(1-d)}$, *then* $n_t^{p^4} > n_t^{r^4}$ $\forall t \in (\mathbb{N} \cup \{0\})$ *and* $\lim_{t\to\infty} a_t^{p^4} = 1$.

Otherwise, $n_t^{p^4} > n_t^{r4}$ *until some period t' where* $n_t^{p^4} = n_t^{r^4}$ *and*

$$\lim_{t\to\infty} a_t^{p^4} = \frac{(1+\beta)(d-1)K - (K - \frac{M}{L}d)\psi L}{K(1+\beta)(1-d)(d-1) + (K - \frac{M}{L}d)\psi(1-L)}, \textit{ where } (\lim_{t\to\infty} a_t^{p^4}) \in (0.5,1].$$

The analysis of this economy is similar to Case 2, Sub-Case 2. The economy has three different steady states for the interest rate. Two are unstable and one is stable. The interest rate that is stable is the one whose value is between the other two steady states. (See Steady States in the appendix for further explanation on stable and unstable steady states.)

As was explained in the previous case, one steady state is $r_t^{p^2}$ in 'Economy 4', and another is $r_t^{r^1}$, where by assumption of Case 3, $r_t^{p^2} > r_t^{r^1}$. However, another steady state is achieved when $\eta^4 = n_t^{p^4} - n_t^{r^4} = \frac{\beta F}{(\beta+\psi+1) r_t^4}(K - \frac{M}{L}d) + \frac{\beta F}{(\beta+\psi+1)}(1-d) = 0$. Denote this steady state $\overline{r^4} = \frac{K - \frac{M}{L}d}{d-1}$. Now, if $< r_t^{r^1} < \overline{r^4} < r_t^{p^2}$ then $\overline{r^4}$ will be the stable steady state interest rate. Otherwise $r_t^{p^2}$ will be the stable steady state interest rate. To see that $\overline{r^4} > r_t^{r^1}$ is true, consider the following. It is clear that $r_0^4 > r_t^{r^1}$. Also, since (6.2.1) is satisfied at period 0, then there are more poor agents born in the next period which puts upward pressure on the interest rate, meaning that $\eta^4 = 0$ at an interest rate higher than $r_t^{r^1}$. Thus, it must be that $\overline{r^4} > r_t^{r^1}$. If $\overline{r^4} < r_t^{p^2}$ (See figure 14), this means that the stable steady state interest rate is $\overline{r^4}$, since it is the middle value. Therefore, $\eta^4 = 0$ at some period t=t'. This means that both the poor and the rich will produce the same number of children in this steady state. Denote this fertility rate by $\overline{n^4}$. This means that $N_t^{p^4} = n_0^{p^4} \dots n_{t-1}^{p^4}$ will grow faster than $N_t^{r^4} = n_0^{r^4} \dots . n_{t-1}^{r^4}$ only until period t', and so $a_t^{p^4}$ will increase only until period t' and then remain constant and equal to $\overline{a^{p^4}} = \frac{(1+\beta)(d-1)K - (K - \frac{M}{L}d)\psi L}{K(1+\beta)(1-d)(d-1) + (K - \frac{M}{L}d)\psi(1-L)}$, where $\overline{a^{p^4}}$ denotes the proportion of poor agents to the total population born in every period t>=t' (See figure 17). It was shown in Case 2, Sub-Case 2 how to get this expression for $\overline{a^{p^4}}$. If, however, $\overline{r^4} > r_t^{p^2}$, then $r_t^{p^2}$ is the stable steady state interest rate, and $n_t^{p^4} > n_t^{r^4}$ in every period t. Therefore $\lim_{t\to\infty} a_t^{p^4} = 1$. This means that, $c_{t[t]}^{p^4}, c_{t[t+1]}^{p^4}, l_t^{p^4}$, and particularly, $n_t^{p^4}$ in their steady states will be the same as in 'Economy 2'. Otherwise, if $\overline{r^4} > r_t^{p^2}$, poor agents will become borrowers, and so $c_{t[t]}^{p^4} > c_{t[t]}^{p^2}, c_{t[t+1]}^{p^4} < c_{t[t+1]}^{p^2}$, and particularly, $n_t^{p^4} > n_t^{p^2}$ (See figure 15). On the other hand, rich agents will become lenders, so $c_{t[t]}^{r^4} < c_{t[t]}^{r^1}$,

$c^{r4}_{t[t+1]} > c^{r1}_{t[t+1]}$, and particularly, $n_t^{r4} < n_t^{r1}$ (See figure 16). This further means that the fertility differential will be lower in the steady state, i.e., $\eta^4 > \eta^{1,2}$.

Figure 14
The interest rate in 'Economy 4'
Case 3

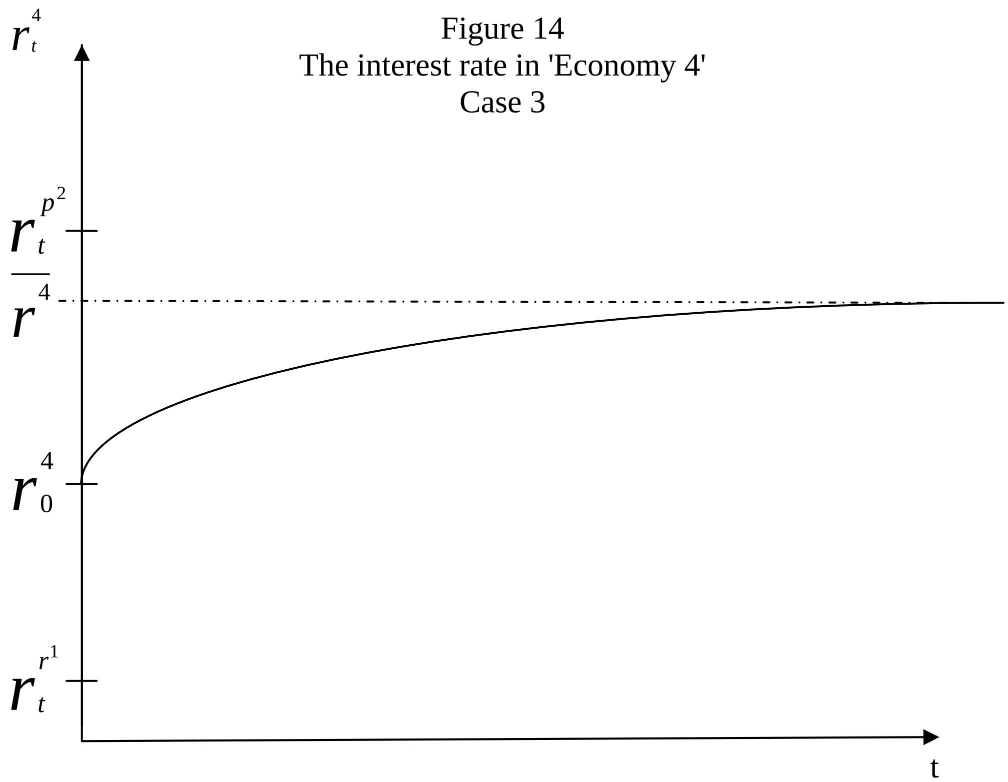


Figure 15
Fertility rates per poor agent in 'Economy 4'
Case 3

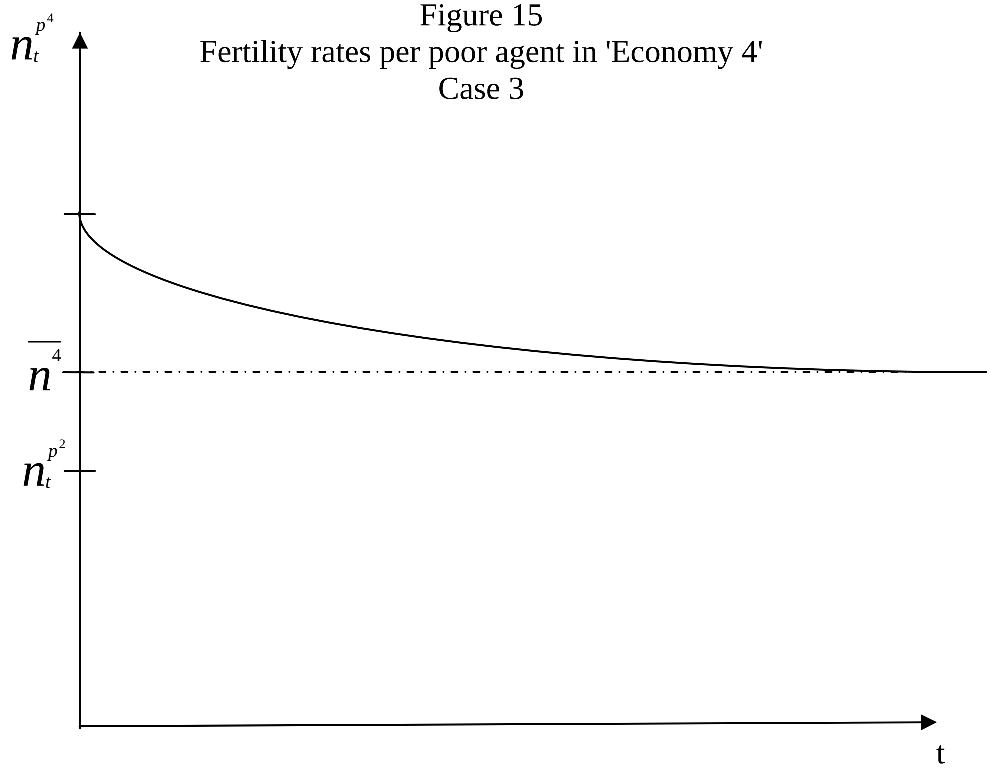
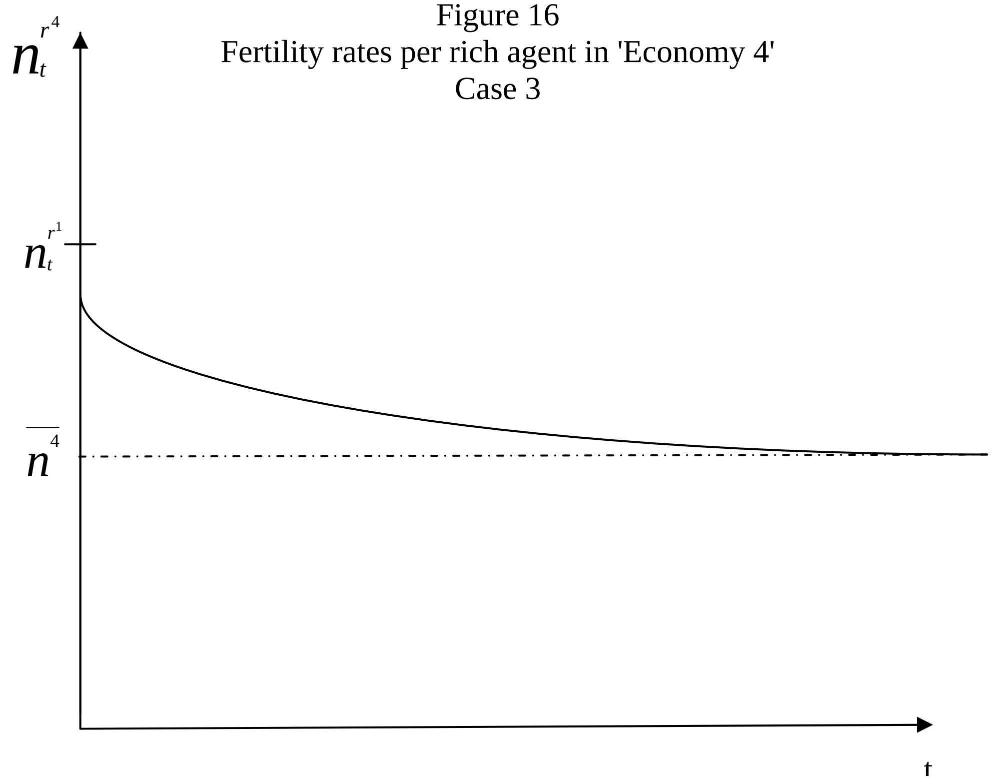


Figure 16
Fertility rates per rich agent in 'Economy 4'
Case 3



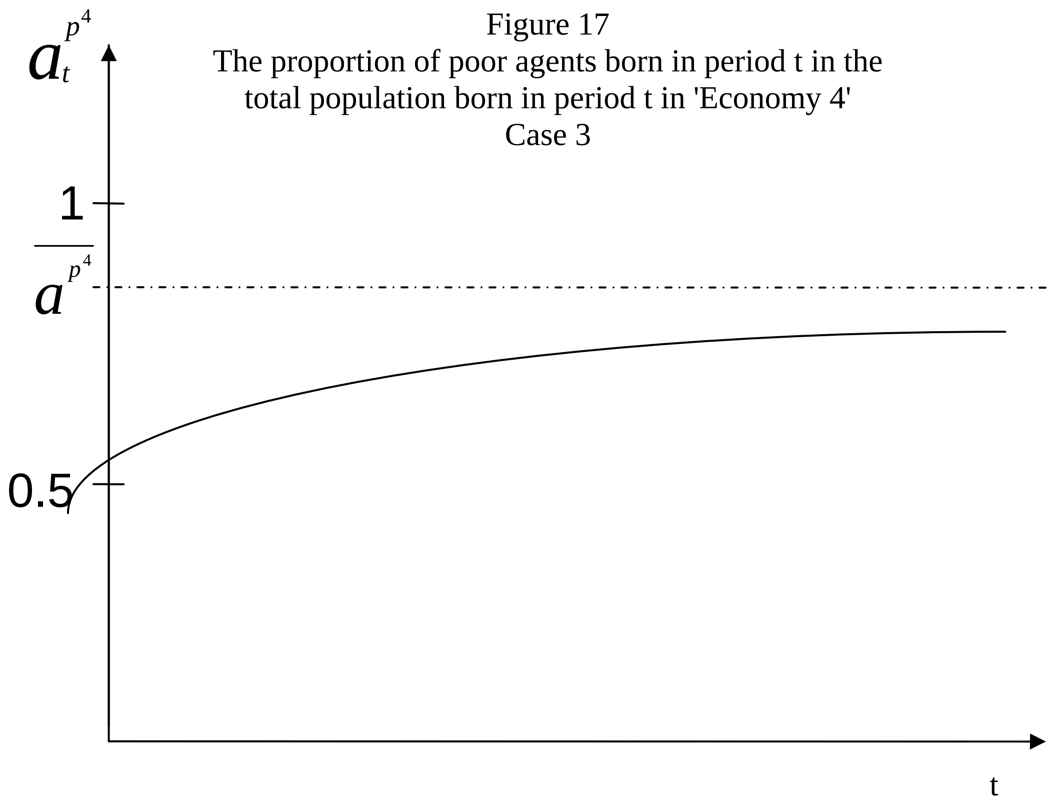


Figure 17
The proportion of poor agents born in period t in the total population born in period t in 'Economy 4'
Case 3

7. Small Open Economy v. Closed Economy

From the previous analysis, it became clear that there are quite a few quantitative and qualitative differences between the effects of credit markets on fertility decisions and the fertility differential in small open economies and in closed economies. In this section these differences will be summarized. The reason for the differences between the two economies lies in the fact that in 'Economy 4' the interest rate r_t^4 is always bounded above and below by $r_t^{r^1}$ and $r_t^{p^2}$. Because of this, one agent is always a net lender while the other is a net borrower, allowing for clear prediction of the fertility differential. On the other hand, in 'Economy 3' the interest rate r_t^4 is exogenous, and thus is not necessarily bounded above and below by $r_t^{r^1}$ and $r_t^{p^2}$. Therefore, there may be cases where both agents are net borrowers, or both are net lenders, making clear-cut conclusions hard to draw.

<u>Case 1:</u>

$$\frac{M}{L} \geq K$$

In this case it was found that both (5.1.7) and (6.2.1) were not satisfied for any period t. Thus, in each period $n_t^{p^3} < n_t^{r^3}$, and $n_t^{p^4} < n_t^{r^4}$ leading $a_t^{p^3}$ and $a_t^{p^4}$ to tend to 0, just as $a_t^{p^{12}}$ does. Moreover, it was found that in 'Economy 3' the change in the fertility differential was ambiguous, while in 'Economy 4' the fertility differential was actually higher, i.e. $\eta^4 < \eta^{1,2}$.

Case 2:

$$\frac{M}{L} < K \text{ and (5.1.7) and (6.2.1) are not satisfied in period 0.}$$

In each period $n_t^{p^3} < n_t^{r^3}$ leading $a_t^{p^3}$ to tend to 0, just as $a_t^{p^{12}}$. However, in the closed economy $n_t^{p^4} < n_t^{r^4}$, generally, but under certain conditions they may equalize at some period t=t'. This also meant that $a_t^{p^4}$ might actually reach its steady state at a number greater than 0. As for the fertility differential, it decreased in the closed economy, $\eta^4 > \eta^{1,2}$, but it was unclear what the changes to it were in the open economy.

Case 3:

$$\frac{M}{L} < K \text{ and (5.1.7) and (6.2.1) are satisfied in period 0.}$$

In this case, it was observed that in each period $n_t^{p^3} > n_t^{r^3}$ leading $a_t^{p^3}$ to tend to 1, however, $n_t^{p^4} > n_t^{r^4}$ was not necessarily satisfied in each period. In fact, it could be that, $n_t^{p^4} > n_t^{r^4}$ only until some period t=t', after which they equalize. In addition, $a_t^{p^4}$ might actually tend to a number less than one. Furthermore, the fertility differential in 'Economy 4' unambiguously decreased, with $\eta^4 > \eta^{1,2}$, whereas there were no clear-cut conclusions about the relationship between η^3 and $\eta^{1,2}$.

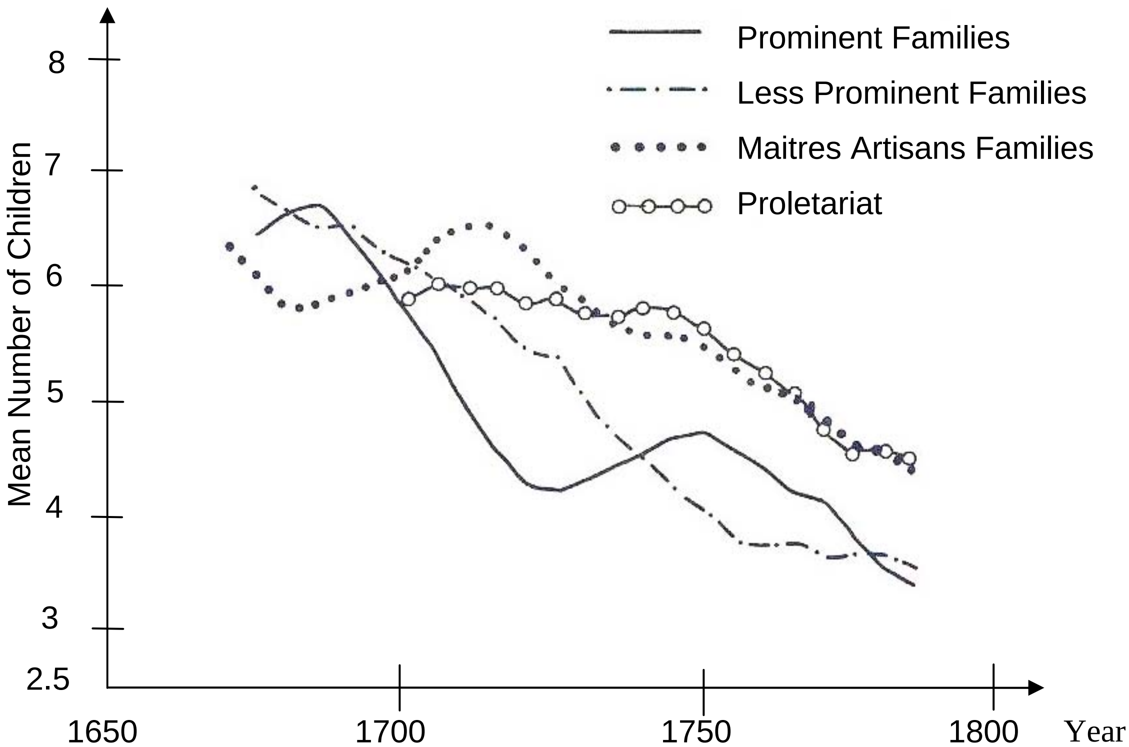
8. Issues of Real-World Data

After having gone through the analyses above, one begins to wonder which of the cases outlined is applicable to the real world. To answer this question one must be able to have data on incomes of poor and rich people (lowest and highest thirds of the income distribution) with the same education level when they are around 20 years old, and then obtain data on the same individuals' incomes 25-35 years later. There seems to be a problem with finding such information since the criteria are very specific. Therefore, instead of using real-world data, one can make use of economic theory and mathematics in order to hypothesize about which case is applicable to the real world.

Suppose that there are two individuals A and B, and that B receives a lower income than A. Suppose also that both get X dollars added to their incomes. It is clear, then, that B had a higher proportion increase in his income, as his starting point was lower. Furthermore, a fundamental concept in economics is decreasing marginal benefits. If B starts with a lower income than A, economic theory will maintain that B's marginal benefit from an extra dollar earned should be, normally, greater than A's. Thus, B is probably more motivated than A to have his income increased, and may try to become a more productive worker than A, rendering B a higher increase in his income over time relative to A. Mathematical analysis and economic theory indicate that the real-world case is probably Case 2 or Case 3, where $\frac{M}{L} < K$. However, neither mathematics nor economic theory can help determine whether (5.1.7) and (6.2.1) are satisfied or not.

As was mentioned in the introduction, and as one can see in Figure 18 (Coale and Watkins 2001), income and fertility used to be positively correlated before the beginning of the 1700s, but have become negatively correlated after that time. In addition, decreasing fertility rates were a trend for all classes well into the end of the 1780s. Interestingly, as was mentioned in the introduction, the banking system in Europe was expanding rapidly in the late 1690s and people were able to obtain loans and lend relatively more easily than ever before. One could argue that Europe between the 17th and 19th centuries resembled a closed economy more than an open economy, mainly because of its large share – about 15-17% – of the world's population (Goldewijk 2005). Thus, 'Economy 4', i.e., a closed economy, is probably a more suitable model for Europe around 200-300 years ago. Now, if one looks at Figures 14 and 15 in Case 3 in ‘Economy 4’, one can see much resemblance between them and the trends seen in Figure 18. Therefore, it could be that Case 3 is the one that is applicable to the real world. This suggests that, although there may have been other factors at play, it is quite possible that credit markets are partially responsible for the negative relationship between income and fertility that has appeared in the last 300 years.

Figure 18
Fertility Rates in Europe by Social Class (1670-1785)



9. Regression to the Mean

As was mentioned in the beginning of the paper, there was a potential problem with the assumption made about intergenerational transmission of ability being perfectly correlated between parents and their children. Although there is support of significant transmission of earnings ability in many different developed countries, the intergenerational elasticity is only estimated around 0.4 (Solon 2002). Thus, in the long run, intergenerational earnings ability regresses to the mean. One way to incorporate regression to the mean in the analysis is to increase recursively the incomes in each period of each poor agent's descendant (by the same amount of X dollars in each period), and similarly, to reduce recursively the income in each period of each rich agent's descendant (by the same X amount of dollars). This will be done until at some period t=t' there will be no income difference between the agents born in period t'. The only difference between them will be their ancestry. The analysis with regression to the mean can be done similarly to the analysis done in the previous sections. The difference lies in the fact that once this construction is introduced, the 'Ratio' of the poor becomes flatter (Since the incomes in periods 1 and 2 increase by the same amount of X dollars, then the ‘Ratio' must decrease), while the 'Ratio' of the rich becomes steeper with time (by the reverse reasoning). However, as the poor become richer in each subsequent period, and the rich become poorer in each subsequent period, the cost per child changes for both the agents. For the poor agents, the cost lowers as a fraction of their income, and for the rich, the cost gets higher. This means that while the steeper profile of a rich agent induces him to have more children, the higher cost per child induces him to have fewer children. The reverse is true for the poor agents. The question then becomes, which force is stronger. This means that there might be fluctuations in the relationship between the fertility rates of the rich and poor from period to period. Hence, the analysis will have to be done in each period until period t', after which both agents will have the same incomes in both periods, and the economy will essentially look like 'Economy 1' or 'Economy 2', i.e., identical agents. This means that after period t', the notion of rich and poor will no longer exist. In other words, adding this construction to our model may change the paths of certain variables somewhat, but will not change the key ideas and essentials about the importance of the relative 'Ratios' picked up in the model without regression to the mean.

10. Comparison with the Becker and Barro Models

Since there are two very prominent OLG models developed by Becker and Barro in the literature investigating the relationship between income and fertility, it is useful to learn the differences between them and the model proposed in this paper. First, it is useful to evaluate the differences in construction and in ideas. Second, it is useful to evaluate the differences between the results of the models once heterogeneity is introduced to the Becker and Barro models. The second evaluation will be done only for the open economy models.

The main difference in construction between the open economy model, the closed economy model, and the model here, lies in the definition of "Rich" and "Poor". In the models by Becker and Barro, a person with a larger non-labor endowment is rich. Thus, being "Rich" or "Poor" is nothing but pure luck, for example, being born to a particular family, or winning the lottery. In this paper’s model, a person with a larger labor income is rich. This means that being rich may be due to luck, but also due to higher innate ability.

Another difference between the models is that agents in the Barro and Becker models have a utility function that depends on their full lineage, whereas in this model, the agents' utility does not depend on their whole pedigree. Although no one can say with certainty, it seems unrealistic to assume that agents believe they can influence or foresee their children's actions and then optimize accordingly with their own choices. Thus, the Becker and Barro utility function may not be a good representation of the real world.

If one introduces heterogeneity of agents in the open economy model from 1988, making half the agents rich and half poor, the prediction of the model is that rich agents will bear more children than the steady state level in the period of increased wealth, while poor agents will have fewer children than the steady state level in the period of decreased wealth. However, both the rich and poor descendants will revert back to the steady state level in the next period, having the same number of children, i.e., there has been a complete regression to the mean.

In this paper's model, as one could observe, in the open economy there was some ambiguity as to which agent would bear more children (although Case 3 was argued for). The result depended on the relative 'Ratios' between them, and the world interest rate. Indeed, there are significant qualitative differences between the models, but this should come as no surprise as the definitions and assumptions of the two models are quite different.

11. Conclusion

In the last 300 years income and fertility have been negatively related. Many different economic theories, such as contraceptive knowledge and female wage rates, have tried to provide insight as to why this is so. These theories seem to have been helpful in explaining this phenomenon. However, even though the banking system had begun expanding around the same time this negative relationship appeared, no theory or paper has looked at the effects credit markets have had on fertility decisions of rich and poor agents.

Through the use of a two-period OLG model with rich and poor agents, it was found that in the absence of credit markets, rich agents have more children than poor agents because of the lower real costs per child rich agents face compared to poor agents. However, it turned out that when agents are granted access to credit, poor agents have more children than rich agents if their second period to first period income is high enough relative to that of the rich, as it induces them to borrow from the rich and use those resources to raise more children, offsetting the higher real cost per child they face. As became evident, the trends observed in 18th century Europe resemble Case 3 of 'Economy 4'. This may indicate that a part of the negative relationship observed between income and fertility for the last 300 years has been due to expansion of the banking system.

That being said, the model is obviously not an ideal representation of the real world. One of its main drawbacks is that even though rich agents are spending nominally more on each child compared to poor agents, they are still receiving the same benefit from each child, *ceteris paribus*, as poor agents do. It seems more realistic that parents who spend more on their children should have, on average, a greater return from them, i.e., greater utility; otherwise it would make more sense for them to use those resources elsewhere, such as on their own consumption. Therefore, a natural extension of the model would be to allow agents to choose the "quality" of their child, and the "quantity" of children they want, in order to create a more realistic economic environment.

References

Becker, G., S. (1981). A Treatise on the Family. Cambridge: *Harvard University Press.*

Becker, G. S., and R. J. Barro. 1988. "A Reformulation of the Economic Theory of Fertility." *The Quarterly Journal of Economics* 103: 1-25.

Becker, G. S., and R. J. Barro. 1989. "Fertility Choice in a Model of Economic Growth." *Econometrica* 57: 481-501.

Coale, A., J., and S. C. Watkins. 2001. *The Decline of Fertility in Europe*. Princeton: Princeton University Press.

Goldewijk, K. K. 2005. "Three Centuries of Population Growth: A Spatial Referenced Population (Density) Database for 1700-2000. *Population and Environment* 26: 343-367.

Hildreth, R. 2001. *The History of Banks: to Which Is Added, a Demonstration of the Advantages and Necessity of Free Competition in the Business of Banking*. Kitchener: Batoche Books.

Jones, L. E., and M. Tertilt. 2007. "An Economic History of Fertility in the U.S.: 1826-1960. Forthcoming in the book *Frontiers of Family Economics*, edited by Peter Rupert.

Lino, M. 2009. *Expenditures on Children by Families, 2009*. Washinton, DC: US Department of Agriculture, Centre for Nutrition Policy and Promotion.

Malthus, T. J. 1798. *An Essay on the Principle of Population*. London: J. Johnson.

Mincer, J. 1963. "Market Prices, Opportunity Costs, and Income Effects." In *Measurement in Economics: Studies in Mathematical Economics and Econometrics in Memory of Yehuda Grunfeld*, edited by C. F. Christ et al., 67-83. Stanford, CA: Stanford University Press.

Scholtz, J. K., and A. Seshardi. 2007. "Children and Household Wealth." Michigan Retirement Research Centre. Working Paper 2007-158.

Solon, G. 2002. "Cross-Country Differences in Intergenerational Earnings Mobility." *Journal of Economic Perspectives* 16: 59-66.

Appendix

Notation

c - Consumption

n - Number of children.

l - Borrowings (Positive values), or lending (Negative values)

r - Interest rate.

Reading the notation

$c^{r^1}_{t[t]}$ - The superscript t indicates the period in which the agent who consumes was born. The second superscript t in the brackets, i.e. [t], indicates that the consumption is done in period t. The superscript r on top indicates that the consumption belongs to a rich agent (If the superscript p is used it means that it is a poor agent's consumption). The superscript 1 above the superscript r indicates that the consumption is done in 'Economy 1' (If the numbers 2, 3, or 4, are used it indicates the consumption is done in the other economies, respectively).

$n^{r^1}_t$ - The superscript t indicates the period in which the agent, who has the children, was born (The children are however, actually born in period t+1). The superscript r on top indicates that the children belong to a rich agent (If the superscript is used p it means that it a poor agent's children.). The superscript 1 above the superscript r indicates that the children live in 'Economy 1' (If the numbers 2, 3, or 4 are used it indicates the children live in the other economies, respectively).

$l^{r^1}_t$ - The superscript t indicates the period in which the loan was taken or made. The superscript r on top indicates that the loan was given or taken by a rich agent (If a p is used it means that the loan was taken or given by a poor agent.). The superscript 1 above the r indicates that the loan was taken or given in 'Economy 1' (If the number 2, 3, or 4 is used, then it indicates the loan was given or taken in the other economies, respectively).

$r^{r^1}_t$ - The superscript t indicates the period in which the interest rate was charged. The superscripts r and 1 on top indicate that the interest rate was in the rich economy, 'Economy 1' (If p, and 2 are used it means it was the interest rate of the poor economy, 'Economy 2'.). If the interest rate is denoted by r^3_t or has a superscript 4 on top, it means that it is the interest rate in 'Economy 3' or 'Economy 4', respectively.

Proof 1

Recall that:

$$\frac{\partial u_t^{r1}}{\partial c_{t[t]}^{r1}} = (c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{\beta} + \beta c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-1)}(-\frac{dF}{L})\text{. Then,}$$

$$\frac{\partial(\frac{\partial u_t^{r1}}{\partial c_{t[t]}^{r1}})}{\partial c_{t[t]}^{r1}} = \beta c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-1)}(-\frac{dF}{L}) + \frac{\partial(\frac{\partial u_t^{r1}}{\partial c_{t[t]}^{r1}})}{\partial n_t^{r1}}\frac{\partial n_t^{r1}}{\partial c_{t[t]}^{r1}} = \beta c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-1)}(-\frac{dF}{L}) +$$
$$[\beta(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-1)} + (\beta-1)c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-2)}(-\frac{dF}{L})](-\frac{dF}{L})$$

But, $\beta c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-1)}(-\frac{dF}{L}) < 0$ since $\beta, c_{t[t]}^{r1}, c_{t[t+1]}^{r1}, n_t^{r1}, d, F$, and $L > 0$ by assumption. Also, $\beta(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-1)} + (\beta-1)c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-2)}(-\frac{dF}{L}) > 0$ by the same reasoning. Thus, when multiplied by $(-\frac{dF}{L})$ it makes the expression negative. Therefore, the second derivative is negative, and the first order condition maximizes the agent's utility.

Now, recall also that:

$$\frac{\partial u_t^{r1}}{\partial c_{t[t+1]}^{r1}} = \psi c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{(\psi-1)}(n_t^{r1})^{\beta} + \beta c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-1)}(-\frac{dF}{r_t^{r1}L})\text{. Then,}$$

$$\frac{\partial(\frac{\partial u_t^{r1}}{\partial c_{t[t+1}^{r1}})}{\partial c_{t[t+1]}^{r1}} = \psi(\psi-1)c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{(\psi-2)}(n_t^{r1})^{(\beta)} + \psi\beta c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{(\psi-1)}(n_t^{r1})^{(\beta-1)}(-\frac{dF}{r_t^{r1}L}) + \frac{\partial(\frac{\partial u_t^{r1}}{\partial c_{t[t+1]}^{r1}})}{\partial n_t^{r1}}\frac{\partial n_t^{r1}}{\partial c_{t[t+1]}^{r1}} =$$
$$\psi(\psi-1)c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{(\psi-2)}(n_t^{r1})^{(\beta)} + \psi\beta c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{(\psi-1)}(n_t^{r1})^{(\beta-1)}(-\frac{dF}{r_t^{r1}L}) +$$
$$[\beta\psi c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{(\psi-1)}(n_t^{r1})^{(\beta-1)} + \beta(\beta-1)c_{t[t]}^{r1}(c_{t[t+1]}^{r1})^{\psi}(n_t^{r1})^{(\beta-2)}(-\frac{dF}{r_t^{r1}L})](-\frac{dF}{r_t^{r1}L})$$

Since r_t^{r1} is positive for any period t by assumption, this expression is negative by the same reasoning applied to the previous case.

Steady States

One can express $r_t^4 = f(r_{t-1}^4)$. Let $\chi_t = \frac{N_t^{p^4}}{N_t^{r^4}}$, and so $\frac{N_t^{p^4}}{N_t^{r^4}} = \chi_{t-1}\frac{n_{t-1}^{p^4}}{n_{t-1}^{r^4}} = \frac{N_{t-1}^{p^4}}{N_{t-1}^{r^4}}\frac{n_{t-1}^{p^4}}{n_{t-1}^{r^4}}$.

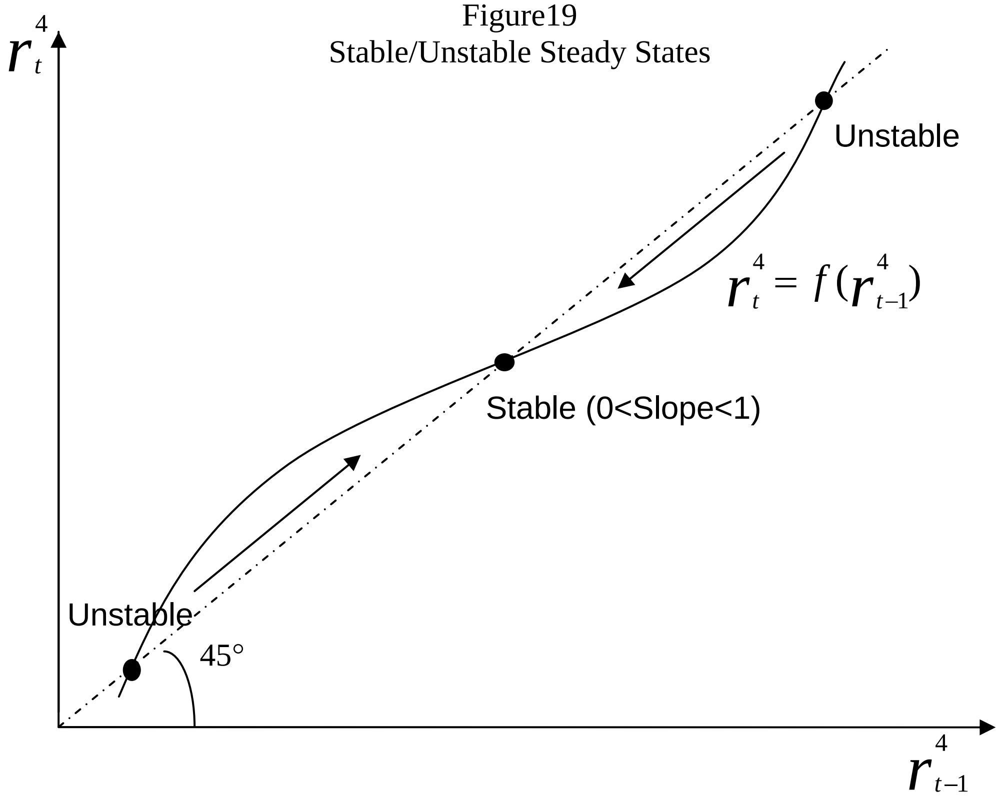
Now, recall that $r_t^4 = \frac{(1+\beta)}{\psi}\frac{(N_t^{p^4}K + N_t^{r^4}M)}{(N_t^{p^4} + N_t^{r^4}L)}$, $n_t^{p^4} = \frac{\beta F(K + r_t^4)}{(\beta+\psi+1)r_t^4}$, and $n_t^{r^4} = \frac{\beta Fd(M + r_t^4 L)}{(\beta+\psi+1)r_t^4 L}$.

Thus, $\frac{N_t^p}{N_t^r} = \frac{M(1+\beta) - r_{t-1}^4\psi L}{r_{t-1}^4\psi - K(1+\beta)}$ and by substituting one can find that

$$r_t^4 = \frac{Md(1+\beta)(M + r_{t-1}^4 L)(r_{t-1}^4\psi - K(1+\beta)) + KL(1+\beta)(K + r_{t-1}^4)(M(1+\beta) - r_{t-1}^4\psi L)}{\psi L(K + r_{t-1}^4)(M(1+\beta) - r_{t-1}^4\psi L) + Ld(M + r_{t-1}^4 L)(r_{t-1}^4\psi - K(1+\beta))} = f(r_{t-1}^4).$$

Now, one can solve for the steady states by replacing r_{t-1}^4 with r_t^4, and solving for r_t^4. One will get a cubic expression equal to 0. One solution will be $r_t^{r^1}$, another will be $r_t^{p^2}$, and the third will be the interest rate where $\eta^4 = n_t^{p^4} - n_t^{r^4} = \frac{\beta F}{(\beta+\psi+1)r_t^4}(K - \frac{M}{L}d) + \frac{\beta F}{(\beta+\psi+1)}(1-d) = 0$. Once the three steady states are found one can find their order from smallest to largest (Note that the cases where there is only one root the stable steady state is the that root, and where there are only two roots the steady states they will be equal $r_t^{r^1}$ and $r_t^{p^2}$, and as was analyzed in the different cases above either, either $r_t^{r^1}$ or $r_t^{p^2}$ will be the stable steady states). Then, one can differentiate $r_t^4 = f(r_{t-1}^4)$ with respect to r_{t-1}^4, and evaluate the derivative at the middle steady state interest rate. If the slope is between 0 and 1 then the stable steady state is the middle steady state. It turns out that this is the case for 'Economy 4'. If one does this procedure for 'Economy 4', one would get the following graph when there are three distinct roots:

Figure19
Stable/Unstable Steady States



This means that the convergence to the steady state is either from below or from above depending on the starting point. Note that middle steady state is stable since the slope is between 0 and 1 (as the graph shows).