# Asset Reallocation and the Business Cycle: A Threshold-Based Regime Switching Approach

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## Abstract

In this paper I examine the optimal reallocation of assets across the business cycle through the lens of modern portfolio theory. By examining past results using recent data, I confirm the earlier finding that mean-variance efficiency can be increased by the reallocation of wealth across economic regimes. Also, I propose an alternative method for asset allocation based on the use of a threshold generalized autoregressive conditional heteroskedasticity model using the CBOE VIX Index as an indicator of economic regime change. Using a selection of mutual funds, observed asset allocation is compared to this model which shows greater promise as an explanatory tool than the standard portfolio model.



## Acknowledgements

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## Introduction

Modern portfolio theory has provided a basic mathematical framework for the optimal allocation of financial assets by using past mean returns and standard deviation of returns as criteria for maximizing the inherent risk and reward relationship. Using past average returns and their associated standard deviation, as is done in the basic model, presents several problems for the maximization of the risk and reward relationship. First is the dependability of past data in a process whose purpose is essentially forward looking. Past performance of an asset, or a class of assets, can only be strictly interpreted as concerning that past performance, but not necessarily the future returns. Second is the assumption surrounding the underlying distribution of returns. Using only the first and second moment of past returns as a measure of future performance carries the assumption that the asset returns follow the normal distribution.[1] However, as I will demonstrate

[1] A normal distribution can be fully defined by its mean and variance. Conversely, if this is the only information used, then it must be assumed that normality holds.

later, as a general rule, asset returns are characterized by negative skewness and positive excess kurtosis. From this fact it is apparent that the simple use of the mean and variance criteria may be insufficient in determining the risk reward relationship. The use of variance or standard deviation as a measure of risk may in this case be biased towards understating risk for any particular asset. Third, and finally, there is the much studied issue of regime changes which is not included in the basic model. From the basic intuition, which will later be confirmed, that risk and returns differ in periods of economic recession, the use of past data without regard for the current economic regime brings into questions the reliability of said data, and by extension, the optimization of the risk reward relationship generated by a model that does not take this into account.

This paper will examine a solution to the third problem as demonstrated by Brocato and Steed (1998) which uses business cycle turning points published by the National Bureau of Economic Research (NBER) as an indicator of a change in economic regime using data from 1973 through 1993 inclusive. Then it will attempt to replicate their results for more recent data encompassing both the "dot-com" expansions and ensuing recession, as well as the recent financial crisis and current recovery. As well, a new method of indicating an economic regime change will be proposed that takes into account a solution to the aforementioned second problem by using the Chicago Board Options Exchange (CBOE) VIX index to indicate changes in regime.

## Literature Review

The original basis for portfolio optimization is Harry Markowitz's article, "Portfolio Selection" (1952), which provides a framework for the selection of portfolio of assets based on their mean return and variance. From this one can solve for an efficient allocation which maximizes the expected return of the portfolio for a given variance, or conversely, minimizes the variance for a given expected return. Mathematically, this problem is represented as follows.

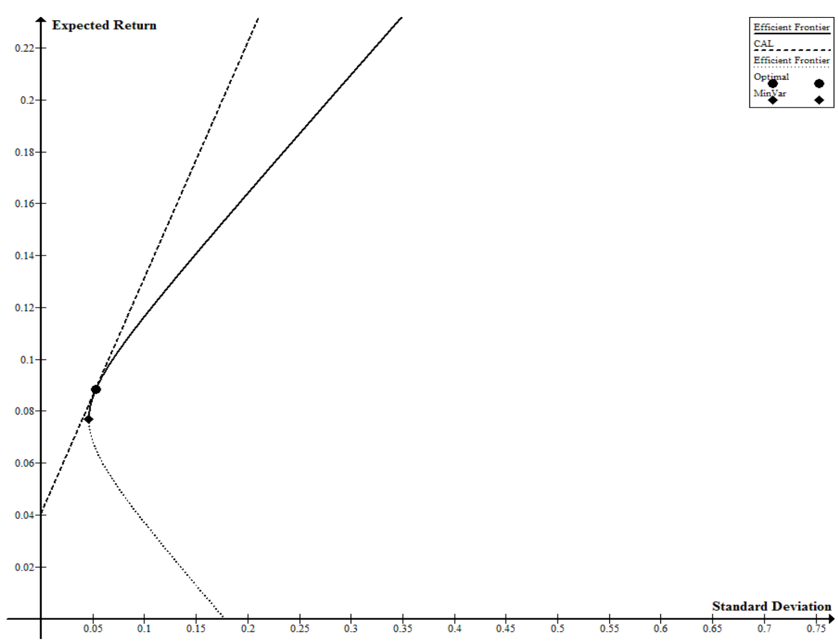
$$Minimize\ \sigma_p^2 = \sum_{i=1}^{N}\sum_{i=1}^{N} x_i x_j \sigma_{ij}\ \ w.r.t.\{x_2, x_2, .., x_N\}$$

$$\text{s.t.}\left\{\sum_{i=1}^{N} x_i = 1, \sum_{i=1}^{N} x_i \mu_i = \mu_p^*\right\}$$

The portfolio's variance, $\sigma_p^2$, is minimized with respect to the weights, $x_i$'s, of $N$ assets which, as each weight represents a percentage of the total wealth invested, must sum to one. The objective expected return, $\mu_p^*$, is the portfolio's expected return as calculated by the weighted sum of the expected return of each asset. The set of portfolio variances and expected returns that solves this problem is the efficient set of portfolios. By allocating assets between one risk-free asset, such as US government Treasury Bills, and the optimal portfolio, one can minimize risk for any given return. This optimal portfolio is generated by maximizing the risk-return ratio, known as the Sharpe ratio, represented mathematically below, where $r_f$ is the risk-free rate.

$$\text{Maximize } \frac{\mu_p^* - r_f}{\sigma_p} \text{ w.r.t} \{x_2, x_2, .., x_N\}$$

The reason that the Sharpe Ratio is taken as the objective for the maximization problem, rather than the world of assets without the risk-free asset, stems from what is known as the "separation property" (Bodie, Kane and Marcus 2003, 234). The portfolio selection problem is essentially separated into two parts: the determination of the optimal portfolio, and the optimal allocation of wealth between said optimal portfolio and the risk-free asset. Shown below is a graphical representation of the problem. The efficient frontier represents the set of attainable portfolio allocations that are mean-variance efficient. That is to say that for a given expected return, variance is minimized, and conversely, for a given variance, expected return in maximized. The capital allocation line (labeled CAL) represents the attainable portfolios by mixing between the risk-free asset and the optimal portfolio. The optimal portfolio is one which is on the efficient frontier whose tangent falls through the risk-free rate (assumed to have zero variance). Since the Sharpe Ratio is the slope of the CAL, maximizing the Sharpe Ratio produces an optimal trade-off between risk and reward, or expected return and variance. The solution to this maximization problem thus generates the optimal portfolio which investors hold in conjunction with the risk-free asset according to their preference for risk. One can see that this mixing between the optimal portfolio and the risk-free asset is superior regardless of preference, since an investor who desires a greater expected return will take on less risk along the CAL rather than the efficient frontier, and conversely, an investor who desires less risk will attain a higher expected return.



Brocato and Steed's (1998) NBER turning points are used to indicate a change in regime in order to achieve a higher mean-variance efficiency across the business cycle. The concept of regime is an important topic to consider. From the observation that "returns and volatility vary through time" (Ang and Bekaert 2004, 97) when examining optimal asset allocation, it is undoubtedly important to consider how the underlying characteristics of returns vary through time. The concept of regime change approaches this problem by grouping data into periods that are alike in these characteristics. Many different approaches have been taken in the literature by using different indicators of a switch in the underlying regime. One such approach is that of Brocato and Steed (1998) by using the business cycle definitions of recession and expansion as two possible regimes. Others have used the monetary cycle or a simple random switching process. One such solution to the regime problem introduced by Hamilton and Susmel (1994) and adapted by Ang and Bekaert (2002) as well as Canarella and Pollard (2007), while not in the context of portfolio optimization, is the use of a Markov chain process that describes the probability of switching from one regime to another. By estimating the probability of switching to regime two, given that one is currently facing regime one, and vice versa, these authors allow a model of returns to take on different parameters, depending on the underlying economic regime. Another aspect of Hamilton and Susmel's paper is the use of an ARCH (autoregressive conditional heteroskedasticity) process and the underlying model behind returns in this regime switching framework.

By comparing the optimal portfolio generated using their full data set as a comparison, Brocato and Steed find that those generated using exclusively expansion or recession data both achieved greater mean-variance efficiency, supporting the intuition that risk and return differ over the business cycle. Not only do they find that expected returns are lower, but also that standard deviations are greater, causing the optimal portfolio for the entire period to differ significantly from that under either regime (144). Also, they find that the gain in efficiency associated with reallocation during recession, in comparison to the use of the full data set, is greater than that from reallocating during expansion. One fundamental issue with this paper is the use of NBER business cycle turning points as an indicator of economic regime change. While these turning points are authoritative indicators of when a recession has occurred, they are only known *ex post*. Brocato and Steed show that portfolio optimization requires the reallocation of assets across the business cycle, yet this is essentially a forward looking task: an indicator that is currently unknown is of little use. However, this paper has been widely cited as a benchmark showing that that reallocation based on economic regime is necessary for mean-variance efficiency.

The ARCH model, originally introduced by Robert Engle (1982), does not assume a constant variance, but rather one that changes over time according the past returns. It was later altered to incorporate past values of the variance as a factor in the current variance in the GARCH model (generalized autoregressive conditional heteroskedasticity). The significance of this model is that it is able to account for several general empirical attributes of asset returns, as described by Aydemir (2002,4): fat tails, volatility clustering, and leverage effects. "Fat tails" refers to the excess kurtosis in the distribution of asset returns which implies that returns extremely above or below the mean are more likely to occur than if returns followed the normal distributions. As noted earlier, the use of a constant variance implies that these extreme values occur at a lower frequency. The implication for the selection of the optimal portfolio is that risk is understated. Volatility clustering refers to the observation that periods of high (low) volatility are more likely to be followed by periods of high (low) volatility. The GARCH model captures this since the estimated variance is in part a function of the preceding variance, a feature that clearly cannot be captured by the assumption of a constant variance. The leverage effect refers to the observation that returns are negatively correlated with volatility, such that periods of high (low) returns tend to be less (more) volatile. The use of the ARCH model takes this feature into account by including past values of returns as a factor in the variance of the series.

To better model these patterns, several flavours of ARCH models have been developed. One such model is the TGARCH model (threshold generalized autoregressive conditional heteroskedasticity) by Zakoian (1994) that allows for changing parameters of the underlying GARCH model based on an observable threshold variable, such that beyond some critical value the model takes on one set of parameters, and below, another. This paper will focus on the implementation of the TGARCH model by Wu (2010) that uses the Chicago Board Options Exchange (CBOE) VIX index as a threshold variable. The VIX index is a measure of market volatility that is measured by the implied volatility of stock options sold on the member stocks of the S&P500 index. This is an attractive threshold variable on account of another of Aydemir's observed features of asset returns: co-movements in volatility. By using this variable that represents general market volatility, this feature can be captured in the model to allow for the conditional volatility of a series of returns to change in response to market volatility.

## The Data

Following the general framework of Brocato and Steed, data were gathered using several broad classes of assets to represent those available to investors. These assets are grouped as follows:

| **Asset group** | **Data** |
|---|---|
| Common equities | S&P500 – total return index |
| Small capitalization equities | S&P600 – total return index |
| Foreign equities | MSCI EAFE (Europe, Australia, Far East) – total return index |
| Government Bonds | Barclays Capital US Treasury Aggregate Index – total return |
| Real estate | National Association of Real Estate Investment Trust – total return index |
| Precious metals | Gold and Silver |
| Cash equivalents | Bank of America 1-3 month US T-Bill – total return index |

All data were gathered on a weekly basis from 1990 through 2010 inclusive from Datastream International (Tomson Reuters 2011). Each set of data shows the total return in percentage terms for internal comparability.[2] Appendix A shows descriptive statistics for these series for the full data set, as well as recession and expansion as defined by the NBER business cycle turning points (National Bureau of Economic Research 2011), and Appendix B shows the associated variance-covariance matrices.

It is notable that this dataset mirrors the earlier observations of Brocato and Steed in that the series of returns differs in nature from recession to expansion, both on an individual basis, and in respect to the covariance between them. The feature of negative skewness is most apparent in the recession data on account of mean negative returns in the case of some assets during recession. Positive excess kurtosis is a striking feature of the full period data, yet somewhat subsides in both the expansion and recession data. This is evidence of the need to differentiate between economic regimes as the extreme deviations from the mean in recession appear to exacerbate the "fat-tails" when viewed using the full data set. Also, the covariance structure changes across regimes, showing higher correlation in recession than would be expected from the full data set. Listed in Appendix C are the NBER business cycle turning points used to determine the regime.

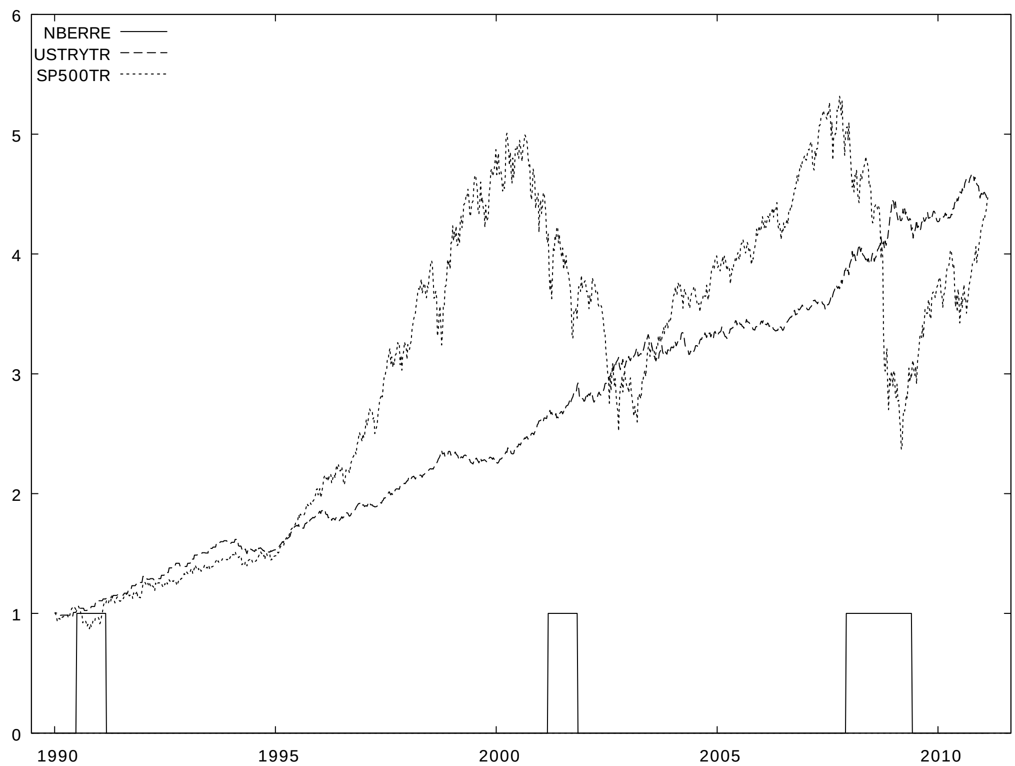
## Optimization Using NBER Turning Points

Using Brocato and Steed's method, I have used the full data set as a benchmark case for determining whether the risk and return relationship can be improved by reallocation over the business cycle. Using the NBER turning points to determine the regime, I have applied Sharpe's optimal portfolio selection technique to construct the mean-variance efficient portfolio using the average annualized three month US Treasury Bill rate as the risk free asset.

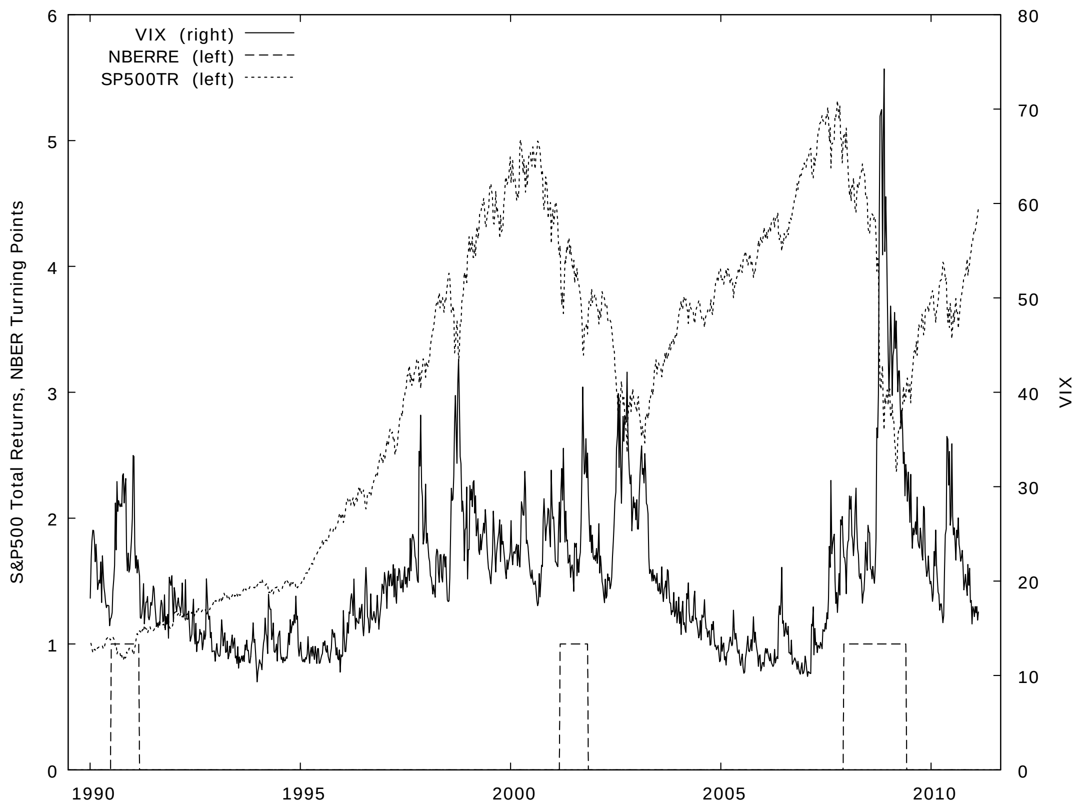
| | **Full Period** | **Recession** | **Expansion** |
|---|---|---|---|
| **Common equities** | 0.00% | 0.00% | 0.34% |
| **Small capitalization stocks** | 12.69% | 0.00% | 21.02% |
| **Foreign equities** | 0.00% | 0.00% | 0.73% |
| **Precious metals** | 2.41% | 5.04% | 1.99% |
| **Real estate** | 0.00% | 0.00% | -2.76% |
| **Government Bonds** | 84.9% | 94.96% | 78.69% |
| **Sharpe Ratio** | **0.813** | **1.543** | **1.129** |
| **Return** | 7.9% | 9.9% | 9.2% |
| **Standard Deviation** | 0.047 | 0.061565 | 0.050 |

[2] For example, the S&P500 total returns index includes the reinvestment of dividends in the index. Likewise, Barclays' US government bonds index includes both the market value of bonds as well as the reinvestment of coupon payments.

From this data it is clear that Brocato and Steed's conclusion that reallocation based on NBER turning points does lead to greater mean-variance efficiency. Featured in Appendix D is a graphical representation of the portfolio allocation problem for the full period, recession, and expansion. A graphical comparison of the Sharpe Ratios using these data is featured in Appendix E. Note that a steeper curve represents a more desirable risk-return relationship. It is notable that common equities are absent or nearly absent from the optimal portfolio in each period. One possible explanation for this result is the domination of the recent financial crisis in the data. Shown below is the result of a $1 investment in US Government bonds (dashes) and the S&P500 index (dots) at the beginning of 1990 with the NBER turning points (solid) showing a value of one during recession and zero otherwise.




From the perspective of mean-variance efficiency, although the initial investment today would yield the same amount, it is clear that the variation in returns from equities greatly exceeds those from government debt. However, when viewed in conjunction with the NBER turning points, a different story emerges. Specifically for the period between 2000 and 2003 the drop in this broad US equity index is only partially captured by the NBER turning points. The most plausible explanation for this result is the predominance of factors other than stock prices in determining dates of the business cycle as a recession is defined as a fall in gross domestic product; stock prices are typically considered a leading indicator, and although related, their rise or fall need not be associated with expansion or recession. As noted earlier, the leverage effect implies that volatility and returns tend to be negatively correlated, which leads to the use of the VIX index as a threshold variable rather than NBER turning points. As a measure of market volatility, the VIX provides a much more robust indicator variable as it updated continuously and can take on a range of values, in contrast to the binary nature of NBER turning points. Shown below is the result of one dollar invested in the S&P500 index in 1990 superimposed against the VIX index and the NBER turning points. It is readily apparent that periods of volatility are much more closely captured by the VIX than the NBER turning points.




## Threshold GARCH

In determining the optimal level of VIX index to use as a threshold I have followed the TGARCH model explicated by Wu (2010). In her paper a series of demeaned returns, $r_t$, is modeled as follows.

$$r_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \begin{cases} \omega_0 + \alpha_0 r_{t-1}^2 + \beta_0 \sigma_{t-1}^2 \text{ if } y_{t-1} < y^* \\ \omega_1 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \text{ if } y_{t-1} \geq y^* \end{cases}$$

Here the variance of the returns process is able to change across time according in proportion to the pervious return and variance, to a differing degree based on an exogenous threshold variable, $y$, which in this case is the VIX index. To estimate this model and the optimal threshold level, the TGARCH model must be estimated several times for different threshold values using maximum likelihood estimation for the following log-likelihood equation given by Wu (14).

$$lnL_T(\theta) = -\frac{1}{2}\sum_{t=1}^{N} ln\sigma_t^2 - \frac{1}{2}\sum_{t=1}^{N}\frac{r_t^2}{\sigma_t^2}$$

$$\text{where } \theta = (\omega_0, \omega_1, \alpha_0, \alpha_1, \beta_0, \beta_1)$$

By using 40 different values corresponding with the 2.5 percentile increments of observed weekly values for the period 1990 through 2010 inclusive, the optimal threshold value for each series of returns was obtained. The values associated with each 2.5 percentile increment are shown in Appendix F, and the relative frequency distribution and autocorrelation function of the VIX index are shown in Appendix G. The relative frequency distribution shows the high frequency of observations around the median of 19.03, and the low incidence but presence of extreme positive values, some in excess of 5 standard deviations from the mean. The autocorrelation function demonstrates that the VIX index captures the high persistence of volatility, as shown by the highly significant lags up to 30 periods in the past.

The application of the aforementioned process to estimate the threshold variable generated the following results of the optimal threshold, coefficient estimates, and standard errors.[3]

| Series | Threshold Value | Percentile Rank | $\omega_0$ | $\omega_1$ | $\alpha_0$ | $\alpha_1$ | $\beta_0$ | $\beta_1$ |
|---|---|---|---|---|---|---|---|---|
| **SP500** | 17.93 | 45.0% | 0.00014 | -4.59E-05 | -0.00047 | -0.00547 | 0.21602 | 0.666372 |
| | | | *2.42E-05* | *2.42E-05* | *0.000764* | *0.000972* | *0.12203* | *0.119477* |
| **SP600** | 28.31 | 87.5% | 0.000188 | 0.000805 | -0.0047 | -0.01154 | 0.633581 | -0.10146 |
| | | | *3.26E-05* | *0.000239* | *0.001083* | *0.004259* | *0.057558* | *0.121535* |
| **MSEAFE** | 24.38 | 77.5% | 0.000234 | 1.97E-06 | -0.00612 | -0.00044 | 0.381468 | 0.46118 |
| | | | *2.52E-05* | *5.00E-05* | *0.000397* | *0.000871* | *0.062183* | *0.07032* |
| **GOLD** | 18.53 | 47.5% | 0.00083 | -0.00067 | -0.0019 | 0.003144 | -0.9035 | 1.63021 |
| | | | *4.76E-05* | *9.12E-05* | *0.000516* | *0.000765* | *0.063077* | *0.149186* |
| **NARETI** | 23.22 | 72.5% | 1.72E-06 | -8.23E-06 | -0.00063 | -0.00452 | 1.00356 | -0.02033 |
| | | | *5.75E-07* | *1.56E-06* | *0.000188* | *0.000383* | *0.00181* | *0.00314* |
| **USTRY** | 28.31 | 87.5% | 3.65E-05 | 0.00012 | -0.00081 | 0.000257 | 0.060986 | -0.89853 |
| | | | *6.73E-06* | *2.26E-05* | *0.000319* | *0.000766* | *0.148342* | *0.210638* |

[3] Standard errors are listed under the corresponding coefficient estimate, italicized and reduced font size.

The coefficient estimates do not correspond directly to those listed in the model above, as those with subscript 1s are akin to dummy variables. For example, to obtain the true $\alpha_1$, one must sum $\alpha_0$ and $\alpha_1$.[4] An examination of these results shows that there are few situations where the coefficient on a factor or the constant changes sign by moving across the estimated threshold. Such cases are indicated by underlined values. Generally, however, the results show evidence of persistence in volatility, through positive betas, as well as evidence of the leverage effect as shown by the mostly negative alphas. Since the optimal threshold differs across the series, I have chosen to initially approximate the six intervals into three groups: 0-45$^{th}$ percentile, 45$^{th}$ -80$^{th}$ percentile, and 80$^{th}$-100$^{th}$ percentile henceforth A, B, and C, respectively. Restricting the dataset based on these groupings and solving for the portfolio optimization problem generates the following results. Descriptive statistics and variance-covariance matrices are shown in Appendix H and I, respectively.

| | **Full Period** | **Recession** | **Expansion** | **A** | **B** | **C** |
|---|---|---|---|---|---|---|
| **SP500** | 0.00% | 0.00% | 0.34% | 50.95% | 0.00% | 0.00% |
| **SP600** | 12.69% | 0.00% | 21.02% | 16.78% | 57.02% | 0.00% |
| **MSEAFE** | 0.00% | 0.00% | 0.73% | 10.27% | 0.00% | 0.00% |
| **GOLD** | 2.41% | 5.04% | 1.99% | 0.00% | 0.00% | 6.55% |
| **NARETI** | 0.00% | 0.00% | -2.76% | 5.13% | 0.00% | 0.00% |
| **USTRY** | 84.90% | 94.96% | 78.69% | 18.58% | 42.98% | 93.45% |
| **Sharpe Ratio** | **0.813** | **1.543** | **1.129** | **2.204** | **2.017** | **0.973** |
| **Return** | 0.079 | 0.099 | 0.092 | 0.22 | 0.237 | 0.103 |
| **Standard Deviation** | 0.047 | 0.062 | 0.050 | 0.08 | 0.093 | 0.061 |

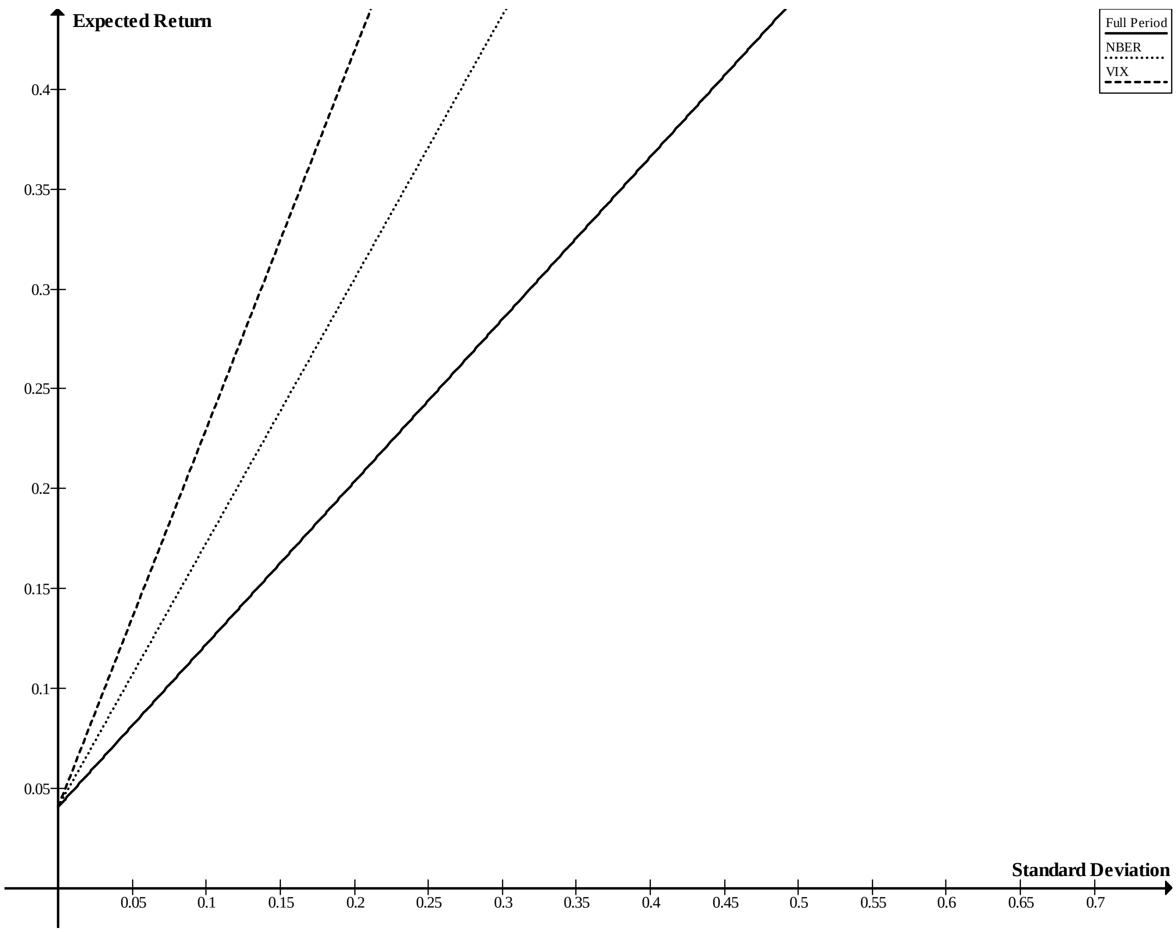
As shown by the Sharpe ratio, the use of the VIX index as a threshold variable for asset reallocation does represent an improvement in the mean-variance efficiency in comparison to using the full data set, as the Sharpe ratio is strictly higher. However, grouping C results in a lower Sharpe ratio than reallocation based on the NBER turning points. One possible explanation for this result is that grouping C is a more accurate representation of the risk to return relationship in periods of high volatility, which have by definition occurred 20% of the time for the period in question. A time weighted average of the Sharpe ratios using NBER turning points and the VIX index shows the VIX index method to be superior.

$$SR_{NBER} = w_{RE}SR_{RE} + w_{EX}SR_{EX} = 1.32$$

$$SR_{VIX} = w_A SR_A + w_B SR_B + w_C SR_C = 1.89$$

[4] I have not done this calculation above since the standard errors correspond to the change in the coefficient.

A graphical representation of the portfolio optimization problem for the VIX groupings is featured in Appendix J, and a comparison of the Sharpe Ratios for the three VIX based groupings is featured in Appendix K. Shown below is a comparison of the Sharpe Ratios for the full period, the NBER turning points, and the VIX based groupings using a time weighted average. As can clearly be seen, the risk-reward relationship is superior using the VIX groupings: more so than using the NBER turning points.




## Fund Data

It has been shown that the use of the VIX index as a threshold confers key advantages over the use of NBER turning points alone. The VIX is an observable indicator and, as it has been shown, is a method that allows for an increase in the risk-return relationship in the portfolio optimization problem. Apart from its use as a tool for investors, it is important to examine its efficacy as a method for understanding investment behaviour. As noted earlier, a peculiar aspect of the use of NBER turning points is the resulting lack of equities in the optimal portfolio. This runs counter to intuition, since this result would imply a drastically lower participation rate in the equity markets. One hypothesis for this observed characteristic is that the period studied has been dominated by the financial crisis which, due to its extent, adversely affects the data. Yet fully accepting this explanation implies the existence of large anomalies need not be incorporated into an analysis of investor behaviour, which is essentially the pricing of risk. For this reason, the following section will be devoted to the use of the VIX threshold model as a means of explaining investor behaviour.

To examine how investors allocate wealth between assets, it is necessary to find a suitable proxy as a source for data in the absence of detailed information. For this purpose, mutual fund allocation will be used as a general proxy for the following reasons. First is availability; funds available to the public typically publish this data for prospective clients, and furthermore are required to file such reports to the Securities and Exchanges Commission. Second is the basic nature of funds. Mutual funds are designed to be held in place of the constituent financial assets on account of the expertise of the fund managers, the increased diversification ability inherent in a larger pool of wealth, and the lower transaction costs to the individual. Since a fund is designed to be held in place of its constituent financial assets, it is a suitable proxy for the optimal portfolio since each individual would presumably hold it only in proportion to their desired level of risk. However, there are several key problems with its use as a proxy. The first is the representativeness of any one fund. There exist many different types of funds available for many different purposes beyond the scope of portfolio theory. For example, some funds carry a specific time horizon and are marketed specifically as a retirement savings product for those who plan to retire in a specific time frame. These funds will tend to become more risk averse as the time horizon approaches. Others are not meant as a general investment proxy, but rather as a method to invest within a specific sector evenly, or in financial assets that are exclusively available to institutional investors on account of size or access restrictions.[5] Thus the use of mutual fund composition as a proxy for an observed optimal portfolio comes with the caveat that it is only a suitable estimate insofar as it can be seen as the only financial asset that an investor holds.

Since this caveat admittedly cannot be fully satisfied, this analysis cannot be said to give definitive answers. However, through the use of widely held and well diversified funds which are explicitly marketed as a general substitute for financial market participation, a reasonable degree of accuracy can be achieved. By selecting only those funds that fit these criteria, I have assembled a sample group of US mutual funds to conduct an analysis of how investment data fit the VIX threshold model. The asset allocation data for a selection of these funds are shown in Appendix L. The average results were approximated such that they fit into the categories used in the portfolio allocation problem. For example, in the fund data US equities were reported as a single asset class,

[5] For example, some issues of sovereign debt may only be available in large minimum denominations, or investment in foreign markets may not be directly available to individuals.

while in the portfolio allocation problem they were split into both small and large capitalization equities. For the purpose of the comparison, the reported datum was divided evenly between the two.[6] Shown below is a comparison between the observed fund data and the estimated optimal portfolio using the full period, the expansion data,[7] and the first grouping based on the VIX threshold.

| | Full Period | Expansion | A | Fund Data |
|---|---|---|---|---|
| **SP500** | 0.00% | 0.34% | 50.95% | 22.72% |
| **SP600** | 12.69% | 21.02% | 16.78% | 22.72% |
| **MSEAFE** | 0.00% | 0.73% | 10.27% | 17.22% |
| **GOLD** | 2.41% | 1.99% | 0.00% | 3.81% |
| **NARETI** | 0.00% | -2.76% | 5.13% | 3.81% |
| **USTRY** | 84.90% | 78.69% | 18.58% | 23.21% |
| **Sharpe Ratio** | **0.813** | **1.129** | **2.204** | **2.064** |
| **Return** | 0.079 | 0.092 | 0.22 | 0.18 |
| **Standard Deviation** | 0.047 | 0.050 | 0.08 | 0.068 |

Upon initial inspection the data appear to fit the VIX threshold model quite well, considering the problems associated with using fund data as a proxy. What is clear is that the VIX threshold model more closely fits the data than both the full period estimation and the NBER turning points indicator. This is very promising since it shows the efficacy of this method as a tool for understanding investor behaviour. An area that this data can be used to explain is the effect of recession on market participation. Using either the standard portfolio model or the NBER turning points as a regime indicator, it appears that information is lost when conducting such analysis. Periods of higher volatility are not fully captured by the NBER turning points and appear to distort the data given their extreme values. This causes “normal” periods to be overshadowed by abnormal events. Since this is not the case in the VIX threshold model, by examining the fit of this data to the model estimate, one can draw some preliminary conclusions with regard to market participation. To the extent that average investment in equities is less than that predicted by the model, there is some indication that there are remaining effects of the financial crisis on investor confidence. The higher than expected proportion of wealth invested in bonds, representing fixed income assets with comparatively little risk, shows that there may be less appetite for risk or general uncertainty surrounding the equity markets. Again, given the limited extent to which fund data can accurately represent investor behaviour, there is a relatively high amount of uncertainty surrounding these conclusions. However the VIX threshold model allows one to begin to examine these issues through the lens of portfolio theory on account of the greater distinction between regimes, while the use of NBER turning points does not.

[6] A similar procedure was applied to the precious metals and real estate categories, as the fund data reported a small but significant “Other” category.
[7] As designated by the NBER turning points.

## Conclusion

Using a more recent dataset than Brocato and Steed's (1998), covering 1990 through 2010 inclusive, their results were shown to hold, demonstrating that the use of NBER turning points as indicators of economic regime were better able to capture the risk-return relationship than the simple use of the full period. However, it was also shown using the same dataset that the use of the CBOE VIX index is better able to capture such changes.

In summary, the use of VIX as a threshold variable to signify changes in regime provides a superior method of approaching the problem of asset allocation across the business cycle. By using an observable indicator variable that is based upon general market volatility, such as the CBOE VIX volatility index, changes in regimes of the risk-return relationship can be better described. This allows for both a greater degree of optimization for a potential investor and a clearer picture of the behaviour of investors on account of the grouping of extreme observations into like groupings. This leaves the general case less affected by periods of extreme volatility, and isolates the periods of extreme volatility to a much greater degree than the use of NBER turning points allows.

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## Appendix A

Descriptive statistics of selected series
Full Period 1990-2010, inclusive

| Series | Mean | Median | Minimum | Maximum | Std. Dev. | Skewness | Ex. kurtosis |
|---|---|---|---|---|---|---|---|
| SP500 | 0.00159438 | 0.003171 | -0.164033 | 0.102211 | 0.023068 | -0.56409 | 4.24635 |
| SP600 | 0.00199247 | 0.005188 | -0.192696 | 0.124239 | 0.029169 | -0.63731 | 3.83228 |
| MSEAFE | 0.000890784 | 0 | -0.160271 | 0.145087 | 0.025842 | -0.60578 | 6.63765 |
| GOLD | 0.00111835 | 0.000966 | -0.123829 | 0.137026 | 0.021576 | 0.030154 | 4.1394 |
| NARETI | 0.00113372 | 0.002411 | -0.244754 | 0.217934 | 0.030584 | -0.8275 | 12.293 |
| USTRY | 0.00139903 | 0.000109 | -0.0323118 | 0.040263 | 0.007165 | 0.269097 | 2.70145 |

Recession (using NBER turning points) 1990-2010, inclusive

| Series | Mean | Median | Minimum | Maximum | Std. Dev. | Skewness | Ex. kurtosis |
|---|---|---|---|---|---|---|---|
| SP500 | -0.00371 | -0.00221 | -0.164033 | 0.096959 | 0.035447 | -0.66433 | 2.76554 |
| SP600 | -0.00406 | -0.0006 | -0.192696 | 0.124239 | 0.044197 | -0.69517 | 2.90179 |
| MSEAFE | -0.00544 | 0 | -0.160271 | 0.145087 | 0.041248 | -0.54938 | 2.85706 |
| GOLD | 0.00173 | 0.001273 | -0.123829 | 0.084346 | 0.033218 | -0.26772 | 0.962456 |
| NARETI | -0.00582 | -0.00168 | -0.244754 | 0.217934 | 0.056938 | -0.67069 | 4.61844 |
| USTRY | 0.001899 | 0.000279 | -0.0202114 | 0.026772 | 0.00898 | 0.144395 | 0.054734 |

Expansion (using NBER turning points) 1990-2010, inclusive

| Series | Mean | Median | Minimum | Maximum | Std. Dev. | Skewness | Ex. kurtosis |
|---|---|---|---|---|---|---|---|
| SP500 | 0.002405 | 0.003544 | -0.09012 | 0.102211 | 0.020383 | -0.15612 | 1.88014 |
| SP600 | 0.002941 | 0.005351 | -0.12239 | 0.116519 | 0.025959 | -0.30663 | 1.34005 |
| MSEAFE | 0.001879 | 0 | -0.13201 | 0.108322 | 0.022396 | -0.22852 | 5.5497 |
| GOLD | 0.001043 | 0.000955 | -0.1027 | 0.137026 | 0.019157 | 0.234009 | 4.79397 |
| NARETI | 0.002221 | 0.002613 | -0.09598 | 0.14264 | 0.023858 | 0.03794 | 4.46146 |
| USTRY | 0.001322 | 9.46E-05 | -0.03231 | 0.040263 | 0.006844 | 0.281749 | 3.51594 |

## Appendix B

Covariance of selected series of returns, annualized

Full Period 1990-2010, inclusive

| SP500 | SP600 | MSEAFE | GOLD | NARETI | USTRY | |
|---|---|---|---|---|---|---|
| 0.027683 | 0.030388 | 0.015045 | 9.84E-05 | 0.024174 | -1.3E-05 | **SP500** |
| | 0.044243 | 0.022239 | 0.002298 | 0.033465 | -3.8E-05 | **SP600** |
| | | 0.034759 | 0.005097 | 0.017993 | -1.1E-05 | **MSEAFE** |
| | | | 0.02421 | 0.003682 | 5.38E-06 | **GOLD** |
| | | | | 0.048639 | -8.2E-06 | **NARETI** |
| | | | | | 5.14E-05 | **USTRY** |

Recession (using NBER turning points) 1990-2010, inclusive

| SP500 | SP600 | MSEAFE | GOLD | NARETI | USTRY | |
|---|---|---|---|---|---|---|
| 0.065338 | 0.076342 | 0.044341 | 0.001212 | 0.078819 | -0.00186 | **SP500** |
| | 0.101575 | 0.068464 | 0.005283 | 0.108036 | -0.00291 | **SP600** |
| | | 0.088474 | 0.014806 | 0.061943 | -0.00254 | **MSEAFE** |
| | | | 0.057379 | 0.009796 | 0.000771 | **GOLD** |
| | | | | 0.168582 | -0.00153 | **NARETI** |
| | | | | | 0.004193 | **USTRY** |

Expansion (using NBER turning points) 1990-2010, inclusive

| SP500 | SP600 | MSEAFE | GOLD | NARETI | USTRY | |
|---|---|---|---|---|---|---|
| 0.021604 | 0.022996 | 0.0102 | -4.5E-05 | 0.015364 | -0.00048 | **SP500** |
| | 0.03504 | 0.014644 | 0.001859 | 0.021548 | -0.00184 | **SP600** |
| | | 0.026081 | 0.003632 | 0.010769 | -0.00025 | **MSEAFE** |
| | | | 0.019084 | 0.002774 | 0.000201 | **GOLD** |
| | | | | 0.029598 | -0.00023 | **NARETI** |
| | | | | | 0.002436 | **USTRY** |

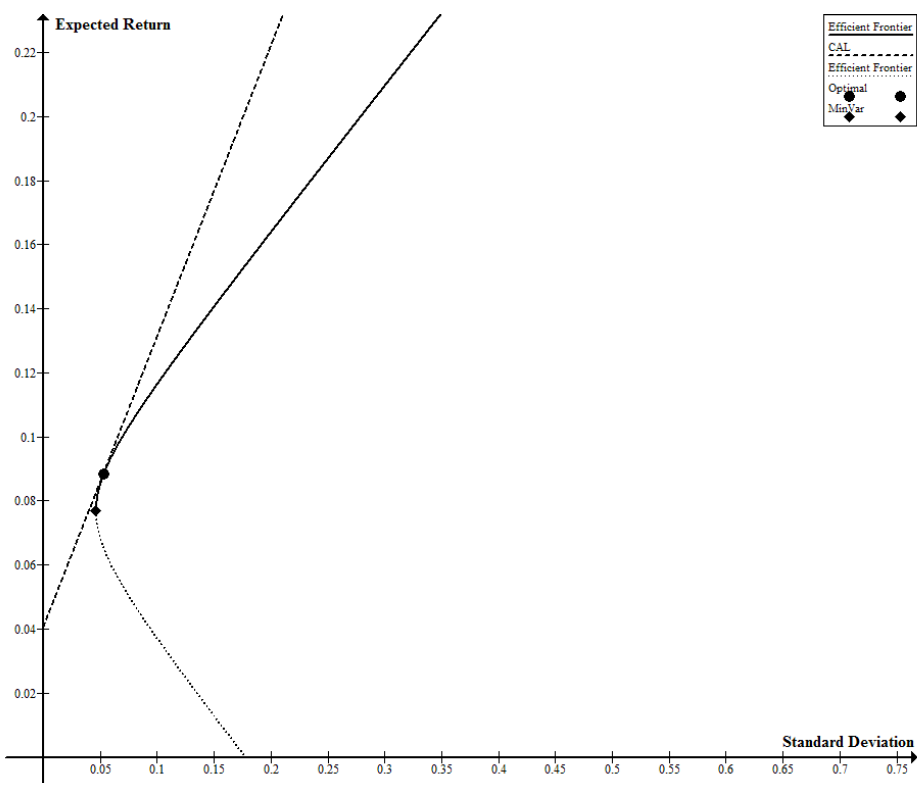
## Appendix C

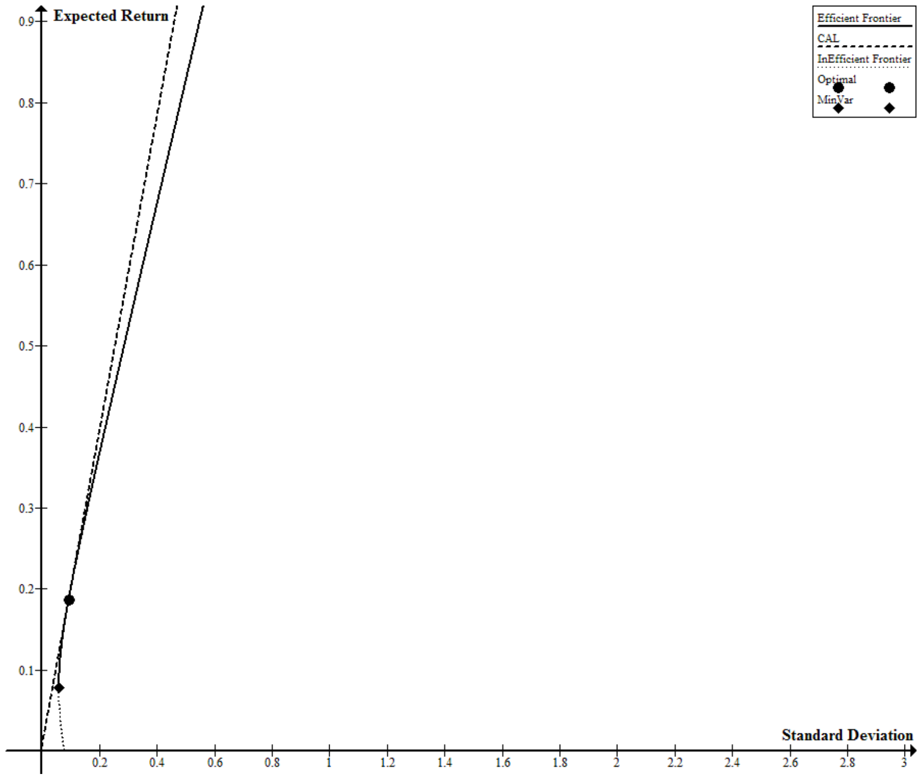
NBER Business Cycle Turning Points, 1990-2010 inclusive

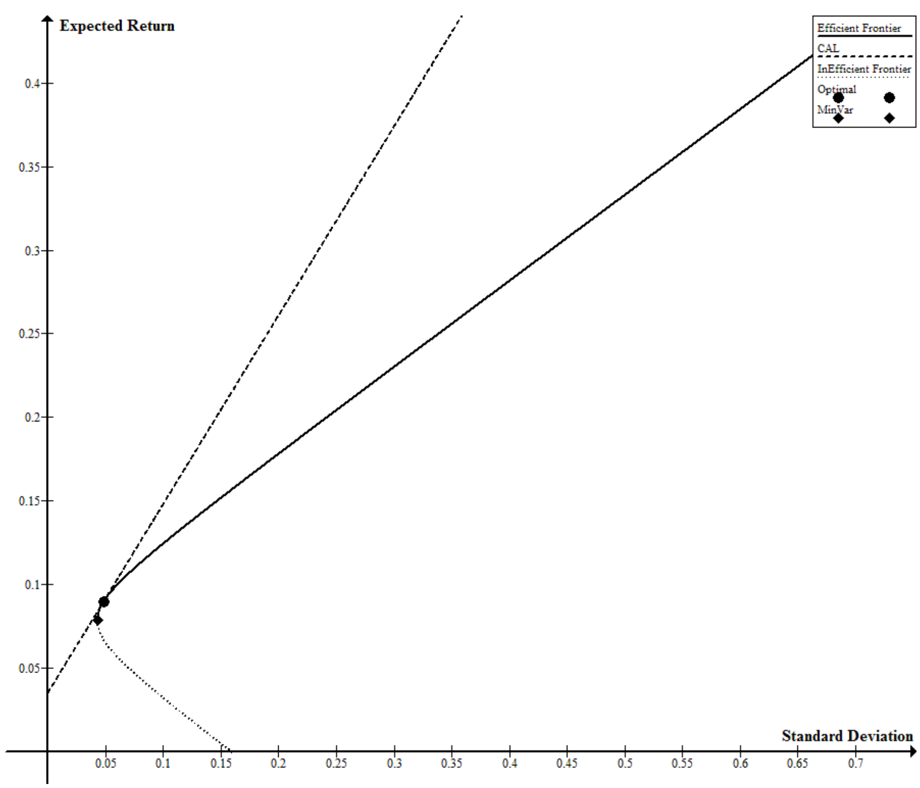
| Recession | | Expansion | |
|---|---|---|---|
| **Beginning** | **Ending** | **Beginning** | **Ending** |
| - | - | November 1982 | June 1990 |
| July 1990 | February 1991 | March 1991 | February 2001 |
| March 2001 | October 2001 | November 2001 | November 2007 |
| December 2007 | May 2009 | June 2009 | - |

## Appendix D

Graphical Representation of the portfolio allocation problem, 1990-2010 inclusive, for the full period (top), periods of recession (middle), and periods of expansion (bottom).

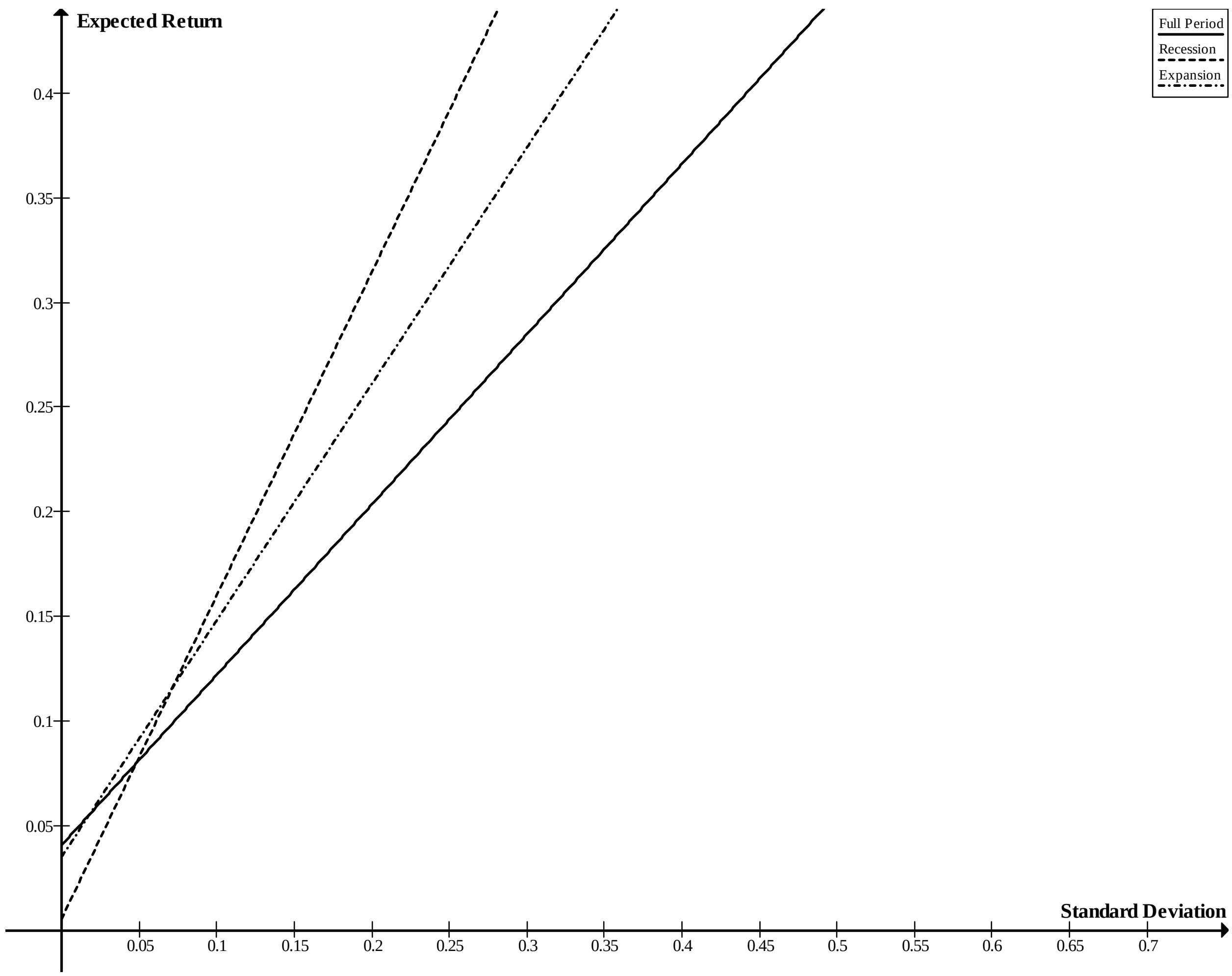



Expected Return
Standard Deviation
Efficient Frontier
CAL
InEfficient Frontier
Optimal
MinVar
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3


Expected Return
Standard Deviation
Efficient Frontier
CAL
InEfficient Frontier
Optimal
MinVar
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7

## Appendix E

Comparison of Sharpe Ratios using NBER turning points versus full period data, 1990 through 2010 inclusive.



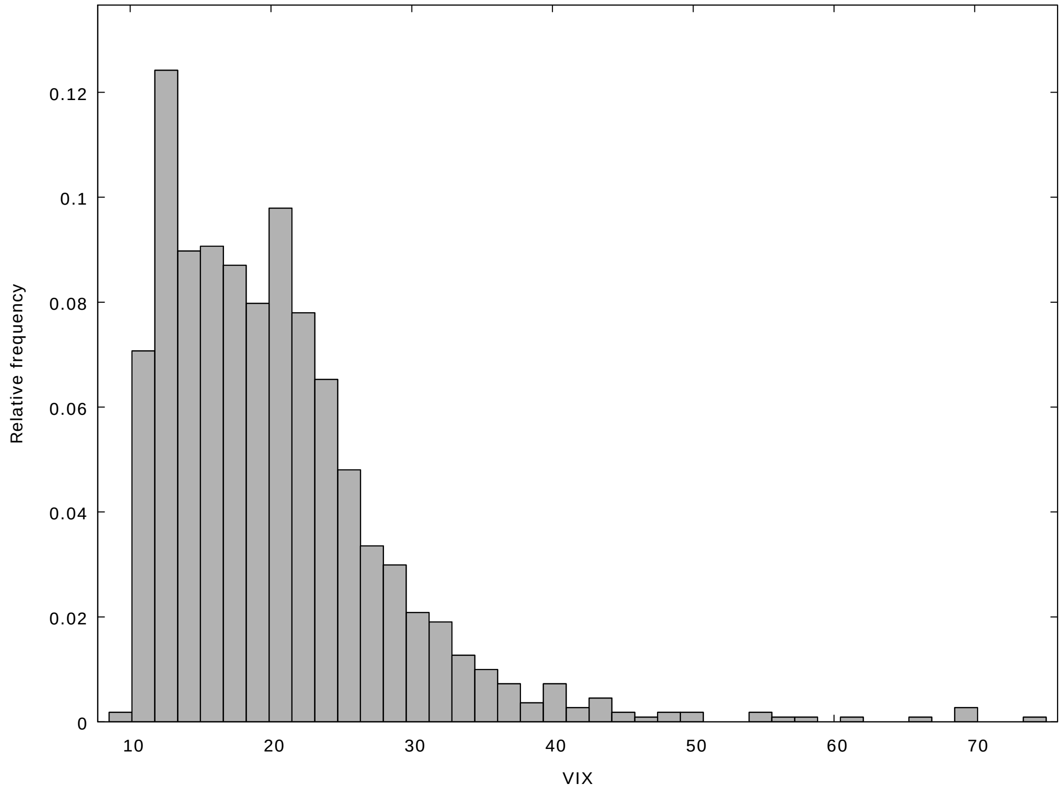
## Appendix F

CBOE VIX index distribution by 2.5 percentile increments, 1990-2010 inclusive

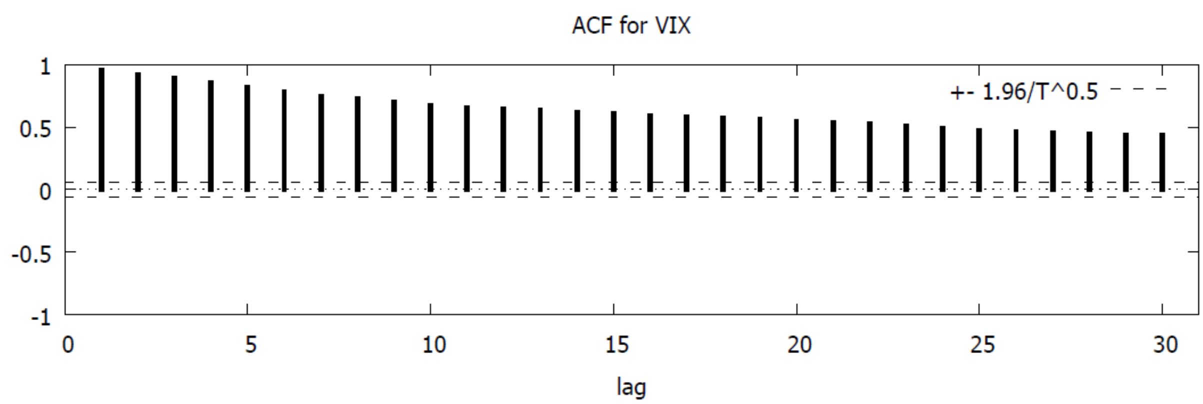
| Percentile | Value | Percentile | Value |
|---|---|---|---|
| 2.5% | 11.00 | 52.5% | 19.58 |
| 5.0% | 11.46 | 55.0% | 20.00 |
| 7.5% | 11.78 | 57.5% | 20.37 |
| 10.0% | 12.11 | 60.0% | 20.80 |
| 12.5% | 12.41 | 62.5% | 21.24 |
| 15.0% | 12.69 | 65.0% | 21.61 |
| 17.5% | 13.08 | 67.5% | 22.23 |
| 20.0% | 13.43 | 70.0% | 22.68 |
| 22.5% | 13.85 | 72.5% | 23.22 |
| 25.0% | 14.35 | 75.0% | 23.90 |
| 27.5% | 14.85 | 77.5% | 24.38 |
| 30.0% | 15.44 | 80.0% | 25.15 |
| 32.5% | 15.76 | 82.5% | 26.06 |
| 35.0% | 16.09 | 85.0% | 27.12 |
| 37.5% | 16.50 | 87.5% | 28.31 |
| 40.0% | 16.95 | 90.0% | 29.84 |
| 42.5% | 17.55 | 92.5% | 31.56 |
| 45.0% | 17.93 | 95.0% | 34.52 |
| 47.5% | 18.53 | 97.5% | 40.03 |
| 50.0% | 19.03 | 100.0% | 74.26 |

## Appendix G

CBOE VIX index relative frequency distribution, 1990-2010 inclusive




CBOE VIX index autocorrelation function, 1990-2010 inclusive



## Appendix H

Descriptive statistics of selected series
Grouping A (VIX 0-45th percentile) 1990-2010, inclusive

| Series | Mean | Median | Minimum | Maximum | Std. Dev. | Skewness | Ex. kurtosis |
|---|---|---|---|---|---|---|---|
| **SP500** | 0.004121 | 0.004236 | -0.04016 | 0.054329 | 0.012821 | 0.0543 | 0.738438 |
| **SP600** | 0.00521 | 0.007081 | -0.06009 | 0.054254 | 0.018488 | 0.3381 | 0.267484 |
| **MSEAFE** | 0.003271 | 0 | -0.09124 | 0.108322 | 0.018329 | 0.694255 | 7.30066 |
| **GOLD** | 0.001343 | 0.001094 | -0.08348 | 0.073942 | 0.017616 | -0.26473 | 2.72347 |
| **NARETI** | 0.003701 | 0.004347 | -0.08648 | 0.064499 | 0.017674 | -0.65273 | 2.80843 |
| **USTRY** | 0.001407 | 0 | -0.03231 | 0.03442 | 0.006581 | 0.783542 | 4.90482 |

Grouping B (VIX 45th-80th percentile) 1990-2010, inclusive

| Series | Mean | Median | Minimum | Maximum | Std. Dev. | Skewness | Ex. kurtosis |
|---|---|---|---|---|---|---|---|
| **SP500** | 0.003988 | 0.00469864 | -0.05207 | 0.075272 | 0.020855 | 0.036502 | -0.04307 |
| **SP400** | 0.00611 | 0.00812449 | -0.05707 | 0.064427 | 0.022795 | -0.08319 | -0.28336 |
| **MSEAFE** | 0.001407 | 0 | -0.10978 | 0.102099 | 0.022342 | -0.40843 | 3.76835 |
| **GOLD** | 0.000537 | 0.000714413 | -0.1027 | 0.085623 | 0.021701 | -0.20325 | 2.04113 |
| **NARETI** | 0.00366 | 0.00217776 | -0.08255 | 0.14264 | 0.025261 | 0.799455 | 3.46237 |
| **USTRY** | 0.001099 | 0.000262668 | -0.02568 | 0.040263 | 0.006823 | 0.340187 | 3.44371 |

Grouping C (VIX 80th to 100th percentile) 1990-2010, inclusive

| Series | Mean | Median | Minimum | Maximum | Std. Dev. | Skewness | Ex. kurtosis |
|---|---|---|---|---|---|---|---|
| **SP500** | -0.00817 | -0.0105125 | -0.164033 | 0.102211 | 0.037482 | -0.01853 | 1.13274 |
| **SP600** | -0.0097 | -0.0100302 | -0.192696 | 0.124239 | 0.043309 | -0.1023 | 1.69954 |
| **MSEAFE** | -0.00545 | 0 | -0.160271 | 0.145087 | 0.040626 | -0.43234 | 2.39961 |
| **GOLD** | 0.001778 | 0.00140032 | -0.123829 | 0.137026 | 0.028374 | 0.34023 | 3.97816 |
| **NARETI** | -0.00899 | -0.00580529 | -0.244754 | 0.217934 | 0.052134 | -0.44062 | 4.64645 |
| **USTRY** | 0.001905 | 0.00240049 | -0.0232316 | 0.022505 | 0.008831 | -0.34659 | 0.002225 |

## Appendix I

Covariance of selected series of returns, annualized

Grouping A 1990-2010, inclusive

| SP500 | SP600 | MSEAFE | GOLD | NARETI | USTRY | |
|---|---|---|---|---|---|---|
| 0.008548 | 0.010179 | 0.003268 | 0.000404 | 0.00588 | 0.000885 | **SP500** |
| | 0.017774 | 0.00732 | 0.003016 | 0.009495 | 0.000297 | **SP600** |
| | | 0.017469 | 0.004555 | 0.003184 | 0.001073 | **MSEAFE** |
| | | | 0.016138 | 0.002074 | -0.00015 | **GOLD** |
| | | | | 0.016243 | 0.001086 | **NARETI** |
| | | | | | 0.002252 | **USTRY** |

Grouping B 1990-2010, inclusive

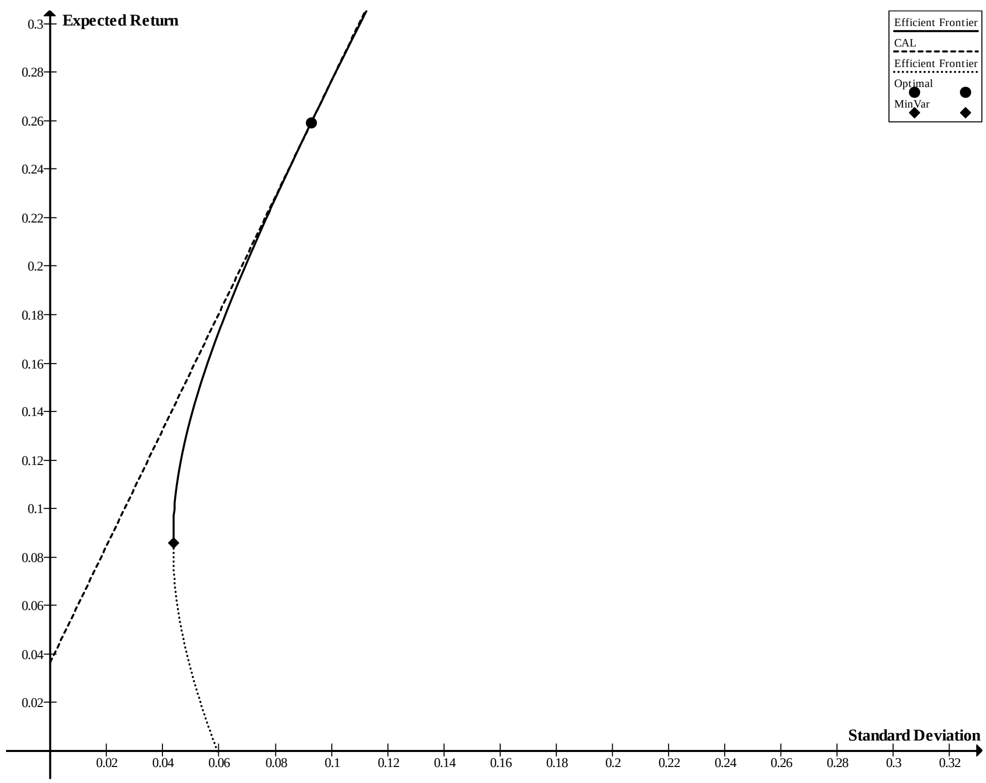
| SP500 | SP600 | MSEAFE | GOLD | NARETI | USTRY | |
|---|---|---|---|---|---|---|
| 0.022617 | 0.019519 | 0.009541565 | -0.000221226 | 0.015821 | 0.000309 | **SP500** |
| | 0.02702 | 0.012235084 | 0.00093376 | 0.018427 | -0.00108 | **SP600** |
| | | 0.025956578 | 0.004565934 | 0.010756 | -0.00053 | **MSEAFE** |
| | | | 0.024489214 | 0.001984 | 0.000585 | **GOLD** |
| | | | | 0.033183 | -3.9E-05 | **NARETI** |
| | | | | | 0.002421 | **USTRY** |

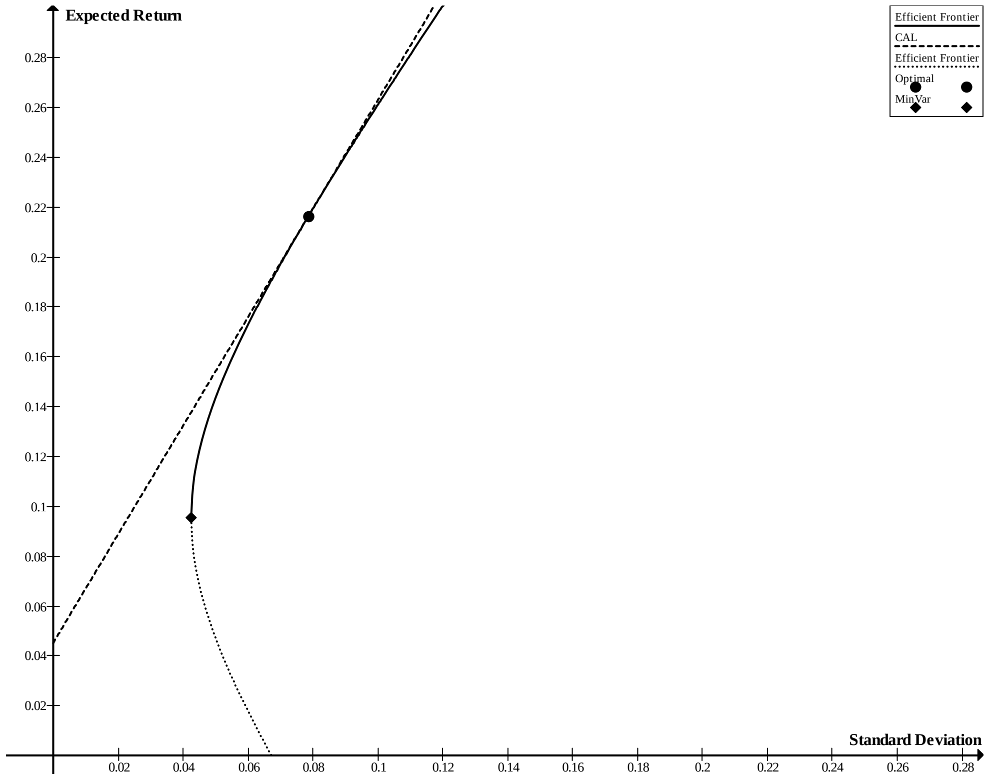
Grouping C 1990-2010, inclusive

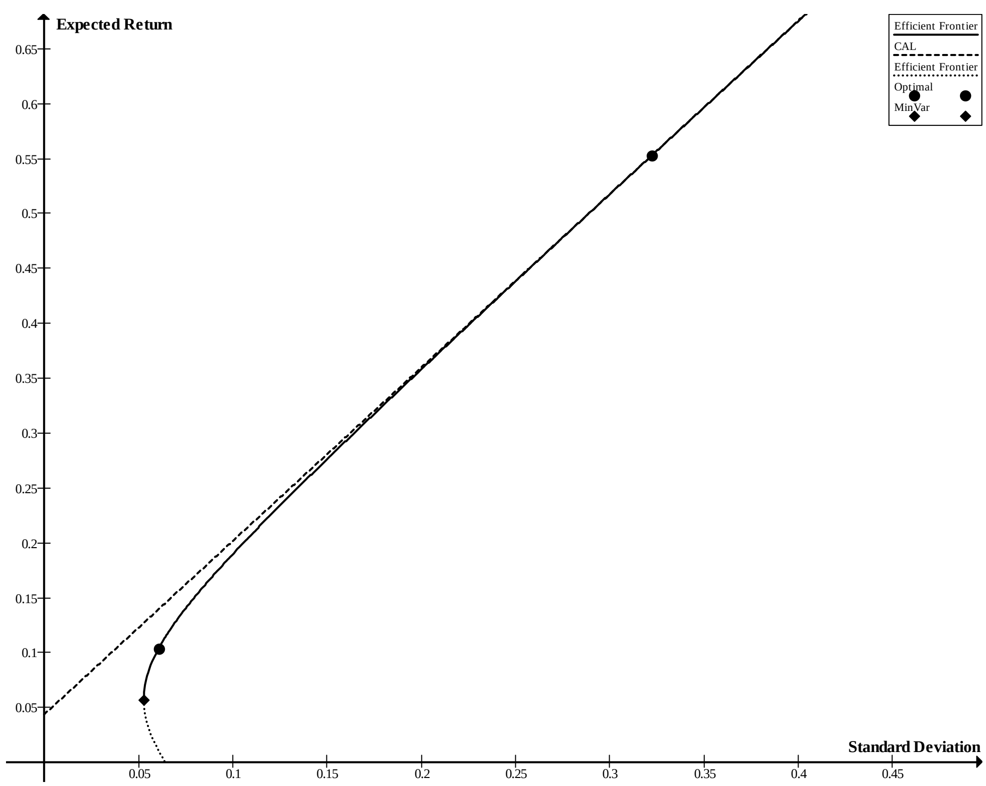
| SP500 | SP600 | MSEAFE | GOLD | NARETI | USTRY | |
|---|---|---|---|---|---|---|
| 0.073053 | 0.076823 | 0.047073 | 0.000337345 | 0.073220712 | -0.00558 | **SP500** |
| | 0.097535 | 0.058335 | 0.003674295 | 0.093058479 | -0.00592 | **SP600** |
| | | 0.085824 | 0.007594611 | 0.059406723 | -0.00419 | **MSEAFE** |
| | | | 0.041864952 | 0.010692054 | 0.000675 | **GOLD** |
| | | | | 0.141332521 | -0.00415 | **NARETI** |
| | | | | | 0.004055 | **USTRY** |

## Appendix J

Graphical Representation of the portfolio allocation problem, 1990-2010 inclusive, for Grouping A (top), Grouping B (middle), and Grouping C (bottom). Not that Grouping C features two optimal solutions. The lower of the two being the solution in which asset weights are restricted to nonnegative values.

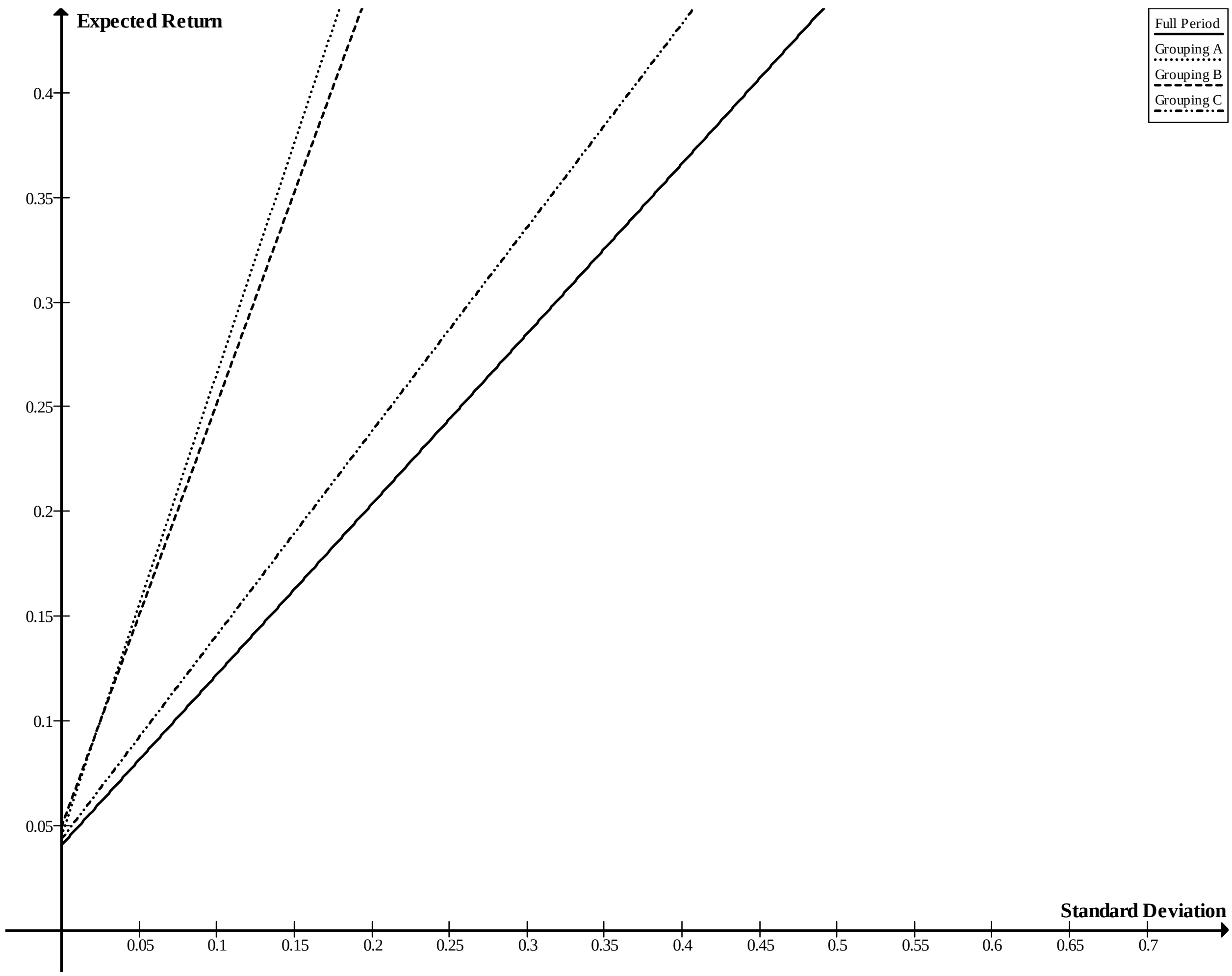



Expected Return
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
Standard Deviation
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0.22
0.24
0.26
0.28
Efficient Frontier
CAL
Efficient Frontier
Optimal
MinVar


Expected Return
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
Standard Deviation
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
Efficient Frontier
CAL
Efficient Frontier
Optimal
MinVar

## Appendix K

Comparison of Sharpe Ratios using VIX based groupings versus full period data, 1990 through 2010 inclusive



## Appendix L

Selection of US mutual fund asset allocation, retrieved March 21, 2011
Source: Morningstar Inc.

| **Fund** | **US Equities** | **Foreign Equities** | **Bonds** | **Cash** | **Other** | **Total** |
|---|---|---|---|---|---|---|
| American Balanced Fund | 61.40% | 4.80% | 28.10% | 5.70% | 0.00% | 100.00% |
| American Funds | 82.30% | 6.20% | 3.20% | 8.30% | 0.00% | 100.00% |
| BlackRock Global Allocation | 40.63% | 32.74% | 21.24% | 0.42% | 4.97% | 100.00% |
| Calamos Growth and Income A | 40.29% | 7.48% | 6.87% | 2.72% | 42.28% | 99.64% |
| Fidelity Puritan | 56.66% | 9.78% | 25.33% | 8.04% | 0.19% | 100.00% |
| Franklin Templeton Foundling Allc. | 42.63% | 30.03% | 18.21% | 4.81% | 4.32% | 100.00% |
| GMO Global Balanced Asset Allocation | 16.82% | 44.49% | 11.76% | 22.76% | 4.18% | 100.01% |
| Invesco Growth and Income | 59.48% | 4.93% | 16.87% | 3.03% | 15.70% | 100.01% |
| Ivy Asset Strategy A | 21.04% | 29.88% | 32.51% | 4.63% | 11.93% | 99.99% |
| Janus Balanced A | 47.24% | 9.52% | 38.38% | 4.38% | 0.49% | 100.01% |
| MFS Total Return A | 56.33% | 3.49% | 37.98% | 1.69% | 0.51% | 100.00% |
| PIMCO All Asset | 43.88% | 12.14% | 28.90% | 9.02% | 6.06% | 100.00% |
| Principle SAM Balanced A | 46.97% | 16.89% | 29.65% | 2.73% | 3.75% | 99.99% |
| Vanguard Wellington ADM | 54.67% | 10.88% | 27.70% | 6.61% | 0.15% | 100.01% |
| Wells Fargo Advantage Asset Allocation | 12.88% | 35.12% | 21.47% | 10.70% | 19.84% | 100.01% |
| ***Average*** | ***45.55%*** | ***17.22%*** | ***23.21%*** | ***6.37%*** | ***7.62%*** | |