How Does Liquidity Risk Disturb Asset Prices? A General Equilibrium Approach

Dai Li

Winner of the Mark K. Inman Senior Essay Prize, 2012

Abstract

Liquidity risk was conspicuous in the recent financial market turbulence. This paper presents a liquidity risk model in which two financial institutions trade an illiquid risky asset. The model develops explicit liquidity demand and supply curves along with analytical solutions, and it inherently generates two types of general equilibrium – liquid and illiquid. Liquidity risk manifests in the illiquid equilibrium to depress the asset price to deviate from the fundamental value. In turn, the model shows that riskier assets have thinner liquidity supply and heavier liquidity demand. The model is able to analyze precautionary hoarding, runs on financial institutions, and loss spiral. The model suggests that hoarding liquidity in turmoil is an effective way for a financial institution to earn profit and also maintain a solid financial condition. Bank-run is an important externality of deteriorating market condition caused by hoarding. It can motivate financial institutions to hoard less liquidity. Lastly, financial institutions should be relieved from marking-to-market to prevent loss spiral, as it may lead to illiquidity and, eventually, to insolvency.

I. Introduction

During the liquidity and credit crunch in 2007 and 2008, the U.S. corporate bond index spread increased to five times its average pre-crisis level, rising from roughly 90 basis points between 2004 and 2006 to a peak of 450 basis points in 2008. Garcia and Prokopiw (2009) used a structural credit-risk model to explain the spread by two factors – credit risk and liquidity risk. They concluded that the increase in the model-implied credit risk explained only a small portion of the spread, most of which was attributed to liquidity risk.

In this paper, I construct a theoretical liquidity risk model incorporating the two characteristics of liquidity risk and their impact on asset prices. The model inherently generates two types of general equilibrium – liquid equilibrium and illiquid equilibrium. The liquid equilibrium is characterized by assets trading at fundamental values. In an illiquid equilibrium, however, asset prices deviate a great deal from the fundamentals and are very sensitive to marginal changes in market liquidity condition.

This model captures two generalized characteristics of liquidity risk implied by Garcia and Prokopiw's study (2009). The first one is that, without any change in the fundamental value of an asset, liquidity risk itself can greatly disturb prices. The second characteristic is that liquidity risk does not manifest itself in normal times; however, it depresses asset prices severely in a distressed market.

Next, I will introduce briefly the theoretical framework, the primary results, and the applications of the model on three liquidity related issues. First, the theoretical framework of the model is depicted in the following. Suppose that there are only two financial institutions (called Bank A and Bank B) in the financial market. Two banks are required to maintain their capital ratios (defined later) above a threshold with a very high probability. Such regulation poses a problem for Bank A, which has experienced an idiosyncratic shock. It needs to sell an illiquid risky asset to reduce the uncertainty of its capital ratio. Liquidation at a fire-sale price may be very costly for Bank A when the market is thin. Bank B, as the only potential buyer in the market, sees this as an opportunity to make profit via buying mispriced assets. Nevertheless, Bank B is also subject to regulations on its capital ratio limiting its ability to inject liquidity into the market to earn profit. As a result, Bank A attempts to minimize the loss by selling only what is necessary at all given prices. The set of Bank A's choices at given prices forms the liquidity demand curve. In compliance with regulations, Bank B utilizes all capital available to maximize profit. Solving Bank B’s problem produces the liquidity supply curve in this market. The general equilibrium occurs when the equilibrium trading price solves both banks’ problems and the market clears.

The primary results of this model are analytical solutions for liquidity demand and supply functions. Unlike conventional supply curves, the liquidity supply curve is downward sloping because lower prices motivate financial institutions to purchase more assets. The regulation on both banks’ capital ratios pins down the position of the demand and supply curve. The relative position of demand and supply curve determines which type of equilibrium occurs. Asset prices in an illiquid equilibrium are very sensitive to liquidity condition mainly because both demand and supply curves are downward sloping.

In terms of the existence and uniqueness of equilibriums, I show that the illiquid equilibrium is unique if it exists under the condition that both banks face required thresholds (thresholds could be different for two banks) on their capital ratio with the **same** probability. In the comparative analysis section of this paper, I will show that riskier assets have thinner liquidity supply and heavier liquidity demand in a time of stress, which means the trading price will deviate from the fundamental value more severely.

The analysis of the model application sheds light on three liquidity related issues: precautionary hoarding runs on financial institutions, and loss spirals.

In the context of my model, Bank B can conduct precautionary hoarding by setting an overly conservative target capital ratio. A higher target ratio reduces liquidity supplied at any given price, i.e., shifts the supply curve downward. My model predicts that financial institutions will hoard liquidity to enhance profit by setting the target ratio as high as possible provided that an illiquid equilibrium occurs. The equilibrium situation deteriorates in the sense that asset prices have larger swings and Bank A suffers huge losses by liquidating all of its risky assets at the lowest acceptable price. Overall, my model suggests that, when facing a desperate seller (Bank A), hoarding is an effective way for the counterparty financial institution to generate profits while maintaining a solid financial condition.

The next application on bank runs captures a noticeable adverse externality of precautionary hoarding. When investors (depositors) are informed about the trade between two banks with a price far below the previously perceived fundamental value, they may mistakenly consider the plunge in Bank A's asset price as a decline in the fundamental value. If panicked investors collectively decide to withdraw investment (deposits), they run indiscriminately on both banks. Further, the lower the trading price, the more likely investors are to run. In the environment with bank run threat, although the decline of asset price is still a profit opportunity for Bank B, it also causes higher expected bank run loss on Bank B as the bank-run probability increases. In terms of the model setup, Bank B's objective changes to maximizing expected net profit rather than the trading profit. My model shows that Bank B is willing to hoard less liquidity and to purchase assets rationally at higher prices in the case where bank run is incorporated. To sum up, if market participants are aware of externalities of declining assets prices, the market liquidity position can be moderately eased to generate higher trading prices in equilibrium.

The final application of my model is on the study of loss spiral. Suppose that after selling a portion of risky assets in an illiquid market with a fire-sale price, Bank A must mark its remaining portfolio to the fire-sale price. The resulting write-down loss would immediately bring down Bank A's capital ratio below regulation threshold again. To be compliant with the regulation, it has to sell more portfolios at even lower prices. It is expected that with reiterated costly liquidations and write-down losses, Bank A's problem would quickly evolve from illiquidity to insolvency. Based on this expectation, financial institutions should be relieved from marking-to-market regulation, at least in the time of stress, to prevent liquidity problems from being transmitted to solvency problems. This is mainly because when a market lacks liquidity, market prices observed from sporadic trades of an asset do not necessarily reflect its fundamental price. My model shows that the market price in a turbulent time may include a large "liquidity risk premium" and it greatly deviates from the fundamental value of an asset.

This paper is structured as follows. Section II is a literature review focusing mainly on two papers that are closely related to my model. Section III builds up the detailed theoretical framework of the model. Section IV first describes the model setup and defines the liquid and illiquid equilibriums. The derivation of liquidity demand and supply functions and the discussion about the equilibrium condition is also included in Section IV. Section V provides numerical examples on liquidity supply and demand curves along with a comparative statics analysis to further illustrate the feature of the model. Section VI addresses how to apply the model to study the three liquidity related issues sketched above. Section VII concludes.

II. Literature Review

In the recent financial crisis, liquidity risk was noticeable in various financial markets. In the debt market, liquidity risk increased due to three factors: falling risk capital, rising repo haircut, and increased counterparty risk (Krishnamurthy 2010). In the money market, banks or investment banks that used off-balance-sheet vehicles faced funding liquidity risk because of the mismatch between the maturity of long-term investment and short-term borrowing (Brunnermeier 2009). In unsecured interbank money markets, Eisenschmidt and Tapking (2009) find that the market spreads have been largely attributable to liquidity risk since the start of the turmoil in 2007.

Cifuentes, Ferrucci, and Shin (2005) study the liquidity-triggered financial contagions using a common illiquid asset as the channel of contagion in a banking system. My model is similar to theirs in terms of the motivation of liquidation – complying with regulatory requirements or internal regulations. Based on regulatory provisions on banks' capital adequacy ratio, an idiosyncratic shock may force one bank to reduce its balance sheet by selling the common illiquid asset that is held by all banks to an external market. In Cifuentes, Ferrucci and Shin's model, the liquidity supply curve in the external market is assumed to be a downward sloping exponential function, so that the price tumbles if one bank is dumping the common illiquid asset. Thus, one bank's behavior may create downward pressure on all other banks' balance sheets, which possibly triggers a wave of liquidation by the other banks. There are two major differences between my model and theirs. First, in my model the transaction price is determined endogenously by two counterparties involved in the trade instead of an external market. Second, their model converts any marginal increase in liquidity demand into a decrease in the asset price. Conversely, in my model, the asset price is invariant to marginal change in supply or demand if the market is awash with liquidity.

My model employs the same measurement device for liquidity risk as Brunnermeier and Pedersen's model (2009). When the market price deviates from the fundamental value, the absolute value of the deviation is defined as the market illiquidity. In addition, they assume that the fundamental value follows a geometric Brownian motion with the volatility following an ARCH process. For simplicity, I assume that it follows a normal distribution, which is sufficient to demonstrate the excessive sensitivity against liquidity supply. One implication of their work is that if the fundamental volatility of an asset is high, then the asset has high market illiquidity. My model implies the same characteristic of illiquid assets.

My liquidity risk model differs from most of the existing literature. Most theoretical models are characterized by similar forms of liquidity shocks – mismatch between stochastic liquidity demand of depositors or consumers and the timing for illiquid investments to pay off (Allen and Gale 2000). Allen, Carletti, and Gale (2009) argue that if banks lack hedging tools, they may hoard liquidity because they face uncertain liquidity demand from depositors, which reduces efficiency in the use of capital. The study also theorizes that the inefficiency should be removed by central banks adopting open market operations. My model suggests that hoarding may be also an effective way for financial institutions to earn profit while maintaining a solid financial condition. Gorton and Huang (2006) justify that banking systems consisting of well-diversified big banks are less prone to idiosyncratic liquidity shocks. Tirole (2011) summarizes the interrelationships among illiquidity, market freezes, fire sales, contagion, insolvency, and bailouts. In terms of empirical studies, Chen, Lesmond and Wei (2007) analyze a comprehensive set of four thousand corporate bonds covering both investment grade and speculative grade bonds, and find that liquidity is a key determinant in yield spreads. Similarly, De Jong and Driessen (2006) find that corporate bond returns have significant exposures to fluctuations in Treasury bond liquidity and equity market liquidity. However, liquidity risk is a minor concern in the credit default swap (CDS) market, which is not surprising because CDS is inherently used to addressing credit risk (Longstaff, Mithal, and Neis 2005).

III. Theoretical Framework

Based on the simple theoretical framework depicted in the introduction, my focus here is on the problems faced by two banks and on the details of the banking regulations.

III.1 Regulatory Environment

In my model, the particular motivation behind Bank A's liquidation is regulation. The regulation stipulates that Banks must maintain their capital ratio (defined later) above a **target level** with a very high **probability**. I employ such a restriction because, usually in a financial crunch, banks must adjust equity capital to keep the probability of financial distress sufficiently low (Krishnamurthy 2010). The "capital ratio" in my model is similar to the capital adequacy ratio used in actual regulation. Although it is not calculated in exactly the same way as capital adequacy ratio, it imposes similar restrictions on financial institutions' behavior. Also, I assume that both Banks are in compliance with the regulation before the idiosyncratic shock hits Bank A.

III.2 Bank A's Problem

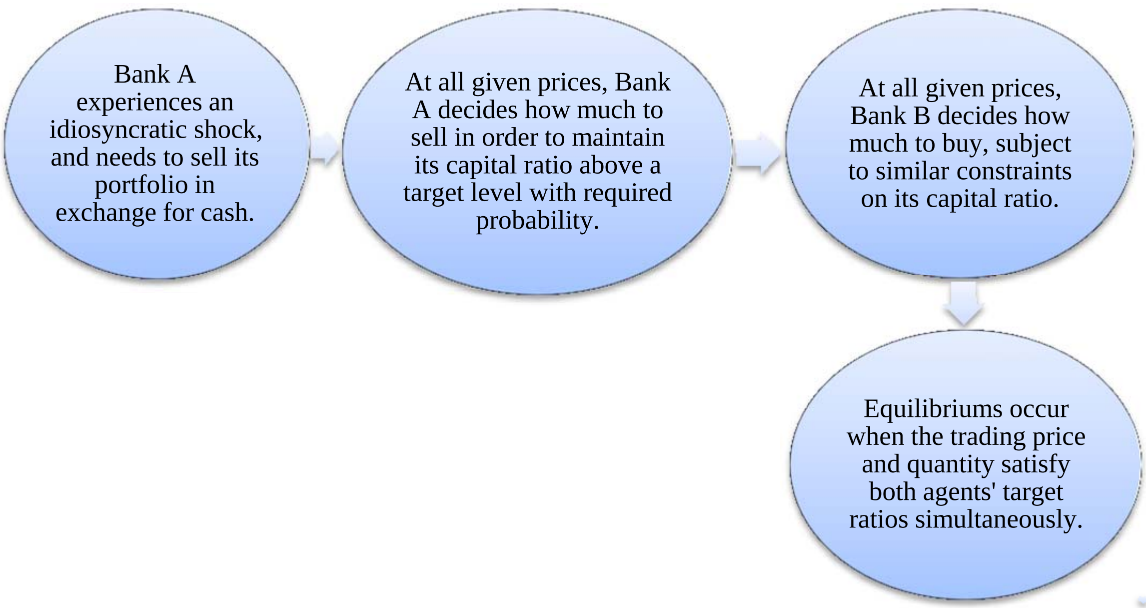
In my model, after the idiosyncratic shock, Bank A wants to lower the holding of its risky portfolio. This is simply because the probability of Bank A's capital ratio falling **below the target level exceeds the required probability**. Selling risky assets helps Bank A reduce the volatility of its capital ratio, and, in turn, the probability of violating the regulation decreases. However, liquidation at a fire-sale price (below fundamental value) is very costly. To minimize the liquidation loss, Bank A calculates the minimum amount of portfolio to sell at all given prices, which reveals the relationship between the liquidity demanded and prices.

III.3 Bank B's Problem

On the buy side of this financial market, Bank B, as the only potential buyer in this market, may want to buy Bank A's portfolio because it makes profit if the portfolio is sold below fundamental value. The regulations on Bank B's capital ratio limit its ability to inject liquidity into the market to earn profit. This is mainly because purchasing the risky asset, even at prices below fundamental value increases the volatility of Bank B's capital ratio. Therefore, Bank B faces a profit maximization problem. It calculates the maximum amount of the risky asset it can purchase at given prices under the constraint of regulations on its capital ratio.

III.4 Results

Solving Bank A's loss minimization and Bank B's profit maximization problems respectively generates liquidity demand and supply curve. The market clears when the liquidity supplied is equal to liquidity demanded at the equilibrium price. The following procedure summarizes how a financial institution may have to liquidate its portfolio in a distressed market.



The first step of modeling this trade is to develop explicitly the demand and supply curve, which in turn will directly determine the equilibrium. In the next section, I will first explain the setup of the model in detail in Section IV.1, the derivation of demand and supply function in Section IV.2, and the equilibrium conditions in Section IV.3.

IV. The Model

IV.1 Model Setup

Notation :

f_A	The fundamental value of Bank A's risky portfolio
e_{AA}	The quantity of Bank A's portfolio
c_A	Bank A's cash
OA_A	Bank A's other illiquid assets
r_A^*	Bank A's target capital ratio
r_A	Bank A's actual capital ratio
μ_A	The expected value of f_A in the next period
σ_A	The volatility of Bank A's portfolio

(Changing subscripts of the above notations to B to obtain all the corresponding notations for Bank B.)

Δ_A	The amount of portfolio sold by Bank A
Δ_B	The amount of portfolio purchased by Bank B
p_A	The trading price of Bank A portfolio

IV.1.1 Definitions and Assumptions

Suppose that at time t=0, two Banks have the same capital structure. Bank A's capital ratio r_A is defined as:

$$r_A = \frac{f_A * e_{AA} + c_A}{f_A * e_{AA} + c_A + OA_A}$$

(Same definition for r_B) In the next period t=1, the fundamental value of Bank A's risky portfolio f_A follows the normal distribution of $N\ [\ \mu_A,\ \sigma_A^2]$. The fundamental value of Bank B's portfolio f_B follows the normal distribution $N\ [\ \mu_B,\ \sigma_B^2]$, and for simplicity f_B is assumed to be uncorrelated with f_A. Suppose that at time t=0 both Banks' portfolios are at their fundamental values, i.e., $f_A = \mu_A$ and $f_B = \mu_B$. Since the fundamental values of portfolios are random variables, the capital ratios are also random variables at time t=1. Based on the randomness of r_A and r_B, the regulation imposes that at time t=1 the probabilities of $r_A \geq r_A^*$ and $r_B \geq r_B^*$ must be at least $q_A > 50\%$ and $q_b > 50\%$ respectively. Put differently, the $1 - q_A$ and $1 - q_B$ percent quantile of r_A and r_B must be at least r_A^* and r_B^* respectively.

IV.1.2 Bank A's Problem

At time t=0, Bank A experiences an idiosyncratic shock that clears out all of its cash c_A. In order to control the probability of violating regulations, Bank A sells some of its risky portfolio at time t=0 to reduce the volatility of its capital ratio at time t=1. After the shock and liquidating Δ_A at $\underline{p_A}$, Bank A's capital ratio changes to:

$$r_A = \frac{f_A(e_{AA} - \Delta_A) + \Delta_A \underline{p_A}}{f_A(e_{AA} - \Delta_A) + \Delta_A\, \underline{p_A} + OA_A} \quad (0 \leq \Delta_A \leq e_{AA}) \qquad (1)$$

The random component in r_A is $f_A(e_{AA} - \Delta_A)$, which follows the normal distribution $N\ [(e_{AA} - \Delta_A)\mu_A, (e_{AA} - \Delta_A)^2\sigma_A^2]$. In expression (1), parameters e_{AA}, OA_A, σ_A^2, and μ_A are constants. The effects of trading price $\underline{p_A}$ and the amount of portfolio sold Δ_A on distribution of r_A are the key determinants of Bank A's decision. Bank A's liquidation decision, which involves a choice of Δ_A at a given $\underline{p_A}$, affects r_A in two ways. On one hand, selling portfolio ($\Delta_A > 0$) reduces the volatility of r_A, which helps Bank A to control its risk. On the other hand, the liquidation decreases the expected value of r_A if the portfolio is sold at a loss ($\underline{p_A} < \mu_A$). To be compliant with the regulatory requirement may be costly for Bank A. Thus, it aims to minimize the loss incurred by liquidation. Bank A's problem is summarized as follows:

$$\underset{\underline{p_A},\Delta_A}{Min} (\mu_A - \underline{p_A})\, \Delta_A$$

$$s.t\ 0 \leq \Delta_A \leq e_{AA}$$

$$Pr\left[r_A\left(\Delta_A, \underline{p_A}\right) \geq r_A^*\right] \geq q_A$$

The first constraint controls the amount portfolio sold below the total amount. The second constraint is the regulatory requirement on r_A. The solution of this optimization problem is the liquidity demand function as a relationship between Δ_A and $\underline{p_A}$. Cast in mathematical form,

$$\Delta_A\left(\underline{p_A}\right) = \Delta_A^*, for\ all\ \underline{p_A} \epsilon [\widetilde{p_A}, \mu_A]$$

where Δ_A^* is given $\underline{p_A}$ the amount of portfolio to sell such that $Pr\left[r_A\left(\Delta_A^*, \underline{p_A}\right) \geq r_A^*\right] = q_A$. $\widetilde{p_A}$ is defined as $Pr[r_A(e_{AA}, \widetilde{p_A}) \geq r_A^*] = q_A$, and it is the lowest trading price that Bank A would accept to sell its entire portfolio. The second constraint always binds in an optimal solution because selling more than the necessary amount causes greater loss. The first constraint is also binding when Bank A sells its entire portfolio at $\widetilde{p_A}$. Economically, the liquidity demand curve represents the required liquidity support of a certain asset at all given prices.

IV.1.3 Bank B's Problem

Bank B does not experience any idiosyncratic shock. After buying Δ_B amount of portfolio at $\underline{p_A}$, Bank B's capital ratio changes from

$$r_B = \frac{f_B * e_{BB} + c_B}{f_B * e_{BB} + c_B + OA_B}$$

to

$$r_B = \frac{f_B e_{BB} + \Delta_B f_A + (c_B - \Delta_B \underline{p_A})}{f_B e_{BB} + \Delta_B f_A + (c_B - \Delta_B \underline{p_A}) + OA_B}.$$

The effects of $\underline{p_A}$ and Δ_B on the distribution of r_B are the key determinants of Bank B's decision. Bank B's capital ratio r_B changes in two ways. First, r_B becomes more volatile due to the purchase of risky assets. Second, Bank B makes profit if the trading price is below the current fundamental value. That is to say, Bank B records a trading profit of $\Delta_B \left(\mu_A - \underline{p_A}\right)$ at t=0 if $\underline{p_A} < \mu_A$, and thus the expected value of r_B at t=1 increases accordingly. This regulation constraint on r_B limits Bank B's ability to inject liquidity and make profit in the market. Bank B's profit maximizing problem is the following:

$$\underset{\underline{p_A}, \Delta_B}{Max} (\mu_A - \underline{p_A})\, \Delta_B$$

$$s.t\ \Delta_B \geq 0$$

$$Pr\left[r_B\left(\Delta_B, \underline{p_A}\right) \geq r_B^*\right] \geq q_B$$

This optimization problem will give the liquidity supply function as a relationship between Δ_B and $\underline{p_A}$. Formally,

$$\Delta_B\left(\underline{p_A}\right)=\begin{cases}[0,\widehat{\Delta_B}] & if\ \underline{p_A}=\mu_A\\ \Delta_B^* & if\ \underline{p_A}<\mu_A\end{cases}$$

where Δ_B^* is given $\underline{p_A}$ the amount of portfolio to buy such that $Pr\left[r_B\left(\Delta_B^*,\underline{p_A}\right)\geq r_B^*\right]=q_B$. By the same token, for $\widehat{\Delta_B}$, $Pr\left[r_B\left(\widehat{\Delta_B},\mu_A\right)\geq r_B^*\right]=q_B$. The second constraint always binds in an optimal solution if $\underline{p_A}<\mu_A$ as Bank B seeks to maximize profit using all resources available. If $\underline{p_A}=\mu_A$, Bank B's trading profit is zero for any Δ_B, so it is indifferent among buying anything between zero and Δ_B^*. As for the economic meaning, the liquidity supply function represents the market capacity of a certain asset at all given prices. For example, $\widehat{\Delta_B}$ stands for the market capacity at fundamental price.

IV.1.4 Definition of Equilibrium

Define the general equilibrium in this financial market:

The general equilibrium is a set of $\{\underline{p_A^e},\ \Delta^e\}$, such that given the trading price $\underline{p_A^e}$, Bank A chooses $\Delta_A^*(\underline{p_A^e})$ to minimize losses, and Bank B chooses $\Delta_B^*(\underline{p_A^e})$ to maximize trading profit. The market clears with $\Delta^e=\Delta_A^*=\Delta_B^*$.

Definition 1: Liquid Equilibrium. If given $\underline{p_A}=\mu_A$, $\Delta_A^*(\mu_A)=\Delta_B^*(\mu_A)$. The Liquid Equilibrium is said to occur at the point where Bank A liquidates with no loss.

Definition 2: Illiquid Equilibrium. If at some $\underline{p_A^e}\in[\widetilde{p_A},\mu_A)$, $\Delta_A^*(\underline{p_A^e})=\Delta_B^*(\underline{p_A^e})$. The Illiquid Equilibrium is said to occur with insufficient liquidity, where Bank A liquidates at a loss of $\left(\mu_A-\underline{p_A^e}\right)*\Delta_A(\underline{p_A^e})$.

Definition 3: No Equilibrium. If at all given $\underline{p_A}\in[\widetilde{p_A},\mu_A]$, $\Delta_A^*\left(\underline{p_A}\right)>\Delta_B^*(\underline{p_A})$. No equilibrium exists. Market is of zero liquidity since Bank A is not able to meet regulatory requirement via liquidating.

IV.2 Derivation of the Demand and Supply Functions

The analytical solutions of Bank A's liquidity demand curve and Bank B's liquidity supply curve are the primary results of this model. The demand function determines the "required liquidity support" at all given price levels.

First, as e_{AA} is the quantity of the portfolio, r_A's expression (1) can be normalized by setting $e_{AA}=1$.

$$\frac{f_A(1-\Delta_A)+\Delta_A \underline{p_A}}{f_A(1-\Delta_A)+\Delta_A\, \underline{p_A}+OA_A} \geq r_A^* \quad (0 \leq \Delta_A \leq e_{AA})$$

Let the component $f_A(1-\Delta_A)+\Delta_A \underline{p_A}$ be a new normal random variable f_{ANew}, with mean $\mu_A + \Delta_A \left(\underline{p_A} - \mu_A\right)$ and variance $(1-\Delta_A)^2\sigma_A^2$. As discussed earlier, when the second constraint of Bank A' problem is binding, it implies

$$Pr\left(1-\frac{OA_A}{f_{ANew}+OA_A} \leq r_A^*\right) = 1-q_A,$$

which is equivalent to

$$Pr\left(f_{ANew} \leq \frac{r_A^* OA_A}{1-r_A^*}\right) = 1-q_A\,. \qquad (2)$$

(See the Appendix I.1 for more discussion about the inequality in equation (2).)

Since f_{ANew} follows the normal distribution$N\{\mu_A + \Delta_A \left(\underline{p_A} - \mu_A\right), (1-\Delta_A)^2\sigma_A^2\}$, equation (2) is equivalent to

$$P\,r\left(z \leq \frac{\frac{r_A^* OA_A}{1-r_A^*} - \left[\mu_A + \Delta_A\left(\underline{p_A}-\mu_A\right)\right]}{(1-\Delta_A)\sigma_A}\right) = 1-q_A \qquad (3)$$

Let $K_A = \Phi^{-1}(1-q_A)$, and thus equation (4) follows equation (3),

$$\frac{\frac{r_A^* OA_A}{1-r_A^*} - [\mu_A + \Delta_A\left(\underline{p_A}-\mu_A\right)]}{(1-\Delta_A)\sigma_A} = K_A. \qquad (4)$$

$\Phi^{-1}(\cdot)$ stands for the CDF of the standard normal distribution. Note that $K_A < 0$ because $q_A > 50\%$. Rearrange the above equation to obtain the inverse demand function

$$\underline{p_A} = \frac{\frac{r_A^* OA_A}{1-r_A^*} - \ \mu_A - K_A\sigma_A}{\Delta_A} + \mu_A + K_A\sigma_A.$$

Further, the demand function

$$\Delta_A = \frac{\frac{r_A^* OA_A}{1-r_A^*} - \mu_A - K_A\sigma_A}{\underline{p_A} - \mu_A - K_A\sigma_A}$$

reveals the “required liquidity support" at all given prices between $[\widetilde{p_A}, \mu_A]$,

Following the similar procedure, I derive the inverse liquidity supply function of Bank B,

$$\underline{p_A} = \mu_A + \frac{K_B\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2} - (\frac{r_B^* OA_B}{1-r_B^*} - c_B - \mu_B)}{\Delta_\mathrm{B}}$$

$where$ $K_B = \Phi^{-1}(1-q_B)$. Note that $K_B < 0$ because $q_B > 50\%$. (See the derivation of the inverse liquidity supply function in Appendix I.2.) Next I will prove an important proposition of this model: unlike conventional supply curves, liquidity supply curves are downward sloping. Put mathematically,

Proposition 1:

$$\frac{\partial \underline{p_A}}{\partial \Delta_\mathrm{B}} = \frac{T_2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2} - K_B\sigma_B^2}{\Delta_B^2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2}} < 0$$

for $\Delta_\mathrm{B} \in [\widehat{\Delta_B}, 1]$, where $T_2 = \frac{r_B^* OA_B}{1-r_B^*} - c_B - \mu_B$.

Proof: It is straightforward that we only need to show that $T_2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2} < K_B\sigma_B^2$. Since $K_B\sigma_B^2 < 0$, so that for $\Delta_\mathrm{B} \in (0, 1]$, the upper bound of $T_2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2}$ is $T_2\sigma_B$ when $\Delta_\mathrm{B} \rightarrow 0$. Therefore, proving $\frac{\partial p_A}{\partial \Delta_\mathrm{B}} < 0$ for all $\Delta_\mathrm{B} \in (0, 1]$ is equivalent to proving $T_2 <$

$K_B\sigma_B$, i.e., $r_B^* < \frac{c_B+\mu_B+K_B\sigma_B}{c_B+\mu_B+OA_B+K_B\sigma_B}$. Before buying additional risk assets, the $1-q_B$ quantile of r_B is $\frac{c_B+\mu_B+K_B\sigma_B}{c_B+\mu_B+OA_B+K_B\sigma_B}$. (See Appendix I.3 for the derivation of this expression.) If the regulatory requirement r_B^* is below $\frac{c_B+\mu_B+K_B\sigma_B}{c_B+\mu_B+OA_B+K_B\sigma_B}$, then Bank B is originally compliant with the regulation, and thus Bank B is eligible to take in more risky assets. In the context of my model, the regulation threshold r_B^* should be below the current $1-q_B$ quantile of r_B. Otherwise, Bank B is supposed to be lowering its holdings of risky assets to comply with regulation as well. Therefore, the condition $r_B^* < \frac{c_B+\mu_B+K_B\sigma_B}{c_B+\mu_B+OA_B+K_B\sigma_B}$ is satisfied, which guarantees $\frac{\partial \underline{p_A}}{\partial \Delta_B} < 0$ for all $\Delta_B \in (0,1]$. Namely, the supply function is monotonically decreasing over $\Delta_B \in (0,1]$.

It may be interesting to delve into the $r_B^* > \frac{c_B+\mu_B+K_B\sigma_B}{c_B+\mu_B+OA_B+K_B\sigma_B}$ case for further study. This is mainly because, if r_B^* is only slightly above $\frac{c_B+\mu_B+K_B\sigma_B}{c_B+\mu_B+OA_B+K_B\sigma_B}$, the supply curve only increases for very small Δ_B and then becomes downward sloping again. This paper focuses on the $r_B^* < \frac{c_B+\mu_B+K_B\sigma_B}{c_B+\mu_B+OA_B+K_B\sigma_B}$ case, which has more sensible financial meaning.

▪

The liquidity supply curve depicts the relationship between the amount of portfolio that Bank B can buy Δ_B and the trading price $\underline{p_A}$. Economically speaking, liquidity supply curves are downward sloping because the larger the spread between fundamental value and trading price, the more Bank B wants to purchase for profit maximization.

IV.3 Existence and Uniqueness of Equilibriums

To study the existence condition of liquid equilibrium, I calculate the $\widehat{\Delta_A}$ such that trading price equal to the fundamental value $\underline{p_A} = \mu_A$,

$$\widehat{\Delta_A} = \frac{\mu_A + K_A\sigma_A - \frac{r_A^* OA_A}{1-r_A^*}}{K_A\sigma_A}$$

$\widehat{\Delta_A}$ is the required liquidity support for fundamental price. When the market capacity for Bank A's portfolio is more than $\widehat{\Delta_A}$, the market is in liquid equilibrium. Then I calculate $\widehat{\Delta_B}$ such that $\underline{p_A} = \mu_A$,

$$\widehat{\Delta_B} = \sqrt{\frac{\left(\frac{\frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B}{K_B}\right)^2 - \sigma_A^2}{\sigma_B^2}}$$

$\widehat{\Delta_{\mathrm{B}}}$ is the market capacity for Bank A's portfolio at fundamental price, which means the market is only able to absorb $\widehat{\Delta_{\mathrm{B}}}$ amount of Bank A's portfolio at fundamental price. For now, simply comparing $\widehat{\Delta_{\mathrm{B}}}$ and $\widehat{\Delta_{\mathrm{A}}}$ reveals the existence of **Liquid Equilibrium**.

Proposition 2**:** If $\widehat{\Delta_{\mathrm{A}}} \leq \widehat{\Delta_{\mathrm{B}}}$, the market has **Liquid Equilibrium**. Otherwise, the market either has **Illiquid Equilibrium** (with liquidity shortage) or **No Equilibrium** (with zero liquidity).

Put differently, in the latter two cases, Bank B either requires a significant liquidity risk premium as compensation for taking over illiquid assets or doesn't take over illiquid assets at all. Also note that increasing Bank B's target ratio reduces market capacity $\widehat{\Delta_B}$ because

$$\frac{\partial \widehat{\Delta_B}}{\partial r_B^*} = \frac{T_2\left(\frac{OA_B}{1 - r_B^*} + \frac{r_B^* OA_B}{(1 - r_B^*)^2}\right)}{\sigma_B K_B^2 \sqrt{\frac{(T_2)^2}{K_B^2} - \sigma_A^2}} < 0$$

where $T_2 = \frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B < 0$. However, the required liquidity support $\widehat{\Delta_A}$ is invariant to changes in r_B^*, namely Bank A's decision is independent of the actual liquidity condition in the market.

Next, in the case of illiquid equilibriums where $\widehat{\Delta_{\mathrm{B}}} < \widehat{\Delta_{\mathrm{A}}}$, I show the condition for the uniqueness of equilibrium. For an **Illiquid Equilibrium**, the equation $\Delta_A = \Delta_B = \Delta^e$ at $\underline{p_A^e} \in [\widetilde{p_A}, \mu_A)$ leads to,

$$\frac{\frac{r_A^* OA_A}{1 - r_A^*} - \mu_A - K_A \sigma_A}{\Delta^e} + \mu_A + K_A \sigma_A = \mu_A + \frac{K_B \sqrt{\sigma_B^2 + (\Delta^e)^2 \sigma_A^2} - \left(\frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B\right)}{\Delta^e}$$

Solving the above equation will give the solution of Δ^e and $\underline{p_A}^e$. When $K_A = K_B$, the above equation has unique solution if it exists in $\Delta^e \in \left(\widehat{\Delta_B}, 1\right]$, $\underline{p_A^e}(\Delta^e) \in [\widetilde{p_A}, \mu_A)$.

Proposition 3: If $K_A = K_B = K$, the Illiquid Equilibrium $(\underline{p_A^e}(\Delta^e), \Delta^e)$, if it exists, must be unique.

$$\Delta^e = \frac{K^2\sigma_B^2 - (T_1 + T_2)^2}{2(T_1 + T_2)K\sigma_A}$$

where $\Delta^e \in \left(\widehat{\Delta_B}, 1\right]$, $\underline{p_A^e}(\Delta^e) \in [\widetilde{p_A}, \mu_A)$.

$$T_1 = \frac{r_A^* OA_A}{1 - r_A^*} - \mu_A - K\sigma_A \text{ and } T_2 = \frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B.$$

Multiple illiquid equilibria may exist if $K_A \neq K_B$, which is equivalent to $q_A \neq q_B$. Since Bank A is selling assets at fire-sale prices to meet regulatory requirements, the equilibrium with lowest Δ^e and highest $\underline{p_A^e}(\Delta^e)$ at the mean time is the most favorable equilibrium for Bank A. Note that for the lowest Δ^e, $\underline{p_A^e}(\Delta^e)$ is assured to be the highest equilibrium trading prices among potential multiple equilibriums because liquidity demand function is monotonically decreasing with respect to Δ^e.

V. Numerical Examples and Comparative Statics Analysis

V.1 Numerical Examples of Liquidity Demand and Supply Curve

Before providing the simulation results of r_A, r_B, liquidity supply curve and liquidity demand curve to further illustrate the model, it is helpful to see how the distributions of r_A and r_B are affected by trading the portfolio.

For r_A, I can first simulate its distribution using expression (1) at any given $\underline{p_A}$ *and* Δ_A. Parameters used for the simulations in Figure 1 are the following: $f_A = \mu_A = f_B = \mu_B = 1$, $OA_A = OB_B = 4, c_B = 1$, $\sigma_A = \sigma_B = 0.2$, $e_{AA} = e_{BB} = 1, q_A = q_B = 99\%$.

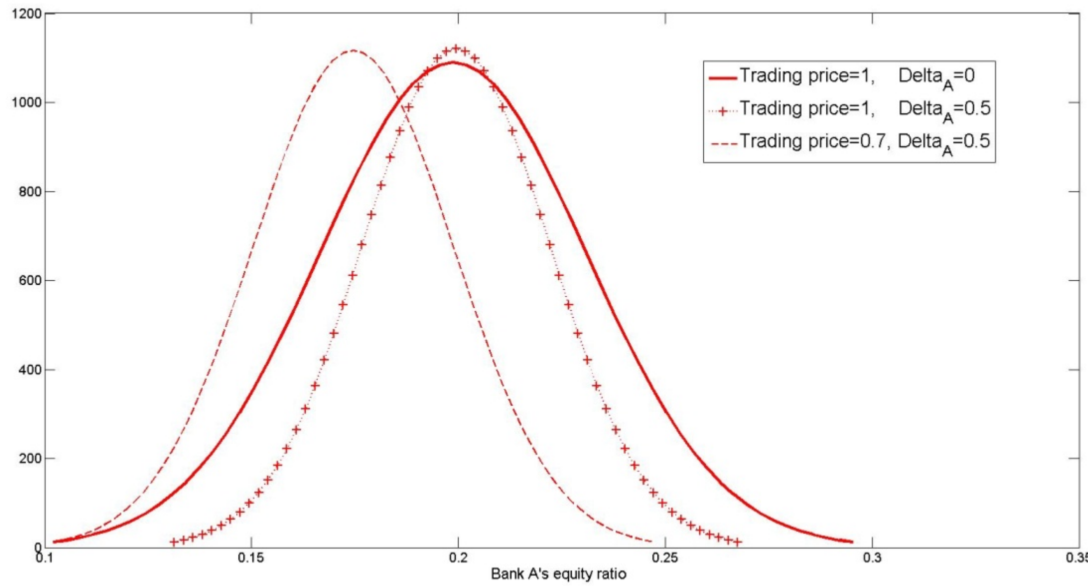


Figure 1: The distributions of Bank A's capital ratios under three conditions.

As shown in Figure 1, if Bank A liquidates half its portfolio at μ_A, then r_A's distribution will become more concentrated. (Figure 1: $\underline{p_A} = 1$ and $\Delta_A = 0.5$) If Bank A must liquidate at a loss, the distribution will still be more concentrated but shifted to the left in parallel. (Figure 1: $\underline{p_A} = 0.7$ and $\Delta_A = 0.5$)

Based on the distribution of r_A, it is easy to obtain the liquidity demand curve of Bank A at a given r_A^*. For any given $\underline{p_A}$, increase Δ_A from zero until the q_A quantile of the distribution is just equal to r_A^*. Repeating such iteration at all $\underline{p_A} \epsilon [\widetilde{p_A}, \mu_A]$ will generate the liquidity demand curve in the situation where the regulatory requirement is (r_A^*, q_A) Here are several examples of simulated liquidity demand curves based on different r_A^*s fitted with analytical solutions. (Fig. 2)

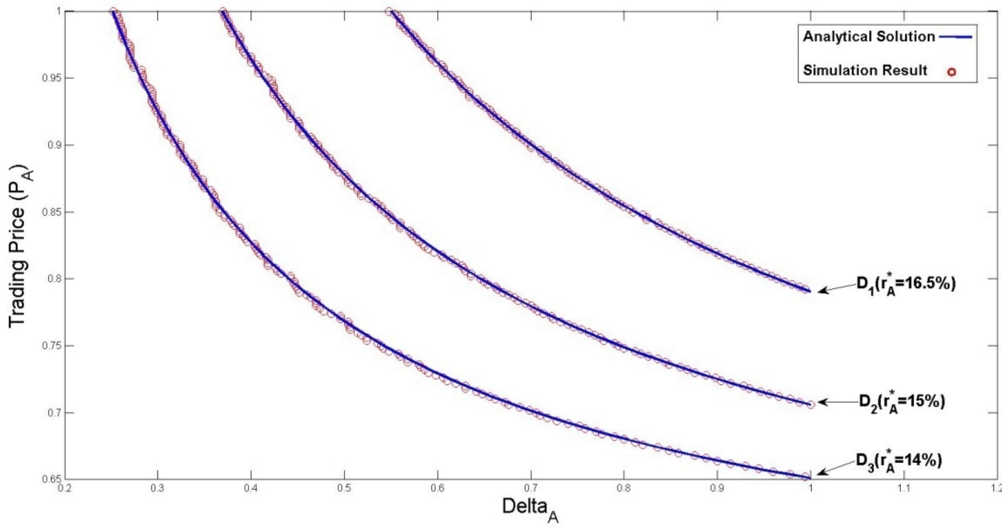


Figure 2: Bank A's liquidity demand curves with different target ratios.

It is clear that increasing r_A^* shifts demand curve rightward. Then I show the relationship between the demand curve and r_A^* more formally.

Proposition 4: $\frac{d\Delta_A}{dr_A^*} > 0$, i.e., liquidity demand curves shift upward if r_A^* increases.

Proof: From Section IV.2, the demand function is

$$\Delta_A = \frac{\frac{r_A^* OA_A}{1 - r_A^*} - \mu_A - K_A\sigma_A}{\underline{p_A} - \mu_A - K_A\sigma_A},$$

and thus

$$\frac{d\Delta_A}{dr_A^*} = \frac{\frac{OA_A}{1 - r_A^*} + \frac{r_A^* OA_A}{(1 - r_A^*)^2}}{\underline{p_A} - \mu_A - K_A\sigma_A}.$$

In order to have $\frac{d\Delta_A}{dr_A^*} > 0$, it must be true that $\underline{p_A} - \mu_A - K_A\sigma_A > 0$ for all $\underline{p_A} \in [\widetilde{p_A}, \mu_A)$, which in turn implies that $\widetilde{p_A} - \mu_A - K_A\sigma_A > 0$ must be true. With the previous result $\widetilde{p_A} = \frac{r_A^* OA_A}{1-r_A^*}$, showing $\widetilde{p_A} - \mu_A - K_A\sigma_A > 0$ is equivalent to showing $r_A^* > \frac{\mu_A + K_A\sigma_A}{\mu_A + K_A\sigma_A + OA_A}$.

Now I employ the similar method used in the proof of **Proposition 1**. As $\frac{\mu_A + K_A\sigma_A}{\mu_A + K_A\sigma_A + OA_A}$ stands for the $1 - q_A$ quantile of r_A before Bank A selling assets, $r_A^* > \frac{\mu_A + K_A\sigma_A}{\mu_A + K_A\sigma_A + OA_A}$ must be true, otherwise Bank A does not have to liquidate its portfolio in the first place. Therefore, $\frac{d\Delta_A}{dr_A^*} > 0$ is true.

▪

When it comes to Bank B, if it buys a half of Bank A's portfolio at μ_A, then the distribution of r_B will be more dispersed. (Figure 3: $\underline{p_A} = 1$ and $\Delta_A = 0.5$) If $\underline{p_A} = 0.7$, Bank B makes a profit of 0.3 from each unit of portfolio bought from Bank A. In this case, the distribution will still be more dispersed but shifted to the right in parallel. (Figure 3: $\underline{p_A} = 0.7$ and $\Delta_A = 0.5$) Based on a similar simulation algorithm, several liquidity supply curves are obtained and fitted with analytical solutions in Fig. 4. For $\underline{p_A} = \mu_A$, Bank B is indifferent among $[0, \widehat{\Delta_B}]$ as its expected profit is zero for all $\Delta_B \in [0, \widehat{\Delta_B}]$. This situation is captured by the horizontal line at $\underline{p_A} = \mu_A = 1$ in Fig. 4.

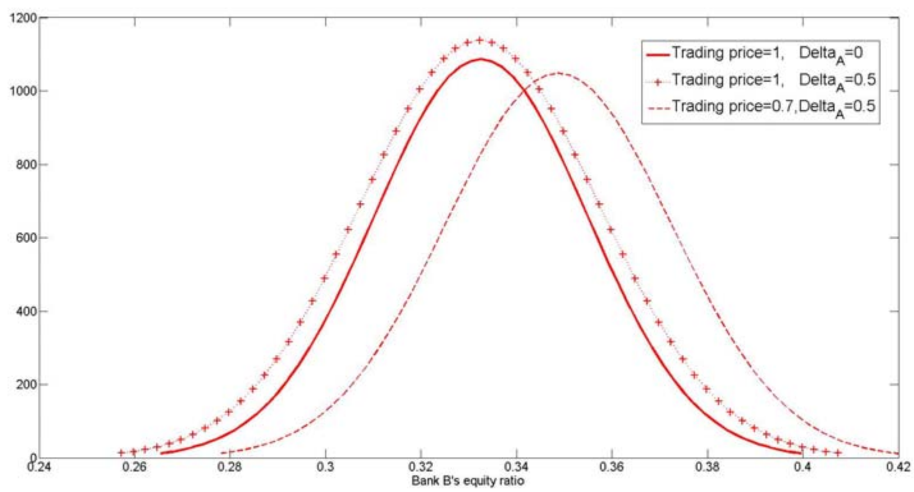


Figure 3: The distributions of Bank B's capital ratios under three conditions.

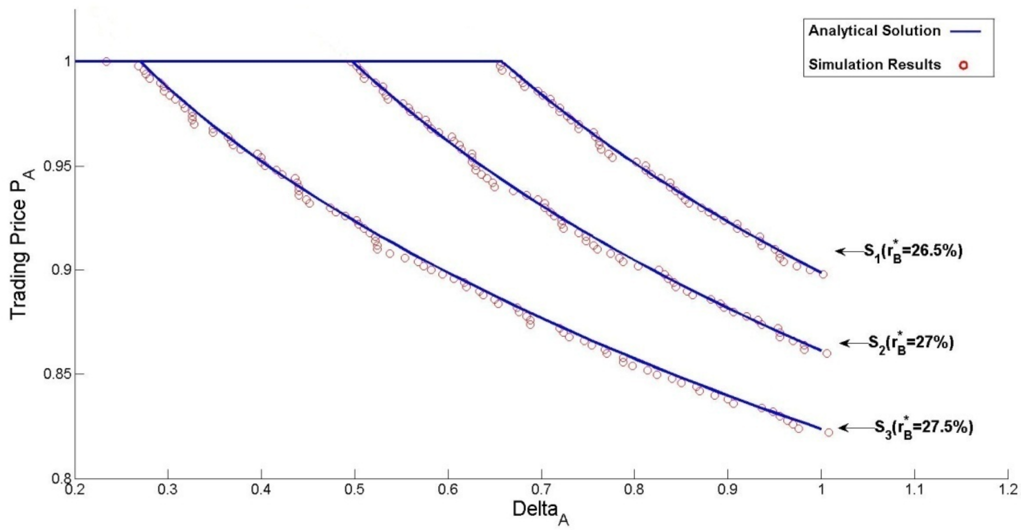


Figure 4: Bank B's liquidity supply curve with different target capital ratios.

Finally, I prove the proposition regarding the relationship between liquidity supply curves and r_B^*.

***Proposition 5*:** $\frac{d\Delta_B}{dr_B^*} < 0$, i.e., liquidity supply curves shift downward as r_B^* increases.

Proof: Define a new implicit function of Δ_B and r_B^*,

$$F(\Delta_\mathrm{B}, r_B^*) = \mu_A + \frac{K_B\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2} - (\frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B)}{\Delta_\mathrm{B}} - \underline{p_A} = 0$$

Thus,

$$\frac{d\Delta_B}{dr_B^*} = -\frac{\frac{\partial F(\Delta_\mathrm{B}, r_B^*)}{\partial r_B^*}}{\frac{\partial F(\Delta_\mathrm{B}, r_B^*)}{\partial \Delta_\mathrm{B}}} = -\frac{\frac{-(\frac{OA_B}{1-r_B^*} + \frac{r_B^* OA_B}{(1-r_B^*)^2})}{\Delta_\mathrm{B}}}{\frac{T_2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2} - K_B\sigma_B^2}{\Delta_B^2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2}}} < 0.$$

(See the proof of $\frac{T_2\sqrt{\sigma_B^2+\Delta_B^2\sigma_A^2} - K_B\sigma_B^2}{\Delta_B^2\sqrt{\sigma_B^2+\Delta_B^2\sigma_A^2}} < 0$ in the proof of **Proposition 1**)

V.2 Numerical Examples of Different Equilibriums

After showing the property of liquidity supply and demand curve, the next two sections focus on different types of equilibria along with a comparative statics analysis regarding the riskiness of Bank A's portfolio. It is found that highly risky assets face more severe liquidity shortage in a stress market, and thus they have large swings in prices.

The demand curve $\mathbf{D_2}$ and the supply curve $\mathbf{S_2}$ in Fig. 5 generate a **Liquid Equilibrium** $\mathbf{E_1}$. When $\underline{p_A} = \mu_A = 1$, the maximum amount that Bank B is willing to take over $\widehat{\Delta_B}$ is more than what Bank A needs to liquidate, i.e. $\widehat{\Delta_A} \epsilon [0, \widehat{\Delta_B}]$. Hence, Bank A is able to liquidate around 40% of its portfolio at the current fundamental price in a liquid market.

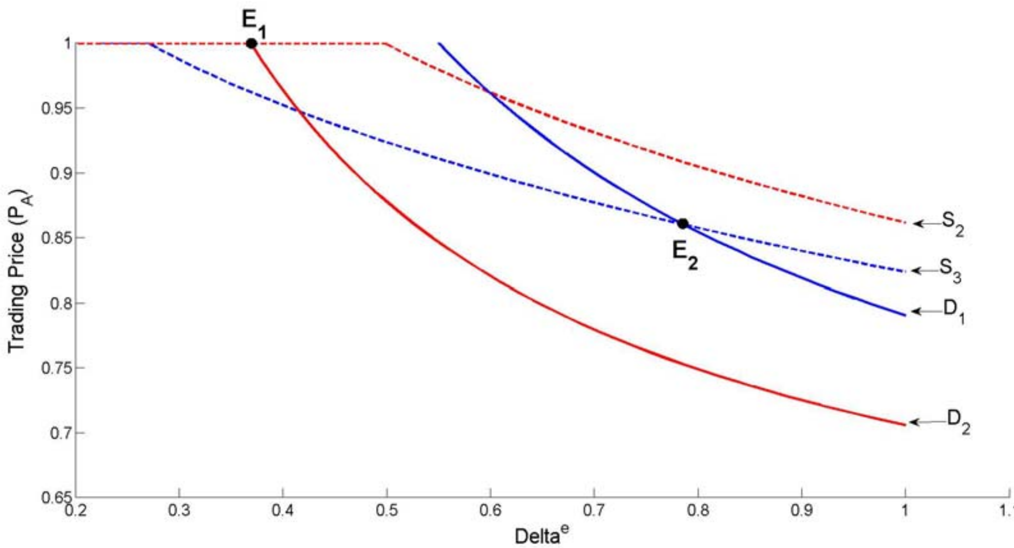


Figure 5: Use supply and demand curves to determine the equilibrium. Note that the demand and supply curves in Fig. 5 are taken directly from Fig. 2 and Fig. 4. Dashed lines are supply curves, and solid lines are demand curves.

Fixing either the demand or supply curve determines a "tipping point" that differentiates liquid market and illiquid market. For example, given a fixed $\mathbf{S}(r_B^*)$, the equilibrium price is invariant to any marginal shift in the liquidity demand curves as long as $\widehat{\Delta_A} \leq \widehat{\Delta_B}$. This property is consistent with the generalized characteristic of liquidity risk that I mentioned in the Introduction: liquidity risk does not manifest itself in normal times, so that assets are traded at fundamental prices in equilibrium. Beyond this critical point ($\widehat{\Delta_A} = \widehat{\Delta_B}$), the equilibrium price will become very sensitive to marginal changes in liquidity demand as the market may fall into an **Illiquid Equilibrium** $\widehat{\Delta_A} < \widehat{\Delta_B}$. Consider the Illiquid Equilibrium $\mathbf{E_2}$ generated by supply curve $\mathbf{S_1}$ and demand curve $\mathbf{D_3}$ in Fig. 5. The equilibrium price drops by roughly fifteen percent with only small changes in r_A^* and r_B^*.

The high sensitivity is mainly due to the similar supply curve slope and demand curves slope. The high price sensitivity confirms the other characteristic of liquidity risk mentioned in the Introduction: without any change in the fundamental value of an asset, liquidity risk itself makes asset prices greatly deviate from the fundamental value, and asset prices are very sensitive to marginal changes in market liquidity condition.

V.3 Comparative Statics Analysis

To better illustrate the characteristics of liquidity risk implied by this model, I will conduct a comparative statics analysis between two portfolios with different volatilities. For the required liquidity support, the first sensitivity analysis regards the standard deviation of f_A. The first order partial derivative of Δ_A with respect to σ_A is,

$$\frac{\partial \Delta_A}{\partial \sigma_A} = \frac{\left(\frac{r_A^* OA_A}{1-r_A^*} - \underline{p_A}\right) K_A}{\left(\underline{p_A} - \mu_A - K_A \sigma_A\right)^2} > 0$$

for $\underline{p_A} \in (\widetilde{p_A}, \mu_A]$. The above inequality holds for all $\underline{p_A} \in (\widetilde{p_A}, \mu_A]$, because $\frac{r_A^* OA_A}{1-r_A^*} = \widetilde{p_A}$. $\frac{\partial \Delta_A}{\partial \sigma_A} > 0$ means that the required liquidity support increases with $\sigma_A$at all given prices. On the other hand, the market capacity reduces with σ_A at all given prices because $\frac{\partial \Delta_B}{\partial \sigma_A} < 0$ (See Appendix II.1 for the proof). Based on the above comparative analysis, I pose the following proposition,

Proposition 6**:** For $\underline{p_A} \in (\widetilde{p_A}, \mu_A]$, $\Delta_A \in [0,1]$, and $\Delta_B \in [0,1]$, higher riskiness of Bank A's portfolio results in leftward shift in liquidity supply curves and rightward shift in liquidity demand curves, i.e. $\frac{\partial \Delta_B}{\partial \sigma_A} < 0$ and $\frac{\partial \Delta_A}{\partial \sigma_A} > 0$.

With the opposite effects of σ_A on Δ_A and Δ_B, it is expected that the market situation will deteriorate if σ_A increases. The following numerical example will demonstrate how the originally liquid equilibrium would become an illiquid equilibrium with the increase in the asset riskiness. I compare the market liquidity condition of highly risky portfolio $\boldsymbol{\sigma'_A} = 25\%$ (Fig. 6) with previous examples of portfolio $\boldsymbol{\sigma_A} = 20\%$ in Figure 5.

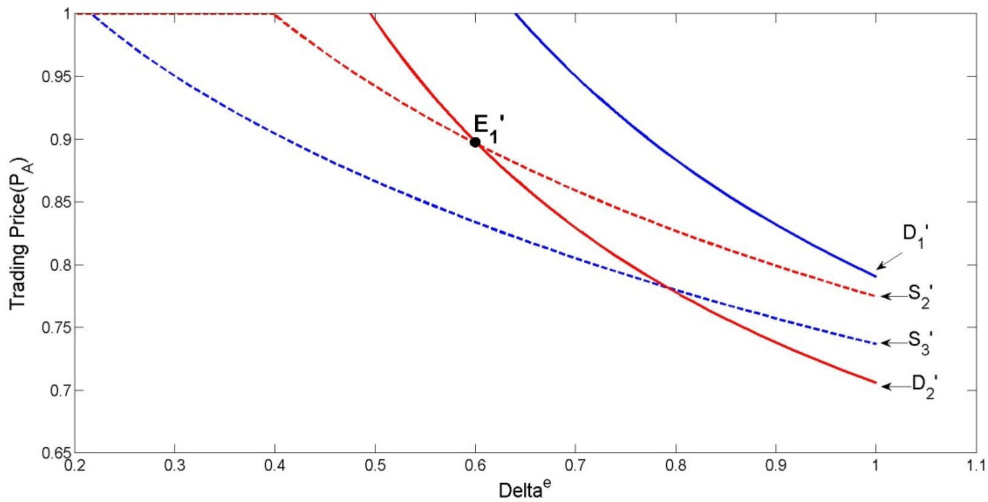


Figure 6: Demand and supply curves with portfolio's volatility equal to 25%.

As shown both analytically and numerically, the increase in riskiness shifts liquidity demand curves rightward, and it shifts liquidity supply curve leftward. Compared with the previous Illiquid Equilibrium $\mathbf{E_2}$ generated by $\mathbf{S_3}$ and $\mathbf{D_1}$ (Fig. 5), two Banks are not able to reach an equilibrium given $\sigma_A' = 25\%$ (See $\mathbf{S_3'}$ and $\mathbf{D_1'}$ in Fig. 6). The market provides zero liquidity for Bank A's highly risky portfolio because the liquidity required at any $\underline{p_A} \in (\widetilde{p_A}, \mu_A]$ is higher than market capacity. With $\mathbf{S_2}$ and $\mathbf{D_2}$ (Fig. 5) shifting to $\mathbf{S_2'}$ and $\mathbf{D_2'}$ (Fig. 6), the originally liquid equilibrium ($\mathbf{E_1}$ in Fig. 5) becomes an illiquid equilibrium ($\mathbf{E_1}'$in Fig. 6). Bank A now has to sell nearly 60% of its portfolio at 10% loss.

The intuitive explanation is as follows. First, investors tend to be reluctant to purchase highly volatile assets in a stressed market, and thus the liquidity supply for such assets is low. Second, in terms of reducing the uncertainty of capital ratio, selling highly volatile assets is more efficient relative to selling less volatile assets. Banks tend to sell risky asset first, which in turn creates higher liquidity demand. The increase in liquidity demand superimposes the decrease of liquidity supply to generate severe downward pressure on the asset price.

VI. Applications on Liquidity Related Issues

Based on the recent crisis in 2007 and 2008, Brunnermeier emphasizes three mechanisms—precautionary hoarding by individual banks, runs on financial institutions and loss spiral, through which mortgage delinquency shock is amplified to a full-blown liquidity crisis (Brunnermeier 2009). The three liquidity-related issues are closely related to my current model.

VI.1 Precautionary Hoarding

The model has an interesting prediction for the impact of precautionary hoarding on asset prices. In the context of my model, Bank B is the financial institution that can conduct precautionary hoarding by setting an overly conservative target ratio. If Bank B is afraid of potential liquidity shocks against itself, it may set its target capital ratio at a very conservative level. Thus, the market capacity reduces as $\frac{\partial \Delta_B}{\partial r_B^*} < 0$ (**Proposition 1**), which may lead Bank A to liquidate at greater losses. Besides dealing with uncertain liquidity demand, Bank B prefers hoarding to supplying sufficient liquidity for another reason. If Bank B realizes its monopolistic power in the market, it is able to make more profit by setting a high r_B^*. To reflect Bank B's decision on the target ratio, I modify Bank B's problem by allowing Bank B to freely set its target ratio above the regulatory requirement. The modified profit maximization problem is shown below.

$$\underset{r_B^*}{Max}\ Z = (\mu_A - \underline{p_A})\ \Delta^e$$

$$s.t.\ \ 0 < \Delta^e \leq 1$$

$$\Delta^e = \frac{K^2\sigma_B^2 - (T_1 + T_2)^2}{2(T_1 + T_2)K\sigma_A}$$

$$\underline{p_A} = \frac{\frac{r_A^* OA_A}{1 - r_A^*} - \mu_A - K\sigma_A}{\Delta^e} + \mu_A + K\sigma_A$$

$$r_B^* \geq R_{regulation}$$

where $T_1 = \frac{r_A^* OA_A}{1 - r_A^*} - \mu_A - K\sigma_A$ and $T_2 = \frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B$

Based on the previous analytical solutions of the general equilibrium, Bank B can choose an optimal r_B^* to achieve an optimal equilibrium with maximum profit. For simplicity, I only study the case where unique illiquid equilibrium occurs under the condition $K_A = K_B = K$. The objective function Z and $\frac{dZ}{dr_B^*}$ have explicit expressions in terms of r_B^*:

$$\frac{dZ}{dr_B^*} = \frac{d(\mu_A - \underline{p_A})\ \Delta^e}{dr_B^*}$$

$$= \frac{d}{dr_B^*}(-T_1 - K\sigma_A\Delta^e)$$

$$= \frac{1}{2}[K^2\sigma_B^2 + (T_1 + T_2)^2]\left[\frac{OA_B}{1 - r_B^*} + \frac{r_B^* OA_B}{(1 - r_B^*)^2}\right] > 0\ for\ all\ r_B^* \in [0,1]$$

Therefore, Bank B will set its target ratio as high possible. It will not choose any r_B^* that makes $\underline{p_A}$ lower than $\widetilde{p_A}$, because such r_B^* leads to a zero-liquidity market in which Bank A will not trade at all. (Recall that $\widetilde{p_A} = \frac{r_A^* OA_A}{1-r_A^*}$, and it is the lowest price that Bank A would accept to sell the entire portfolio.) The relationship between Bank B's profit and its target ratio is summarized in the following proposition.

Proposition 7**:** For $\Delta^e(r_B^*) \in [0,1]$ and $\underline{p_A}(\Delta^e) \in [\widetilde{p_A}, \mu_A]$, Bank B's profit Z increases with r_B^* because $\frac{dZ}{dr_B^*} > 0$ for all $r_B^* \in [0,1]$.

I provide a numerical example (Fig. 7) to demonstrate the effect of r_B^* on $\underline{p_A}$, Δ^e, Z and the expected return $\frac{Z}{\underline{p_A} * \Delta^e}$. I select the demand curve **D₁** (where $r_A^* = 16.5\%$ in Fig. 2), and calculate the equilibrium prices as r_B^* changes from 26% to 28% to obtain Figure 7. When $r_B^* \leq 26.85\%$, there is sufficient liquidity in the market to support trades at the fundamental value. Illiquid Equilibria occur for $26.85\% < r_B^* \leq 27.93\%$, the equilibrium trading price will fall by about twenty percent if r_B^* drops by merely one percent. For $r_B^* > 27.93\%$, no equilibrium occurs and there is zero liquidity in the market. Overall, this high price sensitivity is consistent with the observation—when market participants become concerned about liquidity issues, asset prices fall drastically. The upward sloping Z in illiquid market confirms the result $\frac{dZ}{dr_B^*} > 0$. The upward sloping expected return curve suggests that if Bank B's object is to maximize expected return instead of expected profit, the optimal r_B^* remains the same.

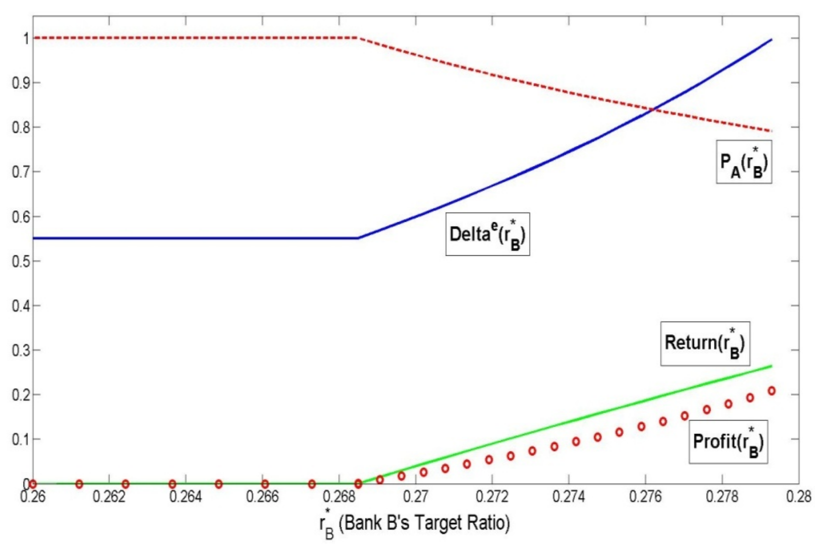


Figure 7: Given a liquidity demand curve, the equilibrium price changes with Bank B's target ratio.

To sum up, my model suggests that when facing a desperate seller (Bank A), hoarding is an effective way for the counterparty financial institution to earn profit and maintain solid financial condition at the same time. The market situation, however, deteriorates in the sense that asset prices have larger swings and the desperate seller suffers a huge loss.

VI.2 Bank Run Loss

The precautionary case does not incorporate any externality of deteriorating market situation. A noticeable adverse consequence is runs on financial institutions. If investors are panicked by the steep decline in trading price, they may trigger another amplifying mechanism—runs on financial institutions. Depositors want to withdraw funds as early as possible if they are afraid that Bank A will become insolvent in the future. There are two scenarios for depositors' behavior:

Scenario 1: Depositors are well informed, namely they know this trade with discounted price is only due to an idiosyncratic shock against Bank A. Since the quality of Bank A's portfolio is fundamentally unchanged, it is not necessary to withdraw deposits immediately. Bank A manages to save itself via liquidation.

Scenario 2: Depositors are not well informed, namely they mistakenly attribute the steeply declining asset price to a downgrade of Bank A's portfolio or a decrease in its fundamental value. Fearful investors may start to dump those holdings that are most correlated with Bank A's portfolio, which aggravates Bank A's financial condition. Or a bank run may drain the cash that Bank A has just acquired. The consequence will spread to Bank B as bank runs are often indiscriminate. Thus, it is assumed that once depositors decide to run, they run on all banks regardless of the financial condition of individual institutions. Another critical property of a bank run assumed in this scenario is that the greater the declines in asset prices, the more likely investors are to run. With an assumed constant bank run loss, the expected bank run loss increases accordingly.

Under **Scenario 2**, although the spread between the trading price and the fundamental value is still a profit opportunity for Bank B, it also causes a higher probability of a bank run. After calculating the expected trading profit and the expected bank-run loss, Bank B may be willing to supply more liquidity to purchasing assets rationally at higher prices. Current literature about bank runs does not provide an explicit quantitative relationship for the probability of bank run and the asset price. The study of such relationship is beyond the scope of this paper. To illustrate, I assume a logistic probability function for the probability of bank run and a lump-sum loss if a bank run occurs. Suppose that bank-run loss is \$0.2, and the probability of a bank run follows

$$Pr(Bankrun) = \frac{1}{1 + exp\left[-\frac{Fall\ in\ asset\ price - 0.17}{0.02}\right]}$$

Then Bank B's problem differs from the last one only in terms of the objective function, and the new one is:

$$\underset{r_B^*}{Max}\ Z = (\mu_A - \underline{p_A})\ \Delta^e - \frac{0.2}{1 + \exp\left[-\frac{(\mu_A - \underline{p_A}) - 0.17}{0.02}\right]}$$

Compared with the no-bank-run case, the $Z(r_B^*)$ is not monotonically increasing anymore as the expected bank-run loss eventually would exceed the trading profit. Based on the results in Fig. 8, it is obvious that focusing only on trading profit is a myopic behavior. Bank B should instead a set lower target ratio to maximize the expected net profit. Bank B's lower target ratio would ease the depressed trading price and liquidity shortage.

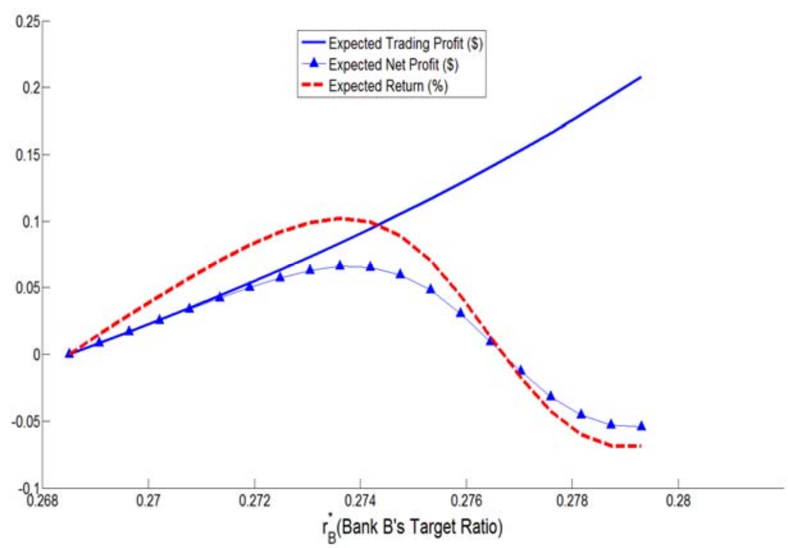


Figure 8: Bank B's expected trading profit, expected net profit, and expected return with r_B^*.

To conclude, my model shows that financial institutions are willing to hoard less liquidity and rationally purchase assets at higher prices in the case where a bank run is incorporated. That is to say, if market participants are provided with clear information about the externalities of declining assets prices, the market liquidity position can be moderately eased to generate higher trading prices in equilibrium.

VI. 3 Loss Spiral

Following the Bank A’s fire sale, a loss spiral is another mechanism that can lead to a deteriorating market situation. Marking-to-market is the mechanism that triggers loss spiral. Suppose that after liquidating $\Delta^e \in (0,1)$ in an illiquid equilibrium with a fire-sale price, Bank A has to mark its remaining portfolio to the equilibrium trading price $\underline{p_A}^e$, then the write-down loss would immediately bring down Bank A's capital ratio. Again, if Bank A has to be compliant with the regulation, it has to sell more assets at even lower prices. It is expected that with reiterated costly liquidations and write-down losses, Bank A's problem would quickly evolve from illiquidity to insolvency. The critical concern about marking-to-market is that it relies on the perception that the market price should reflect the "fair value" of assets. When a market lacks liquidity, however, sporadic trades of an asset do not necessarily reflect its fundamental price. As my model has shown, the trading price may include a large "liquidity risk premium" and it deviates greatly from fundamental value. In general, banks should be relieved from marking-to-market, at least during times of stress, to prevent liquidity problems from being transmitted to solvency problems.

VII. Conclusion

This paper studies the effect of liquidity shortage on asset prices using a general equilibrium approach. Two critical results of this liquidity risk model are as follows. First, this model generates explicit downward sloping liquidity demand and supply curves. Second, this model inherently produces a tipping point that differentiates liquid market from illiquid market. In a liquid market, the equilibrium trading price equals the fundamental value of an asset. In an illiquid market, however, the equilibrium trading price deviates from the fundamental value and becomes very sensitive to marginal changes in market liquidity position. This property is consistent with the observation that liquidity risk only manifest in turbulent markets to depress asset prices.

My model also sheds light on three liquidity related issues, including precautionary hoarding by individual banks, runs on financial institutions and the loss spiral problem. The important conclusions are: although hoarding is an effective way for financial institutions to earn profit, the externality of hoarding should be taken into consideration to achieve a better overall market condition; moreover, financial institutions should be relieved from marking-to-market, at least in the time of stress, to prevent liquidity problems from being transmitted to solvency problems.

For further studies, the model can be expanded to study financial contagion by adding more financial institutions to the market and using the common illiquid risky asset as the channel of contagion. As several other forms of liquidity supply and demand functions are simply assumed in the literature of financial contagion, it may be interesting to construct a financial contagion model based on the liquidity supply and demand function derived in this paper, and then compare the results with existing literature.

References

Allen, F., E. Carletti, and D. Gale. 2009. "Interbank Market Liquidity and Central Bank Intervention." *Journal of Monetary Economics* 56: 639-652.

Allen, Franklin, and Douglas Gale. 2000. "Financial Contagion." *Journal of Political Economy* 108 (1): 1-33.

Brunnermeier, Markus K. 2009. "Deciphering the Liquidity and Credit Crunch 2007-2008." *Journal of Economic Perspectives* 23 (1): 77-100.

Brunnermeier, Markus K., and Lasse Heje Pederson. 2009. "Market Liquidity and Funding Liquidity." *The Review of Financial Studies* 22 (6): 2201–2238.

Chen, L., D. A. Lesmond, and J. Wei. 2007. "Corporate Yield Spreads and Bond Liquidity." *The Journal of Finance* 62 (1): 119-149.

Cifuentes, R., G. Ferrucci, and H. S. 2005. "Liquidity Risk and Contagion." *Journal of the European Economics Association* 3 (2-3): 556-566.

De Jong, Frank, and Joost Driessen. 2006. "Liquidity Risk Premia in Corporate Bond Markets." Working Paper. Available at SSRN: http://ssrn.com/abstract=686681 or http://dx.doi.org/10.2139/ssrn.686681

Eisenschmidt J., and J. Tapking J. 2009. "Liquidity Risk Premia in Unsecured Interbank Money Markets." ECB Working Paper No.1025, European Central Bank, Frankfurt.

Garcia, Alejandro, and Andrei Prokopiw. 2009. "Measure of Aggregate Credit Conditions and Their Potential Use by Central Banks." Bank of Canada Discussion Paper 2009-12, Bank of Canada, Ottawa.

Gorton, Gary, and Lixin Huang. 2006. "Bank Panics and the Endogeneity of Central Banking." *Journal of Monetary Economics* 53: 1613-1629.

Krishnamurthy, Arvind. 2010. "How Debt Market Have Malfunctioned in the Crisis." *Journal of Economic Perspectives*. 24 (1): 3-28.

Longstaff, F., S. Mithal, and E. Neis. 2005. "Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Default Swap Market." *Journal of Finance* 60 (5): 2213-2253.

Tirole, Jean. 2011. "Illiquidity and All Its Friends." *Journal of Economic Literature* 49 (2): 287-325.

Appendix I.1

In order to have $Pr\left(1-\frac{OA_A}{f_{ANew}+OA_A}\le r_A^*\right)=1-q_A$ equivalent to $Pr\left(f_{ANew}\le \frac{r_A^* OA_A}{1-r_A^*}\right)=1-q_A$, $f_{ANew}+OA_A$ must be positive, which is not guaranteed mathematically because f_{ANew} follows a normal distribution. I will discuss the situations that $f_{ANew}+OA_A\le 0$ to show that $f_{ANew}+OA_A>0$ is a reasonable assumption.

Since $f_{ANew}+OA_A$ financially stands for the total asset of Bank A, if $f_{ANew}+OA_A<0$, Bank A should file bankruptcy because it loses all of its assets. Moreover, $f_{ANew}+OA_A=0$ is a zero probability event, so it is impossible to occur. Lastly, if $f_{ANew}+OA_A\to 0$ from the right side, r_A approaches negative infinite, which is included in the case $1-\frac{OA_A}{f_{ANew}+OA_A}\le r_A^*$. Therefore, studying the model under the assumption that $f_{ANew}+OA_A>0$ is reasonable and it does not impede the generality of this model.

Appendix I.2

Bank B's capital ratio constraint says that the capital ratio must be greater than or equal to the target ratio:

$$r_B=\frac{f_B * e_{BB}+\Delta_B f_A+(c_B-\Delta_B\,\underline{p_A})}{f_B * e_{BB}+\Delta_B f_A+(c_B-\Delta_B\,\underline{p_A})+OA_B}\ge r_B^*$$

Set $e_{BB}=1$ to have the normalized expression for this constraint, and define $f_B+\Delta_B f_A$ as a new normal random variable f_{BNew}, with mean of $\mu_B+\Delta_B\mu_A$ and variance $\sigma_B^2+\Delta_B^2\sigma_A^2$. The second constraint of Bank B's problem becomes

$$Pr\left(\frac{f_{BNew}+c_B+\Delta_B\,\underline{p_A}}{f_{BNew}+c_B+\Delta_B\,\underline{p_A}+OA_B}\ge r_B^*\right)=q_B$$

which is equivalent to

$$\Pr(f_{BNew}\le \Delta_B\underline{p_A}-c_B+\frac{r_B^* OA_B}{1-r_B^*}\ )=1-q_B$$

Here I use the property that Bank B's constraint on r_B must be binding in an optimal solution, as well as the argument in Appendix I.1 regarding the sign of the above inequality. Since f_{ANew} follows a normal distribution, the above equation is equivalent to

$$Pr(z\le\frac{\Delta_B\underline{p_A}-c_B+\frac{r_B^* OA_B}{1-r_B^*}-(\mu_B+\Delta_B\mu_A)}{\sqrt{\sigma_B^2+\Delta_B^2\sigma_A^2}}\ )=1-q_B$$

Let $K_B=\Phi^{-1}(1-q_B)$, and thus

$$\frac{\Delta_B \underline{p_A} - c_B + \frac{r_B^* OA_B}{1 - r_B^*} - (\mu_B + \Delta_B \mu_A)}{\sqrt{\sigma_B^2 + \Delta_B^2 \sigma_A^2}} = K_B$$

Rearrange to obtain the liquidity supply curve of Bank B,

$$\underline{p_A} = \mu_A + \frac{K_B \sqrt{\sigma_B^2 + \Delta_B^2 \sigma_A^2} - (\frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B)}{\Delta_\mathrm{B}}$$

Note that

$$\frac{\partial \underline{p_A}}{\partial \Delta_\mathrm{B}} = \frac{T_2 \sqrt{\sigma_B^2 + \Delta_B^2 \sigma_A^2} - K_B \sigma_B^2}{\Delta_B^2 \sqrt{\sigma_B^2 + \Delta_B^2 \sigma_A^2}}$$

where $T_2 = \frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B$.

Appendix I.3

Given that $\widetilde{r_B}$ is the $1 - q_B$ percent quantile of r_B before Bank B purchases any risky portfolio, it follows that

$$Pr\left(\frac{f_B + c_B}{f_B + c_B + OA_B} \le \widetilde{r_B}\right) = 1 - q_B.$$

With similar argument about the sign of $f_B + c_B + OA_B$ in Appendix I.1, rearrange the above expression to be

$$Pr\left(f_B \le \frac{\widetilde{r_B}(OA_B + c_B) - c_B}{1 - \widetilde{r_B}}\right) = 1 - q_B.$$

As $f_B \sim N(\mu_B, \sigma_B)$, the above equation is equivalent to

$$P\,r\left(z \le \frac{\frac{\widetilde{r_B}(OA_B + c_B) - c_B}{1 - \widetilde{r_B}} - \mu_B}{\sigma_B}\right) = 1 - q_B.$$

Since $K_B = \Phi^{-1}(1 - q_B)$, this implies that

$$\frac{\frac{\widetilde{r_B}(OA_B + c_B) - c_B}{1 - \widetilde{r_B}} - \mu_B}{\sigma_B} = K_B.$$

Rearrange to obtain

$$\widetilde{r_B} = \frac{c_B + \mu_B + K_B\sigma_B}{c_B + \mu_B + OA_B + K_B\sigma_B}.$$

Using the same method, one can show that $\widetilde{r_A}$, the $1 - q_A$ percent quantile of r_A before Bank A selling any risky portfolio, is

$$\widetilde{r_A} = \frac{\mu_A + K_A\sigma_A}{\mu_A + K_A\sigma_A + OA_A}.$$

Appendix II. 1

Define a new implicit function of Δ_B and σ_A,

$$F(\Delta_\mathrm{B}, \sigma_A) = \mu_A + \frac{K_B\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2} - (\frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B)}{\Delta_\mathrm{B}} - \underline{p_A} = 0$$

Thus,

$$\frac{d\Delta_B}{d\sigma_A} = -\frac{\frac{\partial F(\Delta_\mathrm{B}, \sigma_A)}{\partial\sigma_A}}{\frac{\partial F(\Delta_\mathrm{B}, \sigma_A)}{\partial\Delta_\mathrm{B}}} = -\frac{\frac{K_B\sigma_A\Delta_B}{\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2}}}{\frac{T_2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2} - K_B\sigma_B^2}{\Delta_B^2\sqrt{\sigma_B^2 + \Delta_B^2\sigma_A^2}}} < 0$$

where only $K_B < 0$ and $T_2 = \frac{r_B^* OA_B}{1 - r_B^*} - c_B - \mu_B < 0$.