The Dynamics of Bertrand Price Competition with Cost-Reducing Investments[†]

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April, 2014

Abstract: We present a dynamic extension of the classic model of Bertrand price competition, that allows competing duopolists to undertake cost-reducing investments in an attempt to "leapfrog" their rival to attain temporary low-cost leadership. We provide analytic characterization of the pay-off set and compute *all* Markov perfect equilibria in this game, and show that the generic equilibrium outcome involves leapfrogging — contrary to the previous literature which has focused on preemptive investments as the generic equilibrium outcome. Unlike the static Bertrand model, the equilibria of the dynamic Bertrand model are generally inefficient due to excessively frequent or duplicative investments. Equilibrium price paths in our model are piece-wise flat and have permanent discontinuous declines that occur when one firm leapfrogs its rival to become the new low cost leader.

Keywords: duopoly, Bertrand-Nash price competition, Bertrand investment paradox, leapfrogging, cost-reducing investments, technological improvement, dynamic models of competition, Markov-perfect equilibrium, tacit collusion, price wars, coordination and anti-coordination games, strategic preemption

JEL classification: D92, L11, L13

[†] We acknowledge helpful comments from Michael Baye, Jeffrey Campbell, Dan Cao, Ulrich Doraszelski, Joseph E. Harrington, Jr., Dan Kovenock, Roger Lagunoff, Stephen Morris, Michael Riordan, Jean-Marc Robin, David Salant, Karl Schmedders, Che-Lin Su, and other participants at seminars at Columbia, University of Copenhagen, CREST, Georgetown, Princeton, the Chicago Booth School of Business, the 3rd CAPCP conference at Pennsylvania State University, the 2012 conferences of the Society of Economic Dynamics, the Society for Computational Economics, the North American Summer Meetings of the Econometric Society, the 4th World Congress of the Game Theory Society, the Initiative for Computational Economics at Chicago and Zurich (ICE 2012 and ZICE 2013), and the NSF/NBER CEME Conference on "The Econometrics of Dynamic Games." Fedor Iskhakov acknowledges support from Frisch Centre project 1307 financed by the Ministry of Labour, Norway.

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1 Introduction

Given the large theoretical literature since the original work of Bertrand (1883), it is surprising that our understanding of price competition in presence of production cost uncertainty is still incomplete. For example, in the introduction to his paper on *static* price competition model, Routlege (2010) states “However, there is a notable gap in the research. There are no equilibrium existence results for the classical Bertrand model when there is discrete cost uncertainty.” (p. 357). Less is known about Bertrand price competition in dynamic models where firms compete by undertaking cost-reducing investments. In these environments the firms face uncertainty about their rivals’ investment decisions as well as uncertainty about the timing of technological innovations that can affect future prices and costs of production.

This paper analyses a dynamic version of the textbook Bertrand-Nash duopoly game, in which firms can make investment decisions as well as pricing decisions. Namely, at any time period, a firm can decide to replace its current production plant with a new state of the art production facility which enables it to produce at a lower marginal cost. We formulate the model in discrete time with infinite horizon. The key assumption of our model is that the state of the art technology evolves stochastically and exogenously, whereas technology adoption decisions are endogenous.

The term *leapfrogging* describes the long run investment competition between the two duopolists where the higher cost firm purchases a state of the art production technology that reduces its marginal cost relative to its rival and allows it to attain, at least temporarily, a position of low cost leadership. The assumption that the state of the art technology evolves exogenously differentiates our model from earlier examples of leapfrogging in the literature by, for example, Fudenberg et. al. (1983) and Reinganum (1985). This earlier work on *patent races* and models of *research and development* focused on firms’ continuous choice of R&D expenditures with the goal of producing a patent or a drastic innovation that could not be easily duplicated by rivals.

However in many industries firms do relatively little R&D but compete to obtain a production cost advantage by investing in state of the art production technology that is produced and sold by other firms. We model this investment as a binary decision: each firm faces a decision of whether or not to incur the substantial investment fixed cost to replace their current legacy production technology with the latest technology in order to become the current low cost leader. Since all firms have equal opportunity to acquire the state of the art production technology the markets we study are different from those studied in the earlier literature on leapfrogging in the context of R&D and patent races. These markets are *contestable due to ease of investment* similar to the way other markets are *contestable due to ease of entry* of Baumol, Panzar and Willig (1982).

If any firm can invest in the state of the art production technology to become the low cost producer, then Bertrand price competition in model where firms produce goods that are perfect substitutes using constant returns to scale production technologies leads to the *"Bertrand investment paradox"*. If more than one firm invests at the same time, Bertrand price competition ensures that *ex post* profits are zero. If the firms expect this, the *ex ante* return on their investments will be negative, so it is possible that no firm would have an incentive to undertake cost-reducing investments. But if no firm invests, it may make sense for at least one firm to invest. This reasoning leads us to conclude that the investment problem has the structure of an *anti-coordination game.*

The Bertrand investment paradox was resolved by Riordan and Salant (1994) (thereafter denoted RS) who analyzed a model of Bertrand price competition where two duopolists make investments to acquire a *deteministically improving* state of the art technology and gain a temporary cost advantage over their rival. RS proved that investment does incur in equilibrium, but *by only one of the firms.* In this *preemption equilibrium* consumers never benefit from technological improvements, because the price remains at the high marginal cost of the non-adopting firm. Further, they showed that the preemption equilibrium is completely inefficient: the preempting firm adopts new technologies so frequently to discourage entry of its rival that all of its profits (and thus all social surplus) is completely dissipated.

Though RS stressed that their investment preemption was "narrow in that it need not hold for other market structures" their analysis "suggests a broader research agenda exploring market structure dynamics" such as "Under what conditions do other equilibrium patterns emerge such as action-reaction (Vickers [1986]) or waves of market dominance in which the identity of the identity of the market leader changes with some adoptions but not others?" (p. 258).

Giovannetti (2001) was the first to show that a particular type of leapfrogging — *alternating* *adoptions* — can be an equilibrium outcome in a discrete time duopoly model of Bertrand price competition under assumptions that are broadly similar to RS. Though Giovannetti did not cite or specifically address RS's work, he showed that both preemption and alternating adoptions can be equilibrium outcomes depending on the elasticity of demand.

Giovannetti's analysis was done in the context of game where firms make *simultaneous* investment decisions, whereas RS modeled the investment choices as an *alternating move game.* The alternating move assumption seems to be a reasonable way to approximate decisions made in continuous time, where it is unlikely that two firms would be informed of a new technological innovation and make investment decisions at precisely the same instant. However the change in timing assumptions could have significant consequences, since Theorem 1 of RS shows that *preemption is the only equilibrium* in the continuous time limit of a sequence of discrete-time alternating move investment games as the time between moves tends to zero. RS conjectured that whether firms move simultaneously or alternately makes no difference with respect to conclusion that preemption is the unique equilibrium of the continuous time limiting game "We believe the same limit holds if the firms move simultaneously in each stage of the discrete games in the definition. The alternating move structure obviates examining mixed strategy equilibria for some subgames of the sequence of sequence of discrete games." (p. 255).

Giovanetti's result (namely that an equilibrium with alternating investments investments is possible if firms move simultaneously) suggests that Riordan and Salant's conjecture is incorrect, though he did not consider whether his results hold if firms move alternately rather than simultaneously, or whether leapfrogging is sustainable in the continuous time limit. Further neither Giovanetti nor RS considered the effect of uncertain technological progress on their conclusions: both assumed that the state of the art production cost declines deterministically over time. Stochastic technological change could create investment opportunities that could upset the preemption equilibrium and lead to more complex adoption dynamics. In particular, deterministic technological progress rules out the possibility of *drastic innovations* in the sense of Arrow (1962), where there is a there is sudden large improvement in technology. Riordan and Salant conjectured that the preemption result was a robust conclusion that would continue to hold in the presence of drastic innovations: “We conjecture that there exists an equilibrium adoption pattern featuring increasing dominance and rent dissipation quite generally. The heuristic reason is the standard one (Gilbert and Newbery [1982]; Vickers [1986]) that the leading firm always has a weakly greater incentive to preempt to protect its incumbent profit flow.” (p. 257).

The main contribution of this paper is the first characterization of the set of pay-offs of *all* Markov perfect equilibria (MPE) — both pure and mixed strategies — of a dynamic duopoly model of Bertrand price competition with stochastic technological progress under both simultaneous and alternating move assumptions (including stochastic alternating move versions of the game). We provide a unifying framework and reconcile the conflicting results of Giovannetti and RS, and by allowing for stochastic technological progress we also study a much wider range of environments that neither of these analyses were able to consider. In particular, by allowing for stochastic technological progress we analyze firm behavior and industry dynamics when there is a possibility of drastic innovations that Arrow (1962) contemplated. Similar to the result of Routledge (2010) in the static context, we establish existence of equilibria in the dynamic Bertrand investment game. Compared to RS we provide a more powerful resolution of Bertrand investment paradox by proving that unless investment cost is prohibitively high (from the point of view of social planner), at least one firm invests in *every* Markov perfect equilibrium of the game.

We confirm the main result of RS in our setting, but show that rent dissipating investment preemption breaks down if any of the three key assumptions (deterministic technological progress, alternating moves, continuous time) is removed, contrary to RS’s conjecture. Instead, we show that very complex patterns of dynamic investment competition are supported, with leapfrogging occurring in many other forms than simple patterns of deterministically alternating investments of Giovannetti (2001). In fact, we show that various types of leapfrogging equilibria constitute the *generic* outcome of the Bertrand investment game. Our finding that investment competition takes the form of leapfrogging seems to be an empirically more realistic than investment preemption, since consumers would never benefit from technological progress if the latter theory were true. However there are numerous examples of consumer electronics and many different physical goods where technological improvements coupled with leapfrogging investments by firms have resulted in dramatic price declines to consumers over time.

In the simultaneous move version of the game the MPE is not unique and under weak conditions we provide a characterization of the set of all equilibrium pay-offs that is reminiscent of the Folk Theorem: the convex hull of set of initial node pay-offs in the game is a triangle, which vertices include two monopoly payoffs, corresponding to RS investment preemption by each of the firms, and an zero profit mixed strategy pay-off. When firms invest in an alternating fashion (under deterministic and stochastically alternating move variations), we show that the convex hull of the set of equilibrium pay-offs is a strict subset of the same triangle, so that neither monopoly nor zero profit mixed strategy outcomes are supportable in this case. We provide a sufficient condition for the uniqueness of equilibrium: in the alternating moves specification when technology improves in every time period with probability one, Bertrand investment game has a unique MPE. This condition is satisfied in RS's and Giovannetti's frameworks where technological progress is deterministic, and thus improves with certainty in every period. However when the probability that the state of the art does not improve in any period is sufficiently large, the set of MPE is no longer a singleton, and will in general include a large number of equilibria that exhibit various types of leapfrogging.

Besides analytic characterization of the set of equilibrium pay-offs, we utilize the *Recursive Lexicographic Search* algorithm of Iskhakov, Rust and Schjerning (2013) to numerically compute *all* MPE in a discretized version of the Bertrand investment game. Then, using a numerical solution to the social planner's problem in the same technological environment, we construct a measure of efficiency for the equilibria in the game as the ratio between social surplus under duopoly and social planner solutions. With this measure, we compute and provide an empirical distribution of the efficiency of all MPE in the Bertrand investment game. We find that the equilibria in our model are typically *inefficient* due to investments that occur *too frequently* relative to the social optimum and due to *duplicative investments* that are a reflection of coordination failures in this game. The most inefficient equilibria are those involving mixed (behavioral) strategies, however we show that there are also *fully efficient equilibria* that take the form of *asymmetric pure strategy equilibria*.

The continuous time limiting preemption equilibrium of RS is fully inefficient, with social surplus completely dissipated due to excessively frequent investments by the preempting firm. Though most of the leapfrogging equilibria display some degree of inefficiency due to duplicative investments, the overall efficiency is generally very high in the examples of the Bertrand investment game we have considered: the median efficiency of all equilibria in examples we provide in section 4 is over 95%. An example of a fully efficient equilibrium is the *monopoly* MPE where one firm never invests and the other does all of the investing and sets a price equal to the marginal cost of production of the high cost, non-investing firm. Although investment competition in the non-monopoly equilibria of the model does benefit consumers by lowering costs and prices in the long run, it does generally come at the cost of some inefficiency due to coordination failures. However we provide examples (and thus establish existence) of perfectly coordinated, fully efficient leapfrogging as well.

Price paths in the equilibria of our model are piece-wise flat, with discontinuous declines just after one of the firms invests and displaces its rival to become a new low cost leader. These large drops in prices could be interpreted as “price wars”. However in our model these periodic price wars lead to a permanent decrease in prices and are part of a fully competitive outcome where the firms are behaving as Bertrand price competitors in every period.

In the next section we present our model and summarize the solution method we used to compute all MPEs of the game. Section 3 discusses the socially optimal investment strategies and solves the social planner’s problem. We present our main results in section 4, and section 5 concludes.

2 The Model

Consider a market consisting of two firms producing an identical good. Assume that the two firms are price setters, have no fixed costs and can produce the good at a constant marginal cost of c_1 and c_2, respectively. Both firms have constant return to scale production technology, so neither of them ever faces a binding capacity constraint.

Under the assumption of perfectly inelastic demand, it is well known that Bertrand equilibrium arises in these settings, leading to the lower cost firm to serve the entire market at a price $p(c_1, c_2)$

equal to the marginal cost of production of the higher cost rival, i.e. $p(c_1,c_2) = \max[c_1,c_2]$. In the case where both firms have the same marginal cost of production we obtain the classic result that Bertrand price competition leads to zero profits for both firms at a price equal to their common marginal cost of production. Normalizing the market size to one, we can write the instantaneous profits of firm 1 as

$$r_1(c_1,c_2) = \begin{cases} 0 & \text{if } c_1 \geq c_2, \\ \max[c_1,c_2] - c_1 & \text{otherwise,} \end{cases} \tag{1}$$

and the profits for firm 2, $r_2(c_1,c_2)$ are defined symmetrically, so we have $r_2(c_1,c_2) = r_1(c_2,c_1)$.

We introduce the dynamics into the model by assuming that at each time period t both firms have the ability to make an investment to acquire a new production facility (plant) to replace their existing technology. Technological progress that drives down the marginal cost of production (while maintaining constant returns to scale) is exogenous and stochastic. Denote c the current state of the art marginal cost of production, and let $K(c)$ be the cost of investing in the plant that embodies this state of the art production technology. If either one of the firms purchases the state of the art technology, then after a one period lag (constituting the "time to build" the new production facility), the firm can produce at the new marginal cost c.

We assume there are no costs of disposal of an existing production plant, or equivalently, the disposal costs do not depend on the vintage of the existing plant and are embedded as part of the new investment cost $K(c)$. Yet, we allow the fixed investment cost $K(c)$ to depend on c. This can capture different technological possibilities, such as the possibility that it is more expensive to invest in a plant that is capable of producing at a lower marginal ($K'(c) < 0$), or situations where technological improvements lower both the marginal cost of production c and the cost of building a new plant ($K'(c) > 0$). Clearly, if investment costs are too high, then there may be a point at which the potential gains from lower costs of production are insufficient to justify incurring the investment cost $K(c)$. Moreover, when the competition between the duopolists leads to leapfrogging behavior, the investing firm will not be able to capture the entire benefit of lowering its cost of production: some of these benefits will be passed on to consumers in the form of lower prices.

Let $c^{(t)}$ denote the marginal cost of production under the state of the art production technology at time period $t \in \{0,1,2,..,\infty\}$.[1] Each period t the firms face a simple binary investment decision: firm j can decide not to invest and continue to produce using its existing production facility at the marginal cost $c_j^{(t)}$. If firm j pays the investment cost $K(c^{(t)})$ and acquires the state of the art production plant with marginal cost $c^{(t)}$, then when this new plant comes on line at $t+1$, firm j will be able to produce at the marginal cost $c_j^{(t+1)} = c^{(t)} < c_j^{(t)}$. If there has been no improvement in the technology and state of the art marginal cost at $t+1$ remains the same, it follows $c^{(t+1)} = c^{(t)} = c_j^{(t+1)}$. Otherwise, if technological innovation occurs at $t+1$, $c^{(t+1)} < c^{(t)} = c_j^{(t+1)}$, and firm j's new plant is already slightly behind the frontier at the moment it comes online.

If c is a continuous stochastic process, the state space for this model which we denote S, is given by the pyramid $S = \{(c_1,c_2,c) : \ c_1 \geq c \text{ and } c_2 \geq c \text{ and } 0 \leq c \leq c_0\}$ in R^3, where $c_0 > 0$ is the initial state, and zero represents the lower bound of the state of the art technology. The choice of lower bound is not essential for any of our results. The Bertrand investment game starts at the apex of the pyramid given by (c_0,c_0,c_0). In cases where for computational reasons we restrict c to a finite set of possible values in $[0,c_0]$, the "discretized" state space is a finite subset of S.

We assume that both firms believe that the state of the art technology for producing the good evolves stochastically according to a Markov process with transition density $\pi(c^{(t+1)}|c^{(t)})$. Specifically, suppose that with probability $\pi(c^{(t)}|c^{(t)})$ there is no improvement in the state of the art technology, and with probability $1-\pi(c^{(t)}|c^{(t)})$ technology improves to marginal cost $c^{(t+1)}$ which is a draw from some distribution over the interval $[0,c^{(t)}]$. An example of a convenient functional form for such a distribution is the Beta distribution. However the presentation of the model and neither of our results do not depend on specific functional form assumptions about π.

The feature of the transition density π that turns out to be crucial for the uniqueness of equilibrium is whether $\pi(c|c) > 0$ for some $c > 0$ or not. We single out a special case of *strictly monotonic* technological progress when $\pi(c|c) = 0$ for all c, i.e. the state of art always improves in every time period.[2] Note that completely deterministic technological progress is characterized by the condi-

[1] We formulate the model in discrete time with infinite horizon, so normally time script is not needed. On rare occasions we use superscript to denote time period of any state variable.

[2] There is a slight abuse of notation because in the absorbing state $\pi(0|0) = 1$. Throughout the paper we use $\pi(c|c) = 0$ to refer to strictly monotonic progress bearing in mind that it only applies for $c > 0$.

tion $\pi(c^{(t+1)}|c^{(t)}) \in \{0,1\}$ for any $c^{(t)}$, $c^{(t+1)}$. Before reaching an absorbing state deterministic technological improvement is strictly monotonic, but not vice versa.

2.1 Timing of Moves

Let $m^{(t)} \in \{0,1,2\}$ be a state variable that governs which of the two firms are "allowed" to undertake an investment at time t. We will assume that $\{m^{(t)}\}$ evolves as an exogenous two state Markov chain with transition probability $f(m^{(t+1)}|m^{(t)})$ independent of the other state variables $(c_1^{(t)}, c_2^{(t)}, c^{(t)})$. While it is natural to assume firms simultaneously set their prices, their investment choices may or may not be made simultaneously. The value $m^{(t)} = 0$ denotes a situation where the firms make their investment choices simultaneously, $m^{(t)} = 1$ indicates a state where only firm 1 is allowed to invest, and $m^{(t)} = 2$ is the state where only firm 2 can invest.

In this paper we analyze two variants of the Bertrand investment game: 1) a *simultaneous move* game where $m^{(t)} = 0$ and $f(0|m^{(t)}) = 1$ (so $m^{(t)} = 0$ with probability 1 for all t), and 2) *alternating move* game, with either deterministic or random alternation of moves, but where there is no chance that the firms could ever undertake simultaneous investments (i.e. where $m^{(t)} \in \{1,2\}$ and $f(0|m^{(t)}) = 0$ for all t). Under either the alternating or simultaneous move specifications, each firm always observes the investment decision of its opponent after the investment decision is made. However, in the simultaneous move game, the firms must make their investment decisions based on their assessment of the probability their opponent will invest. In the alternating move game, since only one of the firms can invest at each time t, the mover can condition its decision on the investment decision of its opponent if it was the opponent's turn to move in the previous period. The alternating move specification can potentially reduce some of the strategic uncertainty that arises in a fully simultaneous move specification of the game.

We interpret random alternating moves as a way of reflecting *asynchronicity* of timing of decisions in a discrete time model that occurs in continuous time models where probability of two firms making investment decisions at the exact same instant of time is zero. There are cases where equilibrium has been shown to be unique (e.g. Lagunoff and Matsui, 1997). We are interested in conditions under which uniqueness emerges in asynchronous move versions of our model.

The timing of events in the model is as follows. At the start of period t each firm knows the costs of production $(c_1^{(t)}, c_2^{(t)})$, and both learn the current values of $c^{(t)}$ and $m^{(t)}$. If $m^{(t)} = 0$, then the firms simultaneously decide whether or not to invest. We assume that both firms know each others' marginal cost of production, i.e. there is common knowledge of state $(c_1^{(t)}, c_2^{(t)}, c^{(t)}, m^{(t)})$. Further, both firms have equal access to the new technology by paying the investment cost $K(c^{(t)})$ to acquire the current state of the art technology with marginal cost of production $c^{(t)}$.

After each firm decides whether or not to invest in the latest technology, the firms then independently and *simultaneously* set the *prices* for their products, where production is done in period t with their existing plant. The Bertrand equilibrium price is the unique Nash equilibrium of the simultaneous move pricing stage game. The one period time-to-build assumption implies that even if both firms invest in new plants at time t, their marginal costs $c_1^{(t)}$ and $c_2^{(t)}$ in period t are unchanged, and enter profit formula (1).

We assume that consumer purchases of the good is a purely static decisions, and consequently there are no dynamic effects of pricing for the firms, unlike in the cases of durable goods where consumer expectations of future prices affects their timing of new durable purchases as in Goettler and Gordon (2011). Thus in our model, the pricing decision is given by the simple static Bertrand equilibrium in every period. The only dynamic decision in our model is firms' investment decisions.

2.2 Solution concept

Assume that the two firms are expected discounted profit maximizers and have a common discount factor $\beta \in (0,1)$. We adopt the standard concept of *Markov-perfect equilibrium* (MPE) for this dynamic game between the two firms. In a MPE, the firms' investment and pricing decision rules are restricted to be functions of the current state, $(c_1^{(t)}, c_2^{(t)}, c^{(t)}, m^{(t)})$. When there are multiple equilibria in this game, the Markovian assumption also restricts the "equilibrium selection rule" to depend only on the current value of the state variable. The firms' pricing decisions only depend on their current production costs $(c_1^{(t)}, c_2^{(t)})$ in accordance with the static Bertrand equilibrium. However the firms' investment decisions also depend on the value of the state of the art marginal cost of production $c^{(t)}$ and the designated mover $m^{(t)}$.

Definition 1. *A Stationary Markov Perfect Equilibrium of the duopoly investment and pricing game consists of a pair of strategies* $(P_j(c_1,c_2,c), p_j(c_1,c_2))$, $j \in \{1,2\}$ *where* $P_j(c_1,c_2,c,m) \in [0,1]$ *is firm* j*'s probability of investing and* $p_j(c_1,c_2) = \max[c_1,c_2]$ *is firm* j*'s pricing decision. The investment rules* $P_j(c_1,c_2,c,m)$ *must maximize the expected discounted value of firm* j*'s future profit stream taking into account the investment and pricing strategies of its opponent.*

We allow the investment strategies of the firms to be probabilistic to allow for the possibility of mixed strategy equilibria.

To derive the functional equations characterizing a stationary Markov-perfect equilibrium, suppose the current state is (c_1,c_2,c,m), i.e. firm 1 has a marginal cost of production c_1, firm 2 has a marginal cost of production c_2, and the marginal cost of production using the current best technology is c and m denotes which of the firms (or both if $m=0$) has the right to make a move and invest. The firms' value functions V_j, $j=1,2$ take the form

$$V_j(c_1,c_2,c,m) = \max[v_{I,j}(c_1,c_2,c,m), v_{N,j}(c_1,c_2,c,m)] \tag{2}$$

where, when $m=0$, $v_{N,j}(c_1,c_2,c,m)$ denotes the expected value to firm j if it does not invest in the latest technology, and $v_{I,j}(c_1,c_2,c,m)$ is the expected value to firm j if it invests. However when $m \in \{1,2\}$, the subscripts N and I refer to whether an investment is made in period t by the firm m, who has the right of move. When $m=1$ (firm 1 has the right to invest), $v_{I,1}(c_1,c_2,c,1)$ and $v_{N,1}(c_1,c_2,c,1)$ denote the expected values to firm 1 from investing and not investing. When $m=2$ (firm 2 has the right to invest), $v_{I,1}(c_1,c_2,c,2)$ and $v_{N,1}(c_1,c_2,c,2)$ denote the expected values to firm 1 from the scenarios when firm 2 makes the investment or does not make the investment. To simplify exposition below, we use the simultaneous move interpretation of N and I ($m=0$), while alternative move interpretation can be reconstructed analogously.

The formula for the expected profits associated with *not* investing is given by:

$$v_{N,j}(c_1,c_2,c,m) = r_j(c_1,c_2) + \beta EV_j(c_1,c_2,c,m,0), \tag{3}$$

where $EV_j(c_1,c_2,m,c,0)$ denotes the conditional expectation of firm j's next period value function

$V_j(c_1,c_2,c,m)$ given that it does not invest this period (represented by the last 0 argument in EV_j), conditional on the current state (c_1,c_2,c,m).

The formula for the expected profits associated with investing is given by

$$v_{I,j}(c_1,c_2,c,m) = r_j(c_1,c_2) - K(c) + \beta EV_j(c_1,c_2,c,m,1), \tag{4}$$

where $EV_j(c_1,c_2,c,m,1)$ is firm j's conditional expectation of its next period value function given that it invests (the last argument is 1), conditional on (c_1,c_2,c,m).

Let $P_1(c_1,c_2,c,m)$ be firm 2's belief about the probability that firm 1 will invest in state is (c_1,c_2,c,m). Consider the simultaneous move case ($m=0$) first. It follows from (2) that

$$P_1(c_1,c_2,c,m) = \mathbb{1}\{v_{I,1}(c_1,c_2,c,m) > v_{N,1}(c_1,c_2,c,m)\}, \tag{5}$$

where $\mathbb{1}\{\cdot\}$ denotes an indicator function, and mixed strategy investment probability arises in the case of equality. Similar formula holds for $P_2(c_1,c_2,c,m)$.

The Bellman equations for firm 1 in the simultaneous move case are as follows.[3] Similar equation for firm 2 are omitted for space considerations.

$$\begin{aligned}
v_{N,1}(c_1,c_2,c) &= r_1(c_1,c_2) + \beta \int_0^c \big[P_2(c_1,c_2,c)\max(v_{N,1}(c_1,c,c'),v_{I,1}(c_1,c,c')) + \\
&\qquad (1-P_2(c_1,c_2,c))\max(v_{N,1}(c_1,c_2,c'),v_{I,1}(c_1,c_2,c'))\big]\,\pi(dc'|c). \\
v_{I,1}(c_1,c_2,c) &= r_1(c_1,c_2) - K(c) + \beta \int_0^c \big[P_2(c_1,c_2,c)\max(v_{N,1}(c,c,c'),v_{I,1}(c,c,c')) + \\
&\qquad (1-P_2(c_1,c_2,c))\max(v_{N,1}(c,c_2,c'),v_{I,1}(c,c_2,c'))\big]\,\pi(dc'|c).
\end{aligned} \tag{6}$$

In the alternating move case, the Bellman equations for the two firms lead to a system of eight functional equations for $\{v_{N,j}(c_1,c_2,c,m), v_{I,j}(c_1,c_2,c,m)\}$ for $j,m \in \{1,2\}$. The Bellman

[3] Variable $m=0$ is omitted for clarity

equations for firm 1 are given below, similar equations for firm 2 are omitted.

$$\begin{aligned}
v_{N,1}(c_1,c_2,c,1) &= r_1(c_1,c_2)+\beta f(1|1)\int_0^c \max(v_{N,1}(c_1,c_2,c',1),v_{I,1}(c_1,c_2,c',1))\pi(dc'|c)+\\
&\quad \beta f(2|1)\int_0^c \rho(c_1,c_2,c')\pi(dc'|c)\\
v_{I,1}(c_1,c_2,c,1) &= r_1(c_1,c_2)-K(c)+\beta f(1|1)\int_0^c \max(v_{N,1}(c,c_2,c',1),v_{I,1}(c,c_2,c',1))\pi(dc'|c)+\\
&\quad \beta f(2|1)\int_0^c \rho(c,c_2,c')\pi(dc'|c)\\
v_{N,1}(c_1,c_2,c,2) &= r_1(c_1,c_2)+\beta f(1|2)\int_0^c \max(v_{N,1}(c_1,c_2,c',1),v_{I,1}(c_1,c_2,c',1))\pi(dc'|c)+\\
&\quad \beta f(2|2)\int_0^c \rho(c_1,c_2,c')\pi(dc'|c)\\
v_{I,1}(c_1,c_2,c,2) &= r_1(c_1,c_2)+\beta f(1|2)\int_0^c \max(v_{N,1}(c_1,c,c',1),v_{I,1}(c_1,c,c',1))\pi(dc'|c)+\\
&\quad \beta f(2|2)\int_0^c \rho(c_1,c,c')\pi(dc'|c).
\end{aligned} \tag{7}$$

where

$$\rho(c_1,c_2,c)=P_2(c_1,c_2,c,2)v_{I,1}(c_1,c_2,c,2)+[1-P_2(c_1,c_2,c,2)]v_{N,1}(c_1,c_2,c,2). \tag{8}$$

Note that $P_2(c_1,c_2,c,1)=0$, since firm 2 is not allowed to invest when it is firm 1's turn to invest, $m=1$, and similarly for $P_1(c_1,c_2,c,c,2)$.

The equilibria of the Bertrand investment game with simultaneous moves are characterized by the large system of non-linear equations composed of equations (6) and (5) written for every combination of (c_1,c_2,c) in a discrete representation of the state space S. Similarly, in the alternating moves game, all quilibria are characterized by the system composed of equations (7) and (5) for every combination of (c_1,c_2,c) and all values of m. Althought contemporary numerical solvers are capable of solving very large systems of non-linear equations, finding of all solutions for such a system is impossible in general.

The key feature of the Bertrand investment model that allows us to compute all MPE in both simultaneous and alternating move specifications of the game is finiteness (on a discrete representation of the state space) and the directionality in the evolution of the cost variables (c_1,c_2,c). Because of the unidirectional evolvement of the state vector, the system of equations characterizing the equilibria of the model turns out to be block-triangular, and thus, it is possible to decompose solving of the whole system into solving a number of much smaller problems.

Recursive lexicographical search algorithm (RLS) developed in Iskhakov, Rust and Schjerning (2013) is guaranteed to find all MPE in a general class of games they call *dynamic directional games* (DDGs), provided there is a finite number of equilibria on every "stage game" defined in this model by a unique combination of (c_1, c_2, c), and that all of them can be computed. These requirements are satisfied in our model, and so using RLS algorithm we are able to compute all MPEs of the Bertrand investment game.

3 Socially optimal production and investment

To assess the efficiency of the outcomes Bertrand investment game, we first derive in this section the social optimum solution to our model that maximizes total expected discounted consumer and producer surplus. In a dynamic model, the social planner has to account for the investment costs. Under our assumptions about constant returns to scale it only makes sense for the social planner to operate a single plant. Thus, the duopoly equilibrium can be inefficient due to duplicative investments that a social planner would not undertake. However we will show that inefficiency in the duopoly equilibrium manifests itself in other ways as well.

Our model of consumer demand is based on the implicit assumption that consumers have quasi-linear preferences; the surplus they receive from consuming the good at a price of p is some initial level of willingness to pay net of p. The social planning solution entails selling the good at the marginal cost of production, and adopting an efficient investment strategy that minimizes the expected discounted costs of production. Let c_ς be the marginal cost of production of the current production plant, and let c be the marginal cost of production of the current state of the art production process, which we continue to assume evolves as an exogenous first order Markov process with transition probability $\pi(c'|c)$ and its evolution is beyond the purview of the social planner. All the social planner needs to do is to determine an *optimal investment strategy* for the production of the good.

Let $C(c_\varsigma, c)$ be the *smallest* present discounted value of costs of investment and production when the plant operated by the social planner has marginal cost c_ς and the state of the art technology has a marginal cost of $c \leq c_\varsigma$. The minimization occurs over all feasible investment and production strategies, but subject to the constraint that the planner must produce enough in every period to satisfy the unit mass of consumers in the market. We have

$$C(c_\varsigma, c) = \min\left\{c_\varsigma + \beta \int_0^c C(c_\varsigma, c')\pi(dc'|c),\ c_\varsigma + K(c) + \beta \int_0^c C(c, c')\pi(dc'|c)\right\}, \tag{9}$$

where the first component corresponds to the case when investment is not made, and cost c_ς is carried in the future, and the second component corresponds to the case when new state of the art cost c is acquired for additional expense of $K(c)$.[4].

It follows that the optimal investment strategy takes the form of a *cutoff rule* where it is optimal to invest in the state of the art technology if the current cost c_ς is above a cutoff threshold $\overline{c}_\varsigma(c)$. Otherwise the drop in expected future operating costs is not sufficiently large to justify undertaking the investment and thus it is optimal to produce the good using the existing plant with marginal cost c_ς. The cutoff rule $\overline{c}_\varsigma(c)$ is the indifference point in (9), and thus it is the solution to the equation

$$K(c) = \beta \int_0^c \left[C(\overline{c}_\varsigma(c), c') - C(c, c')\right] \pi(dc'|c), \tag{10}$$

if it exists, and $\overline{c}_\varsigma(c) = c_0$ otherwise.[5]

We have implicitly assumed that the cost of investment $K(c)$ is not prohibitively high, so that the social planner would always want to invest in a new technology. Theorem 1 provides a bound on the costs of investments that must be satisfied for investment to occur under the socially optimum solution.

Theorem 1 (Necessary and sufficient condition for investment by the social planner)**.** *Let the current costs be (c_ς, c). Investment (at current period or some time in the future) is socially optimal if and only if there exists $c' \in [0, c_\varsigma]$ in the support of the Markov process of the state of the art marginal cost $c^{(t)}$, such that*

$$\frac{\beta(c_\varsigma - c')}{1-\beta} > K(c'). \tag{11}$$

[4]The details about the cost recursion are given in Appendix B

[5]In problems where the support of $\{c_t\}$ is a finite set, the cutoff $\overline{c}_\varsigma(c)$ is defined as the smallest value of c_ς in the support of $\{c_t\}$ such that $K(c) > \beta \int_0^c [C(c_\varsigma, c') - C(c, c')]\pi(dc'|c)$.

The proof of Theorem 1, and all subsequent proofs unless sufficiently short, are provided in Appendix A.

The condition under which it is socially optimal to invest plays a central role when we analyze the duopoly investment dynamics in section 4. We will say that the investment costs are not prohibitively high, or that investment is socially optimal if the condition (11) in Theorem 1 holds with $c_\varsigma = \min[c_1, c_2]$, where c_j denotes the marginal cost of production of firm j in the Bertrand investment game.

As we will show in the next section, Bertrand investment game with simultaneous moves supports a monopoly outcome. The following lemma establishes the efficiency of a monopoly outcome, which is useful for what follows in the next section.

Lemma 1 (Social optimality of monopoly solution)**.** *The socially optimal investment policy is identical to the profit maximizing investment policy of a monopolist who faces the same discount factor* β *and the same technological process* $\{c_t\}$ *with transition probability* π *as the social planner, assuming that in every period the monopolist can charge a price of* c_0 *equal to the initial value of the state of the art production technology.*

Proof. Since the monopolist is constrained to charge a price no higher than c_0 every period, it follows that the monopolist maximizes expected discounted value of profits by adopting a cost-minimizing production and investment strategy as per social planner. □

4 Duopoly Investment Dynamics

We are now in position to solve the model of Bertrand duopoly investment and pricing and characterize the stationary Markov Perfect equilibria of this model. As mentioned above, we used the RLS algorithm from Iskhakov, Rust and Schjerning (2013) to compute all MPEs in the Bertrand investment game. These computations facilitated the illustrative examples below. Yet the majority of our results are based on analytical *proofs* of the general properties of the equilibria of this game.

In the subsequent analysis we focus on a subclass of Bertrand investment games where the support of the Markov process $\{c\}$ representing the evolution of the state of the art production technology is a *finite subset of* R^1. Therefore, as discussed above, the state space of the investment game is a finite subset of S where all of the coordinates c_1, c_2 and c lie in the support of the Markov process $\{c\}$.[6] If we further restrict the set of possible equilibrium selection rules to be *deterministic* functions of the current state (c_1, c_2, c), we can show that there will only be a finite number of possible equilibria in both the simultaneous and alternating move formulations of the game. Yet, the number of the equilibria grows exponentially fast with the number of points in the discretized state space.[7]

4.1 Configuration of the set of equilibrium payoffs

Provided that the investment cost is not prohibitively high, the set of all MPEs in the Bertrand investment game is surprisingly rich. Despite the prevalence of leapfrogging in equilibrium, we show that "monopoly" equilibria is supported in the simultaneous move game.[8] A static Bertrand-like outcome with zero expected payoff for both duopolists is also supported in the simultaneous move game. It is generally not possible to support the neither monopoly nor zero profit outcomes in the alternating move version of the game except for isolated, atypical counterexamples. We summarize these findings in the following theorem what constitutes our main result.

Theorem 2 (Characterization of the set of equilibrium payoffs)**.** *If investments are socially optimal (in the sense of the condition of Theorem 1) at the apex* (c_0, c_0, c_0) *of the state space of the Bertrand investment and pricing game, the following holds:*

1. *No investments by both firms is not supported in any of the MPE equilibria of the game;*

[6] We proved Theorem 2 by mathematical induction, and this is the reason we assume that the support of $\{c\}$ is a finite set. We believe most of the results still hold when the state space is continuous. However in the interest of space we do not attempt to prove this result here and merely state it as a conjecture that we believe to be true.

[7] We can show that if investment is socially optimal and the support of the Markov process $\{c\}$ is the full interval $[0, c_0]$ the simultaneous move Bertrand investment and pricing game has a continuum of MPE.

[8] Note that the monopoly equilibria we characterize below are *not* the preemption equilibrium of Riordan and Salant (1994). In contrast to their rent dissipation result, monopoly profits in our model are positive and are equal to the maximum possible profits subject to the limit on price, by Lemma 1 monopoly outcome is fully efficient.

2. *The simultaneous move game has two fully efficient "monopoly" equilibria in which either one or the other firm makes all the investments and earns maximum feasible profit;*

3. *There exist a symmetric equilibrium in the simultaneous move game that results in zero expected payoffs to both firms at all states $(c,c,c') \in S$ with $c' \in [0,c]$, and zero expected payoffs to the high cost firm and positive expected payoffs to the low cost firm in states (c_1,c_2,c) where $c_1 \neq c_2$;*

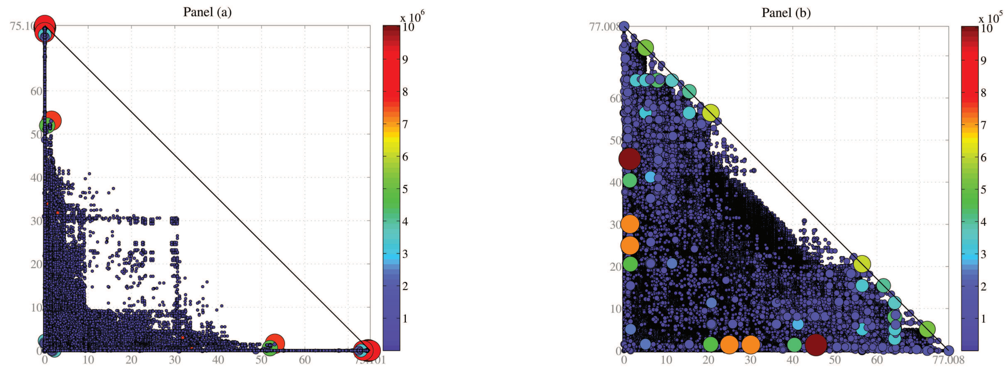
4. *The convex hull of the set of the expected discounted equilibrium payoffs to the two firms in all MPE equilibria of simultaneous move game at the apex is a triangle with vertices $(0,0)$, $(0,V_M)$ and $(V_M,0)$, where $V_M = v_{N,i}(c_0,c_0,c_0)$ is the expected discounted payoff of firm i which makes all investments in the monopoly equilibrium;*

5. *The (convex hull of the) set of expected discounted equilibrium payoffs to the two firms in all possible MPE equilibria at the apex of the alternating move game is a strict subset of the triangle with the same vertices;*

Figure 1 illustrates Theorem 2 by plotting all apex payoffs to the two firms under all possible deterministic equilibrium selection rules in the simultaneous move game where the support of $\{c\}$ is the 5 point set $\{0, 1.25, 2.5, 3.75, 5\}$. Panel (a) plots the set of payoffs that occur when technological progress is deterministic, whereas panel (b) shows the much denser set of payoffs that occur when technological progress is stochastic. Though there are actually a greater total number of equilibria (192,736,405) under deterministic technological progress, many of these equilibria are observationally equivalent *repetitions* of the same payoff point which arise due to our treatment of the equilibrium selection rules that only differ off the equilibrium path as distinct. We indicate the number of repetitions by the size of the payoff point plotted to be proportional to the number of repetitions. Figure 1 shows that when technology is stochastic there are fewer repetitions and so even though there are actually 28 million fewer equilibria, there are actually a substantially greater number (1,679,461 versus 63,676) of distinct payoff points.

It is perhaps not surprising that when firms move in an alternating fashion neither one of them will be able to attain monopoly payoffs in any equilibrium of the alternating move game (except

Figure 1: Initial node equilibrium payoffs in the simultaneous move game



Notes: The panels plot payoff maps of the Bertrand investment game with deterministic (a) and random (b) technologies. Parameters are $\beta = 0.9512$, $k_1 = 8.3$, $k_2 = 1$, $n_c = 5$. Parameters of beta distribution for random technology are $a = 1.8$ and $b = 0.4$. Panel (a) displays the initial state payoffs to the two firms in the 192,736,405 equilibria of the game, though there are 63,676 distinct payoff pairs among all of these equilibria. Panel (b) displays the 1,679.461 distinct payoff pairs for the 164,295,079 equilibria that arise under stochastic technology. The color and size of the dots reflect the number of repetitions of a particular payoff combination.

for some isolated counterexamples we discuss below). When firms make simultaneous investment decisions, the high cost firm has no incentive to deviate from the equilibrium path in which its opponent always invests. However when the firms move in an alternating fashion, the high cost firm will have an incentive to deviate because it knows that its opponent will not be able to invest at the same time (thereby avoiding the Bertrand investment paradox), and once the opponent sees that the firm has invested, it will not have an incentive to immediately invest to leapfrog for a number of periods until it is once again its turn to invest and there has been a sufficient improvement in the state of the art. This creates a temptation for each firm to invest and leapfrog their rival that is not present in the simultaneous move game, and the alternating move structure prevents the firms from undertaking inefficient simultaneous investments, though it also generally prevents either firm from being able to time their investments in a socially optimal way.

Statement 5 in the Theorem 2 states that the zero expected profit mixed strategy equilibrium is not sustainable in the alternating move game either. Though it may seem tempting to conclude that mixed strategies can never arise in the alternating move game, we find that both pure and mixed strategy stage game equilibria are possible in the alternating move game. The intuition as to why this should occur is that even though only one firm invests at any given time, when $\pi(c|c) > 0$ the firms know that there is a positive probability that they will remain in the same state (c_1, c_2, c) for multiple periods until the technology improves. The possibility of remaining in the same state implies that the payoff to each firm from *not investing* depends on their belief about the probability their opponent will invest in this state at its turn.

We formally define the leapfrogging equilibria as those where the high cost firm has a positive probability of investment at least in one point of the state space along the equilibrium path, and thus it can be seen in a realization of such equilibrium that a high cost firm leapfrogs the cost leader. As mentioned above, leapfrogging equilibria are very typical. In all of our numerical solutions of *simultaneous move* game, we found that in the symmetric zero profit mixed strategy equilibrium the high cost firm always has a *strictly higher* probability of investing than the low cost firm, thus satisfying the definition of a leapfrogging equilibrium. We have not been able to prove this result in general, however we did prove it in the end game (when $c = 0$, see Lemma A.2 in the appendix), and in the symmetric, zero expected profit mixed strategy stage game equilibria under a slight strengthening of the condition of social optimality of investment. For the interest of space we don't include this result here.

4.2 Uniqueness

In spite of very large number of MPEs we find in the Bertrand investment game, there is a subclass of games for which the equilibrium is unique, or allowing relabeling of the firms, there are two *asymmetric* equilibria of the game.

Theorem 3 (Sufficient conditions for uniqueness)**.** *In the dynamic Bertrand investment and pricing game a sufficient condition for the MPE to be unique is that (i) firms move in alternating fashion (i.e. $m \neq 0$), and (ii) for each c in the support of π we have $\pi(c|c) = 0$.*

Theorem 3 implies that under strictly monotonic technological improvement the alternating move investment game has a unique Markov perfect equilibrium. This is closely related to, but not identical with an assumption of the *deterministic technological progress* as discussed in section 2. There are specific types of non-deterministic technological progress for which Theorem 3 will still hold, resulting in a unique equilibrium to the alternating move game. In subsection 4.4 we will return to this case, by considering further properties of the unique equilibrium that results when $\pi(c|c) = 0$ including the conditions in which it constitutes a discrete time equivalent to Riordan and Salant's (1994) continuous time preemption equilibrium.

It is also helpful to understand why multiple equilibria can arise in the alternating move game when $\pi(c|c) > 0$. The reason is that when there is a positive probability of remaining in any given given state (assuming firms choose not to invest when it is their turn to invest), it follows that each firm's value of *not investing* depends on their belief about the probability their opponent will invest. Thus, by examining the Bellman equations (7) it not hard to see that for firm 1 the value of not investing when it is its turn to invest, $v_{N,1}(c_1, c_2, c, 1)$, depends on $P_2(c_1, c_2, c, 2)$ when $\pi(c|c) > 0$. This implies that $P_1(c_1, c_2, c, 1)$ will depend on $P_2(c_1, c_2, c, 2)$, and similarly, $P_2(c_1, c_2, c, 2)$ will depend on $P_1(c_1, c_2, c, 1)$. This mutual dependency creates the possibility for multiple solutions to the Bellman equations and the firms' investment probabilities and multiple equilibria at various stage games of the alternating move game.

4.3 Efficiency of equilibria

We evaluated the efficiency of duopoly equilibria by calculating their *efficiency score* which is the ratio of total surplus (i.e. the sum of discounted consumer surplus plus total discounted profits) under the duopoly equilibrium to the maximum total surplus achieved under the social planning solution. We note that the calculation of efficiency is *equilibrium specific* and thus its value depends on the particular equilibrium of the overall game that we select. For example, we have already proved that monopoly investment by one of the firms is an equilibrium in the simultaneous move game, provided the cost of investment is not prohibitively high. This implies immediately that there do exist fully efficient MPE in the simultaneous move game. We now show that the non-monopoly equilibria of either the simultaneous or alternating move investment games are generally *inefficient* and this inefficiency is typically due to two sources a) duplicative investments (valid only in mixed strategy equilibria in the simultaneous move investment game), and b) excessively frequent investments. Note that it is logically possible that inefficiency could arise from *excessively infrequent* *investments* and the logic of the Bertrand investment paradox might lead us to conjecture that we should see investments that are too *infrequent* in equilibrium relative to what the social planner would do. However surprisingly, we find that duopoly investments are generally excessively frequent to the social optimum, with preemptive investments (when they arise) representing the most extreme form of inefficient excessively frequent investment in new technology.

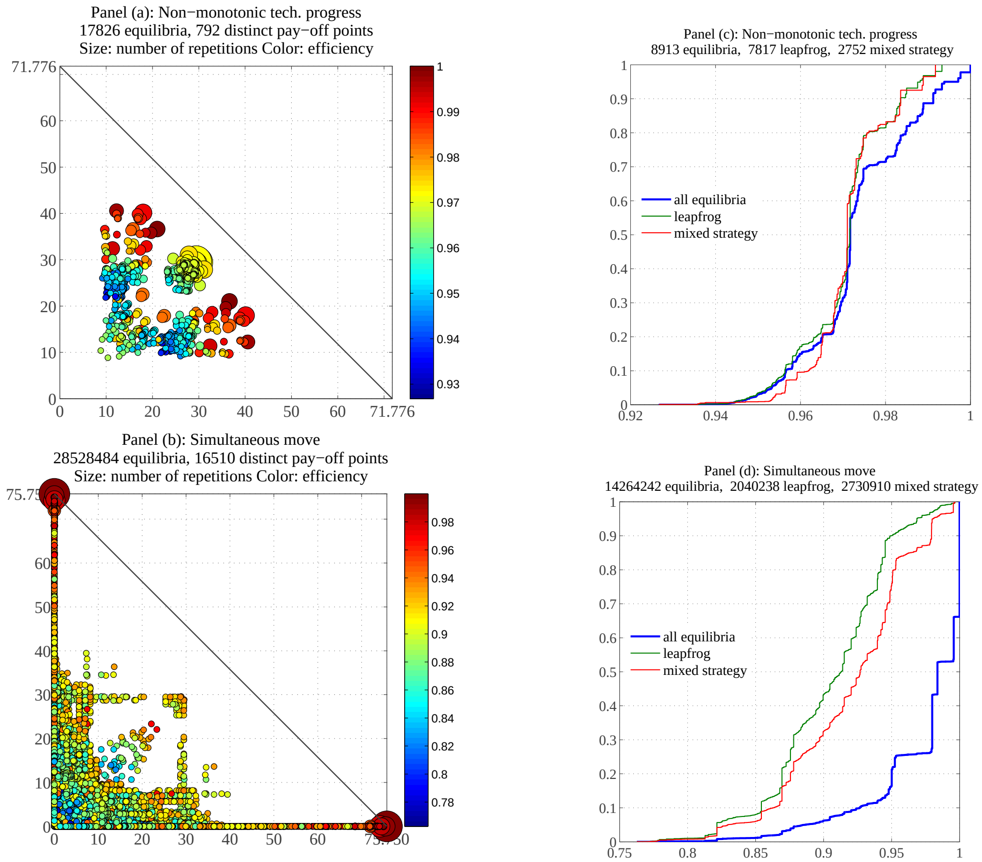
The two panels in the left column in Figure 2 illustrate the set of equilibrium payoffs from all MPE equilibria computed by the RLS algorithm of Iskhakov, Rust and Schjerning (2013). We compute the efficiency of each of the equilibria, and treating the calculated efficiencies as "data", we plot their empirical distribution in the corresponding panels in the right column in Figure 2.

Panels (a) and (c) in Figure 2 represent an alternating move investment game with deterministic alternations of the right to move and the technological progress which is not strictly monotonic, i.e. $\pi(c|c) > 0$ for some c. The opposite of the latter condition ensures unique equilibrium in this game according to Theorem 3, but multiple equilibria is a typical outcome in the alternative move game with "sticky" state of the art technology. Consistent with Theorem 2 the set of equilibrium payoffs is a strict subset of the triangle, showing that it is not possible to achieve the monopoly payoffs (corners) or the zero profit mixed strategy equilibrium payoff (origin) in this case. As before, we have used the size of the plotted payoff points to indicate the number of repetitions of the payoff points, but now we use the color of plotted equilibrium payoffs to indicate the efficiency. Red (hot) indicates high efficiency payoffs, and blue (cool) indicates lower efficiency payoffs.

We see a clear positive correlation between payoff and efficiency in panel (a) — there is a tendency for the points with the highest total payoffs (i.e. points closest to the line connecting the monopoly outcomes) to have higher efficiency indices. The CDFs of efficiency levels in panel (c) shows that 1) overall efficiency is reasonably high, with the median equilibrium having an efficiency in excess of 97%, and 2) the maximum efficiency of the equilibria involving mixed strategies along the equilibrium path is strictly less than 100%.

In panels (b) and (d) of Figure 2 we plot the set of equilibrium payoffs and distribution of equilibrium efficiency for a simultaneous move investment under the deterministic technology process. In accordance with Theorem 2 the monopoly and zero profit outcomes are now present among

Figure 2: Payoff maps and efficiency of MPE in two specifications of the game



Notes: Panel (a)-(b) plots payoff maps and panel (c)-(d) cdf plots of efficiency by equilibrium type for two versions of the Bertrand investment pricing game. In panel (a) and (c) the case of deterministic alternating moves and non-strictly monotonic one step stochastic technological progress. Parameters in this case are $\beta = 0.9592$, $k_1 = 5$, $k_2 = 0$,, $f(1|1) = f(2|2) = 0$, $f(2|1) = f(1|2) = 1$, $c_{tr} = 1$ $n_c = 4$. In panel (b) and (d) we plot the payoffs and the distribution of efficiency for the simultaneous move game with deterministic one step technology. Leapfrog equilibria are defined as having positive probability to invest by the cost follower along the equilibrium path, mixed strategy equilibria are defined as involving at least one mixed strategy stage equilibrium along the equilibrium path.

the computed MPE equilibria of the model. Overall, the equilibria in this game are less efficient compared to the equilibria in the alternating move game displayed in the top row panels, but the tendency of more efficient equilibria to be located closer to the “monopoly” frontier remains. The additional source of inefficiency in this game is redundancy of simultaneous investments, which appear in the mixed strategy equilibria. It is clearly seen in the cumulative distribution plot in panel

(d) that even though more than 30% of the equilibria are approaching full efficiency[9], the mixed strategy equilibria are not among them. Instead, the distribution of their efficiency is stochastically dominated by the distribution of efficiencies in all the equilibria of the game.

We formalize the above discussion in the following theorem.

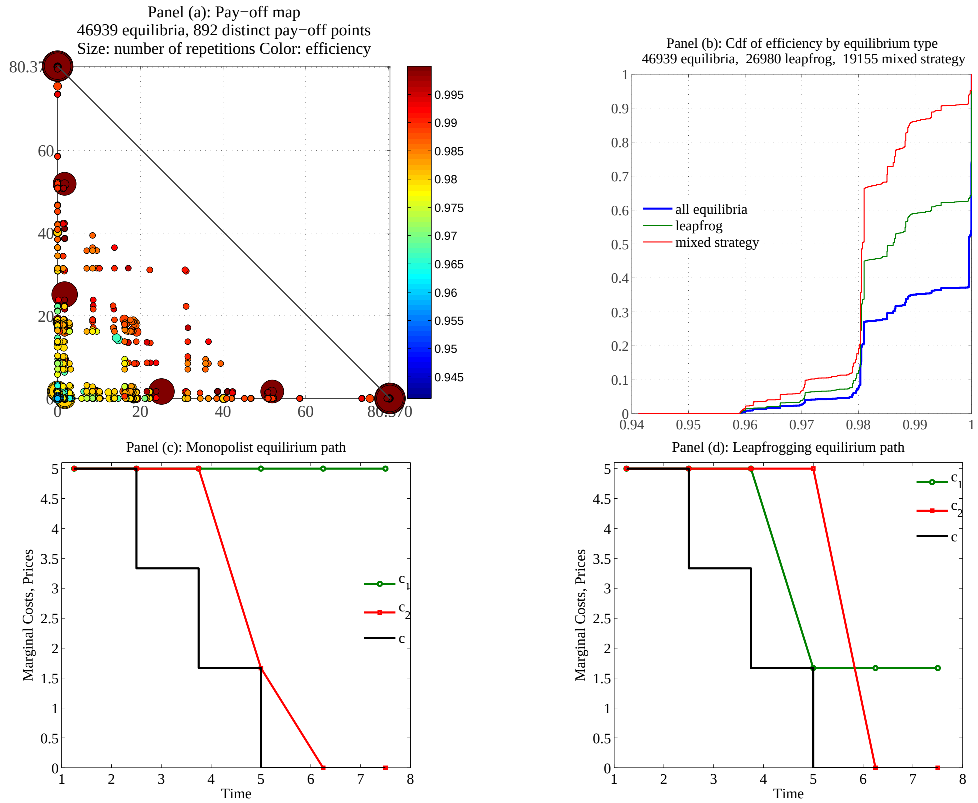
Theorem 4 (Inefficiency of mixed strategy equilibria)**.** *A necessary condition for efficiency in the dynamic Bertrand investment and pricing game is that along MPE path only pure strategy stage equilibria are played.*

Figure 3 establishes the existence of *fully efficient leapfrogging equilibria.* Panel (a) of figure 3 plots the set of equilibrium payoffs in a simultaneous move investment game where there are four possible values for state of the art costs $\{0, 1.67, 3.33, 5\}$ and technology improves deterministically. Recall that the payoff points colored in dark red are 100% efficient, so we see that there are a number of other *non-monopoly equilibria* that can achieve *full efficiency.* The significance of this finding is that we have shown that it is possible to obtain competitive equilibria where leapfrogging by the firms ensures that consumers receive some of the surplus and benefits from technological progress without a cost in terms of inefficient investment such as we have observed occurs in mixed strategy equilibria of the game where socially inefficient excessive investment results in lower prices to consumers but at the cost of zero expected profits to firms. Notice, however, that even the least efficient mixed strategy equilibrium still has an efficiency of 96%, so that in this particular example the inefficiency of various equilibria may not be a huge concern.

Panels (c) and (d) of Figure 3 plot the simulated investment profiles of two different equilibria. Panel (c) shows the monopoly equilibria where firm 2 is the monopolist investor. The socially optimal investment policy is to make exactly two investments: the first when costs have fallen from 5 to 1.67, and the second when costs have fallen to the absorbing value of 0. Panel (d) shows the equilibrium realization from a pure strategy equilibrium that involves leapfrogging, yet the investments are made at exact same time as the social planner would do. After firm 1 invests when costs reach 1.67 (consumers continue to pay the price $p_1 = 5$), in time period 5 it is leapfrogged by

[9]To be exact, 15.22% have efficiency of 0.9878 and the same fraction of equilibria is fully efficient.

Figure 3: Efficiency of equilibria



Notes: Panel (a) and panel (b) plots the apex payoff map and distribution of efficiency indices for the simultaneous move game. 25.88% of all equilibria are fully efficient. The most efficient mixed strategy equilibrium has the efficiency index 0.99998 but does not violate Theorem 4. Panel (c) displays the simulated investment profile from a fully efficient "monopoly" equilibrium, while panel (d) displays the example of fully efficient equilibrium that involves leapfrogging.

firm 2 who becomes the permanent low cost producer. At this point a "price war" brings the price down from 5 to 1.67, which becomes new permanent level.

We conclude that the leapfrogging equilibria may be fully efficient if investments are made in the same moments of time as the monopolist would invest, but in these equilibria consumers also benefit from the investments because the price decreases in a series of permanent drops.

Lemma 2 (Existence of efficient non-monopoly equilibria)**.** *In both the simultaneous move and alternating move investment games, there exist fully efficient non-monopoly equilibria.*

Proof. The proof is by example shown in Figure 3. An example of a fully efficient non-monopoly equilibrium when the firms move alternately (in deterministic fashion) can be constructed as well[10].

□

While we find that efficient leapfrogging occur generically as equilibria in the simultaneous move investment game, the result that there exist efficient leapfrogging equilibria in the alternating move investment game should be viewed as a special counterexample, and that we typically do not get fully efficient leapfrogging equilibria in alternating move games with sufficiently details discritization of the state space and when investment costs are "reasonable" in relation to production costs (i.e. when the cost of building a new plant $K(c)$ significantly different from zero). However due to the vast multiplicity of equilibria in the simultaneous move investment game, we have no basis for asserting that efficient leapfrogging equilibria are any more likely to arise than other more inefficient equilibria.

We conclude by stating that the inefficiency is caused by *excessive frequency of investment* rather than *underinvestment*. In simultaneous move games we already noted that another source of inefficiency is *redundant, duplicative investments* that occur only in mixed strategy equilibria. We noted that while mixed strategy equilibria also exist in the alternating move investment game, duplicative simultaneous investments cannot occur by the assumption that only one firm can invest at any given time. Thus, the inefficiency of the mixed strategy equilibria of the alternating move games is generally a result of excessively frequent investment under the mixed strategy equilibrium. However it is important to point out that we have constructed examples of inefficient equilibria where there is *underinvestment* relative to the social optimum. Such an example is provided in panel (b) of Figure 4 in the next section.

4.4 Leapfrogging, Rent-dissipation and Preemption

In this section we consider the Riordan and Salant conjecture that was discussed in section 2. Riordan and Salant conjectured that regardless of whether the firms move simultaneously or alternately,

[10]Let the possible cost states be $\{0,5,10\}$, assume deterministic technological progress, the cost of investing $K=4$, and the discount factor $\beta=0.95$. Then the socially optimal investment strategy is for investments to occur when $c=5$ and $c=10$, and these investments will occur at those states in the unique equilibrium of the game, but where one firm makes the first investment at $c=5$ and the opponent makes the other investment when $c=0$. These investments clearly involve leapfrogging that is also fully efficient.

or whether technological progress is deterministic or stochastic, the general outcome in all of these environments should be that of *rent-dissipating preemption*, a situation where only one firm invests and does so sufficiently frequently in order to deter its opponent from investing. These frequent preemptive investments fully dissipate any profits the investing firm can expect to earn from preempting its rival (and hence also dissipating all social surplus). We first confirm their main result stated in terms of our model.

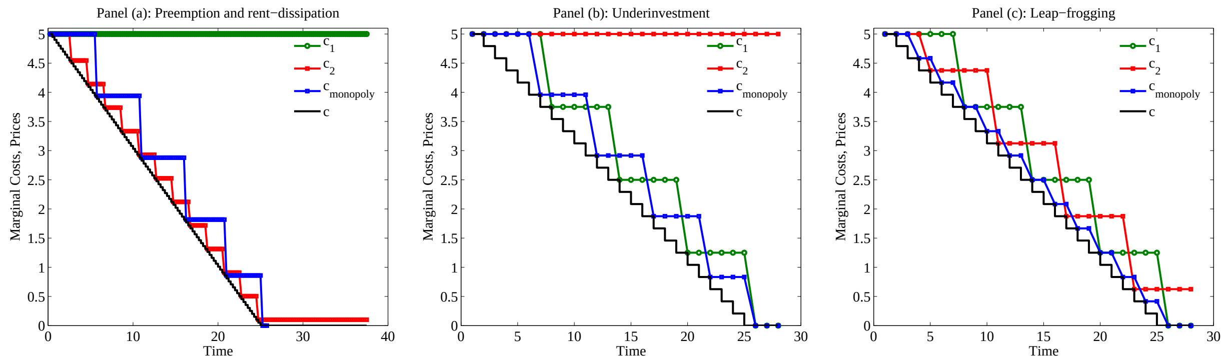
Theorem 5 (Riordan and Salant, 1994). *Consider a continuous time investment game with deterministic alternating moves. Assume that the cost of investment is independent of c, $K(c) = K$ and is not prohibitively high in the sense of inequality (11). Further, assume that technological progress is deterministic with state of the art costs at time $t \geq 0$ given by the continuous, non-decreasing function $c(t)$ and continuous time interest rate $r > 0$. Assume that the continuous time analog of the condition that investment costs are not too high holds, i.e. $C(0) > rK$. Then there exists a unique MPE of the continuous time investment game (modulo relabeling of the firms) that involve preemptive investments by one or the other of the two firms and no investment in equilibrium by its opponent. The discounted payoffs of both firms in equilibrium is 0, so the entire surplus is wasted on excessively frequent investments by the preempting firm.*

Corollary 5.1 (Riordan and Salant, 1994). *The continuous time equilibrium in Theorem 5 is a limit of the unique equilibria of a sequence of discrete time games where $\beta = \exp\{-r\Delta t\}$ and per period profits of the firms, $r_i(c_1, c_2)$, are proportional to Δt and the order of moves alternates deterministically, for a deterministic sequence of state of the art costs given by $(c_0, c_1, c_2, c_3, \dots) = (c(0), c(\Delta t), c(2\Delta t), c(3\Delta t), \dots)$ as $\Delta t \to 0$.*

The proofs of Theorem 5 and Corollary 5.1 is given in Riordan and Salant (1994) who used a mathematical induction argument to establish the existence of the continuous time equilibrium as the limit of the equilibria of a sequence of discrete time alternating move investment games.

In Figure 4 we plot simulated MPE for three versions of the Bertrand investment pricing game with deterministic alternating move and strictly monotonic technological progress. In the panel (a) we let the length of the time periods be relatively small to provide a good discrete time approximation to Riordan and Salant's model in continuous time. In panel (b) we decrease the number points

Figure 4: Production and state of the art costs in simulated MPE: continuous. vs. discrete time

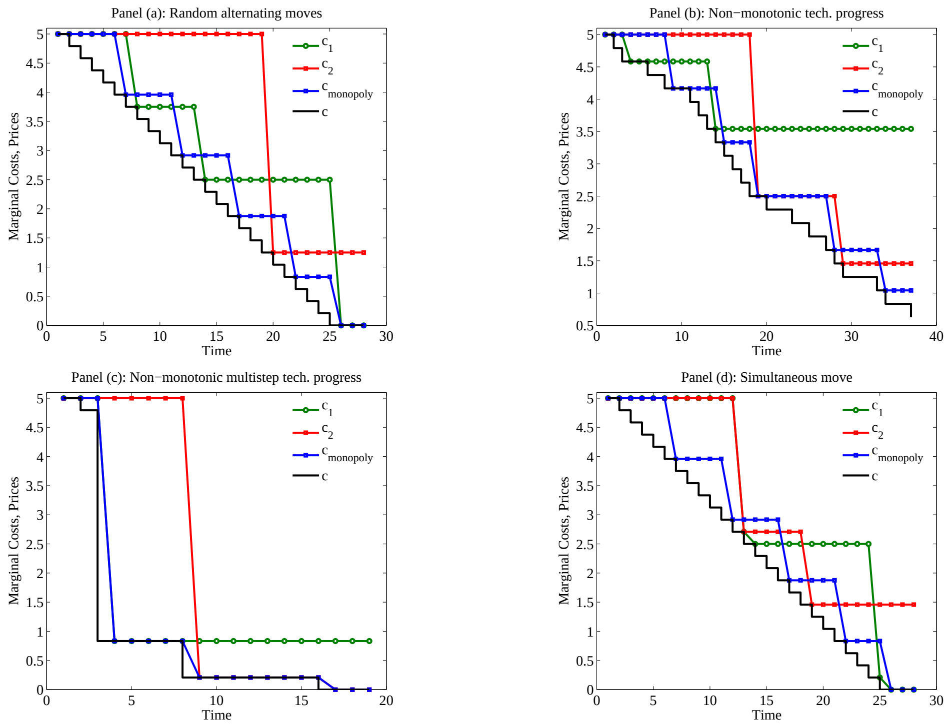


Notes: The figure plots simulated MPE equilibria for three versions of the Bertrand investment pricing game with deterministic alternating move and strictly monotonic technological progress. In panel (a) we present a discrete time approximation to Riordan and Salant's model in continuous time, with parameters $\beta = 0.9512$ $k_1 = 2$, $k_2 = 0$, $\pi(c|c) = 0$, $f(1|1) = f(2|2) = 0$, $f(2|1) = f(1|2) = 1$, $n_c = 100$, $\Delta t = 0.25$. In panel (b) we decrease the number of discrete support points for c to$n_c = 25$ and increase the length of the time period such that $\Delta t = 1$ adjusting per period values. In panel(c) we in addition lower investment costs by setting $k_1 = 0.5$.

of support of the marginal cost and increase the length of the time period. In panel(c) in addition we lower investment cost. These three examples demonstrate that preemptive rent-dissipating investments indeed can happen in discrete time when the cost of investing in the new technology $K(c)$ is large enough relative to per period profits, but fails when the opposite is true as shown in panels (b) and (c). In discrete time, both duopolist have temporary monopoly power that can lead to inefficient under-investment as shown in the equilibrium realization in panel (b) or leapfrogging as shown in panel (c). Since per period profits are proportional to the length of the time period, the latter increases the value of the temporary cost advantage a firm gains after investment in the state of the art technology. If investment costs are sufficiently low relative to per period profits, it can actually be optimal for the cost follower to leapfrog the cost leader, in the limiting case even for a one period cost leadership.

While the Riordan and Salant result of strategic preemption with *full rent dissipation* only holds in the continuous time limit $\Delta t \to 0$, their conclusion that investment preemption will occur is robust to discreteness of time. To this extent we find investment preemption as the only equilibrium in our discrete time numerical solutions when Δt is sufficiently small. Thus, there is a "neighbor-

Figure 5: Production and state of the art costs in simulated MPE under uncertainty



Notes: The figure plots simulated MPE by type for four stochastic generalizations of the model illustrated in Figure 4.b. In panel (a) we consider random alternating moves where $f(1|1) = f(2|2) = 0.2$ and $f(2|1) = f(1|2) = 0.8$. In panel (b) we allow for non-strictly monotonic one step random technological improvement. In panel (c) we allow technological progress to follow a beta distribution over the interval $[c, 0]$ where c is the current best technology marginal cost of production. The scale parameters of this distribution is $a = 1.8$ and $b = 0.4$ so that the expected cost, given an innovation, is $c * a/(a + b)$. Panel (d) plots an equilibrium path from the simultaneous move game. Unless mentioned specifically remaining parameters are as in panel (b) of Figure (4).

hood” of Δt about the limit value 0 for which their unique preemption equilibrium also holds in a discrete time framework. However the conclusion that preemption is fully inefficient and rent dissipating is not robust to discrete time. In discrete time the preempting firm does earn positive profits which results in that the equilibrium is not completely inefficient.

Allowing for *random alternation* in the right to move, we obtain a unique pure strategy equilibrium, since random alternations does not violate the sufficient conditions for uniqueness given in Theorem 3. Yet, random alternation of the right to move destroys the ability to engage in strategic preemption and creates the opportunity for leapfrogging, since firms cannot full control. Figure 5, panel (a) gives an example of a simulated equilibrium path when the right to move alternates randomly. While this equilibrium path depicts a unique pure stately equilibrium, we clearly see the leapfrogging pattern.

From Theorem 3 it follows that if there is positive probability of remaining with the same state of the art cost c for more that one period of time, i.e. $\pi(c|c) > 0$, the main results of Riordan and Salant (1994) will no longer hold in our model. We may have multiple equilibria, there will be leapfrogging, and full rent dissipation fails.

Figure 5 presents simulated equilibrium paths when we introduce randomness in the evolution of the state of the art technology, the order of moves in the alternating move game, or possibility for simultaneous investment. All panels exhibit leapfrogging, reflecting the statement that stochasticity in the model presents the cost follower with more opportunities to leapfrog its opponent and makes it harder for the cost leader to preempt leapfrogging. Overall, in presence of uncertainty, the game becomes much more contestable.

Lemma 3. *(Limits to Riordan and Salant result) Preemption does not hold when (1) cost of investment K(c) is sufficiently small relative to per period profits, (2) investment decisions are made simultaneously, (3) the right to move alternates randomly, (4)* $\pi(c|c) > 0$*, i.e. under other than strictly monotonic technological progress.*

Proof. The proof is by counter examples which are shown in Figure 4 and 5. □

The vast majority of MPE equilibria in the many specifications of the game we have solved using the RLS algorithm exhibited leapfrogging. It appears that Riordan and Salant results are not robust to any of the mentioned assumptions, at least in the discrete time analog of their model. However with the exception of the full rent dissipation result, we believe that there is a *neighborhood* about the limiting set of parameter values that Riordan and Salant used to prove Theorem 5 for which their conjectured preemption equilibrium will continue to hold, at least with high probability.

5 Conclusions

The key contribution of this paper is to provide the first characterization of *all* equilibria of a dynamic duopoly model of Bertrand price competition in the presence of stochastic technological progress. Contrary to the previous literature which has focused on *investment preemption* as the generic equilibrium outcome, we have shown that the generic equilibrium outcome involves various types of *leapfrogging* that result in some of the benefits of technological progress being passed on to consumers in the form of lower prices. We have shown that these dynamic equilibria are generally inefficient due to a combination of excessively frequent investments and duplicative investments resulting from coordination failures between the firms. However we have shown that efficiency is very high and there exist fully efficient asymmetric monopoly equilibria, as well as efficient non-monopoly equilibria involving perfectly coordinated leapfrogging by the two firms.

Our analysis provides a new interpretation of "price wars." In the equilibria of our model, prices are piecewise flat with large sudden declines in prices that occur when a high cost firm leapfrogs its opponent to become the new low cost leader. It is via these periodic price drops that consumers benefit from technological progress and competition between the duopolists. We find surprisingly a large and complex set of equilibria and possible price and investment dynamics from such a simple model. We find a huge number of equilibria ranging from pure strategy monopoly outcomes to highly mixed strategy equilibria where expected profits of both firms are zero. In between are equilibria where leapfrogging investments are relatively infrequent so that consumers see fewer benefits from technological progress in the form of lower prices. We argue that leapfrogging, rather than preemption, is a better description of competitive behavior in actual markets and empirical studies such as Goettler and Gordon (2011) seem to confirm this.

Our analysis also contributes to the long-standing debate about the relationship of market structure and innovation. Schumpeter (1939) argued a monopolist will innovate more rapidly than a competitive industry since the monopolist can fully appropriate the benefits of R&D or other cost-reducing investments, whereas some of these investments would be dissipated in a competitive market. However Arrow (1962) argued that innovation (or new technology adoption) under a monopolist will be slower than would occur in a competitive market which is in turn lower than the rate of innovation that would be chosen by a social planner. Both types of results have appeared in the subsequent literature. For example, in the R&D investment model analyzed by Goettler and Gordon (2011), the rate of innovation under monopoly is higher than under duopoly but still below the rate of innovation that would be chosen by a social planner. These inefficiencies are driven in part by the existence of externalities such as *knowledge spillovers* that are commonly associated with R&D investments.

In a settings where each competing firm can at any time access an *exogenously* developing state of the art technology, we have shown that the rate of adoption of new cost-reducing technologies under the duopoly equilibrium is generally *higher* than the monopoly or socially optimal solution. We showed that equilibria where there is *leapfrogging* and equilibria where there is *investment preemption* both lead the duopolists to collectively invest more in cost reducing technologies than a social planner. Moreover, our model provides an example where monopoly outcome coincides with social optimum investment strategy. This result is rather specialized, and should be checked against some of the restrictive assumptions of the model. In particular, it would be important to extend the model to allow for entry and exit of firms.[11]

A disturbing aspect of our findings from a methodological standpoint is the plethora of Markov perfect equilibria present in a very simple extension of the standard static textbook model of Bertrand price competition, which is reminiscent of the "Folk theorem" for repeated games. Though we have shown that the set of payoffs shrinks dramatically to a strict subset of the triangle under the alternating move game and a unique MPE obtains when the probability that an improvement in the state of the art technology is sufficiently close to one, there will generally be a huge multiplicity of equilibria either when firms move simultaneously, or when the probability of technological improvement is sufficiently low. Thus, though we have demonstrated how leapfrogging can be viewed as an endogenous solution to the "anti-coordination problem" our paper leaves unsolved the more general question of how firms coordinate on a single equilibrium when there is a vast multiplicity of possible equilibria.

[11]We refer readers to the original work by Reinganum (1985) as well as recent work by Acemoglu and Cao (2011) and the large literature they build on. It is an example of promising new models of endogenous innovation by incumbents *and* new entrants. In their model entrants are responsible for more "drastic" innovations that tend to replace incumbents, who focus on less drastic innovations that improve their existing products.

A Proofs of Lemmas and Theorems

Theorem 1 (Necessary and sufficient condition for investment by the social planner).

Proof. Note that the left hand side of inequality (11) is the discounted cost savings from adopting the state of the art technology c when the existing plant has marginal cost c_ς. We first prove that if this inequality holds, then investment will be socially optimal at some state (c'_ς, c') satisfying $c'_\varsigma \in [0, c_\varsigma]$ and $c' \in [0, c'_\varsigma]$. Suppose, to the contrary, that investment is not optimal for the social planner for any value $c'_\varsigma \leq c_\varsigma$ and any $c' \in [0, c'_\varsigma]$. It follows that $C(c'_\varsigma, c') = c'_\varsigma/(1-\beta)$ for all $c'_\varsigma \leq c_\varsigma$ and all $c' \in [0, c'_\varsigma]$. However if the social planner did decide to invest when the state is (c_ς, c) the planner's discounted costs would be $c_\varsigma + K(c) + \beta \int_0^c C(c, c')\pi(dc'|c)$. Since we have assumed it is not optimal for the social planner to invest at any state (c'_ς, c') with $c' \in [0, c'_\varsigma]$, then it cannot be optimal to invest in particular at any state (c_ς, c) with $c \in [0, c_\varsigma]$. It follows that $c_\varsigma/(1-\beta) \leq c_\varsigma + K(c) + \beta \int_0^c C(c, c')\pi(dc'|c)$ for all $c \in [0, c_\varsigma]$. However since $C(c, c') = c/(1-\beta)$ for all $c' \in [0, c]$, it follows that $\beta c_\varsigma/(1-\beta) \leq K(c) + \beta c/(1-\beta)$ for all $c_\varsigma \geq 0$ and all $c \in [0, c_\varsigma]$, but this contradicts inequality (11).

Conversely, suppose inequality (11) does not hold. Then it follows that there is no value of $[c'_\varsigma, c']$ with $c' \in [0, c'_\varsigma]$ for which investment is optimal, since we can show that in this case $C(c'_\varsigma, c') = c'_\varsigma/(1-\beta)$ for all $c'_\varsigma \in [0, c_\varsigma]$. This latter result follows by verifying that it is a solution to the Bellman equation (9), where it follows that the cost of replacing a plant with marginal cost c'_ς in state c' is $c'_\varsigma + K(c') + \beta c'/(1-\beta)$ which exceeds the cost of keeping the existing plant $C(c'_\varsigma, c') = c'_\varsigma/(1-\beta)$ by our assumption that inequality (11) does not hold for any $c'_\varsigma \in [0, c_\varsigma]$ and $c' \in [0, c'_\varsigma]$. Since the solution to the Bellman equation is unique (via the contraction mapping property) and corresponds to an optimal investment policy, we conclude that there is no state (c'_ς, c') with $c'_\varsigma \in [0, c_\varsigma]$ and $c' \in [0, c'_\varsigma]$ for which investment in the state of the art technology c' is socially optimal. □

Theorem 2 (Characterization of the set of equilibrium payoffs).

The proof requires some intermediary results.

Lemma A.1 (Characterization of no investment MPE)**.** *There is a unique no investment equilibrium where neither of the firms invests in state* $(c_1, c_2, c) \in S$*, if* $\forall c_1 \in [0, c_0]$ *and* $\forall c \in [0, c_1]$ *we have*

$$K(c) \geq \frac{\beta(c_1 - c)}{(1-\beta)}. \tag{12}$$

Proof. Consider the simultaneous move game first, i.e. the case $m = 0$. If it is an equilibrium for neither firm to invest, we must have

$$\begin{aligned} P_1(c_1, c_2, c) &= 0 \\ P_2(c_1, c_2, c) &= 0, \end{aligned}$$

$\forall (c_1, c_2, c) \in S$. Let c_0 be the apex point of S, i.e. c_0 is the initial and highest value of the state of the art marginal cost of production at the start of the game. Then, in particular me must have $P_1(c_1, c_0, c) = 0$ for all $c_1 \in [0, c_0]$ and for all $c \in [0, c_1]$. If it is never optimal for either firm to invest in any state, then it follows that

$$\begin{aligned} v_{N,1}(c_1, c_0, c) &= \frac{c_0 - c_1}{(1-\beta)} \\ v_{I,1}(c_1, c_0, c) &= c_1 - K(c) + \frac{\beta(c_0 - c)}{(1-\beta)} \end{aligned}$$

When $\eta = 0$, $P_1(c_1, c_0, c) = 0 \iff v_N(c_1, c_0, c) \geq v_{I,1}(c_1, c_0, c)$. This implies that the following inequality must hold $\forall c_1 \in [0, c_0]$ and $\forall c \in [0, c_1]$

$$\frac{c_0 - c_1}{(1-\beta)} \geq (c_0 - c_1) - K(c) + \beta \frac{(c_0 - c)}{(1-\beta)}. \tag{13}$$

It is easy to see via simple algebra that inequality (13) is equivalent to inequality (12). Now consider the alternating move game, $m \neq 0$. It is not hard to show, using the Bellman equations for the alternating move game (see equation 7 in section 2), that if it is never optimal for either firm to invest, then it follows that for the state $(c_1, c_2, c) = (c_1, c_0, c)$ (where recall that c_0 is the initial value of the state of the marginal production c), that $P_1(c_1, c_0, c) = 0$ for all $c_1 \in [0, c_0]$ and for all $c \in [0, c_1]$. But this will be true if and only if

$$\begin{aligned} v_{N,1}(c_1, c_0, c, 1) &= \frac{c_0 - c_1}{(1-\beta)} \\ v_{I,1}(c_1, c_0, c, 1) &= c_1 - K(c) + \frac{\beta(c_0 - c)}{(1-\beta)} \end{aligned}$$

and $v_{N,1}(c_1,c_0,c,1) \geq v_{I,1}(c_1,c_0,c,1)$. But it is easy to see that this is equivalent to inequality (13) above, which is in turn equivalent to the inequality (12), thereby proving Lemma A.1. $\square$

Lemma A.2 (Leapfrogging in the mixed strategy equilibrium)**.** *Suppose* $\eta = 0$ *and* $m = 0$ *(i.e. simultaneous move game with no investment cost shocks),* $c = 0$*, and the investment is socially optimal, i.e.* $\beta \min(c_1,c_2)/(1-\beta) > K(0)$*. Then if* $c_1 > c_2 > 0$*, in the mixed strategy equilibrium the probability that firm 1 invests exceeds the probability that firm 2 invests,* $P_1(c_1,c_2,0) > P_2(c_1,c_2,0)$*.*

Proof. For convenience, we will drop the arguments in the mixed strategy probabilities and write P_1 and P_2 instead of $P_1(c_1,c_2,0)$ and $P_2(c_1,c_2,0)$. We start by noting that $K(0) < \frac{\beta c_2}{1-\beta}$ ensures that investment is profitable even for firm 1 whose potential pay-off is smaller. In other words, both firms' investment decisions are economically justified. Next, observe that when $\beta = 0$ in the $(c_1,c_2,0)$ end game there is unique pure strategy equilibrium where neither of the companies invests. Thus, we only consider the case $\beta > 0$. Also, to simplify notation we set $K = K(0)$.

The value functions of the two firms in the point $(c_1,c_2,0)$ can be written as

$$\begin{aligned}
V_1 &= P_1 \times \left(P_2 \cdot (-K) + (1-P_2) \cdot \left(\frac{\beta c_2}{1-\beta} - K\right)\right) + \\
&\quad + (1-P_1) \times (P_2 \cdot 0 + (1-P_2) \cdot \beta V_1) \\
V_2 &= P_2 \times \left(P_1 \cdot (c_1 - c_2 - K) + (1-P_1) \cdot \left(c_1 - c_2 + \frac{\beta c_1}{1-\beta} - K\right)\right) + \\
&\quad + (1-P_2) \times (P_1 \cdot (c_1 - c_2) + (1-P_1) \cdot (c_1 - c_2 + \beta V_2))
\end{aligned}$$

where the definition of the probability P_1 of investment by firm 1 in the mixed strategy equilibrium gives

$$P_2 \cdot (-K) + (1-P_2) \cdot \left(\frac{\beta c_2}{1-\beta} - K\right) = P_2 \cdot 0 + (1-P_2) \cdot \beta V_1$$

and thus the value function itself becomes the weighted sum of equal parts, leading to

$$V_1 = P_2 \cdot (-K) + (1-P_2) \cdot \left(\frac{\beta c_2}{1-\beta} - K\right) = P_2 \cdot 0 + (1-P_2) \cdot \beta V_1$$

Using the second equality in the last expression, we find $V_1 = 0$, and then using the first equality in the same expression, we find $1 - P_2 = \frac{K(1-\beta)}{\beta c_2}$.

The definition of the probability P_2 of investment by firm 2 in the mixed strategy equilibrium, similarly gives

$$\begin{aligned} V_2 &= P_1 \cdot (c_1 - c_2 - K) + (1 - P_1) \cdot \left(c_1 - c_2 + \frac{\beta c_1}{1-\beta} - K \right) \\ &= P_1 \cdot (c_1 - c_2) + (1 - P_1) \cdot (c_1 - c_2 + \beta V_2) \end{aligned}$$

Using the second equality in the last expression, we find $V_2 = \frac{c_1 - c_2}{(1-\beta \cdot (1-P_1))}$, and using the it once again we get

$$\begin{aligned} P_1(c_1 - c_2 - K) + (1 - P_1)\left(c_1 - c_2 + \frac{\beta c_1}{1-\beta} - K \right) &= P_1(c_1 - c_2) + (1 - P_1)(c_1 - c_2 + \beta V_2) \\ (1 - P_1)\left(\frac{\beta c_1}{1-\beta} - K \right) - P_1 K &= (1 - P_1)\beta V_2 \\ \frac{c_1}{1-\beta} - \frac{K}{\beta \cdot (1 - P_1)} &= V_2 \end{aligned}$$

Combining the two expressions for the value function V_2, we get the following equation

$$\frac{c_1 - c_2}{1 - \beta \cdot (1 - P_1)} = \frac{c_1}{1-\beta} - \frac{K}{\beta \cdot (1 - P_1)}$$

Multiplying by $1 - \beta$ and inserting the expression for $1 - P_2$, we have

$$\begin{aligned} \frac{c_1 - c_2}{1 + \frac{\beta}{1-\beta} P_1} &= c_1 - \frac{1 - P_2}{1 - P_1} c_2 \\ \frac{c_1 - \frac{1-P_2}{1-P_1} c_2}{c_1 - c_2} &= \frac{1}{1 + \frac{\beta}{1-\beta} P_1} \leqslant 1 \\ c_1 - \frac{1 - P_2}{1 - P_1} c_2 &\leqslant c_1 - c_2 \\ \frac{1 - P_2}{1 - P_1} &\geqslant 1 \\ P_1 &\geqslant P_2 \end{aligned}$$

The inequalities are due to the fact that $0 \leqslant P_1 \leqslant 1$, $\frac{\beta}{1-\beta} > 0$, $c_1 - c_2 > 0$, $c_2 > 0$. The final inequality is strict unless $P_1 = P_2 = 0$, which implies $K = \frac{\beta c_2}{1-\beta}$ thus leading to a contradiction. We conclude then that $P_1 > P_2$. □

Lemma A.3 (Efficiency of equilibria in the simultaneous move end game)**.** *Suppose* $m = 0$ *and* $c = 0$ *(the end game of the simultaneous move game). In states where investment is not socially optimal,* *i.e.* $\beta \min(c_1, c_2)/(1-\beta) < K(0)$, *the investment game has a unique pure strategy equilibrium where neither firm invests. When investment is socially optimal, the investment game has three subgame perfect Nash equilibria: two efficient pure strategy equilibria and an inefficient mixed strategy equilibrium.*

Proof. Recall that we are considering the simultaneous move game and show that there are two possible equilibrium configurations at any end game state $(c_1, c_2, 0)$: either the state admits a unique no-investment equilibrium where neither firm invests, or there are three possible equilibria in the state, two of which are the pure strategy "anti-coordination" equilibria and the third is a mixed strategy equilibrium. We now prove that the no-investment equilibrium obtains if and only if a social planner who is operating two plants with marginal costs (c_1, c_2) when the state of the art marginal cost is 0 does not find it optimal to incur the investment cost $K(0)$ to acquire this state of the art technology. Also the social planner will invest in the state of the art technology if and only if there are the three above mentioned equilibria exist at the end game state $(c_1, c_2, 0)$.

Consider the case where $c_1 \leq c_2$. It is enough to prove this case since it will be clear that the proof in the case where where $c_1 > c_2$ is symmetric to the argument given below. The optimal operating and investment rule for a social planner who controls two plants with costs $c_1 \leq c_2$ is to a) shut down plant 2 since it is obsolete relative to plant 1 and plant 1 can supply the entire market, and b) invest in the state of the art zero marginal cost technology if this lowers the discounted production costs. Since the investment cost of building the new state of the art plant is $K(0)$ and there is a one period time to build it, the discounted costs of investment and production from investing in the state of the art technology is $c_1 + K(0)$. If the social planner does not invest in the state of the art technology and produces forever using the lower cost plant at a marginal cost of c_1 the present value is $c_1/(1-\beta)$. Thus the social planner will invest in the state of the art techology if and only $c_1 + K(0) < c_1/(1-\beta)$, or equivalently,

$$\frac{\beta c_1}{(1-\beta)} > K(0). \tag{14}$$

This condition states that the cost of investing is not too high relative to the discounted marginal cost of production of the lower cost plant c_1, i.e. that the cost of investing does not outweigh the discounted future cost savings resulting from the investment.

Now consider the Nash equilibria of the $(c_1, c_2, 0)$ end game. We show that if investment is not optimal for the social planner, i.e. if $c_1 < (1-\beta)K(0)/\beta$, then there is only a single "no investment" equilibrium of this game. Otherwise there are three equilibria: two pure strategy anticoordination equilibria where either firm 1 invests and firm 2 doesn't (and vice versa), and a zero expected profit mixed strategy equilibria where the two firms invest with probabilities $\pi_1 < \pi_2$, respectively, by Lemma A.2. Consider first the pure strategy equilibrium where firm 1 invests and firm 2 doesn't. The payoff to firm 1 to investing is $c_2 - c_1 + \beta c_2/(1-\beta) - K(0)$ whereas the payoff to to firm 2 from not investing is 0. If firm 2 deviates and chooses to invest, then its payoff is $-K(0)$ because by simultaneous investment both firms 1 and 2 will have acquired the zero marginal cost state of the art technology and the ensuing Bertrand price competition will drive prices and profits of both firms to zero. Thus, the high cost firm (firm 2) does not want to invest if it knows that the low cost firm (firm 1) plans to invest under any circumstances. However the "deviation payoff" to firm 1 involves not investing this period but thereafter "returning to the equilibrium path" and making the investment one period later. The payoff to this one period delay in investing is $c_2 - c_1 + \beta[c_2 - c_1 + \beta c_2/(1-\beta) - K(0)]$. For the conjectured pure strategy equilibrium to actually be possible it must be

$$c_2 - c_1 + \frac{\beta c_2}{(1-\beta)} - K(0) > c_2 - c_1 + \beta\left[c_2 - c_1 + \frac{\beta c_2}{(1-\beta)} - K(0)\right]. \tag{15}$$

But after some simple algebraic rearrangements, this inequality id is equivalent to inequality (14) defining the socially optimal investment condition. Thus, we conclude that the pure strategy equilibrium where firm 1 invests and firm 2 doesn't exists if and only if it is socially optimal for the investment to occur.

Now consider the other pure strategy equilibrium where firm 2 invests and firm 1 doesn't. The payoff to firm 2 from investing is $\beta c_1/(1-\beta) - K(0)$ whereas the payoff to deviating and delaying the investment by one period is $\beta[\beta c_1/(1-\beta) - K(0)]$, so as long as $\beta c_1/(1-\beta) - K(0) > 0$ it will be optimal for firm 2 to invest, but of course this is the same as inequality (14) defining the optimal investment rule for the social planner. For firm 1, the payoff to not investing is $c_2 - c_1$ whereas the payoff to investing given that it knows that firm 2 will also invest is $c_2 - c_1 - K(0)$. Thus, firm 1 will never want to invest if it knows firm 2 will invest, and we have shown that the pure strategy equilibrium where only firm 2 invests exists if and only if it is socially optimal for investment to occur. Notice that even though firm 2 is the high cost firm, the fact that it invests rather than investing being done by the low cost firm does not entail any higher costs because regardless of whether firm 1 or firm 2 invests, both of their existing plants will become obsolete and production will be done using the new state of the art zero marginal cost production technology.

Finally, consider the mixed strategy equilibrium. Following the proof of Lemma A.2, the probability that firm 1 invests in the mixed strategy equilibrium is given by

$$\pi_1 = \frac{\beta c_1/(1-\beta) - K(0)}{\beta c_1/(1-\beta)} \tag{16}$$

and $\pi_2 > \pi_1$. Note that $\pi_1 \geq 0$ if and only if $\beta c_1/(1-\beta) - K(0) \geq 0$ which is the same as inequality (14) for investment to be socially optimal. However this does not imply that the mixed strategy equilibrium is efficient because of the potential *redundant* investment by the two firms in the mixed strategy equilibrium. Let C_m be the present discounted value of investment and production costs under this mixed strategy equilibrium. We have

$$\begin{aligned} C_m &= 2K\pi_1\pi_2 + K\pi_1(1-\pi_2) + K\pi_2(1-\pi_1) + c_1 + \beta(1-\pi_1)(1-\pi_2)C_m \\ &= K(\pi_1\pi_2) + c_1 + \beta(1-\pi_1)(1-\pi_2)C_m \end{aligned}$$

We will now show that $C_m = (K(\pi_1+\pi_2)+c_1)/(1-\beta(1-\pi_1)(1-\pi_2))$ exceeds the socially optimal production and investment costs $c_1 + K(0)$ that a social planner can achieve by undertaking only a single investment in the state of the art technology and avoid the higher costs due to redundant investments (when the two firms invest at the same time) and the costs due to delayed in investment (due to the probability $(1-\pi_1)(1-\pi_2)$ that neither firm invests under the mixed strategy equilibrium. Since the algebra to show that $C_m > c_1 + K(0)$ gets rather messy, we establish this inequality via an indirect argument. Let $p = \pi_1 + \pi_2 - \pi_1\pi_2 \in (0,1)$ be the probability that *at least one of the firms invests in the mixed strategy equilibrium.* Define a new cost value $\underline{C}_m$ by

$$\begin{aligned} \underline{C}_m &= c_1 + pK(0) + \beta(1-p)\underline{C}_m. \\ &= \frac{c_1 + pK(0)}{1-\beta(1-p)}. \end{aligned}$$

Since $\underline{C}_m$ is the present value of costs under a mixed strategy equilibrium that ignores the occurrence of redundant investments by the two firms, it is evident that $\underline{C}_m < C_m$. We now show that

$\underline{C}_m > c_1 + K(0)$, and thus $C_m > c_1 + K(0)$. To see why $\underline{C}_m > c_1 + K(0)$ we write $\underline{C}_m(p)$ to emphasize its dependence on p, the probability that at least one firm invests in the mixed strategy equilibrium. Note that $\underline{C}_m(0) = c_1/(1-\beta) > c_1 + K(0)$, and $\underline{C}_m(1) = c_1 + K(0)$. Since we know that the true value of $p < 1$, it suffices to show that $d/dp\underline{C}_m(p) < 0$. Calculating this derivative, we have

$$\frac{d}{dp}\underline{C}_m(p) = \frac{K(0) - \beta\underline{C}_m(p)}{1 - \beta(1-p)}. \tag{17}$$

Note that since $\underline{C}_m(0) = c1/(1-\beta)$ we have $d/dp\underline{C}_m(p)|_{p=0} < 0$ by inequality (14). Further we have

$$\frac{d}{dp}\underline{C}_m(p)|_{p=1} = K(0)(1-\beta) - \beta c_1 < 0 \tag{18}$$

again by inequality (14). It is not hard to see from the two inequalities above that in fact we also have $d/dp\underline{C}_m(p) < 0$ for each $p \in [0,1]$. Thus, it follows that $C_m > \underline{C}_m > c_1 + K(0)$ which establishes the inefficiency of the mixed strategy equilibrium. □

Lemma A.4 (No investment equilibrium at edge states)**.** *In both the simultaneous and alternating move games with no investment cost shocks (i.e. $\eta = 0$) there is a unique stage equilibrium at all edge states in which neither firm invests.*

Proof. In the absence of random investment costs, once one of the firms has acquired the state of the art technology (i.e. $c_j = c$), it will not want to invest again, but rather wait until a further technological innovation occurs at some time in the future and perhaps invest again at that time. Similarly, the opponent will not have an incentive to invest either even if its plant is not state of the art since it realizes that its investment will only enable it to match the state of the art production cost of its rival, and the resulting Bertrand price competition will ensure that both firms earn zero profits until some technological innovation occurs in the future that would enable one or the other firms to leapfrog its opponent. So the Bertrand investment paradox logic does indeed hold at the edge states and is the reason for no investment by either firm there. □

Lemma A.5 (Necessary and sufficient conditions for investments by social planner at state (c_1, c))**.** *Suppose that it is not optimal for the social planner (or monopolist) to invest at state (c_1, c), with $c_1 \geq c$. Let $\tilde{\tau}$ denote the first passage time from the point (c_1, c) to the set $I = \{(c_1, c) | \iota(c_1, c) = 1\}$,* *i.e.* τ *is the random time until it is optimal for the social planner to invest conditional on starting at state* (c_1, c)*. We then have:*

$$K(c) > \frac{(c_1 - c)(\beta - E\{\beta^{\tilde{\tau}}\})}{(1-\beta)}. \tag{19}$$

Proof. From the Bellman equation (9) for $C(c_1, c)$, defining the cost function corresponding to the socially optimal investment strategy, we see that if $\iota(c_1, c) = 0$ (i.e. investment is not optimal at (c_1, c)), then

$$K(c) > \beta \int_0^c \left[C(c_1, c') - C(c, c')\right] \pi(dc'|c). \tag{20}$$

We also have

$$\begin{aligned} C(c_1, c) &= c_1 + \beta \int_0^c C(c_1, c') \pi(dc'|c) \\ C(c, c) &= c + \beta \int_0^c C(c, c') \pi(dc'|c). \end{aligned} \tag{21}$$

Using equation (21) we can rewrite inequality (20) as

$$K(c) > [C(c_1, c) - C(c, c)] - [c_1 - c]. \tag{22}$$

Let $\tilde{\tau}$ be the first passage time from the point (c_1, c) to the set I, and let $\tilde{c}_{\tilde{\tau}}$ be the value of the $\{c_t\}$ process at the time $\tilde{\tau}$ first enters the set I starting from the point (c_1, c). Let $V_{\tilde{\tau}}(c, c)$ denote the expected discounted value of the policy of starting in state (c, c) and not investing for periods $t = 1, \dots, \tilde{\tau} - 1$ and investing at period $\tilde{\tau}$ and investing in the state of the art technology $\tilde{c}_{\tilde{\tau}}$ in effect at $\tilde{\tau}$ and then following the socially optimal investment policy thereafter. Since $C(c, c)$ is the minimal cost under an optimal investment policy, it follows that

$$C(c, c) \leq V_{\tilde{\tau}}(c, c), \tag{23}$$

and thus,

$$K(c) > [C_1(c_1, c) - V_{\tilde{\tau}}(c, c)] - [c_1 - c]. \tag{24}$$

Since $V_{\tilde{\tau}}(c, c)$ is the discounted expected value of following, with probability 1, the same optimal investment policy that the social planner would follow starting from the point (c_1, c), it follows that

$$C(c_1, c) - V_{\tilde{\tau}}(c, c) = \frac{(c_1 - c)(1 - E\{\beta^{\tilde{\tau}}\})}{(1-\beta)}, \tag{25}$$

i.e. the difference in the values is simply the total expected discounted difference in per period costs, $c_1 - c$, from not investing in periods $t = 1, \ldots, \tilde{\tau}$ when initial production costs are c_1 and c, respectively, and then following the optimal investment policy from the point $\tilde{c}_{\tilde{\tau}} \in I$ thereafter. Substituting the formula for the difference in expected costs in equation (25) and substituting into inequality (22) we obtain inequality (19). $\square$

Proof. We prove Theorem 2 statement by statement.

Statement 1. By Theorem 1, if investment is optimal for the social planner, then inequality (12) cannot hold. However by Lemma A.1, it follows that no investment cannot be an MPE outcome. The result then follows.

Statement 2. The two candidate "monopoly" equilibria are where firm 2 never invests in equilibrium and where firm 1 does all the investing (whenever it is profit-maximizing for firm 1 to do so), and symmetrically, where firm 1 never invests and firm 2 does all the investing (whenever it is profit-maximizing for firm 2 to do so). By "monopoly equilibrium" we mean a situation where the firm that is designated to do all the investing in this duopoly equilibrium will behave exactly the same if this firm were an actual monopolist but constrained to charge a price no higher than the marginal cost of production of its opponent.

Our proof is by induction in the case where the support of the exogenous Markov process $\{c_t\}$ for the evolution of the state of the art production technology is a finite set, $\{c_1, \ldots, c_n\}$ with the normalization that $c_1 = 0$ and $c_n = c_0$ where c_0 is the initial technology level at time $t = 0$. We will prove the result for the case where firm 1 is the "monopolist" and firm 2 never invests. Obviously a symmetric proof holds for the symmetric case where firm 2 is the monopolist and firm 1 never invests.

To start the induction, we refer the reader to Lemma A.3 which establishes that in each endgame state $(c_1, c_2, 0)$ if investment is optimal for the social planner, then there exist three equilibria, one of which is an equilibrium where firm 1 invests and firm 2 does not invest. In any state $(c_1, c_2, 0)$ where investment is not socially optimal, neither firm invests. When investment is socially optimal we choose the equilibrium where firm 1 invests and firm 2 doesn't, and neither firm invests in states where investment is not socially optimal. Thus, we have verified that the result holds in the initial state $c_1 = 0$ of our proof by induction.

Now for inductive step, we prove that if the result holds for $c \in \{c_1, \ldots, c_{j-1}\}$, then it also holds at the state of the art cost c_j, for all points $(c_1, c_2, c) \in S$ where $c_j = c$ and $c_1 \geq c$ and $c_2 \geq c$. We start by considering states $(c_1, c_2, c) \in S$ for which $c_1 \leq c_2$. We now show that for any such state where firm 1 invests in equilibrium, that it is optimal for firm 2 not to invest, and further, firm 1 will only invest in states where it is socially optimal to invest. We will show that $v_{N,2}(c_1, c_2, c) = 0$ and $v_{I,2}(c_1, c_2, c) < 0$ which implies that $P_2(c_1, c_2, c) = 0$.

The fact that firm 1 will adopt a socially optimal investment strategy follows immediately once we prove that firm 2 never invests in equilibrium. Since firm 1 knows that firm 2 will not invest, firm 1 maximizes its profits by adopting an investment strategy that minimizes its present discounted costs of production and investment from any given starting node in the game (c_1, c_2, c). For some of these points, it may be optimal for firm 1 not to invest — both at point (c_1, c_2, c) and all subsequent points (c_1, c_2, c_t) that are reached as that state of the art technology evolves from the point c to other points $\{c_t\}$. However when this is the case, it would not be socially optimal for investment to occur by a social planner who has control of two production plants with marginal costs c_1 and c_2, respectively. As we noted above, the social planner would simply produce from the plant with the lower marginal cost of production and shut the other one down, and if the condition $\beta(\min[c_1, c_2] - c')/(1-\beta) \leq K(c')$ for all $c' \in [0, c]$, then inequality (11) of Theorem 1 implies that it would not be optimal for the social planner to undertake any further investment in the future.

So the remainder of this proof focuses on proving that $P_2(c_1, c_2, c) = 0$. We start with the easy case by showing that it will not be optimal for firm 2 to invest whenever $P_1(c_1, c_2, c) = 1$. From the Bellman equations for $(v_{N,2}, v_{I,2})$ in equation (7) of section 2, we have

$$v_{N,2}(c_1, c_2, c) = \beta \int_0^c \max[v_{N,2}(c, c_2, c'), v_{I,2}(c, c_2, c')] \pi(dc'|c). \tag{26}$$

By the inductive hypothesis we have $v_{N,2}(c, c_2, c') = 0$ and $v_{I,2}(c, c_2, c') < 0$ for $c' < c$. This implies that $v_{N,2}(c_1, c_2, c) = 0$. Now consider $v_{I,2}(c_1, c_2, c)$. For the Bellman equation (7) we have when $P_1(c_1, c_2, c) = 1$ and $c_2 > c_1$

$$v_{I,2}(c_1, c_2, c) = -K(c) + \beta \int_0^c \max[v_{N,2}(c, c, c'), v_{I,2}(c, c, c')] \pi(dc'|c). \tag{27}$$

By the inductive hypothesis, $\max[v_{N,2}(c,c,c'),v_{I,2}(c,c,c')] = 0$ for $c' < c$. Further, by Lemma A.4 it is never an equilibrium for either firm to invest at the corner of the state space, so $\max[v_{N,2}(c,c,c),v_{I,2}(c,c,c)] = 0$. It follows that $v_{I,2}(c_1,c_2,c) = -K(c)$ when $P_1(c_1,c_2,c) = 1$, confirming the claim that $P_2(c_1,c_2,c) = 0$.

Now consider a state (c_1,c_2,c) for which $P_1(c_1,c_2,c) = 0$. The argument is more complicated here since there is a potential for firm 2 to use the non-investment by firm 1 as an opportunity to sneak in and leapfrog firm 1 to become the new low cost leader. We now show that as long as the necessary and sufficient condition for socially optimal investment in inequality (11) of Theorem 1 holds, it will never be optimal for firm 2 to try to exploit firm 1 to become the low cost leader in any state (c_1,c_2,c) where $P_1(c_1,c_2,c) = 0$ (where it is *temporarily* not optimal for firm 1 to invest, but firm 1 will invest at some future state).

Note that though firm 1 does not invest at state (c_1,c_2,c), it will invest at some point in the future at a state (c_1,c') where it is socially optimal (as well as profit maximizing for a monopolist) to invest. Let $\tilde{\tau}$ be the mean first passage time to the set $I = \{(c_1,c)|\psi(c_1,c) = 1\}$ where investment by firm 1 first occurs starting from state (c_1,c_2,c), and let $\tilde{c}_{\tilde{\tau}}$ be the random state of the art cost that induces firm 1 to invest (i.e. for which $\psi(c_1,\tilde{c}_{\tilde{\tau}}) = 1$ under the social planning solution or $P_1(c_1,c_2,\tilde{c}_{\tilde{\tau}}) = 1$ under the posited duopoly equilibrium). It follows that if firm 2 were to invest, it would have temporary low cost leadership over the periods $\{1,2,\ldots,\tilde{\tau}-1\}$ but at period $\tilde{\tau}$ firm 1 will invest and leapfrog firm 2, returning to the firm 1 monopoly investment "equilibrium path" (note that this includes the case $\tilde{\tau} = \infty$ if it is not optimal for firm 1 to invest ever again after firm 2 invests). Once (or if) firm 1 returns to the equilibrium path by investing in the state of the art technology $\tilde{c}_{\tilde{\tau}}$, firm 2 will not invest and earn 0 discounted profits, as per our inductive hypothesis, since $\tilde{c}_{\tilde{\tau}} < c$ with probability 1. Thus, it follows that if firm 2 does not invest at (c_1,c_2,c) it will earn a discounted expected profit of $v_{N,2}(c_1,c_2,c) = 0$, whereas if firm 2 decides to invest, it earns an expected reward equal to

$$v_{I,2}(c_1,c_2,c) = -K(c) + \frac{(c_1-c)(\beta - E\{\beta^{\tilde{\tau}}\})}{(1-\beta)}. \tag{28}$$

However by inequality (19) of Lemma A.5, the hypothesis that it is not optimal for firm 1 to invest at (c_1,c_2,c) implies that the expected profits to firm 2 from this attempt to take advantage of firm

1's non-investment and leapfrog firm 1 is negative.

Statement 3. We use a proof by induction similar to our proof of Statement 2. We have already established that there is a mixed strategy equilibrium in the end game states $(c_1, c_2, 0) \in S$ in Lemma A.3 above. Since the expected payoff to the high cost firm from not investing is 0, it follows that the expected payoff to the high cost firm is zero, but in general the expected payoff to the low cost firm is positive, though in the case where $c_1 = c_2$, it is easy to see that the expected payoffs to both firms are zero. Further, it is not difficult to show that there is symmetry in the payoffs and equilibrium strategies for the two firms in this equilibrium: $v_{N,1}(c_1, c_2, 0) = v_{N,2}(c_2, c_1, 0)$ and $v_{I,1}(c_1, c_2, 0) = v_{I,2}(c_2, c_1, 0)$, and $P_1(c_1, c_2, 0) = P_2(c_2, c_1, 0)$. Thus, we have established the initial induction step of our proof by induction.

Now suppose that the result holds for all state points $(c_k, c_k, c_{j-1}) \in S$ where $c_1 = 0$, and c_j is the j^{th} highest point in the assumed finite support of the process $\{c_t\}$ governing the evolution of the state of the art marginal costs of production and $c_k \in \{c_{j-1}, c_j, \dots, c_n\}$. The theorem will hold if we can prove that $v_{N,1}(c_k, c_k, c_j) = 0$ and $v_{N,2}(c_k, c_k, c_j) = 0$. For notational compactness below, we will let (c, c, c') denote a generic point of the form $(c_k, c_k, c_j) \in S$.

To show that the expected payoffs to both firms are 0 in these "diagonal states" (c, c, c'), it is sufficient to show that $v_{N,i}(c, c, c') = 0$ and $v_{I,i}(c, c, c') \leq 0$, for $i \in \{1, 2\}$. We now show that these payoffs will hold in the two possible equilibria that can hold in the stage game at each of these diagonal states $(c, c, c') \in S$ under the proposed equilibrium: a) a "no investment equilibrium" where there is a unique equilibrium where neither firm invests, not invest or b) an investment equilibrium, where there are three possible equilibria at the stage game, and we select the mixed strategy equilibrium and show it results in zero expected payoffs to both firms.

Suppose there is a unique no investment equilibrium at (c, c, c'). Then from the Bellman equation (7) we have

$$\begin{aligned}
v_{N,1}(c, c, c') &= \frac{\beta \sum_{i=1}^{j-1} \max[v_{N,1}(c, c, c_i), v_{I,1}(c, c, c_i)]}{1 - \beta\pi(c_j|c_j)} = 0 \\
v_{N,2}(c, c, c') &= \frac{\beta \sum_{i=1}^{j-1} \max[v_{N,2}(c, c, c_i), v_{I,2}(c, c, c_i)]}{1 - \beta\pi(c_j|c_j)} = 0,
\end{aligned}$$

by the inductive hypothesis that $\max[v_{N,l}(c,c,c_i), v_{I,l}(c,c,c_i)] = 0$ for $i \in \{1,\ldots,j-1\}$ and $l \in \{1,2\}$. It follows that the claimed result of symmetric, zero expected payoffs holds in this case as claimed.

Now consider the case where there isn't a unique no investment stage game equilibrium at the point (c,c,c'). We now show that there will be three equilibria at such points, two of which are the two pure strategy "anti-coordination" equilibria and the third will be a mixed strategy equilibrium which is the one we select, and will show entails zero expected profits to both firms.

We introduce the notation $v_{I,i}(c_1,c_2,c,P_{-i})$ and $v_{N,i}(c_1,c_2,c,P_{-i})$ for $i \in \{1,2\}$ to represent the values for investing and not investing, respectively, for firm i conditional on the assumption that its opponent will invest in state (c_1,c_2,c) with probability P_{-i}, (possibly a non-equilibrium probability) but return to play equilibrium strategies in all future time periods after this current period. We have already proven in Lemma A.4 above that there is a unique no investment stage game equilibrium at all edge states (c_1,c_2,c) where either c_1 or c_2 equals the current state of the art marginal cost c. It is also possible that there is a unique no investment equilibrium at states (c_1,c_2,c) where $c_1 > c$ and $c_2 > c$ provided c_1 and c_2 are sufficiently close to c.

For other diagonal states (c,c,c') if there is not a unique no investment equilibrium, then it must be the case that for at least one of the firms i we must have $v_{N,i}(c,c,c',1) = 0$ and $v_{I,i}(c,c,c',1) < 0$ (i.e. it is not optimal for firm i to invest if its opponent will invest with probability 1), and $v_{N,i}(c,c,c',0) = 0$ and $v_{I,i}(c,c,c',0) > 0$ (i.e. it is optimal for firm i to invest if it knows its opponent will not invest with probability 1). However by the symmetry of the value functions at states (c,c,c') for $c' < c_j$, it is not hard to show using the Bellman equation (7) that we have

$$v_{I,1}(c,c,c',0) = v_{I,2}(c,c,c',0) > 0 \tag{29}$$

and thus *both firms* will strictly prefer to invest when they are certain that their opponent will not invest. This implies that the reaction functions, or *best response investment probabilities* $P_i(c,c,c',P_{-i})$ for both firms $i \in \{1,2\}$ will be piece-wise flat and non-increasing and jump discontinuously from 1 to 0 at a probability given by

$$P_1(c,c,c') = P_2(c,c,c') = \frac{\beta \sum_{i=1}^{j} v_{N,1}(c,c,c_i) - K(c')}{\beta \sum_{i=1}^{j} v_{N,1}(c,c,c_i)}. \tag{30}$$

These probabilities constitute the unique mixed strategy equilibrium of the stage game at point (c,c,c'). However it is not difficult to show, using the Bellman equation (7), that $v_{N,1}(c,c,c') = v_{N,2}(c,c,c') = 0$, so it follows that the expected payoff to both firms in the mixed strategy equilibrium at the diagonal state (c,c,c') is zero, establishing the induction step.

To complete the proof, we must also show that the value functions and investment probabilities are symmetric in the (c_1,c_2) argument, since we implicitly assumed that this symmetry holds in our assertion that equation (29) holds. By our inductive hypothesis, symmetry holds for all states $(c_1,c_2,c) \in S$ for which $c < c_j = c'$. Now we show that symmetry also holds for all points $(c_1,c_2,c') \in S$ as well. First consider states (c_1,c_2,c') for which the unique equilibrium is the no investment equilibrium, we can use the Bellman equation (7) to express the value functions for not investing for firms 1 and 2 as

$$\begin{aligned} v_{N,1}(c_1,c_2,c') &= \frac{r_1(c_1,c_2) + \beta \sum_{i=1}^{j-1} v_{N,1}(c_1,c_2,c_i)}{1-\beta\pi(c'|c')} \\ v_{N,2}(c_1,c_2,c') &= \frac{r_2(c_1,c_2) + \beta \sum_{i=1}^{j-1} v_{N,2}(c_1,c_2,c_i)}{1-\beta\pi(c'|c')}. \end{aligned}$$

It is not hard to see that the single period profit are symmetric: $r_1(c_1,c_2) = r_2(c_2,c_1)$. Further by our inductive hypothesis, all the functions $v_{N,1}(c_1,c_2,c_i)$ and $v_{N,2}(c_1,c_2,c_i)$ are symmetric functions of their (c_1,c_2) arguments for $i = 1,\ldots,j-1$. Therefore it follows that $v_{N,1}(c_1,c_2,c') = v_{N,2}(c_2,c_1,c')$. The symmetry of $v_{I,1}$ and $v_{I,2}$ follows from the symmetry of $v_{N,1}$ and $v_{N,2}$ in (c_1,c_2) since one can verify from the Bellman equation (7) that the former functions can be written exclusively in terms of the $v_{N,1}$ and $v_{N,2}$ functions, and these latter functions are symmetric.

Finally consider the remaining points $(c_1,c_2,c') \in S$ where it is not the case that a no investment equilibrium holds. We have shown above that at these states there will be 3 equilibria, one of which is a mixed strategy equilibrium which is the "selected" equilibrium in each of these states. We have already shown that the value functions are symmetric along the diagonal states (c,c,c') so we only need to consider the off-diagonal states (c_1,c_2,c') where $c_1 \neq c_2$. When $c_1 > c_2$ we have $v_{N,1}(c_1,c_2,c) = 0$ and when $c_1 < c_2$ we have $v_{N,2}(c_1,c_2,c) = 0$, so symmetry holds for all points $c_1 > c_2$: $v_{N,1}(c_1,c_2,c') = 0 = v_{N,2}(c_2,c_1,c')$.

Finally consider points $(c_1,c_2,c') \in S$ for which $c_1 < c_2$. For these points we have

$v_{N,1}(c_1,c_2,c') > 0$. We need to show that symmetry holds in this region as well. At points in this region, both firms are playing a mixed strategy equilibrium, so the expression for the value functions $v_{N,1}(c_1,c_2,c')$ depends on $P_2(c_1,c_2,c')$ and the expression for $v_{N,2}(c_1,c_2,c')$ depends on $P_2(c_1,c_2,c')$. Using the fact that $v_{N,1}(c_1,c_2,c') = v_{I,1}(c_1,c_2,c')$ (since firm 1 must be indifferent between investing and not investing when it is playing a mixed strategy), we can solve for $P_2(c_1,c_2,c')$ as a ratio of terms involving the value function $v_{N,1}(c_1,c_2,c')$ and a weighted sum of $v_{N,1}(c_1,c_2,c_i)$ at other points (c_1,c_2,c_i), $i = 1,\ldots,j-1$ where our inductive hypothesis holds. We then enter this expression for $P_2(c_1,c_2,c')$ back into the equation for $v_{N,1}(c_1,c_2,c')$, thereby "substituting out" $P_2(c_1,c_2,c')$ from the equation for $v_{N,1}(c_1,c_2,c')$. We omit the tedious and involved algebra here, but when we do this we can express $v_{N,1}(c_1,c_2,c')$ as the solution to a second order polynomial equation in which the coefficients of the polynomial are all symmetric functions of (c_1,c_2), as a result of our inductive hypothesis.

We can also do the same for $v_{N,2}(c_1,c_2,c')$, i.e. first solving for $P_1(c_1,c_2,c')$ using the indifferent condition $v_{N,2}(c_1,c_2,c') = v_{I,1}(c_1,c_2,c')$, and then entering this expression into the Bellman equation for $v_{N,2}(c_1,c_2,c')$, thereby substituting out $P_1(c_1,c_2,c')$ to obtain another second order polynomial expression for $v_{N,2}(c_1,c_2,c')$ whose coefficients are symmetric functions of (c_1,c_2).

Let $Q_1(v,c_1,c_2,c') = 0$ be the second order polynomial equation, one of whose solutions is $v_{N,1}(c_1,c_2,c)$. Similarly let $Q_2(v,c_1,c_2,c') = 0$ be the second order polynomial equation, one of whose solutions is $v_{N,2}(c_1,c_2,c)$. By the symmetry of the coefficients of these polynomials in (c_1,c_2), it follows that $Q_1(v,c_1,c_2,c') = Q_2(v,c_2,c_1,c')$ for all $v \in R$. It follows that the solutions to the equation $Q_2(v,c_2,c_1,c') = 0$ are the same as to the equation $Q_2(v,c_2,c_1,c') = 0$, and this implies that $v_{N,1}(c_1,c_2,c') = v_{N,2}(c_2,c_1,c')$. Since $P_1(c_1,c_2,c')$ and $P_2(c_1,c_2,c')$ can be written as functions of $v_{N,2}(c_1,c_2,c')$ and $v_{N,1}(c_1,c_2,c')$, respectively, and other functions that are symmetric in (c_1,c_2) by our inductive hypothesis, it follows that $P_1(c_1,c_2,c') = P_2(c_2,c_1,c')$, thereby completing our proof by induction.

Statement 4. Statement 2 of this Theorem ensures the existence of two monopoly equilibria in the simultaneous move game, proving that the two corner payoff points $(V_M,0)$ and $(0,V_M)$ exist, where $V_M = v_{N,1}(c_0,c_0,c_0) = v_{N,2}(c_0,c_0,c_0)$ is the monopoly payoff at the initial node (apex)

$(c_0, c_0, c_0) \in S$. Since the monopoly profit equals the full social surplus and is efficient, it is infeasible to obtain any payoff higher than the line segment joining these two monopoly payoff points,and thus all payoffs for all equilibria in the simultaneous move game (which are generally less than 100% efficient) must lie below the line segment joining the two monopoly payoff points. Finally, Statement 5 of this Theorem guarantees the existence of the zero payoff point at the origin $(0,0)$. Obviously the convex hull of these three payoff points equals the full triangle, and thus any point in this triangle can be an expected payoff to the two firms if we allow *stochastic* equilibrium selection rules (i.e. selecting one of these three possible "extremal equilibria" with probabilities (p_1, p_2, p_3) with $p_1 + p_2 + p_3 = 1$ and $p_i \geq 0$, $i \in \{1,2,3\}$).

Statement 5. It is sufficient to show that the origin is not an equilibrium payoff pair at the apex of the alternating move game if investment costs are not too high. By Statement 1 of this Theorem, no investment cannot be a MPE of the full alternating move game if the cost of investment is not too high at the initial apex $(c_0, c_0, c_0) \in S$. However if it is optimal for one of the firms to invest at some point on the equilibrium path, it must be because the firm expects a positive profit from doing so. However from the Bellman equation for the alternating move game, equation (7), if one or the other of the firms expects a positive profit from investing in some stage game on the equilibrium path, the expected profit from that firm at the initial apex of the game $(c_0, c_0, c_0) \in S$ cannot be zero. We note that Theorem 5 implies that a zero payoff for both firms is approached in the limit as $\Delta t \to 0$ when $\pi(c_t|c_t) = 0$ and the order of moves alternates deterministically. However in that case, since the equilibrium is unique, it follows that the monopoly payoff vertices are not supportable in the limit as $\Delta t \to 0$. Thus, even in limiting cases, the set of equilibrium payoffs in the alternating move game will be a strict subset of the triangular payoff region described in Statement 4 of this Theorem for the simultaneous move game.

$\square$

Theorem 3 (Sufficient conditions for uniqueness).

The proof requires some intermediary results.

Lemma A.6 (Efficiency of the alternating move end game)**.** *Suppose* $\eta = 0$ *and* $m \neq 0$ *(i.e. alternating move game with no investment cost shocks), and* $c = 0$*. In every end game state* $(c_1, c_2, 0)$ *there is a unique efficient equilibrium, i.e. both firms invest when it is their turn to invest if and only if investment would be optimal from the point of view of the social planner.*

Proof. Consider the case where $c_1 < c_2$. The proof for the case $c_1 \geq c_2$ is symmetric to the one provided below for $c_1 < c_2$ and is omitted for brevity. Suppose that it is socially optimal to undertake investment, i.e. $\beta c_1/(1-\beta) - K(0) > 0$. We now show that in the unique equilibrium to the alternating move end game, both firms 1 and 2 would want to invest when it is their turn to invest, where uniqueness of equilibrium is a consequence of the uniqueness of the firms' best responses, and the fact that only one of the firm moves at a time. Consider firm 2's decision in this unique equilibrium. If firm 2 chooses to invest, its payoff is $v_{I,2}(c_1, c_2, 0, 2) = \beta c_1/(1-\beta) - K(0)$ and if it chooses not to invest its payoff is $v_{N,2}(c_1, c_2, 0, 2) = 0$ since it believes that firm 1 will invest at its turn with probability 1, which we will verify is true below. Thus, firm 2 will invest in equilibrium if and only if $\beta c_1/(1-\beta) - K(0) > 0$, which is the same condition for optimal investment by the social planner.

Now consider firm 1. At it's turn to move the payoff to investing is

$$v_{I,1}(c_1, c_2, 0, 1) = c_2 - c_1 + \beta c_2/(1-\beta) - K(0). \tag{31}$$

Since $c_2 > c_1$ and by assumption $\beta c_1/(1-\beta) - K(0) > 0$, it is easy to see that the payoff to investing is strictly positive for firm 1. However we must also show that this is higher than the payoff it would get from not investing. Since firm 1 knows that firm 2 will invest when it gets a chance to move, the value to firm 1 to not investing is given by

$$v_{N,1}(c_1, c_2, 0, 1) = c_2 - c_1 + \beta f(1|1)\left[c_2 - c_1 + \frac{\beta}{1-\beta}c_2 - K(0)\right] + \beta f(2|1)[c_2 - c_1]. \tag{32}$$

If the posited equilibrium holds (i.e. it is optimal for firm 1 to invest), then we must have $v_{I,1}(c_1, c_2, 0, 1) > v_{N,1}(c_1, c_2, 0, 1)$, and using the formulas for these values given above, this is equivalent to

$$\frac{\beta}{1-\beta}c_2 - K(0) > \frac{\beta(c_2 - c_1)}{1 - \beta f(1|1)}. \tag{33}$$

Notice that the right hand side of inequality (33) above is maximized when $f(1|1) = 1$ (i.e. when it is always firm 1's turn to invest) and in this case this inequality is equivalent to $\beta c_1/(1-\beta) - K(0) > 0$, confirming that for all $f(1|1) \in [0,1]$ it is strictly optimal for firm 1 to invest when it is its turn to invest when it is socially optimal for this investment to occur.

Now consider the converse situation where it is not socially optimal to invest, and $\beta c_1/(1-\beta) - K(0) < 0$. Following the same reasoning as above, it is easy to see that it is not optimal for firm 2 to invest when it is its turn to invest since firm 2's payoff to investing is $v_{I,2}(c_1,c_2,0,2) = \beta c_1/(1-\beta) - K(0) < 0$ and its payoff to not investing is $v_{N,2}(c_1,c_2,0,2) = 0$. Now we must show that firm 1, knowing that firm 2 will not want to invest at its turn, will also not want to invest when it is its turn. If firm 1 never invests, its payoff is

$$v_{N,1}(c_1,c_2,0,1) = \frac{c_2 - c_1}{1-\beta}, \tag{34}$$

and if it invests, its payoff is given by the same formula for $v_{I,1}(c_1,c_2,0,1)$ as given in equation (31) above. So the condition for investment not to be optimal for firm 1 is $v_{N,1}(c_1,c_2,0,1) > v_{I,1}(c_1,c_2,0,1)$ which is algebraically equivalent to $\beta c_1/(1-\beta) - K(0) < 0$, the condition for when it is not socially optimal for investment to occur.

□

Proof. When $\pi(c|c) = 0$, the probability of remaining in any given state $(c_1,c_2,c) \in S$ is also zero. Using the Bellman equations (7) defining the firms' value functions for investing and not investing when it is their turn to invest, it is not difficult to see that each firm's values are independent of the probability that their opponent will invest in this case. That is, for firm 1 we have $v_{N,1}(c_1,c_2,c,1)$ and $v_{I,1}(c_1,c_2,c)$ are independent of $P_2(c_1,c_2,c,2)$, the probability that firm 2 will invest when it is its turn to invest. This implies that the probability that firm 1 will invest, $P_1(c_1,c_2,c,1)$, is also independent of $P_2(c_1,c_2,c,2)$, as it is given by formula (5) of section 2, which shows that $P_1(c_1,c_2,c,1)$ is a logistic function of $v_{N,1}(c_1,c_2,c,1)$ and $v_{I,1}(c_1,c_2,c,1)$, both of which are independent of $P_2(c_1,c_2,c,2)$. Similar arguments hold for firm 2, so that $P_2(c_1,c_2,c,2)$ is independent of $P_1(c_1,c_2,c,1)$. Since the value functions $(v_{N,1},v_{I,1},v_{N,2},v_{I,2})$ can be calculated recursively using the Bellman equations (5), and since Lemma A.6 establishes that there is always a unique (efficient) equilibrium in the end game states (c_1,c_2,c), it follows that at every state

$(c_1, c_2, c) \in S$ there is a unique stage game equilibrium with probabilities of investing given by $(P_1(c_1, c_2, c, 1), P_2(c_1, c_2, c, 2))$, which depend on the value functions $(v_{N,1}, v_{I,1}, v_{N,2}, v_{I,2})$ that are defined recursively via the Bellman equations (7).

□

B Expected Cost Recursions

This appendix derives the recursion equations that we used to compute expected discounted costs of production for the two firms in any given equilibrium of the model. These expected costs are used in turn to calculate the efficiency (fraction of maximum possible social surplus) that is realized in any given equilibrium of the model.

Let $ECI_j(c_1, c_2, c)$ be the expected discounted costs of production for firm j in state (c_1, c_2, c) given that firm j chooses to invest in this state. Similarly, let $ECN_j(c_1, c_2, c)$ be the corresponding cost for firm j if it chooses not to invest. Let $EPI_j(c_1, c_2, c)$ be the *current period* expected production and investment costs for the firm if it chooses to invest in this state, and $EPN_j(c_1, c_2, c)$ be the expected production and investment costs if firm j chooses not to invest. These latter conditional expectations are the sum of production costs in the current period as well as the conditional expectation of the idiosyncratic, additive, and *IID* costs of investing for firm j. Let $s_j(c_1, c_2)$ be the market share of firm j under the Bertrand price equilibrium when the two firms have production costs (c_1, c_2), respectively. Normalizing market size to 1, then the total production costs for firm j in state (c_1, c_2, c) are simply $s_j(c_1, c_2)c_j$. The idiosyncratic investment costs for firm j corresponding to not investing are $-\varepsilon_{0,j}$ and the idiosyncratic costs corresponding to investing are $-\varepsilon_{1,j}$. Note we put a negative sign in front of the idiosyncratic investment costs since in the presentation of the model in section 2, we included these shocks without the negative signs, meaning we treated them as *benefits* (or net reductions in investment costs) when the shocks were positive and costs when negative.

Using Theorem B.2.2 of Jeffrey A. Dubin *Consumer Durable Choice and the Demand for*

Electricity Amsterdam, North Holland, 1985, we have

$$E\{\varepsilon_{0,j}|\delta_j(c_1,c_2,c)=0\}=\eta\left[\gamma+\eta\log\left(1+\exp\{(v_j(c_1,c_2,c)-w_j(c_1,c_2,c))/\eta\}\right)\right], \tag{35}$$

where $\gamma \simeq 0.5772$ is Euler's constant and v_j and w_j are the conditional value functions for firm j corresponding to investing and not investing at state (c_1,c_2,c), respectively, and $\delta_j(c_1,c_2,c)=0$ denotes firm j's decision not to invest in state (c_1,c_2,c). Similarly, we have

$$E\{\varepsilon_{1,j}|\delta_j(c_1,c_2,c)=1\}=\eta\left[\gamma+\eta\log\left(1+\exp\{(w_j(c_1,c_2,c)-v_j(c_1,c_2,c))/\eta\}\right)\right], \tag{36}$$

Thus we have

$$EPI_j(c_1,c_2,c)=s_j(c_1,c_2)c_j+K(c)-\eta\left[\gamma+\eta\log\left(1+\exp\{(w_j(c_1,c_2,c)-v_j(c_1,c_2,c))/\eta\}\right)\right], \tag{37}$$

and

$$EPN_j(c_1,c_2,c)=s_j(c_1,c_2)c_j-\eta\left[\gamma+\eta\log\left(1+\exp\{(v_j(c_1,c_2,c)-w_j(c_1,c_2,c))/\eta\}\right)\right]. \tag{38}$$

The recursion equations for firm 1 in the simultaneous move game are given by

$$\begin{aligned}
& CN_1(c_1,c_2,c)=EPN_1(c_1,c_2,c)+\\
+\quad & \beta P_2(c_1,c_2,c)\int_0^c[P_1(c_1,c,c')CI_1(c_1,c,c')+(1-P_1(c_1,c,c'))CN_1(c_1,c,c')]\pi(dc'|c)\\
+\quad & \beta(1-P_2(c_1,c_2,c))\int_0^c[P_1(c_1,c_2,c')CI_1(c_1,c_2,c')+(1-P_1(c_1,c_2,c'))CN_1(c_1,c_2,c')]\pi(dc'|c).
\end{aligned}$$

$$\begin{aligned}
& CI_1(c_1,c_2,c)=EPI_1(c_1,c_2,c)+\\
+\quad & \beta P_2(c_1,c_2,c)\int_0^c[P_1(c,c,c')CI_1(c,c,c')+(1-P_1(c,c,c'))CN_1(c,c,c')]\pi(dc'|c)\\
+\quad & \beta(1-P_2(c_1,c_2,c))\int_0^c[P_1(c,c_2,c')CI_1(c,c_2,c')+(1-P_1(c,c_2,c'))CN_1(c,c_2,c')]\pi(dc'|c).
\end{aligned}$$

The recursion equations for CI_2 and CN_2 are defined similarly.

Now consider the random alternating move case. Now there are four value functions for each firm. For example, for firm 1 we have $CN_1(c_1,c_2,c,1)$ and $CI_1(c_1,c_2,c,1)$, which are firm j's expected discounted costs when it is its turn to invest, and $CN_1(c_1,c_2,c,2)$ and $CI_1(c_1,c_2,c,2)$ are firm 1's expected discounted costs when it is firm 2's turn to invest. We use the "trick" of using the CN_1 and CI_1 notation even when it is firm 2's turn to invest. Thus, $CN_1(c_1,c_2,c,2)$ denotes the expected cost for firm 1 when it is firm 2's turn to invest and firm 2 *didn't* invest, whereas $CI_1(c_1,c_2,c,2)$ denotes firm 1's expected discounted costs when it is firm 2's turn to invest and firm 2 *does* invest. Since we assume that the idiosyncratic, additive, *IID* cost shocks for investing and not investing, respectively, are only incurred by the firm in the periods where it is that firm's turn to invest, we define the following functions: $EPI_1(c_1,c_2,c,1)$ and $EPN_1(c_1,c_2,c,1)$ are the expected current production and investment costs for firm 1 when it is firm 1's turn to invest, and these are given by equations (37 and (38) above. When it is not firm 1's turn to invest, then we have $EPI_1(c_1,c_2,c,2) = EPN_1(c_1,c_2,c,2) = s_1(c_1,c_2)c_1$.

$$\begin{aligned}
& CN_1(c_1,c_2,c,1) = EPN_1(c_1,c_2,c,1) \\
+ & \beta f(1|1)\int_0^c \left[P_1(c_1,c_2,c',1)CI_1(c_1,c_2,c',1) + (1-P_1(c_1,c_2,c',1))CN_1(c_1,c_2,c',1)\right]\pi(dc'|c) \\
+ & \beta f(2|1)\int_0^c \left[P_2(c_1,c_2,c',2)CI_1(c_1,c_2,c',2) + (1-P_2(c_1,c_2,c',2))CN_1(c_1,c_2,c',2)\right]\pi(dc'|c).
\end{aligned}$$

$$\begin{aligned}
& CI_1(c_1,c_2,c,1) = EPI_1(c_1,c_2,c,1) \\
+ & \beta f(1|1)\int_0^c \left[P_1(c,c_2,c',1)CI_1(c,c_2,c',1) + (1-P_1(c,c_2,c',1))CN_1(c,c_2,c',1)\right]\pi(dc'|c) \\
+ & \beta f(2|1)\int_0^c \left[P_2(c,c_2,c',2)CI_1(c,c_2,c',2) + (1-P_2(c,c_2,c',2))CN_1(c,c_2,c',2)\right]\pi(dc'|c).
\end{aligned}$$

$$\begin{aligned}
& CN_1(c_1,c_2,c,2) = EPN_1(c_1,c_2,c,2) \\
+ & \beta f(1|2)\int_0^c \left[P_1(c_1,c_2,c',1)CI_1(c_1,c_2,c',1) + (1-P_1(c_1,c_2,c',1))CN_1(c_1,c_2,c',1)\right]\pi(dc'|c) \\
+ & \beta f(2|2)\int_0^c \left[P_2(c_1,c_2,c',2)CI_1(c_1,c_2,c',2) + (1-P_2(c_1,c_2,c',2))CN_1(c_1,c_2,c',2)\right]\pi(dc'|c).
\end{aligned}$$

$$\begin{aligned}
& CI_1(c_1,c_2,c,2) = EPI_1(c_1,c_2,c,2) \\
+ & \beta f(1|2)\int_0^c \left[P_1(c_1,c,c',1)CI_1(c_1,c,c',1) + (1-P_1(c_1,c,c',1))CN_1(c_1,c,c',1)\right]\pi(dc'|c) \\
+ & \beta f(2|2)\int_0^c \left[P_2(c_1,c,c',2)CI_1(c_1,c,c',2) + (1-P_2(c_1,c,c',2))CN_1(c_1,c,c',2)\right]\pi(dc'|c).
\end{aligned}$$

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