Pareto Optimality of Allocating the Bad*

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Abstract

Given quasilinear independent private values, this paper proves a necessary and sufficient condition for all interim Pareto optimal mechanisms to allocate a commonly undesirable item with positive probabilities despite that not allocating it at all is part of an ex ante incentive efficient mechanism. The condition holds when types near the low end carry sufficiently high welfare densities. Replacing the welfare weight distribution by a second-order stochastically dominated one improves the prospect of the condition. The Kuhn-Tucker method in the literature is inapplicable because when our condition holds, the monotonicity constraint the method sets aside is binding unless the method suffers indeterminacy in admitting a continuum of solutions to the relaxed problem.

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1 Introduction

The decision on where to locate a Nimby—a "not in my backyard" type of noxious facility—has been considered as a procurement auction by Kunreuther and Kleindorfer [8], with alternative locations treated as bidders for the contract of hosting the Nimby. What has not been considered theoretically is why the Nimby should exist at all and be located somewhere. An answer to this question in general would require comparison between the Nimby's public good to the society and its private bad to its host. While the private bad is usually apparent, its public good is far from obvious. Debates about the net public benefit of a Nimby are often contentious (e.g., the location of an oil pipeline terminal in Canada), and the conclusion thereof results more from endogenous politics than exogenous nature. In this paper we treat a Nimby as purely a private bad to its host, assuming away its public good and allowing the planner to do away with a Nimby at no cost. Since our conclusion is that it is interim Pareto optimal to allocate such a pure bad to someone sometimes, the argument for the necessity of a Nimby can only strengthen if its public benefit is taken into account.

We thus abstract a Nimby into a *bad*, an item that gives its recipient a utility below the outside option and has no effect on the utility of anyone else. Other than Nimbies, a bad can also be the status of being excluded from an otherwise publicly available service that has been taken for granted. This paper develops a mechanism design method to characterize the set of primitives given which allocating a bad to some society members is necessary to achieve social optimality.

To reduce the arbitrariness in parameters, we assume quasilinear preferences and independent private values (IPV) as usual in mechanism design. Suppose that receiving the bad means a negative payoff $-t_i$ to player i whose realized type is $t_i \in \mathbb{R}_+$ and, if in addition i receives a money transfer m, i's payoff is equal to $-t_i + m$. For anyone to be willing to receive the bad, the society needs to compensate the recipient with money. To raise funds for such compensation, there needs to be a good available for allocation as well that gives its recipient a utility above the outside option. An example can be the privilege of hosting a popular game (whose externalities to other regions we assume away as we do to the Nimby). To minimize the departure from the standard, unidimensional-type framework, we assume that if player i gets the good with probability q_A and the bad with probability q_B, combined with net money receipt $m \in \mathbb{R}$, i's expected payoff is equal to $(q_A - q_B)t_i + m$.[1]

Nothing in this setup forces the allocation of the bad. In fact, one readily sees that never allocating the bad, coupled with allocating the good to the highest realized type, is ex ante incentive efficient.[2] However, even if they have agreed on this allocation ex ante, some players may have second thoughts during the interim, when each is privately informed of one's own type. In the context of universal healthcare coverage, for instance, a player who turns out to be extremely healthy would rather opt out of the coverage in return for money and may push for such an alternative. Thus the allocation that the society ends with depends on the bargaining power across player-types during the interim. In the spirit of Holmström and Myerson's [5] interim incentive efficiency (IIE), let us think of any idealized outcome of this interim bargaining process as if it were an optimum chosen by a social planner whose objective, or social welfare, is a weighted sum of the interim expected payoffs across player-types, with the various welfare weights of player-types capturing their bargaining power relative to one another.[3] Thus the question becomes, in an environment where never allocating the bad is part of an ex ante incentive efficient allocation, under what condition of the welfare weight distribution is the bad allocated with strictly positive probabilities in *all* mechanisms that maximize the social welfare subject to incentive compatibility (IC), individual rationality (IR) and budget balance (BB)?

IIE has been investigated in various models by Dworczak, Kominers and Akbarpour [3], Gresik [4], Laussel and Palfrey [9], Ledyard and Palfrey [10, 11], Pérez-Nievas [20] and Wilson [23]. Although the bad has no counterpart in these models except [11], and allocation of the bad is not the focus in [11],[4] from their characterizations one could get an intuition that introducing a bad might enlarge the social welfare. The typical pattern is that types are

[1] The paper can be easily generalized to the case where $(q_A - q_B)t_i + m$ is replaced by $(q_A - \alpha_i q_B)t_i + m$ for some commonly known, player-specific parameter $\alpha_i \in (0, 1)$.

[2] See Footnote 14 or Corollary 3 for a proof on the ex ante incentive efficiency of this allocation. That implies the allocation is also interim and ex post incentive efficient given appropriate welfare weight distributions (cf. Holmström and Myerson's [5]).

[3] Ledyard and Palfrey [10] provide a forceful motivation for such positive interpretations of IIE. Even from a purely normative viewpoint, one can readily relate to real-world situations where the social planner favors some types against others—such as transferring money from the rich to the poor—and wants to choose an optimal allocation according to her biased value judgement, a particular welfare weight distribution.

[4] Ledyard and Palfrey [11] use the Kuhn-Tucker method. Due to its limitation according to our Theorem 3 (explained later), this method could not have led to our result that the bad is needed for social optimality.

ranked according to their virtual surpluses, with types of higher ranks getting the good with higher probabilities, provided that their virtual surpluses are nonnegative and the incentive constraints non-binding. Thus, it appears natural that types with negative virtual surpluses should be allocated a bad, if available, since the bad is opposite to the good.

For this intuition to lead to a primitive condition under which the bad is allocated, however, one needs to handle the discontinuity of the virtual surplus function at any type whose allocation switches from getting the good in expectation to getting the bad in expectation. Thus, if the designer allocates the bad to the types whose virtual surpluses from getting the good are negative, she may be making a mistake because the virtual surpluses of these types from getting the bad should have been calculated differently. The complication comes from the endogenous buyer-seller role for each player. If a player's type is likely to be allocated the good in the mechanism under consideration, the player acting as a buyer of the good would understate his type. By contrast, if his type is likely a recipient of the bad, the player acting as a seller of such a costly service would exaggerate his type. Consequently, the operation of integrating a player's surplus from an allocation bifurcates between different measures, depending on whether the player is in expectation allocated the good—so his reduced form allocation is positive—or in expectation allocated the bad—with reduced-form allocation negative. In other words, the expected surplus is not a linear functional of allocations.

Without a linear structure, the method to characterize optimal mechanisms is usually a local one à la the Kuhn-Tucker theorem, but the method is usually limited by its local nature and, as part of our results shows, cannot address our question. Since the virtual surplus intuition corresponds to the first-order condition derived from such methods, they cannot warrant its validity here.

This paper thus adopts a global approach. We characterize the optimal mechanisms by a saddle point condition, which is not only sufficient but also necessary for any mechanism to be optimal (Theorem 1). We obtain this condition through formulating the aforementioned integration into a *two-part operator* on the allocations. The operator integrates the positive part of an allocation with one measure and integrates the negative part thereof with another measure. The positive part of an allocation is defined on the types that are more likely to get the good than the bad, and the negative part, more likely to get the bad than the good. Albeit nonlinear, this operator is always concave on the space of allocations. This drives the necessity of the saddle point condition.

From the saddle point condition we obtain a necessary and sufficient condition, in terms of the distribution of types and that of welfare weights, for every optimal mechanism to allocate the bad with a strictly positive probability (Theorem 2). One corollary is that, if the welfare weight distribution is replaced by a second-order stochastically dominated one, the bad is allocated with strictly positive probabilities in all optimal mechanisms if it is so before the substitution (Corollary 1). Another corollary is an unrestrictive condition sufficient for all optimal mechanisms to allocate the bad with strictly positive probabilities: all that we need is that the welfare density (Radon-Nikodym derivative of the welfare weight distribution with respect to the type distribution) around the infimum of the type support be more than twice the average welfare density (Corollary 2). Thus, the social optimality of allocating the bad is not at all sensitive to the particular forms of the welfare density or the type distribution. In particular, a social planner would still allocate the bad even when she assigns most of the welfare weight to high types.[5]

As our model does not force the necessity of the bad, Corollary 3 says that no optimal mechanism allocates the bad at all if the welfare weight distribution second-order stochastically dominates the exogenous distribution of types. Hence a social planner allocates the bad only if she favors the low and high types against the middle ones.

To demonstrate the generality of our method, and to gain understanding of all optimal mechanisms that need the bad as an instrument, we prove that the Kuhn-Tucker method used in the literature could not have led to our finding: The literature applies the method to a relaxed problem that is valid only if its solution happens to satisfy a monotonicity condition—the second-order part of IC—set aside by the relaxed problem. We find that if the bad is allocated with strictly positive probabilities in all optimal mechanisms then the relaxed problem has only two alternatives: either it admits no optimal mechanism as a solution, or it suffers indeterminacy in admitting a continuum of solutions to the relaxed problem (Theorem 3). Furthermore, the first alternative, which means that any solution to the relaxed problem violates the monotonicity constraint, is generic (Corollary 4). Thus, generically speaking, when the bad is needed for social optimality, the condition derived from

[5] For example, suppose that almost the entire welfare weight is assigned to a small interval at the supremum type 1, with a tiny weight say 3ϵ uniformly distributed to the elements of a tiny interval $[0, \epsilon]$ at the infimum. Then Corollary 2 says that the social planner would still allocate the bad with a strictly positive probability.

the Kuhn-Tucker method is vacuous, and any optimal mechanism entails rationing across some types because the monotonicity constraint is binding.

Theorems 2 and 3 are established on the saddle point condition (Theorem 1) of any optimal mechanism. The complication is that the associated Lagrangian is a two-part operator acting on reduced-form allocations with the aforementioned, sign-specific measures. To exploit the saddle point condition despite the complication, we develop a perturbation method. The idea is to perturb an allocation without altering the sign of its reduced form at any type (nonnegative for buyer types, and nonpositive for seller types) so that the measure acting on the reduced form remains unchanged. Then the Lagrangian becomes linear in the perturbations. That allows us to characterize any maximizer of the Lagrangian through perturbations along the direction of the measure, as long as the desired direction can be achieved in an ex post feasible manner. To that end, we formulate a family of ex post feasible, sign-preserving perturbations, thereby obtaining conditions necessary for any maximizer of the Lagrangian associated with the saddle point condition (Section 4.2.1 and Appendix E.2).

By the monotonicity condition of the IC constraint, if the bad is allocated at all, it is allocated to low types. Thus, Corollaries 1 and 2 imply that a social planner should allocate the bad to low types if she either spreads more welfare weights to low types in general, or cares enough about the extreme low types in particular. For instance, since a player's type is equal to his marginal rate of substitution between consumption and money, one may think of a low type as a financially constrained consumer. In such a context, an implication is that a social planner who cares enough about the extreme poor should buy the poor out of the coverage of the benefit under consideration.

We are aware of two other global methods to handle nonlinear problems in mechanism design. One is Toikka's [22] generalized ironing technique to maximize a concave functional of monotone real functions. This could be relevant to the Lagrange problem associated with our saddle point condition, which has absorbed the budget balance constraint of our original problem. However, the Lagrangian does not satisfy the differentiability assumption in Toikka (nor the discreteness assumption in the online supplement). Moreover, it remains to be seen whether the single-agent assumption of the method can be removed to accommodate the ex post feasibility constraint in an auction model such as our Lagrange problem.

The other method is to turn the set of ex post feasible IC reduced form allocations into a family of monotone real functions majorized by a known function and characterize the maxima of a convex functional on this family as its extreme points (Kleiner, Moldovanu and Strack [7]). In our model, since the Lagrangian is not convex, the only case to which this method could be applicable is our original optimization problem subject to an additional restriction that mechanisms be symmetric across players (so that each allocation corresponds to a real function). However, we also require the budget balance condition, and a bad is to be allocated alongside with a good (rather than two goods to be allocated). It remains to be seen whether ex post feasibility can be captured by a single majorization relation and whether the choice set can be equal to a majorization family with respect to such a relation.

Our model shares a similar feature with the partnership dissolution literature in that the buyer or seller role of a player is endogenous (Cramton, Gibbons and Klemperer [2], Chien [1], Loertscher and Wasser [13], Lu and Robert [14], Mylovanov and Tröger [19], and Segal and Whinston [21]). Selling one's partnership share in that framework corresponds to being allocated the bad in our model. However, since partnership dissolution requires market clearance in the trading of shares, it is out of the question in that framework whether a bad should be allocated at all. Nevertheless, our saddle point characterization, giving a necessary and sufficient condition for any interim Pareto optimal mechanism subject to IC, IR and BB, is applicable to partnership dissolution given independent private values (Remark 5, Appendix F). Another difference is in the design objectives. We consider interim Pareto optimality (namely, IIE), which allows for any welfare weights varying across player-types. By contrast, the design objective regarding partnership dissolution is a sum of surpluses with welfare weights uniform across types. It has been the simple sum of surpluses across players with uniform welfare weights in much of the literature.[6] Recently, the objective is a weighted average—with type-independent weights—between the expected revenue and the winner's surplus in Lu and Robert and in Loertscher and Wasser, and the ex ante surplus of one of the players (informed principal) in Mylovanov and Tröger.

With both the buyer and seller roles possible to each player, a player's type at which the participation constraint binds is not determined a priori. Thus our model is somewhat related to the countervailing incentives literature such as Lewis and Sappington [12], Maggi and Rodríguez-Clare [16], and Jullien [6]. Their focus is to address the issue that full participation

[6] Much of the partnership dissolution literature focuses on the implementability of one particular winner-selection rule, the efficient allocation, which is optimal only if it is implementable and only if the design objective is the simple sum of the surpluses across players.

may cause loss of generality and to exploit the curvature of the agent's reservation utility function for explanations of various pooling properties in the principal's optimal mechanism. By contrast, there is no loss to assume full participation in our model, and our focus is to resolve the nonlinearity problem caused by the endogenous buyer-seller role in an otherwise linear structure. In addition, given the endogenous discontinuity of the virtual surplus functions, the Hamiltonian technique in [6] is inapplicable to our model.

2 The Model

2.1 The Good, the Bad, and n Players

There are two items, named A and B, and n players ($n \geq 2$), each of whom can be allocated one or both or none of the items. Each player i's private information at the outset, or type t_i, is independently drawn according to the same cumulative distribution function F with density f strictly positive on the support $[0, 1]$.[7] Given type t_i, if player i gets item A with probability x_{iA}, item B with probability x_{iB}, and delivers money transfer in the amount $y_i \in \mathbb{R}$ (negative y_i meaning i being the recipient of money), player i's payoff is equal to

$$(x_{iA} - x_{iB})\, t_i - y_i. \tag{1}$$

Hence item A is interpreted as a good, and item B a bad, to all players; t_i corresponds to the intensity of player i's preference for the good over the bad.

2.2 Allocations and Mechanisms

An *ex post allocation* means a list $(q_{iA}, q_{iB})_{i=1}^n$ of functions such that $q_{iA}, q_{iB} : [0, 1]^n \to [0, 1]$ for each i and, for each $t \in [0, 1]^n$,

$$\sum_i q_{iA}(t) \leq 1 \quad \text{and} \quad \sum_i q_{iB}(t) \leq 1. \tag{2}$$

An *ex post payment rule* means a list $(p_i)_{i=1}^n$ of functions such that $p_i : [0, 1]^n \to \mathbb{R}$ for each i. By the revelation principle, any equilibrium-feasible mechanism corresponds to a pair of ex post allocation $(q_{iA}, q_{iB})_{i=1}^n$ and ex post payment rule $(p_i)_{i=1}^n$, with $q_{ij}(t)$ interpreted as the

[7] See Appendix F for a generalization that allows for player-specific distributions.

probability with which item j ($j \in \{A, B\}$) is assigned to player i, and $p_i(t)$ the net money transfer from player i to others, when t is the profile of alleged types across players.

For each player i, denote F_{-i} for the product measure on $[0,1]^{n-1}$ generated by F on each subspace $[0,1]$. A *mechanism (in reduced form)* means a list $(Q_i, P_i)_{i=1}^n$, often abbreviated as (Q, P), of functions $Q_i : [0,1] \to \mathbb{R}$ and $P_i : [0,1] \to \mathbb{R}$ ($\forall i = 1, \ldots, n$) such that, for some ex post allocation-payment rule $(q_{iA}, q_{iB}, p_i)_{i=1}^n$, Q_i is the marginal of $q_{iA} - q_{iB}$, and P_i the marginal of p_i, onto the i^{th} dimension. That is, for any i and any $t_i \in [0,1]$,

$$Q_i(t_i) = \int_{[0,1]^{n-1}} (q_{iA}(t_i, t_{-i}) - q_{iB}(t_i, t_{-i}))\, dF_{-i}(t_{-i}) \tag{3}$$

and $P_i(t_i) = \int_{[0,1]^{n-1}} p_i(t_i, \cdot) dF_{-i}$ for any t_i and any i. The part $(Q_i)_{i=1}^n$ in $(Q_i, P_i)_{i=1}^n$ is called (reduced-form) *allocation*. Call $(Q_i)_{i=1}^n$ the *reduced form* of $(q_{iA}, q_{iB})_{i=1}^n$ if and only if (3) holds for all i and all t_i.

Remark 1 Eq. (3) is the construct that sets our model apart from the existing optimal auction framework. The feature of (3) is that the reduced form allocation to a player can be positive or negative, and the sign thereof is endogenous. As we will see in Section 3.1, such endogenous signing of the allocation results in a nonlinear structure. This feature of endogenous signing stems from the assumption that the two items up for allocation point to opposite directions with respect to a player's nonparticipation payoff—which we normalize to zero without loss of generality—with the good generating a payoff larger than, and the bad less than, the nonparticipation payoff. That is why the role of the bad in our model cannot be replaced by a lesser good as long as the utility of the lesser good is still larger than the nonparticipation payoff.

2.3 Constraints

Given any (reduced-form) mechanism $(Q_i, P_i)_{i=1}^n$, it follows from the quasilinear utility function postulated previously that the interim expected payoff for any type t_i of player i to act as a type $\hat{t}_i$, given truthtelling from others, is equal to $Q_i(\hat{t}_i)t_i - P_i(\hat{t}_i)$. Denote

$$U_i(t_i \mid Q, P) := \max_{\hat{t}_i \in [0,1]} Q_i(\hat{t}_i)t_i - P_i(\hat{t}_i). \tag{4}$$

As is routine in auction theory, *incentive compatibility* (IC) of $(Q_i, P_i)_{i=1}^n$ is equivalent to simultaneous satisfaction of two conditions for each player i: (i) Q_i is weakly increasing on $[0,1]$; (ii) for any $t_i, t_i^0 \in [0,1]$,

$$P_i(t_i) - P_i(t_i^0) = \int_{t_i^0}^{t_i} s dQ_i(s). \tag{5}$$

We assume that each player can opt out of a mechanism before it operates thereby getting zero as the outside payoff. Thus $(Q_i, P_i)_{i=1}^n$ is said *individually rational* (IR) if and only if $U_i(t_i|Q,P) \geq 0$ for all i and all $t_i \in T_i$.

For the society consisting of the n players to transfer wealth among themselves without relying on outside subsides, we require that a mechanism be budget-balanced: $(Q_i, P_i)_{i=1}^n$ satisfies *budget balance* (BB) if and only if $\sum_i \int_0^1 P_i(t_i)dF(t_i) \geq 0$.[8]

2.4 Interim Incentive-Constrained Pareto Optimality

While our method applies to cases where players may weigh differently in the social welfare (Appendix F), to focus on transfers across types, the main text presents only welfare weights that are neutral across players, who are assumed ex ante symmetric (again for notational simplicity). Thus, by *welfare density* we mean a function $w : [0,1] \to \mathbb{R}_{++}$ for which $\int_0^1 w dF = 1$. Given any welfare density w, the mechanism design problem is to maximize

$$\sum_{i=1}^n \int_0^1 U_i(t_i \mid Q, P) w(t_i) dF(t_i) \tag{6}$$

among all mechanisms (Q, P) that are IC, IR and BB.[9] It is obvious that any solution say (Q^*, P^*) to this problem is interim incentive-constrained Pareto optimal.[10] That is, there does not exist another IC, IR and BB mechanism (Q, P) for which $U_i(\cdot|Q,P) \geq U_i(\cdot|Q^*, P^*)$ a.e. on $[0,1]$ for all players i and, for some player i, $U_i(\cdot|Q,P) > U_i(\cdot|Q^*, P^*)$ on a positive-measure subset of $[0,1]$.[11]

[8] There is no substantive difference between such ex ante condition for budget balance and its ex post counterpart. Mimicking the proof of Lemma 4 of Cramton, Gibbons and Klemperer [2], one can prove that if $\sum_i \int_0^1 P_i(t_i)dF(t_i) \geq 0$ then $(P_i)_{i=1}^n$ is the profile of the marginals of an ex post payment profile $(p_i)_{i=1}^n$ for which $\sum_i p_i(t) \geq 0$ for all $t \in [0,1]^n$.

[9] By definition, any mechanism is ex post feasible in the sense of being the reduced form of some ex post allocation-payment rule (Section 2.2).

[10] Interim incentive-constrained Pareto optimality is the same as interim incentive efficiency (IIE) if the IR and BB constraints are added to the IIE framework, which usually considers only IC.

[11] Since F is absolutely continuous in Lebesgue measure by assumption, the notion of measure zero with respect to F is equivalent to that with respect to Lebesgue measure restricted to $[0,1]$.

3 Saddle Point Characterization

To state the saddle point characterization for all *optimal mechanisms* (maximizers of (6) subject to IC, IR and BB), we need to introduce a notation for a nonlinear, *two-part operator* on allocations. A crucial property of this operation is its concavity, which drives the necessity of the saddle point condition. Section 3.1 motivates the notation from the endogenous discontinuity of the virtual surplus function in our model. Section 3.2 defines the notation.

3.1 Nonlinearity from Having Both a Good and a Bad

Given any welfare density w and any IC mechanism (Q, P), the objective (6) is equal to

$$\sum_i \int_0^1 Q_i(t_i)t_i dW(t_i) - \sum_i \int_0^1 P_i(t_i)dW(t_i), \tag{7}$$

where we define

$$W(t_i) := \int_0^{t_i} w(s)dF(s) \tag{8}$$

for any $t_i \in [0, 1]$. Note that W is a cdf with support $[0, 1]$. We shall call W *welfare weight distribution*.

By the routine of envelope theorem and integration by parts, one obtains that (Appendix A), for any $t^0 \in [0, 1]$ and any IC mechanism (Q, P),

$$\begin{aligned}\int_0^1 P_i(t_i)dW(t_i) &= \int_0^1 Q_i(t_i)t_i dW(t_i) + \int_0^{t^0} Q_i(t_i)W(t_i)dt_i - \int_{t^0}^1 Q_i(t_i)\left(1 - W(t_i)\right)dt_i \\ &\quad -U_i(t^0|Q, P).\end{aligned} \tag{9}$$

To keep track of the IR and BB constraints, this equation is useful only when $U_i(t^0|Q, P) = \min_{[0,1]} U_i(\cdot|Q, P)$. By IC and the envelope equation (5), $U_i(\cdot|Q, P)$ is convex and its derivative is equal to Q_i a.e. Thus, if $U_i(\cdot|Q, P)$ attains its minimum at t^0 then $Q_i \le 0$ on $[0, t^0)$, and $Q_i \ge 0$ on $(t^0, 1]$. Hence (9) implies

$$\begin{aligned}\int_0^1 P_i\, dW &= \int_0^1 Q_i(t_i)t_i\, dW(t_i) + \int_0^1 \left(-Q_i^-(t_i)\right)W(t_i)dt_i + \int_0^1 Q_i^+(t_i)\left(-1 + W(t_i)\right)dt_i \\ &\quad - \min_{[0,1]} U_i(\cdot|Q, P),\end{aligned} \tag{10}$$

where

$$Q_i^-(s) := \max\{-Q_i(s), 0\} \quad \text{and} \quad Q_i^+(s) := \max\{Q_i(s), 0\}.$$

Eq. (10) reveals that the cdf W acts on an allocation Q_i in a bifurcated manner. When Q_i is positive (player i acting like a buyer in expectation), the marginal utility t_i is reduced by $1 - W(t_i)$; when Q_i is negative (i acting like a seller in expectation), the marginal cost t_i is increased by $W(t_i)$. Thus, $1 - W(t_i)$ and $W(t_i)$ correspond to a type-t_i player's information rent, bifurcated between the player's buyer and seller roles, that takes into account the weight $w(t_i)$ in the social welfare.[12] It is important to note that this action of W, or the right-hand side on the first line of (10), is not a linear functional on the space of Q_i unless the space is restricted by either nonnegativity (availability of only goods) or nonpositivity (availability of only bads). Such nonlinearity is precisely the effect of having both a good and a bad for allocation in our otherwise linear, standard model.

3.2 Two-Part Operators

To focus attention to the nonlinear action of W, and to apply the same kind of actions to other distributions, we abstract from (10) a two-part operator defined below. For any function $\varphi : \mathbb{R} \to \mathbb{R}^2$ denoted by $\varphi := (\varphi_+, \varphi_-)$ such that $\varphi_+, \varphi_- : \mathbb{R} \to \mathbb{R}$, denote

$$\langle Q_i : \varphi| := \int_0^1 Q_i^+(s)\varphi_+(s)ds + \int_0^1 \left(-Q_i^-(s)\right)\varphi_-(s)ds.$$

Note that the operator $Q_i \mapsto \langle Q_i : \varphi|$ acts on the function Q_i in two parts, one on the positive part Q_i^+, the other on the negative part $-Q_i^-$. Hence we call $\langle \cdot : \varphi|$ *two-part operator*. The asymmetric bracket of Q_i and φ is to highlight the asymmetry between the two arguments: $\langle Q_i : \varphi|$ is a nonlinear functional of Q_i and yet a linear functional of φ.

The φ in $\langle Q_i : \varphi|$ corresponds to the information rent density derived from an underlying distribution. In general, by *distribution* on $[0, 1]$ we mean a function $G : \mathbb{R} \to \mathbb{R}_+$ that is weakly increasing, right-continuous, vanishing on $(-\infty, 0)$, and equal to $\max_{\mathbb{R}} G$ on $[1, \infty)$. For any distribution G on $[0, 1]$, define a function $\rho(G) : \mathbb{R} \to \mathbb{R}^2$ by

$$\rho(G) := (\rho_+(G), \rho_-(G))$$

such that, for all $s \in \mathbb{R}$,

$$\rho_+(G)(s) := -G(1) + G(s) \quad \text{and} \quad \rho_-(G)(s) := G(s). \tag{11}$$

[12] In the special case where $w = 1$ on $[0, 1]$, $W = F$ and $1 - W(t_i)$ and $W(t_i)$ become the recognizable information rents in the optimal auction and optimal procurement models.

Thus, for any distribution G on $[0,1]$ and any $Q_i : [0,1] \to \mathbb{R}$, the above notation implies

$$\langle Q_i : \rho(G)| = \int_0^1 Q_i^+(s)\rho_+(G)(s)ds + \int_0^1 \left(-Q_i^-(s)\right)\rho_-(G)(s)ds. \tag{12}$$

By (12), we generalize (10) to all distributions G on $[0,1]$ and all IC mechanisms (Q,P):

$$\int_0^1 P_i\, dG = \int_0^1 Q_i(t_i)t_i\, dG(t_i) + \langle Q_i : \rho(G)| - G(1) \min_{[0,1]} U_i(\cdot|Q,P). \tag{13}$$

Comparing (13) with (10), we see that $\rho_+(G)$ reflects i's information rent density when i acts as a buyer in expectation, and $\rho_-(G)$, i's information rent density when i acts as a seller in expectation, had i's type been measured by G.[13]

3.3 The Lagrangian

Denote $\mathscr{Q}$ for the space of all reduced-form allocations $(Q_i)_{i=1}^n$ (each being the reduced form of an ex post allocation according to (3)). Denote $\mathscr{Q}_{\text{mon}}$ for the set of $(Q_i)_{i=1}^n \in \mathscr{Q}$ such that Q_i is weakly increasing on $[0,1]$ for any i. For any welfare density w, define W according to (8). For any $Q := (Q_i)_{i=1}^n \in \mathscr{Q}$ and any $\lambda \in \mathbb{R}_+$, define

$$\mathscr{L}(Q,\lambda) := \sum_i \int_0^1 Q_i(t_i)\left((1+\lambda)t_i - \frac{W(t_i)-F(t_i)}{f(t_i)}\right)dF(t_i) + \lambda \sum_i \langle Q_i : \rho(F)| . \tag{14}$$

Theorem 1 *Given any welfare density w, there exists a payment rule P^* with which a mechanism (Q^*,P^*) maximizes (6) subject to IC, IR and BB if and only if there exists $\lambda \in \mathbb{R}_+$ such that, for all $Q \in \mathscr{Q}_{\text{mon}}$ and all $\lambda' \in \mathbb{R}_+$,*

$$\mathscr{L}(Q^*,\lambda') \geq \mathscr{L}(Q^*,\lambda) \geq \mathscr{L}(Q,\lambda). \tag{15}$$

Proof First, the problem of maximizing (6) subject to IC, IR and BB is equivalent to

$$\max_{Q\in\mathscr{Q}_{\text{mon}}} \quad \sum_i \int_0^1 Q_i(t_i)\left(t_i f(t_i) - W(t_i) + F(t_i)\right)dt_i \tag{16}$$

$$\text{s.t.} \quad \sum_i \int_0^1 Q_i(t_i)t_i\, dF(t_i) + \sum_i \langle Q_i : \rho(F)| \geq 0. \tag{17}$$

The domain $\mathscr{Q}_{\text{mon}}$ captures the ex post feasibility requirement (that $Q \in \mathscr{Q}$) and the second-order part of IC (monotonicity of Q). Ineq. (17) is the joint constraint of IR, BB and the

[13] In the main text, G can be W as in (10), or F as in the next section. In general (Appendix F), G can be any multiple of F or W.

first-order part of IC (Lemma 5, Appendix B.1). It is obtained through applying (13) to the case $G = F$ for all players i, summing such equations across i, and then use IR and BB. The objective (16) is obtained through calculating the social welfare (6) generated by a mechanism (Q, P) that takes into account of the optimal choice of the payment rule among those that implement any given Q. The proof amounts to calculating the optimal amount of lump sum transfers to be redistributed (Lemma 6, Appendix B.2).

Second, the set of all $(Q_i)_{i=1}^n \in \mathscr{Q}_{\text{mon}}$ that satisfy (17) is a convex set. That is because the domain $\mathscr{Q}_{\text{mon}}$ is convex (Appendix B.3), and the mapping $(Q_i)_{i=1}^n \mapsto \sum_i \langle Q_i : \rho(F) |$ a concave functional on $\mathscr{Q}$ (Lemma 7, Appendix B.4). Such concavity is driven by the fact that a player tends to shade the marginal value by a price discount $\rho_+(F)$ when acting like a buyer (when $Q_i = Q_i^+$), and exaggerate the marginal cost by a price markup $\rho_-(F)$ when acting like a seller (when $Q_i = -Q_i^-$). One can also show that there exists a $(Q_i)_{i=1}^n \in \mathscr{Q}_{\text{mon}}$ such that (17) is satisfied strictly (Appendix B.5). Consequently, the conditions corresponding to those in Luenberger [15, Corollary 1, p219] are satisfied. Hence the saddle point condition is necessary and sufficient for any solution to Problem (16)–(17). ■

Remark 2 To appreciate the succinct two-part operator notation, consider the counterparts of (17) in the literature. In the bilateral trade model of Myerson and Satterthwaite [18], where a player's buyer or seller role is exogenous, the counterpart to this constraint is their Ineq. (2), the right-hand side of which can be split into two integrals, one being an integral of the valuations v_2 and v_1, the other an integral of the information rents $(1 - F_i(v_i))/f_i(v_i)$ and $F_i(v_i)/f_i(v_i)$. The first integral corresponds to our first integral in (17), and the second integral, our two-part operation (17). Note, however, that their counterpart to our two-part operation is a linear functional of their allocation p. That is because a player in their model is a priori either a buyer or a seller, hence the two-part operation reduces to

$$\int Q_{\text{buyer}}(t_{\text{buyer}}) \left(-1 + F_{\text{buyer}}(t_{\text{buyer}})\right) dt_{\text{buyer}} - \int Q_{\text{seller}}(t_{\text{seller}}) F_{\text{seller}}(t_{\text{seller}}) dt_{\text{seller}},$$

which is linear in $(Q_{\text{buyer}}, Q_{\text{seller}})$. By contrast, in the partnership dissolution model of Cramton et al. [2], where a player's buyer or seller role is endogenous, the counterpart to our (17) is their Ineq. (I), which is nonlinear in their allocation S_i. It is nonlinear in S_i because their integration depends on the upper or lower limit v_i^*, which depends on S_i. Their counterpart to the first integral in (17) is zero because of their market clearance condition.

Remark 3 For any $\lambda \in \mathbb{R}_+$, define $\left(V_+^\lambda, V_-^\lambda\right) : [0,1] \to \mathbb{R}^2$ by, for any $t_i \in [0,1]$,

$$V_+^\lambda(t_i) \quad := \quad (1+\lambda)t_i - \frac{W(t_i) - F(t_i)}{f(t_i)} - \lambda \cdot \frac{1 - F(t_i)}{f(t_i)}, \tag{18}$$

$$V_-^\lambda(t_i) \quad := \quad (1+\lambda)t_i - \frac{W(t_i) - F(t_i)}{f(t_i)} + \lambda \cdot \frac{F(t_i)}{f(t_i)}. \tag{19}$$

The counterpart when $\lambda = 0$ is, for any $t_i \in [0,1]$,

$$V(t_i) := t_i - \frac{W(t_i) - F(t_i)}{f(t_i)}. \tag{20}$$

Denote $(V_+^\lambda, V_-^\lambda)f := (V_+^\lambda f, V_-^\lambda f)$. Then Eq. (14) is equivalent to

$$\mathscr{L}(Q, \lambda) = \sum_i \left\langle Q_i : (V_+^\lambda, V_-^\lambda)f \right| . \tag{21}$$

Hence the vector-valued function $(V_+^\lambda, V_-^\lambda)$ is the virtual surplus in our model.

Remark 4 When the constraint (17)—the joint constraint of IR, BB and the first-order part of IC—is non-binding, $\lambda = 0$ and the virtual surplus is reduced to the real-value function V in (20), which is similar to those in standard models except for the influence $W(t_i)$ from the welfare weights.[14] When the constraint (17) is binding, however, $\lambda > 0$ and the meaning of virtual surplus is enriched by (18) and (19): The marginal contribution of type t_i is equal to $1 + \lambda$ times its marginal gain t_i of trade subtracted by its net information rent $\frac{W(t_i) - F(t_i)}{f(t_i)}$ and plus its marginal contribution $\lambda\rho(F)(t_i)$ to budget balancing, with the information rent skewed by the welfare weight W, and the budget-balancing contribution determined by the vector-value function $\rho(F)$.

4 The Condition for the Bad to Be Needed

Given any welfare density w, we say that *the bad is needed* if and only if, for any (Q^*, P^*) that maximizes (6) subject to IC, IR and BB, $Q_i^* < 0$ on a positive-measure subset of $[0,1]$

[14] To illustrate the impact of the welfare weights, suppose that the welfare density is uniform across types, namely, $w = 1$ on $[0,1]$. In that case, $W = F$ by (8), and the Lagrangian (14) becomes $(1 + \lambda) \sum_i \int_0^1 Q_i(t_i) t_i \, dF(t_i) + \lambda \sum_i \langle Q_i : \rho(F)|$. Consequently, if the constraint (17) were set aside, the optimal allocation given the fact $t_i > 0$ for all $t_i \in (0,1]$ is to allocate the good to a player with the highest realized type and never allocate the bad at all. Auctioning off only the good and not at all the bad, the allocation satisfies (17) and hence is indeed optimal (Lemma 2). This confirms the intuition that the bad is not needed at all if the welfare density is constant across types. That is, not allocating the bad at all is ex ante efficient.

for some player i. In other words, the bad is needed if and only if every socially optimal mechanism given welfare density w allocates the bad with a strictly positive probability.[15]

Theorem 2 *The bad is needed if and only if*

$$\exists t_i \in (0,1) : \int_0^{t_i} V(s)dF(s) < 0. \quad (22)$$

The "if" part of Theorem 2 is proved in Section 4.1, and the "only if" part in Section 4.2. Before proving it, we present three corollaries of the theorem, each proved in Appendix C.

Condition (22) both necessary and sufficient, we obtain a sharp comparative statics result regarding the prospect that the bad is needed for social optimality. From any welfare density function $w : [0,1] \to [0,1]$ (with $\int_0^1 wdF = 1$), the corresponding welfare weight distribution W is derived according to (8). For any such welfare weight distributions W and $\widetilde{W}$, W is said to *second-order stochastically dominate* $\widetilde{W}$ if and only if

$$\forall r \in [0,1] : \int_0^r \widetilde{W}(s)ds \geq \int_0^r W(s)ds. \quad (23)$$

Corollary 1 *If the bad is needed given welfare weight distribution W and if W second-order stochastically dominates $\widetilde{W}$, then the bad is also needed given $\widetilde{W}$.*

The corollary implies that, if the social planner moves some welfare weight from the middle types to the low and high types, she would allocate the bad to someone if she did so previously.

Corollary 2 shows how little the condition (22) requires for the bad to be needed. All that it takes is for the types near 0 to carry welfare densities above twice the average density:

Corollary 2 *If f is differentiable at 0, the welfare density w is continuous at zero, and $w(0) > 2$, then the bad is needed.*

[15] The definition rules out the uninteresting case where an optimal mechanism allocates the bad with a strictly positive probability just to cancel out any such allocation outcome by simultaneously allocating the good so that $Q_i^* = 0$ whenever the bad is allocated. To see why this case does not count as the bad being needed, modify the mechanism so that it allocates neither item to i whenever $Q_i^* = 0$. The modification is ex post feasible, and it preserves the reduced form of the original mechanism. Thus it is an optimal mechanism that does not allocate the bad at all. That violates the condition that *every* optimal mechanism allocates the bad with a strictly positive probability.

We see from both Corollaries 1 and 2 that the bad is allocated when the welfare densities on low types are sufficiently high. The intuition is that we can think of a players type as the player's marginal rate of substitution between the net utility from the items and the money transfer, and hence low types value money transfers more than the cost of receiving a bad. Given the IC and IR constraint, whenever a bad is allocated, it is allocated to some low types with positive money transfers. Thus, when the planner puts a relatively high weight on low types, a bad should be allocated.

It should be noted that our model does not force the result that the bad is needed. As the next corollary shows, the model allows for a nondegenerate set of welfare densities given which the bad is not needed at all. It says that any welfare distribution that second-order stochastically dominates the exogenous distribution of types renders the bad unnecessary for social optimality. The corollary follows from Corollary 1 and the fact that a social planner who weighs all types uniformly does not need the bad for optimality.[16]

Corollary 3 *If the welfare weight distribution W second-order stochastically dominates the exogenous distribution F of types, the bad is not needed.*

4.1 Why the Bad Is Needed if (22) holds

The argument that (22) implies the necessity of the bad is a proof by contrapositive. Suppose that an optimal mechanism does not allocate the bad at all. Then there is no need to raise funds to pay someone to receive the bad. Thus budget balancing becomes a nonissue. That is, the constraint (17) is non-binding (Lemma 2) and so the Lagrangian (14) reduces to a linear form. It then follows from the saddle point characterization that the mechanism is a solution to a linear programming problem. Thus, one can apply the optimal auction technique to show that the mechanism would allocate the bad with a strictly positive probability unless the ironed copy of the virtual surplus is not negative enough for (22) to hold.

To formalize this argument, let us recall the notations of hierarchical allocations and ironing in the optimal auction theory. For each item j (which can be the good or the bad) and any function $\phi : [0,1] \to \mathbb{R}$, an allocation of item j is said *hierarchical according to ϕ* if and only if, for almost every $(t_k)_{k=1}^n \in [0,1]^n$, item j is allocated to player i if $\phi(t_i) > \max\{0, \max_{k\neq i} \phi(t_k)\}$, and the item is not allocated if $\phi(t_i) < 0$ for all players i.

[16] This fact is verified in the proof of Corollary 3 (Appendix C) succinctly, and in Footnote 14 intuitively.

For any integrable function $g:[0,1]\to\mathbb{R}$, define $H_g:[0,1]\to\mathbb{R}$ by

$$H_g(r) := \int_0^r g\left(F^{-1}(s)\right) ds \tag{24}$$

for all $r\in[0,1]$ and denote $\widehat{H}_g$ for the convex hull of H_g on $[0,1]$. The function $\overline{g}:[0,1]\to\mathbb{R}$ such that

$$\overline{g}(t_i) = \frac{d}{dr}\widehat{H}_g(r)\bigg|_{r=F(t_i)} \tag{25}$$

a.e. $t_i\in[0,1]$ is called *ironed copy* of g.

Note that the V defined by (20) is integrable, with both W and F continuous. Hence its ironed copy $\overline{V}$ is well-defined. The next lemma is a straightforward extension of Myerson's [17, §6] ironing technique. Hence we omit its proof.

Lemma 1 *If Q^* maximizes $\mathscr{L}(Q,0)$ among all $Q\in\mathscr{Q}_{\text{mon}}$, then Q^* is the reduced form of an ex post allocation $(q^*_{iA},q^*_{iB})_{i=1}^n$ such that $(q^*_{iA})_{i=1}^n$ is a hierarchical allocation of the good according to $\overline{V}$, and $(q^*_{iB})_{i=1}^n$ a hierarchical allocation of the bad according to $-\overline{V}$.*

The next lemma, proved in Appendix D, formalizes the aforementioned intuition that if the bad is not allocated at all then budget balancing becomes a nonissue.

Lemma 2 *If $Q\in\mathscr{Q}_{\text{mon}}$ and if $Q_i\geq 0$ on $[0,1]$ for any i, then Q satisfies (17). If, in addition, Q solves Problem (16)–(17), then Q satisfies (17) strictly.*

Proof of the "If" Part of Theorem 2 First, suppose (22) and, to the contrary of the claim, that for some optimal mechanism (Q^*,P^*), $Q^*_i\geq 0$ a.e. on $[0,1]$ for all players i. Then Lemma 2 implies that Q^* satisfies the constraint (17) strictly. Thus, by the saddle point condition in Theorem 1, $\lambda=0$ and Q^* maximizes $\mathscr{L}(\cdot,0)$ on $\mathscr{Q}_{\text{mon}}$. Then Lemma 1 implies that Q^* entails a hierarchical allocation of the bad according to $-\overline{V}$. That is, for almost every $(t_1,\ldots,t_n)\in[0,1]^n$, Q^* awards the bad to a player i if $\overline{V}(t_i)<\min\{0,\min_{k\neq i}V(t_k)\}$. Thus, Q^* allocates the bad with a strictly positive probability if $\overline{V}<0$ on a nondegenerate interval of $[0,1]$ (as F is assumed strictly increasing on $[0,1]$). By (22) and the definition of H_V ((20) and (24)), $H_V(F(t_i))<0$ for some $t_i\in(0,1)$. This, coupled with the fact $H_V(0)=0$ (due to (24)) implies that the convex hull $\widehat{H}_V$ of H_V is negatively sloped on $[0,F(t_i)]$. Then, by (25), $\overline{V}<0$ on the nondegenerate interval $[0,t_i]$. Thus, in the positive-probability event where some player's type belongs to $[0,t_i]$, the bad is allocated to someone. This contradicts the supposition that $Q_i\geq 0$ a.e. on $[0,1]$ for all i. ■

4.2 Why (22) Is True if the Bad Is Needed

Suppose that (22) does not hold and yet the bad is needed. We shall derive a contradiction from this hypothesis through perturbing any optimal mechanism that allocates the bad with a strictly positive probability. The perturbation either enlarges the Lagrangian or renders another optimal mechanism that does not allocate the bad at all, hence a contradiction obtains in either case. The complication is that the Lagrangian is a two-part operation that switches between two integrations (V_-^λ versus V_+^λ, cf. (21)) depending on the signs of the reduced forms. Thus we want the perturbation to preserve the sign of each player's reduced-form allocation. Hence we start with Section 4.2.1 to formalize such perturbations.

4.2.1 Sign-Preserving Perturbations of Allocations

For any $Q := (Q_i)_{i=1}^n \in \mathscr{Q}$, a vector $(c_i)_{i=1}^n \in [0,1]^n$ is called *crossing point* of Q if and only if, for each i, $Q_i \le 0$ a.e. on $[0, c_i]$ and $Q_i \ge 0$ a.e. on $[c_i, 1]$. Obviously, if $Q \in \mathscr{Q}_{\text{mon}}$, then a crossing point of Q exists, each Q_i being weakly increasing. If $Q \in \mathscr{Q}$ has a crossing point $(c_i)_{i=1}^n \in [0,1]^n$ then, for any $\lambda \ge 0$, Eqs. (18), (19) and (21) together imply that

$$\mathscr{L}(Q,\lambda) = \sum_i \int_0^{c_i} Q_i(t_i) V_-^\lambda(t_i) dF(t_i) + \sum_i \int_{c_i}^1 Q_i(t_i) V_+^\lambda(t_i) dF(t_i). \tag{26}$$

For any $Q \in \mathscr{Q}$ with any crossing point $c \in [0,1]^n$, we are interested in perturbing the negative part of Q without upsetting its crossing point or the second sum in (26). Such perturbations transform Q into an element of—

$$\mathscr{Q}(Q,c) := \left\{ (Q_i')_{i=1}^n \in \mathscr{Q} \mid \forall i \, [Q_i' \le 0 \text{ on } [0, c_i), Q_i' = Q_i \text{ on } (c_i, 1]] \right\}. \tag{27}$$

For now, we need only to define one kind of such perturbations (more in Appendix E.2):

Reservation $R_{i,T}$: For any player i, any $T \subseteq [0,1]^n$, and any $Q \in \mathscr{Q}$ that is the reduced form of an ex post allocation $(q_{kA}, q_{kB})_{k=1}^n$, define $R_{i,T}(Q)$ to be the reduced form of the ex post allocation $(\tilde{q}_{kA}, \tilde{q}_{kB})_{k=1}^n$ that is the same as $(q_{kA}, q_{kB})_{k=1}^n$ except $\tilde{q}_{ij}(t) := 0$ for any $t \in T$ and any item $j \in \{A, B\}$. That is, when T occurs, the planner keeps any item to herself whenever the original allocation would award it to player i.

Denote π_i for the projection of any n-vector $(t_1, \ldots, t_n)$ onto its i^{th} component t_i. The next lemma follows directly from the definition of $R_{i,T}$ and hence we omit its proof:

Lemma 3 *For any $Q \in \mathscr{Q}$ with crossing point $c := (c_i)_{i=1}^n \in [0,1]^n$, any player i and any $T \subseteq [0,1]^n$ for which $\pi_i(T) \subseteq [0, c_i]$, if $Q' := R_{i,T}(Q)$, then:*

a. $Q'_i = 0$ on $\pi_i(T)$, $Q'_i = Q_i$ on $[0,1] \setminus \pi_i(T)$, and $Q'_k = Q_k$ on $[0,1]$ for all $k \neq i$;

b. $Q' \in \mathscr{Q}(Q, c)$ and, if in addition $Q \in \mathscr{Q}_{\text{mon}}$ and $\pi_i(T) \supseteq [0, c_i]$, $Q' \in \mathscr{Q}_{\text{mon}}$;

c. Eq. (26) holds when Q is replaced by Q'.

4.2.2 Proof of the "Only If" Part of Theorem 2

Suppose, to the contrary, that the bad is needed and yet (22) does not hold. Then

$$\forall t_i \in (0,1) : \int_0^{t_i} V(s) dF(s) \geq 0. \tag{28}$$

Pick any optimal mechanism (Q^*, P^*). The saddle point condition in Theorem 1 implies that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}_{\text{mon}}$ for some $\lambda \geq 0$. By (19) and (20),

$$V_-^\lambda \begin{cases} = & V \text{ on } [0,1] \quad \text{if } \lambda = 0 \\ > & V \text{ on } (0,1] \quad \text{if } \lambda > 0. \end{cases} \tag{29}$$

Thus (28) implies

$$\forall \lambda > 0 : \forall t_i \in (0,1) : \int_0^{t_i} V_-^\lambda(s) dF(s) > 0. \tag{30}$$

By hypothesis, Q^* allocates the bad with a strictly positive probability. Thus $Q_i^* < 0$ on a positive-measure subset of $[0,1]$ for some player i. Since $Q^* \in \mathscr{Q}_{\text{mon}}$, Q_i^* is weakly increasing, hence this subset is an interval $[0, c_i)$ or $[0, c_i]$ for some $c_i \in (0,1]$. Without loss of generality, let c_i be the maximum among all such upper bounds so that $Q_i^* < 0$ on $[0, c_i)$, and $Q_i^* \geq 0$ on $(c_i, 1]$. For any $k \neq i$, with Q_k^* weakly increasing, there exists $c_k \in [0,1]$ for which $c := (c_i, (c_k)_{k \neq i})$ is a crossing point of Q^*.

Consider an alternative allocation $R_{i,T}(Q^*)$ for which $T = \{(t_k)_{k=1}^n \in [0,1]^n \mid t_i \in [0, c_i]\}$. That is, modify Q^* by reserving both items from player i when i's type belongs to $[0, c_i]$. By Lemma 3.b, $R_{i,T}(Q^*) \in \mathscr{Q}_{\text{mon}}$ and also has c as a crossing point. Thus, given the same c, (26) holds whether $Q = R_{i,T}(Q^*)$ or $Q = Q^*$. Then by Lemma 3.a,

$$\mathscr{L}(Q^*, \lambda) - \mathscr{L}(R_{i,T}(Q^*), \lambda) = \int_0^{c_i} Q_i^*(t_i) V_-^\lambda(t_i) dF(t_i). \tag{31}$$

Since Q_i^* is weakly increasing, by Fubini's theorem we have

$$\int_0^{c_i} Q_i^*(t_i) V_-^\lambda(t_i) dF(t_i) = \left(\lim_{s \uparrow c_i} Q_i^*(s)\right) \int_0^{c_i} V_-^\lambda(r) dF(r) - \int_0^{c_i} \int_0^s V_-^\lambda(r) dF(r) dQ_i^*(s). \quad (32)$$

By (28), (29) and (30), together with Q_i^* being weakly increasing, the right-hand side of (32) is nonpositive. This coupled with (31) implies that

$$\mathscr{L}(R_{i,T}(Q^*), \lambda) \geq \mathscr{L}(Q^*, \lambda). \quad (33)$$

Furthermore, if $\lambda > 0$, the right-hand side of (32) is negative. That is because, by the choice of c_i, either (i) $\lim_{s \uparrow c_i} Q_i^*(s) < 0$ or (ii) $\lim_{s \uparrow c_i} Q_i^*(s) = 0$. In Case (i), the first term on the right-hand side of (32) is negative due to (30). In Case (ii), $\lim_{s \uparrow c_i} Q_i^*(s) - Q_i^*(0) = 0 - Q_i^*(0) > 0$ (since $Q_i^* < 0$ on $[0, c_i)$) and so Q_i^* as a distribution assigns a positive measure on $[0, c_i)$; hence the double integral on the right-hand side of (32) is positive. Thus, by (31),

$$\lambda > 0 \Rightarrow \mathscr{L}(R_{i,T}(Q^*), \lambda) > \mathscr{L}(Q^*, \lambda).$$

Consequently, by the saddle point condition that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}_{\text{mon}}$, $\lambda = 0$. This coupled with (33) means that $R_{i,T}(Q^*)$ is a maximizer of $\mathscr{L}(\cdot, 0)$ on $\mathscr{Q}_{\text{mon}}$.

If there is another player $k \neq i$ to whom $R_{i,T}(Q^*)$ allocates the bad with a strictly positive probability, perturb $R_{i,T}(Q^*)$ by the reservation operator R_{k,T_k} such that $T_k = \{(t_l)_{l=1}^n \in [0,1]^n \mid t_k \in [0, c_k]\}$. By the previous reasoning, $R_{k,T_k}(R_{i,T}(Q^*))$ is a maximizer of $\mathscr{L}(\cdot, 0)$ on $\mathscr{Q}_{\text{mon}}$. Repeating this reservation procedure, we eventually obtain an allocation $\tilde{Q}$ that allocates the bad with zero probability and maximizes $\mathscr{L}(\cdot, 0)$ on $\mathscr{Q}_{\text{mon}}$. Since $\tilde{Q}$ is entirely nonnegative (c a crossing point of Q^*), the left-hand side of (17) is nonnegative (Lemma 2) and so $\lambda = 0$ is a minimum of the Lagrangian $\mathscr{L}(\tilde{Q}, \cdot)$ on $\mathbb{R}_+$. Thus $(\tilde{Q}, 0)$ is a saddle point and hence, by the sufficiency part of Theorem 1, $\tilde{Q}$ constitutes an optimal mechanism subject to IC, IR and BB. Since $\tilde{Q}$ does not allocate the bad at all, we obtain a contradiction to the premise that the bad is allocated with a strictly positive probability in every optimal mechanism. ■

5 Why the Kuhn-Tucker Method Does Not Deliver

To solve a constrained optimization problem such as (16) with the Kuhn-Tucker theorem, a typical approach is to apply the theorm to a relaxed problem, which sets aside the monotonicity constraint (the second-order part of IC). For the solution thereby obtained to be valid to the original problem, the method would have to assume that any solution obtained through this method happens to satisfy the set aside monotonicity constraint (e.g., Ledyard and Palfrey [11], who refer to this assumption "regular case"). Could this method have delivered some counterpart to Theorem 2 such as the bad being needed given a nondegenerate set of parameter values? The answer is No. The next Theorem 3 says that if the bad is needed then either every solution to the relaxed problem violates the monotonicity constraint, or the relaxed problem suffers indeterminacy in the sense that it has a continuum of solutions.

To state the theorem, recall that an *optimal mechanism* means any maximizer of (6) subject to IC, IR and BB. As shown in Section 3, maximizing (6) subject to IC, IR and BB is equivalent to maximizing (16) among all $Q \in \mathscr{Q}$ subject to (17) and the monotonicity constraint $Q \in \mathscr{Q}_{\text{mon}}$. Call an allocation *optimal* if and only if it is a solution to this maximization problem. The *relaxed problem*, by contrast, is to maximize (16) among all $Q \in \mathscr{Q}$ subject to only (17). Explicitly put, the relaxed problem is

$$\begin{array}{ll} \max_{Q\in\mathscr{Q}} & \sum_i \int_0^1 Q_i(t_i)\left(t_i f(t_i) - W(t_i) + F(t_i)\right) dt_i \\ \text{s.t.} & \sum_i \int_0^1 Q_i(t_i) t_i \, dF(t_i) + \sum_i \langle Q_i : \rho(F) | \geq 0. \end{array} \tag{34}$$

The next lemma, proved in Appendix E.1, provides the basis for the theorem.

Lemma 4 *Given any welfare density w:*

a. Q^* is a solution to (34) if and only if there exists $\lambda \in \mathbb{R}_+$ such that (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$ *in that (15) holds for all $Q \in \mathscr{Q}$ and all $\lambda' \in \mathbb{R}_+$;*

b. if Q^* maximizes $\mathscr{L}(Q, \lambda)$ among all $Q \in \mathscr{Q}$, then:

i. if $\lambda = 0$ then, for almost all $(s, s') \in [0,1]^2$, $V(s) > V(s') \Rightarrow Q_i^*(s) > Q_i^*(s')$;

ii. if $\lambda > 0$, then $Q_i^*(s) Q_j^*(s) \geq 0$ for all players i and j and almost every $s \in [0,1]$;

iii. if $Q_i^* < 0$ on $[0, c_i)$ for some $c_i \in (0,1]$ and some player i, then $V_-^\lambda \leq 0$ on $(0, c_i)$.

Theorem 3 *Assume that f is differentiable on $[0,1]$. For any welfare density w given which the bad is needed, the relaxed problem (34) either (i) does not have any optimal allocation as a solution or (ii) has a continuum of solutions.*

Alternative (ii) in Theorem 3, where the relaxed problem admits a continuum of solutions, corresponds to a condition that the virtual surplus is constantly zero on a nondegenerate interval $(0, c_*)$. This condition can be violated with slight perturbations of the type-density f or the welfare density w near 0. Thus the next corollary obtains (proved in Appendix E.4).

Corollary 4 *In the parameter space consisting of all pairs (f, w) of type-density function f and welfare density w such that f is differentiable, it is generically true that if the bad is needed then the constraint $Q \in \mathscr{Q}_{\text{mon}}$ is binding for any optimal mechanism.*

Proof of Theorem 3 It suffices to prove that if statement (i) is not true then statement (ii) is true. Thus, let Q^* be an optimal allocation that is also a solution to (34), and we shall prove that (34) has a continuum of solutions. As part of the definition of optimality, $Q^* \in \mathscr{Q}_{\text{mon}}$. With Q^* a solution to (34), there exists a $\lambda \in \mathbb{R}_+$ for which (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$ (Lemma 4.a). Thus, Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$.

We claim that $\lambda > 0$. Suppose not, then Lemma 4.b.i implies that $Q_i^*(t_i)$ is a strictly increasing function of $V(t_i)$ a.e. $t_i \in [0, 1]$. Since the bad is needed by hypothesis, Theorem 2 implies that (22) holds, which in turn implies $V < 0$ somewhere in $[0, 1]$. This, coupled with the fact that $V(0) = 0$ and V is differentiable (as f is differentiable), implies that V is negative and strictly decreasing on (a, b) for some $0 \leq a < b \leq 1$. Then Q_i^* is strictly decreasing a.e. on (a, b), contradicting the monotonicity condition $Q^* \in \mathscr{Q}_{\text{mon}}$.

By the hypothesis that the bad is needed and the fact $Q^* \in \mathscr{Q}_{\text{mon}}$, $Q_i^* < 0$ on $[0, x)$ for some $x \in (0, 1]$ and some player i. Let

$$c_* := \max_{i=1,\ldots,n} \sup \left\{x \in [0, 1] : Q_i^* < 0 \text{ on } [0, x)\right\}.$$

Note $c_* > 0$. Let i_0 be a player that attains this maximum, so $Q_{i_0}^* < 0$ on $[0, c_*)$. Since $\lambda > 0$, Lemma 4.b.ii applies. Thus, for any player $k \neq i_0$, $Q_k^* Q_{i_0}^* \geq 0$ a.e., and hence $Q_k^* \leq 0$ a.e. on $[0, c_*)$. The definition of c_*, coupled with $Q^* \in \mathscr{Q}_{\text{mon}}$, also implies $Q_k^* \geq 0$ on $(c_*, 1]$ for all players k. Thus $c := (c_k)_{k=1}^n$ defined by $c_k := c_*$ for all k is a crossing point of Q^*.

There are only two possible cases: (i) $V_-^\lambda(x) < 0$ for some $x \in (0, c_*)$, or (ii) $V_-^\lambda \geq 0$ on $(0, c_*)$. The rest of the proof is to establish two observations:

a. Case (i) implies that Q^* violates the monotonicity constraint and hence (34) admits no optimal allocation as a solution.

b. Case (ii) implies that (34) has a continuum of solutions.

Both observations are based on the fact that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}(Q^*, c)$, where $\mathscr{Q}(Q^*, c)$, according to (27) and the definition of c here, is the set of $Q \in \mathscr{Q}$ that result from some sign-preserving perturbations of Q^* that leave the positive part of Q_i^* ($\forall i$) unchanged. Note, for any $Q \in \mathscr{Q}(Q^*, c)$, (26) holds and

$$\mathscr{L}(Q, \lambda) - \mathscr{L}(Q^*, \lambda) = \sum_i \int_0^{c_*} \left(Q_i(s) - Q_i^*(s)\right) V_-^{\lambda}(s) dF(s). \tag{35}$$

Thus, plug "$Q_i^- = -Q_i$ and $(Q_i^*)^- = -Q_i^*$ on $[0, c_*)$" into (35) to get that Q^* solves

$$\max_{Q \in \mathscr{Q}(Q^*, c)} \sum_i \int_0^{c_*} Q_i^-(s) \left(-V_-^{\lambda}(s)\right) dF(s). \tag{36}$$

In Case (i), since V_-^{λ} is differentiable and $V_-^{\lambda}(0) = 0$, there is a nondegenerate interval $I \subseteq (0, c_*)$ on which V_-^{λ} is negative and strictly decreasing. Since Q^* is a solution to (36), it does not allocate the good to any player-type in I, nor the bad to any player-type in $(c_*, 1]$; furthermore, if the bad is to be allocated to some player-types in $[0, c_*]$, the bad goes to the one whose V_-^{λ}-value is the lowest among all negative ones. If these three properties are not all satisfied, one can construct a $Q \in \mathscr{Q}(Q^*, c)$ that outperforms Q^* in terms of the objective of (36) (Lemma 9 in Appendix E.2, where $g = -V_-^{\lambda}$). Thus, by (3), for any i and any $t_i \in I$, $Q_i^*(t_i)$ is equal to the negative of the marginal of the ex post allocation $q_{iB}^*(t_i, \cdot)$ of the bad, and $Q_i^*(t_i)$ is strictly increasing in $V_-^{\lambda}(t_i)$ for a.e. $t_i \in I$ (Lemma 10 in Appendix E.2, where $g = -V_-^{\lambda}$). But then Q_i^* is strictly decreasing a.e. on I, violating the monotonicity constraint. Hence Q^* cannot be an optimal allocation.

In Case (ii), $V_-^{\lambda} = 0$ on $[0, c_*)$ by Lemma 4.b.iii and the definition of c_*. Then (35) implies $\mathscr{L}(Q, \lambda) = \mathscr{L}(Q^*, \lambda)$ for any $Q \in \mathscr{Q}(Q^*, c)$. Thus, any $Q \in \mathscr{Q}(Q^*, c)$ is also a maximizer of $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$. By Lemma 4.a, any such Q is a solution to the relaxed problem if λ minimizes $\mathscr{L}(Q, \cdot)$ on $\mathbb{R}_+$, which is true if the constraint in the relaxed problem (34) is binding for Q. By the definition of $\langle Q_i : \rho(F)|$, the constraint being binding for Q is equivalent to

$$\sum_i \int_0^1 Q_i^-(s) \left(sf(s) + F(s)\right) dF(s) = \sum_i \int_0^1 Q_i^+(s) \left(sf(s) + F(s) - 1\right) dF(s).$$

Since c is a crossing point for all $Q \in \mathscr{Q}(Q^*, c)$, this equation is the same as

$$\sum_i \int_0^{c_*} Q_i^-(s)\,(sf(s)+F(s))\,dF(s) \;=\; \sum_i \int_{c_*}^1 Q_i^+(s)\,(sf(s)+F(s)-1)\,dF(s) \tag{37}$$

$$= \underbrace{\sum_i \int_{c_*}^1 (Q_i^*(s))^+\,(sf(s)+F(s)-1)\,dF(s)}_{=:z},$$

with the second line due to the fact that $Q_i(s) = Q_i^*(s)$ whenever $Q_i^*(s) > 0$ (by the definition of $\mathscr{Q}(Q^*, c)$). Thus, any $Q \in \mathscr{Q}(Q^*, c)$ for which

$$\sum_i \int_0^{c_*} Q_i^-(s)\,(sf(s)+F(s))\,dF(s) = z \tag{38}$$

is a solution to the relaxed problem. Since $c_* > 0$, $\mathscr{Q}(Q^*, c)$ is a convex set with nonempty interior. The left-hand side of (38) is a linear functional on $\mathscr{Q}(Q^*, c)$. Thus one can show that the set of $Q \in \mathscr{Q}(Q^*, c)$ that satisfies (38) is a hyperplane intersection of the interior of $\mathscr{Q}(Q^*, c)$ (Lemma 11, Appendix E.3). Hence there is a continuum of solutions to the relaxed problem, as asserted. ■

6 Conclusion

This paper asks a novel question: Under what primitive condition in a quasilinear independent private values model is a commonly undesirable item needed as an instrument to achieve interim Pareto optimality? The answer is a necessary and sufficient condition, which holds if the extreme low types weigh in the social welfare more than twice the average weight, or if the welfare weight distribution spreads out sufficiently. This result holds regardless of the particular functional forms of the social welfare distribution and the type distribution. The finding sheds a new light on policy issues regarding the location decision of Nimbies. Even if the public good effect of a Nimby were assumed away and there were no cost to do away with the Nimby completely, the Nimby is still needed to optimize the social welfare when the welfare weights of the high and low types are sufficiently large. Put differently, if we think of welfare weights as the bargaining power among various players in the interim, in any idealized outcome of the interim bargaining process, some low types have to end with the bad if the low and high types are sufficiently powerful relative to the middle types.

It is important to note that the purpose of this paper is not to find a tractable model where a bad is needed for social optimality. Our purpose, rather, is to identify the condition under which the bad is needed in an environment that does not at all force the usage of the bad, with never allocating the bad part of an ex ante incentive efficient allocation. Nevertheless, some models where the usage of the bad arises more easily are also interesting to study and could also use our method. For example, consider the provision of healthcare with congestion such that everyone is endowed with a basic amount of healthcare service. If someone wants a premium service, someone else has to give up the basic service thereby freeing up the facility for the former. The latter's action can be interpreted as receiving a unit of the bad, and this setup is subject to the market clearing condition that the quantity of the good (premium service) awarded be equal to the quantity of the bad received. Ex ante incentive efficiency would then entail assignment of the bad to some types. Given such a setup, our saddle point characterization remains valid.

This paper contributes a new method to the mechanism design problems where a player's role in the market is not exogenous but rather determined by the mechanism and the player's action. Such endogeneity upsets the linearity of a player's ex ante surplus as a function of the allocation in the mechanism. We restore the structure to a tractable, concave two-part operator thereby characterizing all the optimal mechanisms with a saddle point condition. Furthermore, to derive properties of all optimal mechanisms from the saddle point condition, we develop a perturbation method that uses a family of ex post feasible, sign-preserving perturbations of any optimal allocation. Preserving every player's endogenous role in the market in a type-by-type manner, such perturbations affect the associated Lagrangian linearly, because they do not alter the measure with which the two-part operator acts on the allocation. Considering such perturbations in the direction of the fixed measure, we obtain necessary conditions for all—rather than only for some—optimal mechanisms. Our method proves more applicable than the Kuhn-Tucker method given our environment, as it is generically impossible for the Kuhn-Tucker method to obtain a counterpart to our result.

A Proof of (9): An Integration-by-Part Routine

Pick any $t^0 \in [0,1]$. Since (Q,P) is IC, (5) implies

$$\begin{aligned}\int_0^1 P_i\, dW &= \int_0^1 \left(t_i Q_i(t_i) - \int_{t^0}^{t_i} Q_i(s)ds - U_i(t^0 \mid Q,P) \right) dW(t_i) \\ &= \int_0^1 t_i Q_i(t_i) dW(t_i) - U_i(t^0 \mid Q,P) - \int_0^1 \int_{t^0}^{t_i} Q_i(s)ds\, dW(t_i).\end{aligned}$$

Decompose the last double integral and use Fubini's theorem to obtain

$$\begin{aligned}\int_0^1 \int_{t^0}^{t_i} Q_i(s) ds\, dW(t_i) &= -\int_0^{t^0} \int_{t_i}^{t^0} Q_i(s) ds\, dW(t_i) + \int_{t^0}^1 \int_{t^0}^{t_i} Q_i(s) ds\, dW(t_i) \\ &= -\int_0^{t^0} \int_0^s Q_i(s) dW(t_i) ds + \int_{t^0}^1 \int_s^1 Q_i(s) dW(t_i) ds \\ &= -\int_0^{t^0} Q_i(s) W(s) ds + \int_{t^0}^1 Q_i(s) \left(1 - W(s)\right) ds.\end{aligned}$$

Plugging the second multiline formula into the first one for $\int_{T_i} P_i dW$, we get (9).

B Details of Theorem 1

B.1 The Joint Constraint for IC, IR and BB

Lemma 5 *For any allocation $(Q_i)_{i=1}^n$ such that Q_i is weakly increasing on $[0,1]$ for any i, there exists a payment rule $(P_i)_{i=1}^n$ with which $(Q_i)_{i=1}^n$ constitutes an IC, IR and BB mechanism if and only if (17) is true.*

Proof Applying (13) to the case $G = F$ for all players i and summing the equations thereby obtained across i, we get the total expected money surplus from any IC mechanism (Q, P):

$$\sum_i \int_0^1 P_i\, dF = \sum_i \int_0^1 Q_i(t_i) t_i\, dF(t_i) + \sum_i \langle Q_i : \rho(F)| - \sum_i \min_{[0,1]} U_i(\cdot|Q, P). \tag{39}$$

Thus, BB ($\sum_i \int_0^1 P_i\, dF \geq 0$), IR ($\min_{[0,1]} U_i(\cdot|Q, P) \geq 0$ for all i) and IC together imply (17).

Conversely, suppose (17). With Q_i weakly increasing, pick $t_i^0 \in [0,1]$ for which $Q_i(t_i) \geq 0$ for all $t_i \in (t_i^0, 1]$ and $Q_i(t_i) \leq 0$ for all $t_i \in [0, t_i^0)$. With such t_i^0, construct P_i via (5) so that $U_i(t_i^0|Q, P) = 0$ for all i. This, with Q_i weakly increasing, implies IC. Since P_i is constructed via (5), the derivative of $U_i(\cdot|Q, P)$ is equal to Q_i and hence, by the choice of t_i^0, $U_i(\cdot|Q, P)$ attains its minimum at t_i^0. Hence IR obtains by construction of P. Now that $(Q_i, P_i)_{i=1}^n$ is IC, (39) holds. Then (17) coupled with $\sum_i \min_{[0,1]} U_i(\cdot|Q, P) = 0$ implies BB. ∎

B.2 The Social Welfare with Optimal Lump Sum Rebate

Lemma 6 *Any maximand of (6) subject to IC, IR and BB is equal to (16).*

Proof Let (Q, P) be a maximizer of (6) subject to IC, IR and BB. By IC, (10) holds for all i. Plug (10) into (7) and apply the two-part operator notation to see that the social welfare (6) generated by (Q, P) is equal to

$$\sum_i \min_{[0,1]} U_i(\cdot|Q,P) - \sum_i \langle Q_i : \rho(W)| \,. \tag{40}$$

By (39) and BB ($\sum_i \int_0^1 P_i \, dF \geq 0$),

$$\sum_i \min_{[0,1]} U_i(\cdot|Q,P) \leq \sum_i \int_0^1 Q_i(t_i) t_i \, dF(t_i) + \sum_i \langle Q_i : \rho(F)| \,. \tag{41}$$

The right-hand side of (41) can be attained by a payment rule that implements $(Q_i)_{i=1}^n$: Construct a payment rule P_i^Q via (5) such that $\min_{[0,1]} U_i(\cdot|Q, P^Q) = 0$ for all i. With (Q, P) IC, Q_i is weakly increasing for each i. This coupled with the construction of P_i^Q implies that $(Q_i, P_i^Q)_{i=1}^n$ is IC. Thus $\sum_i \int_0^1 P_i^Q dF$ is equal to the right-hand side of (39) with the role of P there played by P^Q here. Consequently, since $\min_{[0,1]} U_i(\cdot|Q, P^Q) = 0$, $\sum_i \int_0^1 P_i^Q dF$ is equal to the right-hand side of (41). Then define P^* to be the payment rule that combines $(P_i^Q)_{i=1}^n$ with the lump sum transfer back to the players in the amount equal to $\sum_i \int_0^1 P_i^Q dF$. With $P := P^*$, the left-hand side of (41) is equal to $\sum_i \int_0^1 P_i^Q dF$. It follows that P^* satisfies (41) as an equality.

Thus, given any implementable Q, the maximand of $\sum_i \min_{[0,1]} U_i(\cdot|Q, P)$ is equal to the right-hand side of (41). Substitute the right-hand side of (41) for the $\sum_i \min_{[0,1]} U_i(\cdot|Q, P)$ in (40) to see that the social welfare (6) generated by an optimal (Q, P) is equal to

$$\begin{aligned} & \sum_i \int_0^1 Q_i(t_i) t_i \, dF(t_i) + \sum_i \langle Q_i : \rho(F)| - \sum_i \langle Q_i : \rho(W)| \\ = \; & \sum_i \int_0^1 Q_i(t_i) t_i \, dF(t_i) + \sum_i \langle Q_i : \rho(F) - \rho(W)| \,, \end{aligned}$$

with the equality due to linearity of $\varphi \mapsto \langle Q_i : \varphi|$. The right-hand side of the above equation, by (11), is equal to (16). ■

B.3 Convexity of $\mathscr{Q}_{\text{mon}}$

Let $\gamma \in [0, 1]$ and $Q, \hat{Q} \in \mathscr{Q}_{\text{mon}}$. Since $Q \in \mathscr{Q}_{\text{mon}}$, it is generated by a $(q_{iA}, q_{iB})_{i=1}^n$ with $\sum_i q_{iA}(\cdot) \leq 1$ and $\sum_i q_{iB}(\cdot) \leq 1$ via (3), and Q_i is weakly increasing for all i. Likewise, $\hat{Q} = (\hat{Q}_i)_{i=1}^n$ is generated by a $(\hat{q}_{iA}, \hat{q}_{iB})_{i=1}^n$ with each $\hat{Q}_i$ weakly increasing. Then

$\sum_i (\gamma q_{iA} + (1-\gamma)\hat{q}_{iA}) \leq 1$ and $\sum_i (\gamma q_{iB} + (1-\gamma)\hat{q}_{iB}) \leq 1$; furthermore, for each i, $\gamma Q_i + (1-\gamma)\hat{Q}_i$ satisfies (3) with respect to $(\gamma q_{iA} + (1-\gamma)\hat{q}_{iA}, \gamma q_{iB} + (1-\gamma)\hat{q}_{iB})$, and is weakly increasing because both Q_i and $\hat{Q}_i$ are so. Thus $(\gamma Q_i + (1-\gamma)\hat{Q}_i)_{i=1}^n \in \mathscr{Q}_{\text{mon}}$.

B.4 Concavity of Two-Part Operators

Lemma 7 $(Q_i)_{i=1}^n \mapsto \sum_i \langle Q_i : \rho(F)|$ *is a concave functional on* $\mathscr{Q}$. *Furthermore, for any* $Q, Q' \in \mathscr{Q}$, *if* $Q_i(t_i)Q_i'(t_i) < 0$ *for all* t_i *in a positive-measure subset of* $[0,1]$ *for some* i *and if* $\alpha \in (0,1)$, *then*

$$\alpha \sum_i \langle Q_i : \rho(F)| + (1-\alpha) \sum_i \langle Q_i' : \rho(F)| < \sum_i \langle \alpha Q_i + (1-\alpha) Q_i' : \rho(F)| . \tag{42}$$

Proof It suffices to prove, for each i, that $\langle Q_i : \rho(F)|$ is a concave functional of Q_i, and strictly so if $Q_i^+ \neq 0$ on a positive-measure subset of $[0,1]$. By (11), the definition of $\rho(F)$, $\rho_+(F) < \rho_-(F)$ on $[0,1]$. By the definition of two-part operators and the fact $Q_i = Q_i^+ - Q_i^-$,

$$\begin{aligned}
\langle Q_i : \rho(F)| &= \int_0^1 Q_i^+(t_i)\rho_+(F)(t_i)dt_i - \int_0^1 Q_i^-(t_i)\rho_-(F)(t_i)dt_i \\
&= \int_0^1 Q_i(t_i)\rho_-(F)(t_i)dt_i + \int_0^1 Q_i^+(t_i)\left[\rho_+(F)(t_i) - \rho_-(F)(t_i)\right]dt_i.
\end{aligned}$$

On the second line, the first integral is linear in Q_i; the second integral is concave in Q_i because $Q_i(t_i) \mapsto Q_i^+(t_i)$ is convex, $\rho_+(F) - \rho_-(F) \leq 0$ on $[0,1]$, and hence $Q_i(t_i) \mapsto Q_i^+(t_i)(\rho_+(F)(t_i) - \rho_-(F)(t_i))$ is concave for all $t_i \in [0,1]$. Thus $\langle Q_i : \rho(F)|$ is concave in Q_i. To prove the second statement of the lemma, note that the convex mapping $x \mapsto x^+$ is strictly convex for those $x, y \in \mathbb{R}$ such that $xy < 0$ in the sense that $xy < 0$ implies

$$\alpha x^+ + (1-\alpha)y^+ > (\alpha x + (1-\alpha)y)^+$$

for all $\alpha \in (0,1)$. This coupled with the fact $\rho_+(F) - \rho_-(F) < 0$ on $[0,1]$ implies that

$$\left(\alpha Q_i^+ + (1-\alpha)(Q')_i^+\right)(\rho_+(F) - \rho_-(F)) < (\alpha Q_i + (1-\alpha)Q_i')^+ (\rho_+(F) - \rho_-(F))$$

on the subset of $[0,1]$ where $Q_i Q_i' < 0$. Thus, if this subset, denoted by E, is of positive measure, the above strict inequality is preserved by integration on E. When the integration domain extends from E to $[0,1]$, the strictly inequality is again preserved because $Q_i(t_i) \mapsto Q_i^+(t_i)(\rho_+(F)(t_i) - \rho_-(F)(t_i))$ is concave for every $t_i \in [0,1]$. Thus (42) follows. ■

B.5 Existence of Interior Solutions for (17)

Let $(Q_i)_{i=1}^n$ be the allocation of auctioning off the good through an expected-revenue-maximizing auction (cf. Myerson [17]) and never assigning the bad at all. Hence Q_i is never negative, $\langle Q_i : \rho(f)| = \int_0^1 Q_i(s)\rho_+(F)(s)ds$, and so the left-hand side of (17) is equal to

$$\sum_i \int_0^1 Q_i(t_i)\left(t_i - \frac{1-F(t_i)}{f(t_i)}\right) dF(t_i),$$

which, by Myerson [17], is equal to the expected value of the pointwise maximum among the nonnegative ironed virtual utilities of the good. Since F has no gap in $[0,1]$, this expected value is strictly positive. Thus (17) is satisfied strictly.

C Proof of Corollaries 1, 2 and 3

Corollary 1 Use (20) and integration-by-parts to obtain

$$\begin{aligned}\int_0^{t_i} V(r)dF(r) &= t_iF(t_i) - t_iW(t_i) + \int_0^{t_i} rdW(r) \\ &= t_iF(t_i) - \int_0^{t_i}(t_i - r)w(r)f(r)dr\end{aligned}$$

for any $t_i \in [0,1]$, with the second line due to (8). For each $t_i \in [0,1]$, define

$$\begin{aligned}R(t_i) &:= \int_0^{t_i}(t_i - r)w(r)f(r)dr, \\ \widetilde{R}(t_i) &:= \int_0^{t_i}(t_i - r)\widetilde{w}(r)f(r)dr.\end{aligned}$$

By this definition and the above calculation, (22) is equivalent to "$t_iF(t_i) < R(t_i)$ for some $t_i \in (0,1)$." By hypothesis, the bad is allocated with a strictly positive probability given welfare density w, thus the "only if" part of Theorem 2 implies that $t_iF(t_i) < R(t_i)$ for some $t_i \in (0,1)$. The "if" part of the theorem implies that the bad is also allocated with a strictly positive probability given $\widetilde{w}$ if $t_iF(t_i) < \widetilde{R}(t_i)$ for some $t_i \in (0,1)$. Thus, we complete the proof by showing that $\widetilde{R} \geq R$ on $(0,1]$. To show that, pick any $t_i \in (0,1]$. We have:

$$\begin{aligned}
\widetilde{R}(t_i) - R(t_i) &= \int_0^{t_i} (t_i - r)\left(\widetilde{w}(r) - w(r)\right) f(r)dr \\
&= \int_0^{t_i} (t_i - r)\, d\left(\widetilde{W}(r) - W(r)\right) \\
&= -t_i\left(\widetilde{W}(0) - W(0)\right) - \int_0^{t_i} \left(\widetilde{W}(r) - W(r)\right) d(t_i - r) \\
&= \int_0^{t_i} \left(\widetilde{W}(r) - W(r)\right) dr,
\end{aligned}$$

which is nonnegative by (23), as $\widetilde{W}$ is second-order stochastically dominated by W. ■

Corollary 2 By Theorem 2, it suffices to prove (22). By (20) and differentiability of f at 0, the derivative of V at 0 is equal to $2 - w(0)$, which is negative by the hypothesis of the corollary. Thus, with $V(0) = 0$ by (20), (22) holds for some t_i near 0. ■

Corollary 3 By Corollary 1, it suffices to prove that the bad is not needed when the welfare weight distribution is F. Given such welfare weight distribution, $V(t_i) = t_i$ for all $t_i \in [0,1]$ by (20). Hence (22) does not hold. By Theorem 2, the bad is not needed. ■

D Details of Theorem 2

Proof of Lemma 2 Since $Q_i \geq 0$ by hypothesis, $\langle Q_i : \rho(f)| = \int_0^1 Q_i(s)\rho_+(F)(s)ds$, and so the left-hand side of (17) is equal to

$$\sum_i \int_0^1 Q_i(s)\left(sf(s) - (1 - F(s))\right) ds = \sum_i \int_0^1 Q_i(s) d\left(-s(1 - F(s))\right).$$

With integration by parts,

$$\int_0^1 Q_i(s) d\left(-s(1 - F(s))\right) = \int_0^1 s(1 - F(s)) dQ_i(s).$$

Since $s(1 - F(s)) > 0$ for all $s \in (0,1)$, and Q_i weakly increasing, the above integral is nonnegative for all i. Hence (17) is satisfied. Furthermore, the above integral is strictly positive unless Q_i is constant a.e. on $[0,1]$. Thus, the proof is complete if it is impossible to have a solution $(Q_i)_{i=1}^n$ to problem (16) such that, for each i, Q_i is equal to a nonnegative constant a.e. on $[0,1]$. To that end, let Q be such an allocation: for each i, $Q_i = a_i$ a.e. on $[0,1]$ for some $a_i \in [0,1]$. Then, by (3),

$$\begin{aligned}\sum_i a_i &= \sum_i \int_0^1 \int_{[0,1]^{n-1}} \left(q_{iA}(t_i,t_{-i}) - q_{iB}(t_i,t_{-i})\right) dF_{-i}(t_{-i}) dF(t_i) \\ &\leq \sum_i \int_0^1 \int_{[0,1]^{n-1}} q_{iA}(t_i,t_{-i}) dF_{-i}(t_{-i}) dF(t_i) \\ &= \int_{[0,1]^n} \sum_i q_{iA}(t_1,\ldots,t_n)\, dF(t_1)\cdots dF(t_n) \\ &\leq 1.\end{aligned}$$

Given this allocation, the objective in problem (16) is equal to

$$\sum_i a_i \int_0^1 V(t_i)dF(t_i) = \left(\sum_i a_i\right) \int_0^1 V(t_i)dF(t_i) \leq \int_0^1 V(s)dF(s) = \int_0^1 \overline{V}(s)dF(s), \quad (43)$$

with the last "=" due to the definition of ironing, (24)—(25). By contrast, consider the allocation that never allocates the bad and allocates the good hierarchically according to the ironed copy $\overline{V}$ of V (cf. Section 4.1). Never allocating the bad, this allocation satisfies (17) by the previous reasoning; since $\overline{V}$ is weakly increasing by definition, this allocation belongs to $\mathscr{Q}_{\text{mon}}$. Thus the allocation is feasible. Furthermore, given this allocation, which chooses the largest nonnegative $\overline{V}(t_i)$ almost surely, the objective in problem (16) is equal to

$$\int_{[0,1]^n} \left(\max_{i=1,\ldots,n} \overline{V}(t_i)^+ \right) dF(t_1)\cdots dF(t_n),$$

which is larger than (43). Thus $(Q_i)_{i=1}^n = (a_i)_{i=1}^n$ (a.e.) cannot be a solution to (16). ∎

E Details of Theorem 3

E.1 Proof of Lemma 4

Claim (a): Since $\mathscr{Q}$ is convex, the proof of Theorem 1 remains valid when $\mathscr{Q}_{\text{mon}}$ is replaced by $\mathscr{Q}$, so the saddle point characterization applies to any solution to problem (34).

Claim (b.i): Plug $\lambda = 0$ into (14) to see that the $\mathscr{L}(Q,0)$ is equal to $\sum_i \int_0^1 Q_i(s)V(s)dF(s)$, a linear functional on the convex domain $\mathscr{Q}$. Thus, maximization of $\mathscr{L}(\cdot,0)$ on $\mathscr{Q}$ is a linear programming, hence any solution Q^* thereof entails a hierarchical allocation of the good according to V, and a hierarchical allocation of the bad according to $-V$. Thus, for almost all t_i, t_i', $V(t_i) > V(t_i') \Rightarrow Q_i^*(t_i) > Q_i^*(t_i')$.

Claim (b.ii): Suppose, to the contrary, that $Q_i^*(s)Q_j^*(s) < 0$ for all s on a positive-measure subset of $[0,1]$. Let $Q' \in \mathscr{Q}$ be the same as Q^* except that $Q_i' = Q_j^*$ and $Q_j' = Q_i^*$. Then (42) holds. By $\lambda > 0$ and (14), it follows that, for any $\alpha \in (0,1)$,

$$\mathscr{L}\left(\alpha Q^* + (1-\alpha)Q', \lambda\right) > \alpha \mathscr{L}(Q^*, \lambda) + (1-\alpha)\mathscr{L}(Q', \lambda) = \mathscr{L}(Q^*, \lambda),$$

where the equality comes from the fact that the permutation "$Q_i' = Q_j^*$ and $Q_j' = Q_i^*$" renders $\mathscr{L}(Q', \lambda) = \mathscr{L}(Q^*, \lambda)$ since the players' types are symmetrically distributed by F and weighed by the same W. Thus, Q^* does not maximize $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$, contradiction.

Claim (b.iii): Suppose, to the contrary, that $V_-^\lambda(x) > 0$ for some $x \in (0, c_i)$, with V_-^λ continuous due to (19), there is an interval $(a,b) \subseteq (0, c_i)$, with $a < x < b$, on which $V_-^\lambda > 0$. Perturb Q^* by the reservation operator $R_{i,T}$ such that $T = \{(t_k)_{k=1}^n \in [0,1]^n \mid t_i \in (a,b)\}$, namely, reserve both items from player i when i's type belongs to (a,b). By Lemma 3, $R_{i,T}(Q^*) \in \mathscr{Q}(Q^*, c)$ and we can apply (26) to both Q^* and $R_{i,T}(Q^*)$ to obtain

$$\mathscr{L}(Q^*, \lambda) - \mathscr{L}(R_{i,T}(Q^*), \lambda) = \int_a^b Q_i^*(t_i) V_-^\lambda(t_i) dF(t_i) < 0,$$

where the strictly inequality follows from $Q_i^* < 0$ on $[0, c_i) \supset (a,b)$ and $V_-^\lambda > 0$ on (a,b). Thus we have another allocation in $\mathscr{Q}$ that generates larger Lagrangian than Q^* given λ, contradicting the hypothesis that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$. ∎

E.2 Lemma 9 and 10: The Perturbation Method

We need to introduce two additional kinds of sign-preserving perturbations:

Reservation $R_{i,T}^A$ of the good: This is the same as $R_{i,T}$ (Section 4.2.1) except that the only modification of the original ex post allocation $(q_{kA}, q_{kB})_{k=1}^n$ is to set $\tilde{q}_{iA}(t) := 0$ for any $t \in T$ without altering the allocation of item B. That is, when T occurs, the planner keeps the good to herself if the original allocation would award the good to i.

Merge $M_{i,k,T}$: For any two distinct players i and k, any $T \subseteq [0,1]^n$, and for any $Q \in \mathscr{Q}$ that is the reduced form of an ex post allocation $(q_{lA}, q_{lB})_{l=1}^n$, define $M_{i,k,T}(Q)$ to be the reduced form of the ex post allocation $(\tilde{q}_{lA}, \tilde{q}_{lB})_{l=1}^n$ that is the same as $(q_{lA}, q_{lB})_{l=1}^n$

except that, for any $t \in T$, $\tilde{q}_{iB}(t) := q_{kB}(t) + q_{iB}(t)$ and $\tilde{q}_{kB}(t) := 0$. Namely, when T occurs, award the bad to player i whenever it is originally allocated to players i or k.

Recall that π_i denotes the projection from $\mathbb{R}^n$ onto the i^{th} dimension. The next lemma follows directly from the definitions of $R^A_{i,T}$ and $M_{i,k,T}$ and hence we omit its proof.

Lemma 8 *For any $Q \in \mathscr{Q}$ with crossing point $c := (c_l)_{l=1}^n \in [0,1]^n$, any players i and k with $i \neq k$, and any $T \subseteq [0,1]^n$ for which $\pi_i(T) \subseteq [0, c_i]$, denote $Q' := R^A_{i,T}(Q)$ and $Q'' := R^A_{k,T}(M_{i,k,T}(Q))$. Then:*

a. $Q' \in \mathscr{Q}(Q, c)$, $Q'_i = Q_i$ on $[0,1] \setminus \pi_i(T)$, and $Q'_k = Q_k$ on $[0,1]$ for all $k \neq i$;

b. if $\pi_k(T) \subseteq [0, c_k]$, then $Q'' \in \mathscr{Q}(Q, c)$, $Q''_i = Q_i$ on $[0,1] \setminus \pi_i(T)$, $Q''_k = Q_k$ on $[0,1] \setminus \pi_k(T)$, and $Q''_l = Q_l$ on $[0,1]$ for all $l \notin \{i, k\}$.

Recall from Section 4.2.1 the definitions of crossing point and $\mathscr{Q}(Q^*, c)$. As explained around (35), any solution to the relaxed problem (34) is also a solution to (36). The next two lemmas characterize any solution to (36), where the $-V^\lambda_-$ corresponds to the g in the lemmas. Since the objective in the problem (36) is a linear functional of the reduced form Q when Q ranges in $\mathscr{Q}(Q^*, c)$, the intuition of these lemmas is to perturb Q toward the direction of $-V^\lambda_-$ (or that of g here). However, the perturbation needs to be ex post feasible—obeying (3)—and remain within $\mathscr{Q}(Q^*, c)$. The key of the proof is to guarantee such conditions with the sign-preserving perturbations defined earlier.

Lemma 9 *For any integrable $g : [0,1] \to \mathbb{R}$, any $(c_i)_{i=1}^n \in [0,1]^n$ and any $Q^* \in \mathscr{Q}$, denote*

$$E_i := \{t_i \in [0, c_i] \mid g(t_i) > 0\}$$

for each player i, and suppose:

i. $c := (c_i)_{i=1}^n$ is a crossing point of Q^*, and

ii. Q^* maximizes $\sum_i \int_0^{c_i} Q_i^-(s) g(s) dF(s)$ among all $(Q_i)_{i=1}^n \in \mathscr{Q}(Q^*, c)$.

Then each of the following sets is of zero measure for any players i and k with $i \neq k$ (where $(q^*_{lA}, q^*_{lB})_{l=1}^n$ denotes the ex post allocation the reduced form of which is Q^*):

a. $T_i := \{(t_i, t_{-i}) \in E_i \times [0,1]^{n-1} \mid q^*_{iA}(t_i, t_{-i}) > 0\}$;

b. $Z_i := \{(t_i, t_{-i}) \in E_i \times [0,1]^{n-1} \mid \sum_{k=1}^n q^*_{kB}(t_i, t_{-i}) < 1\}$;

c. $O_{ik} := \left\{(t_i, t_k, t_{-(i,k)}) \in E_i \times (c_k, 1] \times [0,1]^{n-2} \mid q^*_{kB}(t_i, t_k, t_{-(i,k)}) > 0\right\}$;

d. $S_{ik} := \left\{(t_i, t_k, t_{-(i,k)}) \in E_i \times E_k \times [0,1]^{n-2} \mid g(t_i) > g(t_k), q^*_{kB}(t_i, t_k, t_{-(i,k)}) > 0\right\}$.

Proof Category (a): Suppose that T_i for some i is of positive measure. Then perturb Q^* by the reservation operator R^A_{i,T_i} of the good. Denote $Q := R^A_{i,T_i}(Q^*)$. By definition of R^A_{i,T_i}, Q is the same as Q^* except that, for all $t_i \in \pi_i(T_i)$,

$$Q_i(t_i) \overset{(3)}{=} \int_{[0,1]^{n-1}} (-q^*_{iB}(t_i, t_{-i}))\, dF_{-i}(t_{-i}) < \int_{[0,1]^{n-1}} (q^*_{iA}(t_i, t_{-i}) - q^*_{iB}(t_i, t_{-i}))\, dF_{-i}(t_{-i}) \overset{(3)}{=} Q^*_i(t_i).$$

Since $\pi_i(T_i) \subseteq E_i \subseteq [0, c_i]$ by definition, Q also has c as a crossing point, namely, $Q \in \mathscr{Q}(Q^*, c)$. But

$$\sum_k \int_0^{c_k} Q^-_k(s) g(s) dF(s) - \sum_k \int_0^{c_k} (Q^*_k(s))^- g(s) dF(s) = \int_{\pi_i(T_i)} (Q^*_i(s) - Q_i(s))\, g(s) dF(s)$$

is positive because the measure of $\pi_i(T_i)$ is positive and because $g > 0$ and $Q_i < Q^*_i$ on $\pi(T_i)$. This, with $Q \in \mathscr{Q}(Q^*, c)$, contradicts hypothesis (ii).

Category (b): Suppose that Z_i for some i has a positive measure. Let $(q_{kA}, q_{kB})_{k=1}^n$ be the ex post allocation that is the same as $(q^*_{kA}, q^*_{kB})_{k=1}^n$ except that

$$q_{iB}(t) := q^*_{iB}(t) + 1 - \sum_{k=1}^n q^*_{kB}(t)$$

for all $t \in Z_i$. That is, if Z_i occurs, allocate the bad to player i if the original allocation would reserve it from all players. Denote Q for the reduced form of $(q_{kA}, q_{kB})_{k=1}^n$. Clearly Q is the same as Q^* except that (by (3)) $Q_i < Q^*_i$ on $\pi_i(Z_i)$. As in Category (a), $Q \in \mathscr{Q}(Q^*, c)$ and $\sum_k \int_0^{c_k} Q^-_k(s) g(s) dF(s) > \sum_k \int_0^{c_k} (Q^*_k(s))^- g(s) dF(s)$, again a contradiction to (ii).

Category (c): Suppose that O_{ik} is of positive measure for some $i \neq k$. Let $t_k \in \pi_k(O_{ik})$. By the supposition,

$$\int_{[0,1]^{n-1}} q^*_{kB}(t_k, t_{-k}) dF_{-k}(t_{-k}) \geq \int_{\pi_{-k}(O_{ik})} q^*_{kB}(t_k, t_{-k}) dF_{-k}(t_{-k}) > 0,$$

where π_{-k} denotes the projection $(t_l)_{l=1}^n \mapsto (t_l)_{l \neq k}$. By definition of O_{ik}, $t_k \in (c_k, 1]$. This coupled with c being a crossing point of Q^* implies that $Q^*_k(t_k) \geq 0$. Thus, by (3),

$$\int_{[0,1]^{n-1}} q^*_{kA}(t_k, t_{-k}) dF_{-k}(t_{-k}) \geq \int_{[0,1]^{n-1}} q^*_{kB}(t_k, t_{-k}) dF_{-k}(t_{-k}) \geq \int_{\pi_{-k}(O_{ik})} q^*_{kB}(t_k, t_{-k}) dF_{-k}(t_{-k}).$$

Thus there exists $O^*(t_k) \subseteq \{t_{-k} \in [0,1]^{n-1} \mid q^*_{kA}(t_k, t_{-k}) > 0\}$ such that

$$\int_{O^*(t_k)} q^*_{kA}(t_k, t_{-k}) dF_{-k}(t_{-k}) = \int_{\pi_{-k}(O_{ik})} q^*_{kB}(t_k, t_{-k}) dF_{-k}(t_{-k}). \tag{44}$$

Denote

$$O := \bigcup_{t_k \in \pi_k(O_{ik})} \left(\{t_k\} \times O^*(t_k)\right).$$

Now perturb Q^* by the merge operator $M_{i,k,O_{ik}}$ together with the reservation operator $R^A_{k,O}$ of item A. That is, allocate the bad to player i instead of k in the event of O_{ik}, and reserve the good from player k in the event of O. Denote $Q := R^A_{k,O}\left(M_{i,k,O_{ik}}(Q^*)\right)$. By (44), the perturbation leaves $Q^*_k(t_k)$ unchanged for every $t_k \in \pi_k(O_{ik})$, namely, $Q_k = Q^*_k$ on $\pi_k(O_{ik})$. Thus, $Q_l = Q^*_l$ on $(c_l, 1]$ for all players l, $Q_l = Q^*_l$ on $[0, c_l]$ for all $l \neq i$, $Q_i = Q^*_i$ on $[0, c_i] \setminus \pi_i(O_{ik})$, and $Q_i < Q^*_i$ on $\pi_i(O_{ik})$. Hence $Q \in \mathscr{Q}(Q^*, c)$. But

$$\sum_l \int_0^{c_l} Q^-_l(s) g(s) dF(s) - \sum_l \int_0^{c_l} \left(Q^*_l(s)\right)^- g(s) dF(s) = \int_{\pi_i(O_{ik})} \left(Q^*_i(s) - Q_i(s)\right) g(s) dF(s)$$

is positive, as in Category (a). Again we have a desired contradiction to (ii).

Category (d): Suppose that S_{ik} for some $k \neq i$ is of positive measure. Perturb Q^* by the merge operator $M_{i,k,S_{ik}}$ together with the reservation operator $R^A_{k,S_{ik}}$ of the good. Denote Q for the outcome of the perturbation, i.e., $Q := R^A_{k,S_{ik}}\left(M_{i,k,S_{ik}}(Q^*)\right)$. That is, in the event S_{ik}, assign the bad to player i whenever it is originally allocated to players i or k, and reserve the good from player k. Let $(q_{lA}, q_{lB})^n_{l=1}$ be the ex post allocation the reduced form of which is Q. By definition of the perturbation, $q_{iB}(t) = q^*_{iB}(t) + q^*_{kB}(t)$ and $q_{kB}(t) = q_{kA}(t) = 0$ for all $t \in S_{ik}$. Since $\pi_i(S_{ik}) \subseteq E_i \subseteq [0, c_i]$ and $\pi_k(S_{ik}) \subseteq E_k \subseteq [0, c_k]$, Lemma 8.b implies that Q also has c as a crossing point, namely, $Q \in \mathscr{Q}(Q^*, c)$. By the definitions of $R^A_{k,S_{ik}}$ and $M_{i,k,S_{ik}}$ and the notations $F^n(t) := F(t_1) \cdots F(t_n)$, $q^*_l(t) := q^*_{lA}(t) - q^*_{lB}(t)$ and $q_l(t) := q_{lA}(t) - q_{lB}(t)$,

$$\begin{aligned}
& \sum_l \int_0^{c_l} Q^-_l(s) g(s) dF(s) - \sum_l \int_0^{c_l} \left(Q^*_l(s)\right)^- g(s) dF(s) \\
= \ & \sum_l \int_0^{c_l} \int_{[0,1]^{n-1}} \left(-q_l(t_l, t_{-l}) + q^*_l(t_l, t_{-l})\right) g(t_l) dF_{-l}(t_{-l}) dF(t_l) \\
= \ & \int_{S_{ik}} \left(\left(-q_i(t) + q^*_i(t)\right) g(t_i) + \left(-q_k(t) + q^*_k(t)\right) g(t_k)\right) dF^n(t) \\
= \ & \int_{S_{ik}} \left(q^*_{kB}(t) g(t_i) + \left(q^*_{kA}(t) - q^*_{kB}(t)\right) g(t_k)\right) dF^n(t) \\
= \ & \int_{S_{ik}} \left(q^*_{kB}(t) \left(g(t_i) - g(t_k)\right) + q^*_{kA}(t) g(t_k)\right) dF^n(t) \\
> \ & 0,
\end{aligned}$$

where the first equality is due to the definition of Q_l^- (and $(Q_l^*)^-$) and (3), the second and third equalities are due to the definitions of $R^A_{k,S_{ik}}$ and $M_{i,k,S_{ik}}$, and the inequality due to the fact that $g(t_i) > g(t_k) > 0$ on S_{ik} and the hypothesis that S_{ik} is of positive F^n-measure. Again we obtain a contradiction to the hypothesis (ii), as desired. ■

Lemma 10 *For any integrable $g : [0, 1] \to \mathbb{R}$, any $(c_i)_{i=1}^n \in [0, 1]^n$ and any $Q^* \in \mathscr{Q}$, denote*

$$E := \left\{ s \in \left[0, \min_{i=1,\dots,n} c_i\right] \;\middle|\; g(s) > 0 \right\}$$

and suppose:

i. $c := (c_i)_{i=1}^n$ is a crossing point of Q^*, and

ii. Q^* maximizes $\sum_i \int_0^{c_i} Q_i^-(s)g(s)dF(s)$ among all $(Q_i)_{i=1}^n \in \mathscr{Q}(Q^*, c)$.

Then for any player i and almost every $t_i, t_i' \in E$, $g(t_i) > g(t_i') \Rightarrow Q_i^*(t_i) < Q_i^*(t_i')$.

Proof For any player i and any $t_i \in E$, define:

$$\mathscr{B}_i(t_i, \succ) := \left\{ (t_k)_{k\neq i} \in \prod_{k\neq i}[0, c_k] \;\middle|\; g(t_i) > \max\left\{0, \max_{k\neq i} g(t_k)\right\} \right\},$$

$$\mathscr{B}_i(t_i, \sim) := \left\{ (t_k)_{k\neq i} \in \prod_{k\neq i}[0, c_k] \;\middle|\; g(t_i) = \max\left\{0, \max_{k\neq i} g(t_k)\right\} \right\},$$

and $\mathscr{B}_i(t_i) := \mathscr{B}_i(t_i, \succ) \cup \mathscr{B}_i(t_i, \sim)$. We have

$$\begin{aligned} Q_i^*(t_i) &= -\int_{[0,1]^{n-1}} q^*_{iB}(t_i, t_{-i})dF_{-i}(t_{-i}) \\ &= -\int_{[0,c_*]^{n-1}} q^*_{iB}(t_i, t_{-i})dF_{-i}(t_{-i}) \\ &= -\int_{\mathscr{B}_i(t_i)} q^*_{iB}(t_i, t_{-i})dF_{-i}(t_{-i}), \end{aligned}$$

with the first line due to (3) and Lemma 9.a, the second line due to Lemma 9.c, and the third line Lemma 9.d. Also observe that, for almost every $t_i \in E$,

$$q^*_{iB}(t_i, \cdot) = 1 \quad \text{a.e. on} \quad \mathscr{B}_i(t_i, \succ). \tag{45}$$

That is because $q^*_{iB}(t_i, t_{-i}) = \sum_{k=1}^n q^*_{kB}(t_i, t_{-i})$ for almost all $t_{-i} \in \mathscr{B}_i(t_i, \succ)$ by Lemma 9.d, and $\sum_{k=1}^n q^*_{kB}(t) = 1$ for almost all $t \in \bigcup_{t_i\in E} (\{t_i\} \times \mathscr{B}_i(t_i, \succ))$ by Lemma 9.b. Thus, for almost every $t_i, t_i' \in E$ such that $g(t_i) > g(t_i')$, which means $\mathscr{B}_i(t_i') \subsetneq \mathscr{B}(t_i, \succ)$ and, because F has no gap in $[0,1]$, $\mathscr{B}(t_i, \succ) \setminus \mathscr{B}_i(t_i')$ is of positive measure, we have

$$\begin{aligned}
\int_{\mathscr{B}_i(t_i)} q^*_{iB}(t_i, t_{-i}) dF_{-i}(t_{-i}) &\geq \int_{\mathscr{B}_i(t_i, \succ)} q^*_{iB}(t_i, t_{-i}) dF_{-i}(t_{-i}) \\
&= \int_{\mathscr{B}_i(t_i, \succ)} dF_{-i}(t_{-i}) \\
&> \int_{\mathscr{B}_i(t_i')} dF_{-i}(t_{-i}) \\
&\geq \int_{\mathscr{B}_i(t_i')} q^*_{iB}(t_i', t_{-i}) dF_{-i}(t_{-i}),
\end{aligned}$$

with the second line due to (45), and the third line due to $\mathscr{B}(t_i, \succ) \setminus \mathscr{B}_i(t_i')$ having positive measure. Thus, $Q^*_i(t_i) < Q^*_i(t_i')$, as asserted. ■

E.3 Lemma 11: The Indeterminacy Case of the Relaxed Problem

Lemma 11 *For any $\lambda > 0$, $c_* \in (0,1)$ and $Q^* \in \mathscr{Q}_{\text{mon}}$, if (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$, $Q^*_i < 0$ on a positive-measure subset of $[0, c_*]$ for some i, and $c := (c_i)_{i=1}^n$ with $c_i := c_*$ for all i is a crossing point of Q^*, then*

$$\left\{ Q \in \mathscr{Q}(Q^*, c) \;\middle|\; \sum_i \int_0^{c_*} Q_i^-(s) \left(sf(s) + F(s)\right) ds = z \right\} \tag{46}$$

contains a continuum.

Proof Since $\lambda > 0$ and (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$, the constraint (17) in the relaxed problem (34) is binding for Q^*. This, combined with (37) and the hypothesis that c is a crossing point of Q^*, means

$$\sum_i \int_0^{c_*} \left(Q_i^*(s)\right)^- \left(sf(s) + F(s)\right) ds = z. \tag{47}$$

Note that $\mathscr{Q}(Q^*, c)$ is a convex set and the mapping

$$\phi : Q \longmapsto \sum_i \int_0^{c_*} Q_i^-(s) \left(sf(s) + F(s)\right) ds$$

is linear on $\mathscr{Q}(Q^*, c)$. Denote $\mathbf{0}$ for the element of $\mathscr{Q}(Q^*, c)$ that assigns zero to $Q_i(s)$ for all $s \in [0, c_*]$ and all i. Note that $\phi(\mathbf{0}) = 0$.

First we claim $z > \min_{\mathscr{Q}(Q^*,c)} \phi$. By hypothesis, $(Q_i^*)^- > 0$ on a positive-measure subset of $[0, c_*]$ for some i, it follows from (47) that $z > 0$. Thus the claim follows from the fact that $0 = \phi(\mathbf{0})$ and $\mathbf{0} \in \mathscr{Q}(Q^*, c)$.

Second, we claim $z < \max_{\mathscr{Q}(Q^*,c)} \phi$. Otherwise, $z = \max_{\mathscr{Q}(Q^*,c)} \phi$. Then by (47) Q^* maximizes $\sum_i \int_0^{c_*} Q_i^-(s)\,(sf(s) + F(s))\,ds$ among all $Q \in \mathscr{Q}(Q^*, c)$. Let $g(s) := sf(s)+F(s)$ for all $s \in [0,1]$ and apply Lemma 10. Note that the set E in Lemma 10 is $(0, c_*]$ here. Thus, for every player i and almost every $s, s' \in (0, c_*]$, $sf(s) + F(s) > s'f(s') + F(s') \Rightarrow Q_i^*(s) < Q_i^*(s')$. Note that $sf(s) + F(s)$ is equal to 0 when $s = 0$ and strictly positive when $s > 0$. Thus, by differentiability of f, there exists an interval $I \subseteq [0, c_*]$ on which $sf(s) + F(s)$ is strictly increasing in s. Hence $Q_i^*(s)$ is a strictly decreasing function of s almost everywhere on I. But then $Q^* \notin \mathscr{Q}_{\text{mon}}$, contradiction. Thus $z < \max_{\mathscr{Q}(Q^*,c)} \phi$.

Third, there exist at least two distinct elements of the set (46). Let $\overline{Q}$ be a maximizer of ϕ on $\mathscr{Q}(Q^*, c)$. Thus,

$$0 = \phi(\mathbf{0}) < z = \phi(Q^*) < \phi(\overline{Q}).$$

Since $sf(s) + F(s) > 0$ for all $s \in (0, 1]$, $\left(\overline{Q}_i\right)^- > 0$ a.e. on $(0, c_*]$ for all i. For any $\theta \in [0, c_*]$ and for any player i, let $T_i^\theta := \{(t_k)_{k=1}^n \in [0,1]^n \mid t_i \in [0, \theta]\}$ and perturb $\overline{Q}$ iteratively by $R_{1,T_1^\theta}, R_{2,T_2^\theta}, \ldots, R_{n,T_n^\theta}$. That is, let

$$Q^{>\theta} := R_{n,T_n^\theta}\left(\cdots\left(R_{2,T_2^\theta}\left(R_{1,T_1^\theta}\left(Q^*\right)\right)\right)\cdots\right),$$

which results from modifying Q^* by reserving both items from any player whose type is in $[0, \theta]$. Then $Q^{>0} = \overline{Q}$, $Q^{>c_*} = \mathbf{0}$, and

$$\phi(Q^{>\theta}) = \textstyle\sum_i \int_\theta^{c_*} \left(\overline{Q}_i(s)\right)^- (sf(s) + F(s))\,ds,$$
$$0 = \phi(Q^{>c_*}) < z = \phi(Q^*) < \phi(Q^{>0}). \tag{48}$$

By continuity of the integration operator, there exists a $\theta_* \in [0, c_*]$ for which $\phi(Q^{>\theta_*}) = \phi(Q^*)$. Furthermore, for any i, since $\left(\overline{Q}_i\right)^- > 0$ a.e. on $(0, c_*]$, Ineq. (48) implies $0 < \theta_* < c_*$. Note, for any i, that $\left(Q_i^{>\theta_*}\right)^- = 0$ on $[0, \theta_*]$ and $\left(Q_i^{>\theta_*}\right)^- > 0$ on $(\theta_*, c_*]$.

Analogously, for any $\tau \in [0, c_*]$ and any i, perturb $\overline{Q}$ iteratively by the reservation operators that reserve both items from any player whose type belongs to $[\tau, c_*]$. By the same reasoning as above, there exists a $\tau_* \in (0, c_*)$ and a $Q^{<\tau_*} \in \mathscr{Q}(Q^*, c)$ for which $\phi(Q^{<\tau_*}) = \phi(Q^*)$ and, for all i, $(Q_i^{<\tau_*})^- > 0$ on $[0, \tau_*)$ and $(Q_i^{<\tau_*})^- = 0$ on $[\tau_*, c_*]$.

Finally, note that $Q^{>\theta_*}$ and $Q^{<\tau_*}$ are two distinct elements of the set (46). Since the set is convex, any convex combination between the two elements also belongs to the set. Thus, the set (46) contains a continuum of elements, as asserted. ∎

E.4 Proof of Corollary 4

Given any (f, w) such that the bad is needed, according to the proof of Theorem 3, the only case where the relaxed problem (34) admits an optimal mechanism as a solution is $V_-^\lambda = 0$ on $(0, c_*)$ for some $c_* \in (0, 1)$ and some $\lambda > 0$. By (19), that means $\frac{W(s)}{F(s)+sf(s)} = 1 + \lambda$ for all $s \in (0, c_*)$, with W derived from w by (8). Thus, this case means

$$\exists c_* \in (0,1) : \forall s \in (0, c_*) : \frac{d}{ds} \ln \left(\frac{W(s)}{F(s) + sf(s)} \right) = 0. \tag{49}$$

This condition can be violated with slight perturbations of f or w at points near 0. Thus one can formalize the space of (f, w) such that the contrary of (49) is generic. ∎

F Generalization to ex ante Asymmetric Players

Here we sketch how to generalize the saddle point characterization (Theorem 1), and briefly indicate generalization of the other two theorems, to the asymmetric-player model: each player i's type is independently drawn according to a commonly known, possibly player-specific, cdf F_i with density f_i positive on its support $[0, 1]$; and player i is weighed in the social welfare function according to a possibly player-specific *welfare distribution* $W_i : \mathbb{R} \to \mathbb{R}_+$, which is a weakly increasing function generated by a Radon measure, such that the social welfare from a mechanism (Q, P) is equal to

$$\sum_i \int_0^1 U_i(t_i \mid Q, P) dW_i(t_i). \tag{50}$$

There is no loss of generality to assume (50) as the social welfare, because any interim Pareto optimal mechanism in this environment is a maximizer of (50) subject to IC, IR and BB, for some profile $(W_i)_{i=1}^n$ of distribution functions across players.[17]

[17] The proof is in Zheng [24], available upon request.

Given the general model, it is easy to generalize (9) to

$$\begin{aligned}\int_0^1 P_i(t_i)dW_i(t_i) &= \int_0^1 Q_i(t_i)t_i dW_i(t_i) + \int_0^{t_i^0} Q_i(t_i)W_i(t_i)dt_i - \int_{t_i^0}^1 Q_i(t_i)\left(W_i(1) - W_i(t_i)\right)dt_i \\ &\quad -W_i(1)U_i(t_i^0|Q,P),\end{aligned}$$

where $W_i(1)$ need not be equal to one for all i, because the welfare distributions $(W_i)_{i=1}^n$ may assign different average weights to different players. It is also easy to generalize (17) to

$$\sum_i \int_0^1 Q_i(t_i)t_i\, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)| \geq 0. \tag{51}$$

The first nontrivial difference due to the generalization is that the optimal social welfare (16) becomes the following nonlinear functional of the allocation Q:

$$\sum_i \int_0^1 Q_i(t_i)t_i\, d\left(\omega F_i(t_i)\right) + \sum_i \langle Q_i : \rho(\omega F_i) - \rho(W_i)|\,, \tag{52}$$

where $\omega := \max_i W_i(1)$.

First,. we observe nonlinearity of (52). By the definition of two-part operators, (52) is linear in Q if and only if $\langle Q_i : \rho(\omega F_i) - \rho(W_i)|$ is linear in Q_i for each i, and the latter is linear if and only if $\rho_+(\omega F_i) - \rho_+(W_i) = \rho_-(\omega F_i) - \rho_-(W_i)$. By (11), that means $W_i(1) = 1$ for all i, which is not necessarily true when W_i is a *distribution* but not a cdf.

Second, we explain why (52) is true for the generalization of Lemma 6. Mimicking the proof of Lemma 6, one readily sees that the social welfare (50) generated by any IC mechanism (Q, P) is equal to

$$\sum_i W_i(1) \min_{[0,1]} U_i(\cdot|Q,P) - \sum_i \langle Q_i : \rho(W_i)|\,, \tag{53}$$

and BB implies

$$\begin{aligned}\sum_i W_i(1) \min_{[0,1]} U_i(\cdot|Q,P) &\leq \left(\max_i W_i(1)\right) \sum_i \min_{[0,1]} U_i(\cdot|Q,P) \\ &\leq \left(\max_i W_i(1)\right) \left(\sum_i \int_0^1 Q_i(t_i)t_i\, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|\right).\end{aligned}$$

By the reason analogous to the proof of Lemma 6, the above-displayed weak inequality holds as equality when P is optimally chosen among those that implement Q: Let $(P_i^Q)_{i=1}^n$ be the payment rule that implements Q with $\min_{[0,1]} U_i(\cdot|Q,P) = 0$ for all i. The ex ante expected revenue generated by $(P_i^Q)_{i=1}^n$ is equal to $\sum_i \int_0^1 Q_i(t_i)t_i\, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|$. Thus, combining $(P_i^Q)_{i=1}^n$ with distributing $\sum_i \int_0^1 Q_i(t_i)t_i\, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|$ to any member of $\arg\max_i W_i(1)$ as lump sums, we obtain a payment rule with which

$$\sum_i W_i(1) \min_{[0,1]} U_i(\cdot|Q,P) = \left(\max_i W_i(1)\right)\left(\sum_i \int_0^1 Q_i(t_i)t_i\, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|\right).$$

Plug this equation into (53) and set $\omega := \max_i W_i(1)$ to get (52).

Thus, due to asymmetric welfare weights across players, the lump sum transfer in an optimal mechanism is not rebated to players indiscriminately, but rather distributed only to those players whose ex ante expected welfare weights, $W_i(1)$, are largest among all.

Based on the reasoning sketched above, any interim Pareto optimal mechanism is a solution of maximizing (52) among $Q \in \mathscr{Q}_{\text{mon}}$ subject to (51). As in the proof of Theorem 1, the set of $Q \in \mathscr{Q}_{\text{mon}}$ subject to (51) is convex and contains an interior point. The only difference from that proof is that the objective (52) is nonlinear in general. However, the objective (52) one can prove is a concave functional on $\mathscr{Q}_{\text{mon}}$, hence the conditions corresponding to those in Luenberger [15, Corollary 1, p219] are met, and so the saddle point condition is necessary and sufficient for any solution to this constrained optimization problem.

To prove concavity of the objective (52), it suffices to prove that $\langle Q_i : \rho(\omega F_i) - \rho(W_i)|$ is a concave functional of Q_i for each i. The proof is similar to that of Lemma 7. By the definition of two-part operators, we need only to show $\rho_+(\omega F_i) - \rho_+(W_i) \leq \rho_-(\omega F_i) - \rho_-(W_i)$ on $[0,1]$ for all i: for any $t_i \in [0,1]$, by (11),

$$\begin{aligned}
\left(\rho_+(\omega F_i)\right)(t_i) - \left(\rho_+(W_i)\right)(t_i) &= \omega\left(-1 + F_i(t_i)\right) - \left(-W_i(1) + W_i(t_i)\right) \\
&= \omega F_i(t_i) - W_i(t_i) - \left(\omega - W_i(1)\right) \\
&\leq \omega F_i(t_i) - W_i(t_i) \\
&= \omega\left(\rho_-(F_i)\right)(t_i) - \left(\rho_-(W_i)\right)(t_i),
\end{aligned}$$

with the inequality due to $\omega = \max_i W_i(1)$.

In sum, in the general asymmetric model, any interim Pareto optimal mechanism is a solution of maximizing (52) among $Q \in \mathscr{Q}_{\text{mon}}$ subject to (51), and hence satisfies the saddle point condition with respect to the Lagrangian

$$\begin{aligned}\mathscr{L}(Q,\lambda) &:= \sum_i \int_0^1 Q_i(t_i)t_i \, d\left(\omega F_i(t_i)\right) + \sum_i \langle Q_i : \rho(\omega F_i) - \rho(W_i)| \\ &\quad + \lambda \left(\sum_i \int_0^1 Q_i(t_i)t_i \, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)| \right) \\ &= \sum_i \int_0^1 Q_i(t_i)t_i \, d\left((\omega+\lambda)F_i(t_i)\right) + \sum_i \langle Q_i : \rho\left((\omega+\lambda)F_i\right) - \rho(W_i)| \,, \quad (54)\end{aligned}$$

defined for all $Q \in \mathscr{Q}$ and all $\lambda \in \mathbb{R}_+$.

Remark 5 This saddle point characterization is also a necessary and sufficient condition for any interim Pareto optimal mechanism in the partnership dissolution IPV environment where player i's initial share is θ_i. To see that, interpret the $x_{iA} - x_{iB}$ in (1) as player i's net gain in i's share of the partnership, and hence (1) is i's *net* payoff from acquiring a net amount $x_{iA} - x_{iB}$ of shares and paying an amount y_i of money. (This payoff is net in the sense that if player i vetoes the dissolution plan then i keeps i's initial share θ_i thereby getting the payoff $\theta_i t_i$.). The only modification on the model is to define an ex post allocation as a function $(q_i)_{i=1}^n : [0,1]^n \to \prod_i [-\theta_i, 1-\theta_i]$ such that $q_i(t)$ is player i's net gain in shares given realized type profile t, with the feasibility condition (2) replaced by $\sum_i q_i(t) = 0$ to reflect the market clearance condition on the net trades of shares. This modification, however, has no effect on the saddle point characterization.

The "if" part of Theorem 2 can also be generalized. The reasoning is analogous to that in Section 4.1. For simplicity of exposition, assume that the welfare distributions W_i are all absolutely continuous in F_i with density w_i so that $W_i(1) = 1$ for all i. Suppose that the bad is not allocated at all in an optimal mechanism. Then the generalized saddle point characterization implies that $\lambda = 0$ and so the Lagrangian (54) is reduced to (52). As noted previously, the assumption $W_i(1) = 1$ for all i implies that (52) is a linear functional of Q and hence the proof of the "if" part of Theorem 2 can be easily extended. Thus, any optimal mechanism allocates the bad with a strictly positive probability if

$$\int_0^{t_i} \left(s - \frac{W_i(s) - F_i(s)}{f_i(s)} \right) dF_i(s) < 0$$

for some $t_i \in (0,1)$ and some player i. One can see that this condition is satisfied if $w_i(0) > 2$ for some player i, which is hence sufficient for the bad to be allocated sometimes given asymmetric players. Thus Corollary 2 is generalized. In the more general case where $W_i(1)$ need not be equal to one for all i, the Lagrangian (54) remains to be a nonlinear functional of Q. The argument in that case is much more involved. Nevertheless, one can obtain in that case a sufficient condition for the bad to be allocated sometimes by any optimal mechanism: $w_i(0) > 2 \max_k W_k(1)$ for some player i.

The "only if" part of Theorem 2, as well as Theorem 3, relies on conditions necessary for all—rather than only for some—optimal mechanisms. These conditions we obtain through the perturbation method presented in Appendix E.2. There, Lemmas 9 and 10 allow for reduced-form allocations whose cutoffs c_i between positive and negative domains to be different across players i. In addition, one can generalize the two lemmas so that the function g there is player-specific. Thus it is possible that both theorems are generalizable.

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