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Trilateral escalation in the dollar auction

Fredrik Ødegaard[1] · Charles Z. Zheng[2]

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Abstract

We find a new set of subgame perfect equilibria in a dollar auction that involves three active bidders. The player who falls to the third place continues making efforts to catch up until his lag from the frontrunner widens to a critical distance beyond which the catchup efforts become unprofitable. At that juncture the second-place player pauses bidding thereby bettering the chance for the third-place one to leapfrog to the front so as to perpetuate the trilateral rivalry. Once two players have emerged as the top two rivals, any such trilateral rivalry equilibrium produces larger total surplus for the three players than its bilateral rivalry counterpart does, where anyone who falls to the third place immediately drops out.

Keywords Dollar auction · Three-player bidding dynamics · Leapfrog

JEL Classification D44 · D74

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✉ Charles Z. Zheng
charles.zheng@uwo.ca
https://economics.uwo.ca/faculty/zheng/

Fredrik Ødegaard
fodegaard@ivey.uwo.ca
https://www.ivey.uwo.ca/faculty/directory/fredrik-odegaard/

[1] Ivey Business School, The University of Western Ontario, London, ON N6A 3K7, Canada

[2] Department of Economics, The University of Western Ontario, London, ON N6A 5C2, Canada

1 Introduction

This paper analyzes a three-player Shubik's (1971) dollar auction game and uncovers a cyclic dynamic pattern where the players perpetuate their trilateral rivalry through accommodating the player who falls behind. Albeit a classroom favorite for decades, the game in its general form of more than two bidders has not been analyzed before. We find a class of subgame perfect equilibria (SPE) where three active bidders escalate their competition in a cyclic pattern, wherein the third-place bidder occasionally manages to leapfrog to the front, and the second-place bidder stays put to accommodate the leapfrog. Moreover, such trilateral rivalry arises endogenously in a model where bilateral rivalry is also equilibrium-feasible, and we find that the trilateral equilibrium generates larger total surplus for all the bidders than its bilateral rivalry counterpart does.

This finding is largely due to a dynamic aspect of the dollar auction that has not been considered in previous studies of the game: Different from the conventional clock model of ascending auctions, the dollar auction allows a bidder to refrain from raising his bid while others are raising theirs and to join the race later through making a leapfrogging bid that tops the frontrunner's, as long as the race has not ended. Theoretical studies of the dollar auction, O'Neill (1986), Leininger (1989), Demange (1992), Hörner and Sahuguet (2011), and Dekel et al. (2006, 2009), all assume only two bidders. Thus leapfrogging bid is impossible because if one bidder does not top the other player's bid immediately, the other player wins right away.[1]

Another kind of dynamic auctions that has been found to also exhibit leapfrogging bidding patterns is online penny auctions, considered by Augenblick (2016), Hinnosaar (2016), Kakhbod (2013) and Ødegaard and Anderson (2014). As in the dollar auction, bidders in a penny auction incur a sunk cost for each bid increment they submit. The main difference between the two games is that the sunk cost in the dollar auction is counted as part of a bidder's eventual payment but not so in a penny auction, where the sunk bidding cost is only a fee and a winner still needs to pay the entire price for the good. Another difference is with regard to their tie-breaking rules when multiple players bid simultaneously. In both games a random frontrunner is selected from the multiple bidders, but the difference is that while all bidders incur the sunk bidding cost in online penny auctions, only the chosen frontrunner incurs the sunk cost in the dollar auction. Given these differences the bidding incentive in a penny auction is different from that in the dollar auction.

With three players, the dollar auction game still has bilateral rivalry as some of its SPEs, where the third player quits forever once the other two have emerged as the top two rivals. This paper, differently, presents a new kind of SPEs, featuring trilateral rivalry, where a player who happens to fall behind tries to leapfrog after

[1] The same applies to two-player models in the literature of all-pay auctions and wars of attrition. Although n-player models in the all-pay and war-of-attrition literature have been considered by Baye et al. (1996), Bulow and Klemperer (1999), Krishna and Morgan (1997), and Siegel (2009), they are either static games where all players simultaneously choose their one-shot bids, or stopping games where anyone who stops raising his bid quits irrevocably, without possibility of leapfrogging.

he failed to raise his bid in the past while his rivals have emerged as the top two. Since leapfrog is possible, a player's equilibrium action depends on the current distances among the players and also on whether a player who falls behind still wants to leapfrog. Since leapfrogging takes sunk costs, a player's willingness to leapfrog in turn depends not only on the endogenous value of being in the frontrunner position but also on the current distance between the player's currently committed bid and the frontrunner's bid. Thus, construction of such an equilibrium requires a recursive method more complicated than steady state dynamics. We obtain explicit characterization of the strategy profiles of these equilibria (Theorem 1). In any equilibrium of such there is an endogenous node at which the third-place bidder falls so far behind that he is indifferent between leapfrog and dropout. At that node, the second-place bidder pauses raising her bid to better the chance for the third-place bidder to leapfrog into the front. Thus, our equilibrium construction shows that it is a best response for a rival near the top to accommodate, rather than deter, the leapfrogging efforts of an underdog.

Furthermore, we compare the bidders' welfare in our trilateral rivalry equilibria with that in bilateral rivalry equilibria. In each bilateral rivalry equilibrium, once the game has started so that the highest and second-highest bidders have emerged, the subgame equilibrium takes away almost the entire surplus from the bidders. In each trilateral rivalry equilibrium constructed here, by contrast, once the top two bidders have emerged, the subgame equilibrium yields a larger expected payoff for each of the three players than the bilateral rivalry counterpart does (Theorem 3). Thus, should the top two bidders have a chance to choose which equilibrium to play thereafter, each would prefer trilateral rivalry to bilateral rivalry. That is, each would include rather than exclude the third player.

The game is presented in Sect. 2. Then Sect. 3 sets up the notation and equilibrium concept for the recursive analysis necessary to construct trilateral rivalry equilibria. To minimize the notation, we present the game and the equilibrium concept in an informal—and self-contained—manner and leave the formal presentation to Appendix A, which can be skipped without loss. Section 4 presents the main result, which explicitly derives the strategy profile for trilateral rivalry. The exposition there shows our method in this paper. Section 5 verifies that such strategy profiles do constitute SPEs. Section 6 points out their superiority over bilateral rivalry equilibria. Section 7 concludes and outlines how similar dynamic patterns are preserved in several extensions of the model, which are detailed in our Supplementary Information. Proofs are in the appendix, in the order of appearance of the corresponding claims.

2 The dollar auction

There is one indivisible good and three risk-neutral players. The value of the good, commonly known, is equal to v for every player. The good is to be auctioned off via an ascending-bid procedure with bid increment fixed at a positive constant δ such that $2\delta < v$. In the initial round, all players simultaneously choose whether to bid or stay put; if all stay put then the game ends with the good not sold, else one among those who bid is chosen randomly, with equal probability, to be the *frontrunner*, whose committed payment becomes δ, with everyone else's committed payment being zero, and the current price of the good becomes δ. Suppose that the game continues to any subsequent round, with p being the current price and b_i player i's committed payment ($b_i \leq p$ and strictly so unless i is the frontrunner), all players but the frontrunner simultaneously choose whether to bid or stay put. If all stay put then the game ends, the good is sold to the frontrunner, who pays the price p, and every other player i pays b_i; else the current price becomes $p + \delta$ and one among those who bid in this round is chosen randomly, with equal probability, to be the frontrunner, whose committed payment becomes $p + \delta$, with the committed payments of others unchanged. Then the game continues to the next round. If the game never ends, then each bidder pays the supremum of his committed payment, and the good is randomly assigned to one of those whose supremum committed payments reach infinity. (See Appendix A.1 for a formal definition of the game.)

Our assumption that price ascends in a fixed increment δ, dating back to Shubik's (1971) original formulation, idealizes the notion that a player cannot easily preempt competition through making the price jump to a level that renders rivals' entry unprofitable. The anticompetitive effect of such price jumping has been observed by Dekel et al. (2006). Our tie-breaking rule for the case of multiple bids being submitted within a round again follows Shubik's formulation. However, the finding can be partially extended to other tie-breaking rules, outlined in the Conclusion and detailed in our Supplementary Information.

3 Preliminary analysis

3.1 The state variable and equilibrium concept

Denote α for the frontrunner, whose committed payment is the current price p ($b_\alpha = p$), and β the *follower*, whose committed payment, by the rule of the game, is always just δ below the frontrunner's ($b_\beta = p - \delta$); denote γ for the *underdog*, whose committed payment b_γ is the lowest. A feature that sets the dollar auction apart from the stopping game models of war of attrition is that the active bidders' committed prices are not necessarily close to one another: $b_\alpha - b_\gamma$ is not only larger than the constant $b_\alpha - b_\beta$ but also is variable. Thus, let the state variable of the game be represented by the frontrunner-underdog lag

$$s := \left(b_\alpha - b_\gamma\right)/\delta,$$

i.e. $b_\gamma = p - s\delta$. Note that $s \geq 2$ once a third-place bidder has emerged (or once the game has entered the third round). From then on, $s \geq 2$ as long as the game continues. Consider subgame perfect equilibria in the form of

$$\left(\pi_0, \pi_1, \left(\pi_{\beta,s}, \pi_{\gamma,s}\right)_{s=2}^{\infty}\right)$$

such that π_0 is every bidder's probability of bidding at the initial round, π_1 the probability of bidding at the second round for everyone but the current α player and, for every $s \geq 2$ and each $i \in \{\beta, \gamma\}$, $\pi_{i,s}$ is the probability with which the current i player bids at state s. (See Appendix A.2 for the formal definitions of the strategy and the equilibrium concept.)

Implicit in the above formulation are two conditions that we use to restrict the set of subgame perfect equilibria. First is *symmetry* in the sense that a player's equilibrium strategy depends not on his name but rather on his relative position with respect to other players. The second one is *history-independence*, that each player's equilibrium strategy depends only on the current state ($s \in \{2, 3, \dots, \}$) when the game has entered the third round or thereafter, or on the current round ($t \in \{0, 1\}$) if it is the initial or the second round. In addition, we restrict the set of equilibria with a third condition:

$$\left[\forall s \geq s_* \,:\, \pi_{\gamma,s} = 0\right] \Rightarrow \left[\forall s, s' \geq s_* \,:\, \pi_{\beta,s} = \pi_{\beta,s'}\right]. \tag{1}$$

That is, once the third-place bidder has become inactive from now on, the other two bidders play a steady state subgame equilibrium. One may think of (1) as an independence condition of non-participants. It rules out the equilibria where the active bidders choose their actions based on their distance from the player who is no longer active.

3.2 The surplus-dissipating subgame equilibrium

Our first observation is that bilateral rivalry is always equilibrium-feasible. Consider any subgame with state variable $s \geq 2$. That is, the price p has risen to at least 2δ, with the frontrunner having committed a payment p, the follower having committed $p - \delta$, and the underdog having committed at most $p - 2\delta$.

Lemma 1 *In any subgame starting with $s \geq 2$:*

i. there is an equilibrium where, in each round, the current follower bids with probability $1 - 2\delta/v$, the current underdog stays put, the continuation value is equal to 2δ for the current frontrunner, and zero for the other two players;
ii. this is the only equilibrium where any player in the underdog position stays put forever.

This equilibrium we shall call *surplus-dissipating subgame equilibrium*, because a player in climbing up to the frontrunner position has already incurred a sunk cost no less than 2δ.

Due to Lemma 1.ii and condition (1), we consider only equilibria that, to any subgame where any player in the underdog position stays put forever, prescribe the surplus-dissipating subgame equilibrium. That provides a basis to compare one equilibrium with another.

3.3 The value functions

Let any equilibrium in the form of $\left(\pi_0, \pi_1, \left(\pi_{\beta,s}, \pi_{\gamma,s}\right)_{s=2}^{\infty}\right)$ be given. For any $s \geq 2$ and any $i \in \{\beta, \gamma\}$, denote $q_{i,s}$ for the probability with which the current i player becomes the α player in the next round. By the uniform tie-breaking rule,

$$q_{i,s} = \pi_{i,s}\left(1 - \pi_{-i,s}/2\right) \tag{2}$$

at any $s \geq 2$, with $-i$ being the element of $\{\beta, \gamma\} \backslash \{i\}$. Given this equilibrium and any state s, denote V_s for the expected payoff for the current α player, M_s the expected payoff for the current β, and L_s that for the current γ. Denote V_0 for everyone's expected payoff at the initial round, V_1 for the initial frontrunner's the expected payoff, and $M_1 = L_1$ for every non-α player's, at the second round. The law of motion is described below:

$$V_0 \longrightarrow \begin{cases} 0 & \text{prob. } \left(1 - \pi_0\right)^3 \\ V_1 - \delta & \text{prob. } \pi_0^3/3 + \pi_0^2(1 - \pi_0) + \pi_0(1 - \pi_0)^2 \\ M_1 & \text{prob. } 2\pi_0^3/3 + 2\pi_0^2(1 - \pi_0) + 2\pi_0(1 - \pi_0)^2; \end{cases} \tag{3}$$

$$V_1 \longrightarrow \begin{cases} v & \text{prob. } \left(1 - \pi_1\right)^2 \\ M_2 & \text{prob. } 1 - \left(1 - \pi_1\right)^2; \end{cases} \tag{4}$$

$$M_1 \longrightarrow \begin{cases} 0 & \text{prob. } \left(1 - \pi_1\right)^2 \\ V_2 - 2\delta & \text{prob. } \pi_1\left(1 - \pi_1/2\right) \\ L_2 & \text{prob. } \pi_1\left(1 - \pi_1/2\right); \end{cases} \tag{5}$$

and, for each $s \geq 2$:

$$V_s \longrightarrow \begin{cases} v & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ M_{s+1} & \text{prob. } q_{\beta,s} \\ M_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{6}$$

$$M_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ V_{s+1} - 2\delta & \text{prob. } q_{\beta,s} \\ L_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{7}$$

$$L_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ L_{s+1} & \text{prob. } q_{\beta,s} \\ V_2 - (s+1)\delta & \text{prob. } q_{\gamma,s}. \end{cases} \tag{8}$$

3.4 The dropout state

Since v is finite, we have, at any equilibrium in the form of $\left(\pi_0, \pi_1, \left(\pi_{\beta,s}, \pi_{\gamma,s}\right)_{s=2}^{\infty}\right)$, that V_2 is finite and hence $V_2 < s\delta$ for all sufficiently large s; thus for the equilibrium

$$s_* := \max\left\{s \in \{1, 2, 3, \dots\} \mid V_2 \geq s\delta\right\} \tag{9}$$

exists and is unique. Call s_* the *dropout state* of the equilibrium. The next lemma explains the appellation.

Lemma 2 *If the dropout state is s_*, then an underdog (γ player) (i) stays put for sure at state s if and only if $s \geq s_*$, and (ii) bids for sure at state s if $2 \leq s < s_* - 1$.*

As explained in Sect. 3.2, we shall construct all equilibria based on the condition that, in any subgame where the underdog never bids, the other two bidders play the surplus-dissipating subgame equilibrium defined there, which yields a continuation value 2δ for the current frontrunner, and zero for the other two. That is,

$$\left[\forall s \geq \tilde{s}\left[\pi_{\gamma,s} = 0\right]\right] \Longrightarrow \left[V_{\tilde{s}} = 2\delta,\ M_{\tilde{s}} = L_{\tilde{s}} = 0\right]. \tag{10}$$

In particular, once the game enters the dropout state s_*, the player currently in the underdog position will stay put forever (Lemma 2). Thus (10) implies $V_s = 2\delta$ and $M_s = L_s = 0$ for all $s \geq s_*$.

4 Derivation of trilateral rivalry equilibria

By (9) and Lemma 2.i, if the dropout state is $s_* = 1$, then everyone stays put, thereby ending the game, after the initial round; and if $s_* = 2$ then starting from the third round the underdog quits, with the subgame equilibrium being the surplus-dissipating one played by the frontrunner and the follower that emerge at the second round. The focus of this paper, however, is *trilateral rivalry*, where the underdog does not quit forever after the second round. In other words, we look for equilibria with dropout states $s_* \geq 3$. Following is our finding of what such equilibria necessarily entail.

Theorem 1 *If the dropout state $s_* \geq 3$ then:*

i. *s_* is an even number;*
ii. *at the initial round everyone bids for sure;*
iii. *at each state $s \in \{1, 2, \dots, s_* - 2\}$ every non-α player bids for sure;*
iv. *at state $s_* - 1$ the β player stays put and the γ player bids with probability $\pi_{\gamma,s_*-1} \in (0, 1)$ such that $V_2 = s_*\delta$, where V_2 is derived through the law of motion, (3)–(8);*
v. *at any state $s \geq s_*$, the γ player stays put and the β player bids with probability $1 - 2\delta/v$.*

Theorem 1.iv reveals a striking feature of a trilateral rivalry equilibrium: At the state $s_* - 1$, when the current underdog's lag is just one step shy of the dropout state of the equilibrium, the currently higher bidder β stays put for sure so that the underdog γ can leapfrog into the front if γ chooses so. This coupled with Claims (ii), (iii) and (v) of the theorem depicts the trajectory of trilateral rivalry: First, as the escalation continues, the gap between the frontrunner and the underdog may collapse or expand, depending on whether the latter manages to leapfrog thereby replacing the frontrunner. Second, the escalation may end only when this gap reaches $s_* - 1$, called *critical state*, at which the follower lets the underdog decide whether to continue the escalation through leapfrogging. Third, bilateral rivalry never occurs on path, as the game ends if the underdog does not leapfrog at the critical state. Finally, a rather peculiar feature of all these equilibria is, according to Claim (i), that these equilibria support only even numbered dropout states. This section outlines the reasoning for the theorem and leaves the details to Appendices B.3 and B.4.

4.1 Why the underdog leapfrogs at all

Here we explain why the γ player at state $s_* - 1$ bids at all, i.e., $\pi_{\gamma,s_*-1} > 0$. First note $s_* \geq 4$ from the hypothesis $s_* \geq 3$ and Claim (i) of the theorem, which we will explain in Sect. 4.3. Second, we shall use a Lemma 5 in Appendix B.3:

$$L_2 \leq M_2 < V_3. \tag{11}$$

Now suppose, to the contrary of the claim, that $\pi_{\gamma,s_*-1} = 0$ at equilibrium. Then, since $\pi_{\gamma,s} = 0$ at all $s > s_* - 1$ (Lemma 2.ii), the γ player quits forever starting from the state $s_* - 1$. Thus (10) implies $V_{s_*-1} = 2\delta$ and $M_{s_*-1} = 0$. Combine (6), (7) and Lemma 2.ii to get the following chains of transition:

$$\begin{array}{ccccccccc} \cdots \rightarrow & M_{s_*-4} & \xrightarrow{-2\delta} & V_{s_*-3} & \rightarrow & M_{s_*-2} & \xrightarrow{-2\delta} & V_{s_*-1} \\ & \downarrow & & \downarrow & & \downarrow & & \\ & L_2 & & M_2 & & L_2 & & \end{array} \tag{12}$$

$$\begin{array}{ccccccccc} \cdots \rightarrow & V_{s_*-4} & \rightarrow & M_{s_*-3} & \xrightarrow{-2\delta} & V_{s_*-2} & \rightarrow & M_{s_*-1} \\ & \downarrow & & \downarrow & & \downarrow & & \\ & M_2 & & L_2 & & M_2 & & \end{array} \tag{13}$$

Reason backward along the two chains, starting from $V_{s_*-1} = 2\delta$ and $M_{s_*-1} = 0$, and repeatedly use the fact $L_2 \leq M_2$ in (11). Then we obtain two chains of inequalities:

$$\begin{aligned} V_{s_*-1} = 2\delta \Rightarrow & M_{s_*-2} \le \max\{-2\delta + V_{s_*-1}, L_2\} = L_2 \le M_2 \\ & \Rightarrow V_{s_*-3} \le \max\{M_{s_*-2}, M_2\} \le M_2 \\ & \Rightarrow M_{s_*-4} \le \max\{-2\delta + V_{s_*-3}, L_2\} \le \max\{M_2, L_2\} \le M_2 \ \Rightarrow \cdots \\ M_{s_*-1} = 0 \Rightarrow & V_{s_*-2} \le \max\{M_{s_*-1}, M_2\} = M_2 \\ & \Rightarrow M_{s_*-3} \le \max\{-2\delta + V_{s_*-2}, L_2\} \le \max\{M_2, L_2\} \le M_2 \\ & \Rightarrow V_{s_*-4} \le \max\{M_{s_*-3}, M_2\} \le M_2 \ \Rightarrow \cdots \end{aligned}$$

The two chains combined imply $V_s \le M_2$ for all $s \le s_* - 1$. But then $V_3 \le M_2$, contradicting (11). Thus, $\pi_{\gamma,s_*-1} > 0$.

4.2 Why the follower accommodates the underdog

Here we explain why the β player at state $s_* - 1$ stays put for sure, i.e., $\pi_{\beta,s_*-1} = 0$. First, observe that if $s_* \ge 4$ then $L_2 > 0$. That is because the underdog γ at state $s = 2$ can bid thereby securing a fraction of the payoff $V_2 - 3\delta$, which is strictly positive by $s_* \ge 4$ and (9). (In bidding at state $s = 2$, γ also gets a fraction of L_3, which is nonnegative as he can always stay put thereafter, cf. (8).) Second, since $\pi_{\gamma,s} = 0$ for all $s \ge s_*$ (Lemma 2.i), (10) implies $V_{s_*} - 2\delta = 0$. Thus, bidding is worse than staying put for the β player at state $s_* - 1$: Staying put gives him $\pi_{\gamma,s_*-1}L_2$, which is strictly positive because $L_2 > 0$ and, as explained in Sect. 4.1, $\pi_{\gamma,s_*-1} > 0$; by contrast, bidding gives him $(V_{s_*} - 2\delta)\pi_{\gamma,s_*-1}/2 + L_2\pi_{\gamma,s_*-1}/2 = L_2\pi_{\gamma,s_*-1}/2$, strictly less than $L_2\pi_{\gamma,s_*-1}$. Hence $\pi_{\beta,s_*-1} = 0$.

4.3 Why dropout states cannot be odd numbers

The proof of Claim (i) of the theorem, relegated to Appendix B.3, follows a method similar to that illustrated in Sect. 4.1. Here we provide its intuition. Note that in a trilateral rivalry equilibrium the game ends only when the state is $s_* - 1$: When $s < s_* - 1$, the underdog bids for sure (Lemma 2.ii and Claim (ii) of this theorem); when $s = s_* - 1$, the underdog may bid while the follower stays put. If the underdog bids (thereby becoming the next α) then the state returns to $s = 2$, else the game ends and the current α wins the good. Thus, in order to win, a player needs to be the α player at the critical state $s_* - 1$. Consequently, if s_* is an odd number then, on the path to winning, a bidder must have in previous rounds been the β player for all odd states $s < s_* - 1$, and the α player for all even states $s \le s_* - 1$. Figure 1 illustrates the case of $s_* = 7$, where solid lines represent possible transitions if one bids, dashed lines if he stays put, and the thick gray states and arrows indicate the path to winning.

Thus, if s_* is odd, a player in the β position at any even state $s < s_* - 1$ would rather, in order to reach the winning path, become the γ player in state $s = 2$ (through staying put) than become the superfluous α player in the odd state $s + 1$ at the cost of 2δ (through bidding). In particular, in state $s = 2$, the β player would never bid while the γ player would always bid; hence the state $s = 2$ repeats itself, with the players

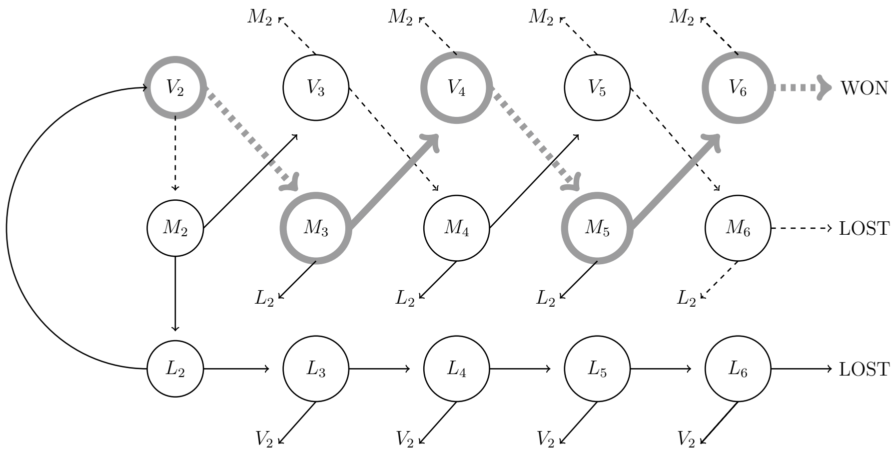

Fig. 1 The law of motions and equilibrium winning path if $s_* = 7$

switching roles according to $\gamma \to \alpha \to \beta \to \gamma$, thereby trapping them in an infinite bidding loop.[2]

4.4 Why even number dropout states are possible

When the dropout state s_* is an even number, by contrast, a β player is not in the predicament as in the previous case. First, in any even state $s < s_* - 1$ the β player wants to bid in order to stay on the winning path and become the α in the odd state $s + 1$. Second, in any odd state $s < s_* - 1$ the β player would rather bid and become the α in the even state $s + 1$ than stay put thereby becoming the γ player in state 2. With the former option, it takes a cost of 2δ (to become α in $s + 1$) and two rounds for the player to have a chance to become the β player in state $s = 2$ thereby landing on the winning path. With the latter option, it takes a cost of 3δ and three rounds for him to have such a chance of reaching the winning path. In Fig. 2, with $s_* = 6$, the situation of this odd-state β player is illustrated by the node M_3, from which the former option (becoming the next α) reaches the winning path state M_2 via $M_3 \to V_4 \to M_2$, while the latter option (being the next γ) reaches M_2 via the more roundabout route $M_3 \to L_2 \to V_2 \to M_2$.[3] That is the intuitive reason for the part of Theorem 1.iii on the β player's strategy.

[2] The odd-vs-even contrast in Sects. 4.3 and 4.4 echoes an odd-vs-even contrast observed by Kilgour and Brams (1997) on three-player duels. An interesting difference is that the odd-vs-even contrast in three-player duels is about the exogenous length of a finite game, whereas our contrast is about the *endogenous* length of the on-path rivalry of the equilibrium in our potentially infinite game.

[3] In the more roundabout route, the last step, from V_2 to M_2, is preferable to a player because of a nontrivial Lemma 9, saying that in the consecutive configuration it is better-off to be the follower than the frontrunner.

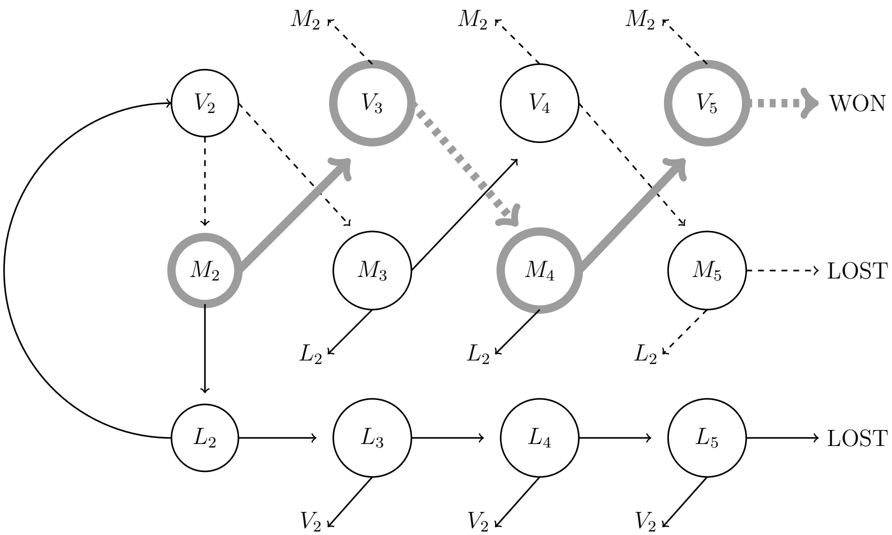

Fig. 2 The law of motions and equilibrium winning path if $s_* = 6$

5 Verification of trilateral rivalry equilibria

By Theorem 1.ii, Eq. (2) and the uniform tie-breaking rule,

$$2 \le s \le s_* - 2 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{14}$$

Given any $\pi_{\gamma,s_*-1} \in [0,1]$, the value functions $\left(V_s, M_s, L_s\right)_s$ associated to the strategy profile specified by Theorem 1 can be calculated based on (14) and the law of motion, (6)–(8). The question is whether such a strategy profile constitutes an equilibrium. The crucial step in answering this question is to verify that, given the strategy profile in Theorem 1, bidding is a best response for the β player at every state below $s_* - 1$. Verification for all such states might sound cumbersome, but it turns out that we need only to check two inequalities:

Lemma 3 *For any even number $s_* \ge 4$ and any strategy profile specified by Theorem 1, bidding is a best response for the β player at state $s \in \{1, 2, \ldots, s_* - 2\}$ if either (i) s is even and $V_3 - 2\delta \ge L_2$, or (ii) s is odd and $V_{s_*-2} - 2\delta \ge L_2$.*

Thus we obtain an equilibrium with an even number dropout state $s_* \ge 4$ if the following conditions all hold: (a) the bidding probability π_{γ,s_*-1} at the critical state solves the equation $V_2 = s_*\delta$ (Theorem 1.iv), with V_2 as well as other value functions derived from the law of motion (6)–(8) according to the strategy profile in Theorem 1; (b) $V_3 - 2\delta \ge L_2$; and (c) $V_{s_*-2} - 2\delta \ge L_2$. Condition (a), by Lemma 17 in the appendix, is equivalent to

$$\begin{aligned} &\frac{3\mu_* v}{\delta}(1-\pi_{\gamma,s_*-1})(2-\mu_*)+(2-\mu_*)^2(s_*-6+\mu_*) \\ &\quad=\big(2(1+\mu_*)-3\mu_*\pi_{\gamma,s_*-1}\big)\big(3s_*+2(1-2\mu_*) \\ &\qquad-(s_*-4+\mu_*)(1-2\mu_*+3\mu_*\pi_{\gamma,s_*-1})\big), \end{aligned} \tag{15}$$

where $\mu_* := 2^{-s_*+3}$. Condition (b), by Lemma 16 in the appendix, turns out to be redundant, implied by Condition (c), which by Lemma 18 in the appendix is equivalent to

$$\pi_{\gamma,s_*-1} \geq 1 - \frac{3(2-\mu_*)}{2(1-2\mu_*)(s_*-4+\mu_*)}. \tag{16}$$

When $s_* \leq 6$, (16) is guaranteed as long as (15) admits a solution $\pi_{\gamma,s_*-1} \in [0,1]$, which in turn is guaranteed if $v/\delta > 35/2$ when $s_* = 4$, and $v/\delta > 6801/120$ when $s_* = 6$ (Lemma 21). When $s_* > 6$, to guarantee both (15) and (16) we need a stronger condition (cf. the end of Appendix B.6),

$$\frac{v}{\delta} \geq \left(\frac{1}{3}s_*^2 + \frac{5}{3}s_* - 8\right)2^{s_*-3}. \tag{17}$$

In sum, trilateral rivalry equilibria with even-number dropout states exist provided that the parameter v/δ is sufficiently large:

Theorem 2 *A trilateral rivalry equilibrium exists if v/δ is sufficiently large. In particular*:

i. *if $v/\delta > 35/2$ then one exists with dropout state $s_* = 4$;*
ii. *if $v/\delta > 6801/120$ then one exists with dropout state $s_* = 6$;*
iii. *if an even number $s_* \geq 8$ satisfies (17), then one exists with dropout state s_*.*

Numerical Illustration

Consider the case where $\delta = 1$, v varies from 0 to 1000, and $s_* \in \{4, 6, 8, 10\}$. Figure 3 shows the equilibrium bidding probability of the underdog (the γ player) in the critical state $s_* - 1$ as a function of the underlying value v. The vertical lines indicate the point at which additional equilibria for $s_* > 4$ are admitted. For instance, starting at $v = 57$ ($\approx$ 6801/102) the equilibrium corresponding to the dropout state $s_* = 6$ is permissible. Note that within each equilibrium the bidding probability is increasing in v (or v/δ as $\delta = 1$ in this example). Interestingly, when a new equilibrium with a higher dropout state becomes permissible the corresponding equilibrium bidding probability drastically reduces. Furthermore, each additional equilibrium requires an order of magnitude increase in v, consistent with the exponential growth rate of the right-hand side of (17).

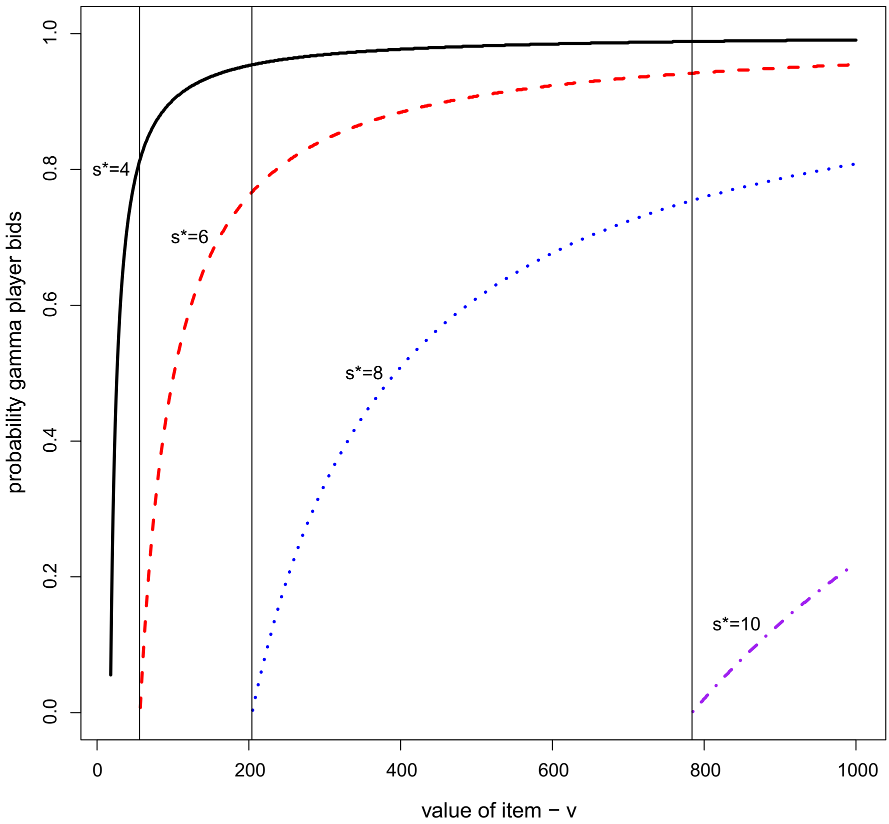

Fig. 3 Equilibrium bidding probability for the underdog in the critical state $s_* - 1$; $\delta = 1$

6 Welfare properties

6.1 Bilateral rivalry equilibria

Different from the equilibria characterized above, where the dropout states are $s_* \geq 3$, the equilibria with dropout states $s_* = 2$ all exhibit bilateral rivalry on path. In each of them, once the game has reached the third round, where two players have emerged as the top two rivals, the surplus-dissipating subgame equilibrium is played from then on, with the underdog dropping out for good and the follower topping the frontrunner with probability $1 - 2\delta/v$ in any round until the game ends (cf. Ødegaard and Zheng 2020). Thus, any such equilibrium exhibits only bilateral rivalry and, once the game has entered the third round, only the top two bidders may remain in the race and only the current frontrunner gets a nonzero payoff, 2δ.

6.2 Pareto superiority of trilateral rivalry

From the viewpoint before the start of the game, there is a continuum of bilateral rivalry equilibria that yield zero expected payoff to all bidders, but there is also a bilateral rivalry one that yields the highest total expected payoff, $v - \delta$, for the bidders, where escalation does not occur and the randomly selected initial frontrunner wins (cf. Ødegaard and Zheng 2020). However, given that players, be they in classroom experiments or in the real world, are so easily trapped in escalating wars of attrition, it makes sense to consider their welfare *after* escalation has started so that the top two rivals have emerged, that is, when the state of the game is such that $s \geq 2$. In any bilateral rivalry equilibrium, if $s \geq 2$ then the subgame equilibrium is the surplus-dissipating one, giving an expected payoff 2δ to the current frontrunner and zero to each of the other two (Sect. 6.1). In any trilateral rivalry equilibrium with dropout state s_*, by contrast, the surplus-dissipating subgame equilibrium does not occur on path at all, whereas the path—when the state $s \in \{2, \ldots, s_* - 1\}$—exhibits trilateral rivalry that yields a total expected payoff larger than 2δ:

Theorem 3 *In any trilateral equilibrium with dropout state* s_*, *in any subgame where the state* $s \in \{2, \ldots, s_* - 1\}$, *we have* $V_s > 2\delta$, $M_s > 0$ *and* $L_s \geq 0$, *with* $L_s > 0$ *if* $s < s_* - 1$.

The theorem is based on the law of motion of the value functions (6)–(8), coupled with the strategy profile specified by Theorem 1. The case of $V_2 > 2\delta$, however, is obvious: By $V_2 = s_*\delta$ (Theorem 1.iv) and $s_* \geq 4$ (Theorem 1.i), $V_2 = s_*\delta \geq 4\delta$.

Theorem 3 implies that, once the first two rounds of the game have elapsed so that two players have emerged as the top two rivals, the two can avoid the detrimental bilateral rivalry if they could engage the third player in a trilateral rivalry equilibrium.

7 Conclusion

In considering the dollar auction—a dynamic game that albeit specific has deep roots in the conflict literature—this paper demonstrates a novel dynamic pattern where three contestants perpetuate the trilateral rivalry through a cyclic pattern of leapfrog and accommodation. Importantly, the trilateral rivalry is not assumed in our model but rather arises as an equilibrium while there is another equilibrium of bilateral rivalry, where any player who happens to fall behind the top two immediately drops out. The normative advantage of the trilateral-rivalry equilibrium is that it generates larger total surplus across all players than the bilateral-rivalry equilibrium does. Given the current polarizing political climate in the United States, our normative result could be taken as a timely—though stylized—suggestion that adding a third political party to the US two-party system might help to mitigate the more and more acute conflict between the two sides.

To sustain such collectively advantageous trilateral rivalry, the contestants accommodate one another at critical junctures of the equilibrium: when the third-place player lags behind the frontrunner by a distance beyond which he will drop out forever, the rival near the top stays put to better the chance that the third-place player can catch up through a leapfrog bid. Thus the game either ends, when the latter fails to make the leapfrog bid, or repeats the trilateral rivalry cycle, when he makes the leapfrog.

Although the main results are based on the specific model originally formulated by Shubik (1971), similar dynamic patterns can be found in variants of the model. One direction of extension is to consider alternative tie-breaking rules in the event where multiple players bid simultaneously. In that event, instead of randomly selecting one among the simultaneous bidders to increment his bid and become the frontrunner, we can have all simultaneous bidders increment their bids—thereby all incurring the sunk costs—and become tying frontrunners with the proviso that the prize is not allocated unless only a single frontrunner remains and is not outbid immediately. While this tie-breaking rule complicates the analysis, given a nondegenerate range of parameter values there exists a trilateral rivalry equilibrium that exhibits cyclic patterns of leapfrog and accommodation, as we have shown in our Supplementary Information, Prop. 1. Not only is there an event where the second-place bidder pauses bidding to accommodate the leapfrog efforts from the lowest bidder, but there is also an event where the lowest bidder pauses bidding—to see the top two rivals bid against each other—and then makes a leapfrog bid.

Another alternative tie-breaking rule is to have all simultaneous bidders increment their bids—thereby incurring the sunk costs—but designate only one of them, selected randomly, as the provisional winner, who would win the prize if not outbid immediately. Such modification is consistent with the mechanism of most online penny auctions. While this rule imposes a sunk cost on a bidder who ties with others and hence may discourage an underdog from making a leapfrog effort, players can avoid such costly ties in equilibrium with a sunspot coordination device à la Shell (1977) and Cass and Shell (1983), thereby replicating all the trilateral rivalry equilibria characterized in the main model (Supplementary Information).

The other direction of extension is to have more than three active bidders in an equilibrium. For example, one can extend the model to have four players and construct an equilibrium of quadrilateral rivalry analogous to a trilateral rivalry equilibrium (Supplementary Information). One can furthermore consider an n-player model where m of them ($m = 2, \ldots, n$) are involved in an indefinitely long loop of competition and accommodation. If the top m bidders are positioned consecutively, the subgame admits an equilibrium where only the player in the mth-position bids at all and all the other players stay put to accommodate his leapfrog efforts (Supplementary Information).

A formal definition of the model

A.1 The dollar auction game

Let $I := \{1, 2, 3\}$ denote the set of players; and $w_0 := b_{i,0} := 0$ for all $i \in I$. Let

$$n_0 := \left(w_0, (b_{i,0})_{i\in I}\right) \quad (= (0, 0, 0, 0))$$

be the initial node of the game. Given n_0, each player i's set of feasible actions is $A_i(n_0) := \{0, 1\}$, with 1 signifying "to bid" and 0 signifying "to stay put."

Pick any $t = 1, 2, 3, \ldots$. Suppose that a node $n_{t-1} := (w_{t-1}, (b_{i,t-1})_{i\in I}) \in (I \cup \{0\}) \times \mathbb{R}^3_+$ and each player's set of feasible actions at n_{t-1} have been defined. For any $a_{t-1} := (a_{i,t-1})_{i\in I}$ such that $a_{i,t-1}$ is a feasible action for player i at node n_{t-1} ($\forall i \in I$), define the set of immediate successors of n_{t-1} to be

$$N(n_{t-1}, a_{t-1}) := \left\{ (w_t, (b_{i,t})_{i\in I}) \in (I \cup \{0\} \times \mathbb{R}^3_+ \;\middle|\; \begin{array}{l} w_t \neq w_{t-1} \Rightarrow \left[\begin{array}{l} a_{w_t,t-1} = 1, \\ b_{w_t,t} = \delta + \max_{i\in I} b_{i,t-1} \end{array}\right] \\ \left[i \neq w_t \text{ or } i = w_t = w_{t-1}\right] \Rightarrow b_{i,t} = b_{i,t-1} \end{array} \right\}.$$

At any node $n_t := (w_t, (b_{i,t})_{i\in I}) \in N(n_{t-1}, a_{t-1})$, which descends immediately from (n_{t-1}, a_{t-1}), define the set $A_i(a_{t-1}, n_t)$ of feasible actions for player i by:

$$A_i(a_{t-1}, n_t) := \begin{cases} \{0, 1\} & \text{if } w_t \neq 0 \text{ and } i \neq w_t \textit{ and } \exists j \in I : a_{j,t-1} = 1 \\ \{0\} & \text{if } w_t = 0 \text{ or } i = w_t \textit{ or } a_{1,t-1} = a_{2,t-1} = a_{3,t-1} = 0. \end{cases}$$

The transition probability from (n_{t-1}, a_{t-1}) to any element $n_t := (w_t, (b_{i,t})_{i\in I})$ of $N(n_{t-1}, a_{t-1})$ is defined by:

$$\{i \in I \mid a_{i,t-1} = 1\} = \varnothing \Rightarrow \Pr\{w_t = w_{t-1} \mid n_{t-1}, a_{t-1}\} = 1;$$
$$\{i \in I \mid a_{i,t-1} = 1\} \neq \varnothing \Rightarrow \Pr\{w_t = i \mid n_{t-1}, a_{t-1}\} = \frac{a_{i,t-1}}{\left|\{i \in I \mid a_{i,t-1} = 1\}\right|} \quad (\forall i \in I).$$

The initial node n_0 is the 0-*history*. For any $t = 1, 2, \ldots$, a t-*history* is a finite sequence $((n_k, a_k)_{k=0}^{t-1}, n_t)$ that satisfies all the following conditions: (i) n_0 is the 0-history; (ii) $((n_k, a_k)_{k=0}^{t'-1}, n_{t'})$ is a t'-history for any $t' \in \{1, \ldots, t-1\}$; (iii) $t \geq 2 \Rightarrow a_{t-1} \in \prod_{i\in I} A_i(a_{t-2}, n_{t-1})$, and $t = 1 \Rightarrow a_{t-1} \in \prod_{i\in I} A_i(n_0)$; (iv) $n_t \in N(n_{t-1}, a_{t-1})$; and (v) the transition probability from (n_{t-1}, a_{t-1}) to n_t is positive.

A *feasible path* is an infinite sequence $(n_0, a_0, (n_k, a_k)_{k=1}^{\infty})$ in which $((n_k, a_k)_{k=0}^{t-1}, n_t)$ is a t-history for any $t = 0, 1, 2, \ldots$.

Given any feasible path $(n_0, a_0, (n_k, a_k)_{k=1}^{\infty})$, which contains the sequence $(n_k)_{k=1}^{\infty}$, namely $(w_k, (b_{i,k})_{i\in I})_{k=0}^{\infty}$, the ex post payoff to any player $i \in I$ is defined to be equal to

$$v \liminf_{t\to\infty} \mathbf{1}_{w_t=i} - \sup_{t=1,2,\ldots} b_{i,t}, \tag{18}$$

where $\mathbf{1}_{w_t=i} := 1$ if $w_t = i$, and $\mathbf{1}_{w_t=i} = 0$ if $w_t \neq i$.

Example 1 of a feasible path:

$$((0;0,0,0),(1,1,0);(2;0,\delta,0),(0,0,1);(3;0,\delta,2\delta),(0,0,0);$$
$$(3;0,\delta,2\delta),(0,0,0);\cdots),$$

where $w_t = 3$ for all $t = 2$, namely, player 3 wins the prize in the third round of the game.

Example 2 of a feasible path:

$$((0;0,0,0),(1,1,0);(2;0,\delta,0),(1,0,0);(1;2\delta,\delta,0),(0,1,0);$$
$$(2;2\delta,3\delta,0),(1,0,0);(1;4\delta,3\delta,0),\cdots),$$

where players 1 and 2 outbid each other, and hence take turn to be the frontrunner, in alternate order. Thus $\liminf_{t\to\infty} \mathbf{1}_{w_i=i} = 0$ for both players (and trivially so for player 3 as he never bids). According to (18), the payoff is equal to $-\infty$ for both players 1 and 2, and zero for player 3.

A.2 Formalization of the equilibrium concept

A *strategy* σ_i for player i is a mapping that assigns, for any $t = 0, 1, 2, \ldots$, to each t-history $((n_k, a_k)_{k=0}^{t-1}, n_t)$ a lottery on the set $A_i(n_{t-1}, a_{t-1})$ (or on the set $A_i(n_0)$ if $t = 0$) of feasible actions for i. A *strategy profile* is a profile $(\sigma_i)_{i\in I}$ of strategies across all players. A *subgame perfect equilibrium (SPE)* is a strategy profile that satisfies sequential rationality at each t-history for every $t = 0, 1, 2, \ldots$.

An SPE $(\sigma_i)_{i\in I}$ is said to be *history independent* iff for any $i \in I$, any $t, t' \in \{0, 1, 2, \ldots\}$, and any t- and t'-histories $((n_k, a_k)_{k=0}^{t-1}, n_t)$ and $((n'_k, a'_k)_{k=0}^{t'-1}, n'_{t'})$,

$$n_t = n'_{t'} \Longrightarrow \sigma_i((n_k, a_k)_{k=0}^{t-1}, n_t) = \sigma_i((n'_k, a'_k)_{k=0}^{t'-1}, n'_{t'}).$$

Denote

$$N := \{n_0\} \cup \{n_t \mid t = 1, 2, \ldots; ((n_k, a_k)_{k=0}^{t-1}, n_t) \text{ is a } t\text{-}history\}.$$

In any history-independent SPE, each player i's strategy is function of only the last node of any t-history. Thus we write

$$\sigma_i(n_t) := \sigma_i((n_k, a_k)_{k=0}^{t-1}, n_t)$$

for any $n_t \in N$ such that $((n_k, a_k)_{k=0}^{t-1}, n_t)$ is a t-history for some t. It follows that we can suppress the subscript t to write any element of N as $n = (w, (b_i)_{i\in I})$.

For any $n, n' \in N$ such that $n = (w, (b_i)_{i\in I})$ and $n' = (w', (b'_i)_{i\in I})$, we say that n is *isomorphic* to n', or $n \equiv n'$, iff there exists a permutation $\psi : I \to I$ such that $b_i - b_j = b'_{\psi(i)} - b'_{\psi(j)}$ for all $i, j \in I$. This ψ is called a permutation associated with $n \equiv n'$.

A history-independent SPE $(\sigma_i)_{i\in I}$ is said to be *symmetric* iff $\sigma_i(n) = \sigma_{\psi(i)}(n')$ for any $n \equiv n'$ with any associated permutation ψ.

For any $n \in N$ such that $n = (w, (b_i)_{i\in I})$, define $\phi_n : I \to \{\alpha, \beta, \gamma\}$ by

$$\phi_n(i) := \begin{cases} \alpha & \text{if } i = w \\ \gamma & \text{if } i \in \arg\min_{j \in I} b_j \\ \beta & \text{if } i \in I \backslash \left(\{w\} \cup \arg\min_{j \in I} b_j\right). \end{cases}$$

Examples: At the initial node n_0, since $w = 0$, $\phi_{n_0}(i) = \gamma$ for all players i. At a period-1 node say $n_1 := (2;0,\delta,0)$, $\phi_{n_1}(2) = \alpha$ and $\phi_{n_1}(1) = \phi_{n_1}(3) = \gamma$. At any node n_t with $t \geq 2$ such that $a_{t-1} \neq (0,0,0)$, ϕ_{n_t} is one-to-one.

For any $n, n' \in N$ such that $n \equiv n'$, there exists a permutation $\psi : I \to I$ such that

$$\phi_n(i) = \phi_{n'}(\psi(i))$$

for any $i \in I$. Since $n \equiv n'$, if $n = (w, (b_i)_{i \in I})$ and $n' = (w', (b'_i)_{i \in I})$ then (by the definitions of ϕ_n and $\equiv$) we have $b_i - b_j = b'_k - b'_l$ whenever $\phi_n(i) = \phi_{n'}(k)$ and $\phi_n(j) = \phi_{n'}(l)$. Thus ψ is a permutation associated with $n \equiv n'$. Consequently, by the definition of symmetry,

$$\phi_n(i) = \phi_{n'}(j) \Rightarrow \sigma_i(n) = \sigma_j(n'). \tag{19}$$

For any $n \in N$ such that $n = (w, (b_i)_{i \in I})$, denote

$$\tilde{s}(n) := \frac{1}{\delta}\left(\max_{i \in I} b_i - \min_{i \in I} b_i\right).$$

Claim: For any $n, n' \in N$, $n \equiv n' \iff \tilde{s}(n) = \tilde{s}(n')$. This follows from two observations, each proved inductively. First, for any $n \in N$ such that $n = (w, (b_i)_{i \in I})$ and any $i, j \in I$, $b_i - b_j = m\delta$ for some integer m. Second, for any $n \in N$ such that $n = (w, (b_i)_{i \in I})$,

$$\tilde{s}(n) \neq 0 \Longrightarrow \left[w \in I \text{ and } b_w - \max_{j \in I \backslash \{w\}} b_j = \delta\right].$$

That is, the gap of committed payments between the frontrunner and any second-place bidder is always δ. Thus, the only payment-gap that may be different between any two elements of N is the gap between the frontrunner and the lowest bidder.

The claim and (19) together imply that, for any symmetric history-independent SPE $(\sigma_i)_{i \in I}$,

$$\left[\phi_n(i) = \phi_{n'}(j) \text{ and } \tilde{s}(n) = \tilde{s}(n')\right] \Longrightarrow \sigma_i(n) = \sigma_j(n').$$

Let $\sigma_i(n)(1)$ denote the probability of playing the action 1—to bid—at the node n by player i according to strategy σ_i. By the above-displayed fact, there is no loss of generality to represent a symmetric history-independent SPE by a list

$$\left(\pi_0, \pi_1, (\pi_{\beta,s}, \pi_{\gamma,s})_{s=1}^{\infty}\right)$$

such that $\pi_0 = \sigma_i(n_0)(1)$, $\pi_1 = \sigma_i(n)(1)$ such that $\tilde{s}(n) = 1$ for any $i \in I$ with $\phi_n(i) \neq \alpha$, and, for any $s = 2, 3, \ldots$ and any $r \in \{\beta, \gamma\}$, $\pi_{r,s} = \sigma_i(n)(1)$ such that $\tilde{s}(n) = s$ and $\phi_n(i) = r$.

A symmetric history-independent SPE is said to be *independent of non-participants* iff (1) holds in the π-notation introduced above.

Our equilibrium concept is therefore defined to be any symmetric history-independent SPE that is independent of non-participants.

B Proofs

B.1 Lemma 1

Claim (i) has been proved in Ødegaard and Zheng (2020). Here we prove (ii), the uniqueness claim. Consider any equilibrium where the underdog stays put forever. Then only the follower's strategy needs to be specified. By independence condition (1) of nonparticipants, the follower's strategy is independent of the state variable, which measures merely the distance from the forever inactive underdog. Thus, the follower's strategy is a constant probability π of bidding. Let V_* denote the expected payoff for the frontrunner in any such equilibrium. In the current round, he either wins the prize v with probability $\tilde{\sigma}_2$ or becomes the follower next round. In the latter case, by the symmetry condition in our equilibrium concept, he bids with probability π thereby paying 2δ to become the frontrunner again. Thus, the Bellman equation is

$$V_* = (1-\pi)v + \pi^2\left(-2\delta + V_*\right).$$

Note that $\pi > 0$, otherwise ($\pi = 0$), $V_* = v$; with $v > 2\delta$ by assumption, the follower would bid for sure, contradicting the supposition $\pi = 0$. Also note that $\pi < 1$, otherwise the Bellman equation implies $V_* = -2\delta + V_*$, contradiction. Now that $0 < \pi < 1$, the follower is indifferent about bidding, hence $V_* = 2\delta$. Plug this into the Bellman equation to obtain $\pi = 1 - 2\delta/v$. That proves Claim (ii). □

B.2 Lemma 2

Lemma 2

By definition of L_s, the equilibrium expected payoff for an underdog whose lag from the frontrunner is s, we know that $L_s = 0$ for all $s \geq v/\delta$. Starting from any such s and use backward induction towards smaller s, together with the law of motion (8) and the fact $V_2 - (s+1)\delta < 0$ for all $s \geq s_*$ due to the definition of s_*, we observe that $L_s = 0$ for all $s \geq s_*$. At any state $s \geq s_*$, by (8), an underdog gets zero expected payoff if he does not bid; if he bids then by Eq. (2) there is a positive probability with which he gets a negative payoff $V_2 - (s+1)\delta$; hence his best response is uniquely to not bid at all. Hence

$$s \geq s_* \implies \quad L_s = 0 \text{ and } \pi_{\gamma,s} = q_{\gamma,s} = 0, \tag{20}$$

which proves Claim (i) of the lemma. Apply backward induction to (8) starting from $s = s_*$ and we obtain

$$2 \leq s \leq s_* - 1 \Longrightarrow V_2 - (s+1)\delta \geq L_s \geq L_{s+1} \geq 0, \tag{21}$$

with the inequality $L_s \geq L_{s+1}$ being strict whenever $s < s_* - 1$. Thus, for any $s < s_* - 1$, $V_s - (s+1)\delta > L_{s+1} \geq 0$; hence Eqs. (2) and (8) together imply that an underdog's best response is uniquely to bid for sure:

$$2 \leq s < s_* - 1 \Longrightarrow \quad L_s > 0 \text{ and } \pi_{\gamma,s} = 1, \tag{22}$$

which proves Claim (ii) of the lemma. □

B.3 Claims (i) and (iv) of Theorem 1

First, we make several observations first. By (8) and (21), L_2 is a convex combination between L_3 and $V_2 - 3\delta$, with $V_2 - 3\delta \geq L_3$ when $s_* \geq 3$. Thus,

$$s_* \geq 3 \Longrightarrow L_2 \leq V_2 - 3\delta. \tag{23}$$

Equation (10), combined with (7) and (8), implies

$$M_{s_*-1} = q_{\gamma,s_*-1} L_2 \overset{(23)}{\leq} \left(V_2 - 3\delta\right)^+. \tag{24}$$

Lemma 4 $s_* \neq 3$

Proof Suppose, to the contrary, that $s_* = 3$. Hence $0 \leq V_2 - 3\delta < \delta$. Thus, by Eq. (24), $M_2 < \delta$. Then (4) requires that $\pi_1 < 1$, otherwise $V_1 = M_2 < \delta$, implying a contradiction that no one would bear the sunk cost δ to become the initial α player. Now consider the decision of any non-α player at the state $s = 1$, as depicted by (5). Since $V_2 - 2\delta > V_2 - 3\delta \geq L_2$, with the second inequality due to (23), each non-α player at $s = 1$ would maximize the probability of becoming the α in the next round, i.e., $\pi_1 = 1$, contradiction. □

Lemma 5 *If* $s_* \geq 4$ *then* $V_3 - 2\delta \geq M_2 \geq L_2 > 0$.

Proof Suppose that $V_3 - 2\delta < L_2$. Then, by the fact $\pi_{\gamma,2} = 1$ (Lemma 2.ii and $s_* \geq 4$) and Eq. (2), the β player at state $s = 2$ would rather stay put than bid, hence $\pi_{\beta,2} = 0$. This, combined with (6) in the case $s = 2$ and the fact $\pi_{\gamma,2} = 1$, implies that $V_2 = M_2$. Since $V_3 - 2\delta < L_2$ coupled with (7) implies $M_2 \leq L_2$, we have a contradiction $V_2 \leq L_2 < V_2$, with the last inequality due to (8). Thus we have proved $V_3 - 2\delta \geq L_2$. Therefore, with M_2 a convex combination between $V_3 - 2\delta$ and L_2 (since $\pi_{\gamma,2} = 1$), $V_3 - 2\delta \geq M_2 \geq L_2$. Finally, to show $L_2 > 0$, note from the hypothesis $s_* \geq 4$ and definition of s_* that $V_2 - 3\delta > 0$. This positive payoff the underdog at state $s = 2$ can secure with a positive probability through bidding. Hence $L_2 > 0$ follows from (8). □

Lemma 6 *If* $s_* \geq 4$ *then* $\pi_{\gamma,s_*-1} > 0$.

Proof Section 4.1. □

Proof of Theorem 1.i (impossibility of odd dropout states)

Suppose, to the contrary, that the dropout state s_* of an equilibrium is an odd number. Since $s_* \geq 3$ by hypothesis of the lemma, $s_* \geq 5$. By Lemma 5, $L_2 \leq M_2 \leq V_3 - 2\delta$. Apply (6) to the case $s = s_* - 2$ and use the fact that $\pi_{\gamma,s_*-2} = 1$ (thereby ruling out $V_{s_*-2} \to v$) due to Lemma 2.ii. Then we have $M_{s_*-1} \leq L_2$, due to (24), and hence $M_{s_*-1} \leq M_2 \leq V_3 - 2\delta$. Reason backward along the transition chain (13), starting from $M_{s_*-1} \leq V_3 - 2\delta$ and using the fact $L_2 \leq M_2 \leq V_3 - 2\delta$ (Lemma 5), to obtain

$$
\begin{aligned}
M_{s_*-1} \leq V_3 - 2\delta \Rightarrow & V_{s_*-2} \leq \max\{V_3 - 2\delta, M_2\} \leq V_3 - 2\delta \\
& \Rightarrow M_{s_*-3} \leq \max\{-2\delta + V_{s_*-2}, L_2\} \leq \max\{V_3 - 2\delta, L_2\} \leq V_3 - 2\delta \\
& \Rightarrow V_{s_*-4} \leq \max\{M_{s_*-3}, M_2\} \leq V_3 - 2\delta \\
& \Rightarrow \cdots
\end{aligned}
$$

Since s_* is an odd number and $s_* \geq 5$, this chain eventually reaches V_3, i.e., $3 = s_* - 2m$ for some positive integer m. Hence we obtain the contradiction $V_3 \leq V_3 - 2\delta$. □

Proof of Theorem 1.iv

That the β player stays put for sure at state $s_* - 1$ has been proved in Sect. 4.2, where the hypothesis $s_* \geq 4$ is true due to Lemma 4. For the rest of the claim, we first prove $0 < \pi_{\gamma,s_*-1} < 1$. The first inequality follows from Lemma 6 since $s_* \geq 4$. To prove $\pi_{\gamma,s_*-1} < 1$, suppose to the contrary that $\pi_{\gamma,s_*-1} = 1$. Then by the fact $\pi_{\beta,s_*-1} = 0$ and (6) applied to the case $s = s_* - 1$, we have $V_{s_*-1} = M_2$ and $M_{s_*-1} = L_2$. Reason backward along the transition chains (12) and (13), starting from $V_{s_*-1} = M_2$ and $M_{s_*-1} = L_2$ and using the fact $L_2 \leq M_2$ (Lemma 5). Then we have two chains of inequalities:

$$
\begin{aligned}
V_{s_*-1} = M_2 \Rightarrow & M_{s_*-2} \leq \max\{-2\delta + M_2, L_2\} \leq M_2 \\
& \Rightarrow V_{s_*-3} \leq \max\{M_{s_*-2}, M_2\} \leq M_2 \\
& \Rightarrow M_{s_*-4} \leq \max\{-2\delta + V_{s_*-3}, L_2\} \leq \max\{M_2, L_2\} \leq M_2 \;\Rightarrow \cdots \\
M_{s_*-1} = L_2 \Rightarrow & V_{s_*-2} \leq \max\{L_2, M_2\} \leq M_2 \\
& \Rightarrow M_{s_*-3} \leq \max\{-2\delta + V_{s_*-2}, L_2\} \leq \max\{M_2, L_2\} \leq M_2 \\
& \Rightarrow V_{s_*-4} \leq \max\{M_{s_*-3}, M_2\} \leq M_2 \;\Rightarrow \cdots
\end{aligned}
$$

The two chains combined lead to $V_s \leq M_2$ for all $s \leq s_* - 1$. Hence $V_3 \leq M_2$, contradicting Lemma 5. Thus we have proved that $\pi_{\gamma,s_*-1} < 1$.

With $\pi_{\gamma,s_*-1} < 1$, bidding is not the unique best response for the γ player at state $s_* - 1$, hence $V_2 \leq s_*\delta$ (otherwise the bottom branch of (8) in the case $s = s_* - 1$ is strictly positive and, by (20), is strictly larger than the middle branch, so the γ player would strictly prefer to bid). By definition of s_*, $V_2 \geq s_*\delta$. Thus $V_2 = s_*\delta$. □

B.4 Claims (ii) and (iii) of Theorem 1 (action of β when $s \leq s_* - 2$)

The part of Theorem 1.iii on the γ player follows from Lemma 2.ii. We prove the part on the β player here, through proving Lemmas 8 and 10. The former shows that bidding is a follower's unique best response at even-number states, and the latter, odd-number states. Lemma 10 uses a Lemma 9, which also implies Claim (ii) of the theorem.

Lemma 7 *If the dropout state is an even number* $s_* \geq 4$, *then* $L_2 < V_2 \leq V_3 - 2\delta$.

Proof Since $\pi_{\beta,s_*-1} = 0$ (Theorem 1.iv), $M_{s_*-1} \leq L_2$. Thus, since $\pi_{\gamma,s_*-2} = 1$ (Lemma 2.ii), V_{s_*-2} is a convex combination between M_{s_*-1}, which is less than L_2, and M_2, which is a convex combination between $V_3 - 2\delta$ and L_2, as $\pi_{\gamma,2} = 1$. Thus V_{s_*-2} is between L_2 and $V_3 - 2\delta$. Consequently, M_{s_*-3}, a convex combination between L_2 and $V_{s_*-2} - 2\delta$ (since $\pi_{\gamma,s_*-3} = 1$), is between L_2 and $V_3 - 2\delta$. Repeating this reasoning, with s_* being an even number, we eventually reach $2 = s_* - 2m$ for some integer $m \geq 1$, and obtain the fact that V_2 is a number between L_2 and $V_3 - 2\delta$. Thus, $L_2 < V_3 - 2\delta$, otherwise the fact $L_2 < V_2$ by (8) would be contradicted. Hence $L_2 < V_2 \leq V_3 - 2\delta$. □

B.4.1 Bidding at even states

Lemma 8 *If the dropout state is an even number* $s_* \geq 4$, *then* $\pi_{\beta,s} = 1$ *for any even number* s *such that* $2 \leq s \leq s_* - 2$.

Proof First, by Lemma 7, $L_2 < V_3 - 2\delta$. Thus at state $s = 2$ the β player strictly prefers to bid, i.e., $\pi_{\beta,2} = 1$. Second, pick any even number s such that $4 \leq s \leq s_* - 2$ and suppose, to the contrary of the lemma, that $\pi_{\beta,s} < 1$, which means that the β player at state s does not strictly prefer to bid. Thus $M_s \leq L_2$ (as the transition $M_s \to 0$ is ruled out by the fact $\pi_{\gamma,s} = 1$). Consequently, V_{s-1}, a convex combination between M_s and M_2, is weakly less than M_2, as $L_2 \leq M_2$ by Lemma 5. Furthermore, M_{s-2}, a convex combination between $V_{s-1} - 2\delta$ and L_2, is less than M_2, and that in turns implies $V_{s-3} \leq M_2$. Repeating this reasoning, with s an even number, we eventually obtain the conclusion that $V_3 \leq M_2$, which contradicts Lemma 5. Thus, $\pi_{\beta,s} = 1$. □

If the dropout state is an even number $s_* \geq 4$, since $\pi_{\gamma,s} = 1$ for all $s \leq s_* - 2$ (Lemma 2.ii), Eq. (2) and the equal-probability tie-breaking rule together imply

$$\forall s \in \{2, 3, 4, \ldots, s_* - 2\} : \left[\pi_{\beta,s} = 1 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2\right]. \tag{25}$$

By Lemma 8,

$$2 \leq s \leq s_* - 2 \text{ and } s \text{ is even } \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{26}$$

B.4.2 Bidding at odd states

In the following, we extend the summation notation by defining, for any sequence $(a_k)_{k=1}^{\infty}$,

$$i > j \Longrightarrow \sum_{k=i}^{j} a_k := 0. \tag{27}$$

In particular, $\sum_{k=1}^{0} a_k = 0$ according to this notation.

Lemma 9 *If the dropout state is an even number $s_* \geq 4$, $M_2 > V_2 + \delta/2$.*

Proof Let $m := \min\{k \in \{0, 1, 2 \dots\} : V_{2k+4} - 2\delta \leq L_2\}$. Note that m is well-defined because $s_*/2 - 2$ belongs to the set, as $V_{s_*} - 2\delta = 0 \leq L_2$ (Eq. (10)). At any odd state $2k + 1 \leq 2m + 1$ (hence $k - 1 < m$) we have $V_{2k+2} - 2\delta = V_{2(k-1)+4} - 2\delta > L_2$, with the last inequality due to the definition of m; hence by (6) in the state $s = 2k + 1$ the β player bids for sure, i.e. $\pi_{\beta,2k+1} = 1$. Thus, (25) implies that $q_{\beta,s} = q_{\gamma,s} = 1/2$ at any such odd state. Coupled with (26), that means the transition at every state s from 2 to $2m + 2$ is that the current β and γ players each have probability 1/2 to become the next α player. Thus,

$$V_2 = M_2 \sum_{k=0}^{m} 2^{-2k-1} + L_2 \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} z_m \right) - 2\delta \sum_{k=1}^{m} 2^{-2k}, \tag{28}$$

where $z_m := 1$ if $2m + 2 < s_* - 2$, and $z_m := 2\pi_{\gamma,s_*-1} - 1$ if $2m + 2 = s_* - 2$; and the last series $\sum_{k=1}^{m}$ on the right-hand side uses the summation notation defined in (27) when $m = 0$.

To understand the term for M_2 on the right-hand side, note that M_2 enters the calculation of V_2 at the even states $s = 2, 4, 6, \dots, 2m - 2$, and upon entry at state s and in every round transversing from states s to 2, the M_2 is discounted by the transition probability 1/2. The term for L_2 is similar, except that L_2 enters at the odd states $s = 3, 5, 7, \dots, 2m - 1$, and that the transition probability for the L_2 at the last state $2m - 1$ is equal to one if $2m - 1 < s_* - 1$, and equal to π_{γ,s_*-1} if $2m - 1 = s_* - 1$. That is why the last two terms within the bracket for L_2 are

$$2^{-2m-2} + 2^{-2m-2} z_m = \begin{cases} 2^{-2m-2} + 2^{-2m-2} = 2^{-2m-1} & \text{if } z_m = 1 \\ 2^{-2m-2} + 2^{-2m-2}\left(2\pi_{\gamma,s_*-1} - 1\right) = 2^{-2m-1}\pi_{\gamma,s_*-1} & \text{if } z_m = 2\pi_{\gamma,s_*-1} - 1. \end{cases}$$

The term for -2δ is analogous to that for M_2.

With $s_* \geq 4$, $V_2 - 4\delta \geq 0$. Thus, by the above-calculated transition probabilities,

$$L_2 = \frac{1}{2}(L_3 + V_2 - 3\delta) \leq \frac{1}{2}(V_2 - 4\delta + V_2 - 3\delta) = V_2 - \frac{7}{2}\delta.$$

This, combined with Eq. (28) and the fact $z_m \leq 1$ due to its definition, implies that

$$
\begin{aligned}
V_2 \leq & M_2 \sum_{k=0}^{m} 2^{-2k-1} + \left(V_2 - \frac{7}{2}\delta\right)\left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) - 2\delta \sum_{k=1}^{m} 2^{-2k} \\
< & M_2 \sum_{k=0}^{m} 2^{-2k-1} + V_2\left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) - \frac{7}{8}\delta.
\end{aligned}
$$

Thus, the lemma is proved if

$$
1 - \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) = \sum_{k=0}^{m} 2^{-2k-1}, \tag{29}
$$

as the left-hand side of this equation is clearly strictly between zero and one. To prove (29), we use induction on m. When $m = 0$, (29) becomes $1 - 2^{-2} - 2^{-2} = 2^{-1}$, which is true. For any $m = 0, 1, 2, \ldots$, suppose that (29) is true. We shall prove that the equation is true when m is replaced by $m + 1$, i.e.,

$$
1 - \left(\sum_{k=0}^{m+1} 2^{-2k-2} + 2^{-2(m+1)-2}\right) = \sum_{k=0}^{m+1} 2^{-2k-1}. \tag{30}
$$

The left-hand side of (30) is equal to

$$
\begin{aligned}
& 1 - \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) + 2^{-2m-2} - 2^{-2(m+1)-2} - 2^{-2(m+1)-2} \\
&= \sum_{k=0}^{m} 2^{-2k-1} + 2^{-2m-2} - 2^{-2(m+1)-1} \quad \text{(the induction hypothesis)} \\
&= \sum_{k=0}^{m} 2^{-2k-1} + 2^{-2m-3},
\end{aligned}
$$

which is equal to the right-hand side of (30). Thus (29) is true in general, as desired. □

Lemma 10 *If the dropout state is an even number* $s_* \geq 4$, *then* $\pi_{\beta,s} = 1$ *at any odd number state* s *such that* $1 \leq s \leq s_* - 2$.

Proof Pick any odd number s such that $s \leq s_* - 2$. It suffices to prove that $V_{s+1} - 2\delta > L_2$. Since $s + 1$ is even, it follows from (26) that

$$
V_{s+1} = \frac{1}{2}\left(M_2 + M_{s+2}\right) \geq \frac{1}{2}\left(M_2 + L_2\right),
$$

with the inequality due to the fact $M_{s+2} \geq L_2$, which in turn is due to the fact that the β player at state $s + 2$ can always secure the payoff L_2 through not bidding at all. Thus,

$$\begin{aligned}
V_{s+1} - 2\delta - L_2 &\geq \frac{1}{2}\left(M_2 + L_2\right) - 2\delta - L_2 \\
&= \frac{1}{2}M_2 - \frac{1}{2}L_2 - 2\delta \\
&= \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}L_3 + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\
&\geq \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}(V_2 - 4\delta) + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\
&= \frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta,
\end{aligned}$$

with the second inequality due to the definition of L_s and the fact $V_2 - 4\delta \geq 0$ ($s_* \geq 4$). Since $\frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta > 0$ by Lemma 9, $V_{s+1} - 2\delta - L_2 > 0$, as desired. □

Proof of Claims (ii) and (iii) of Theorem 1
Claim (iii) of the theorem follows directly from combining Lemmas 2.ii, 8 and 10. With Claim (iii) established, every non-α player in the state $s = 1$ (i.e., in the second round) bids for sure, hence (4) implies that the transition $V_1 \to M_2$ happens for sure. Thus, the payoff from becoming the initial frontrunner is equal to $M_2 - \delta$. By contrast, in failing to become the initial frontrunner a player gets the payoff M_1, which by (5) and Claim (iii) of the theorem is equal to $\frac{1}{2}\left(V_2 - 2\delta + L_2\right)$. Thus, the net gain from becoming the initial frontrunner is equal to

$$\begin{aligned}
M_2 - \delta - \frac{1}{2}(V_2 - 2\delta + L_2) &> V_2 + \delta/2 - \delta - \frac{1}{2}(V_2 - 2\delta + L_2) \\
&= \frac{1}{2}\left(V_2 - L_2 + \delta\right) > 0,
\end{aligned}$$

with the first ">" due to Lemma 9, and the last ">" due to (23). Thus, every player at the initial round strictly prefers being the frontrunner. Hence Claim (ii) of the theorem follows.

B.5 Lemma 3

All lemmas in this subsection assume the hypotheses in Lemma 3, that $s_* \geq 4$ is an even number and a strategy profile $\left(\pi_0, \pi_1, \left(\pi_{\beta,s}, \pi_{\gamma,s}\right)_{s=2}^{\infty}\right)$ according to Theorem 1 is given, with the associated value functions $\left(V_s, M_s, L_s\right)_s$ derived from (6)–(8) and Eq. (14).

Lemma 11 *For any positive integer m such that $2m + 1 \leq s_* - 1$, if $V_{2m+1} - 2\delta \leq L_2$ then $V_3 - 2\delta < L_2$.*

Proof Pick any m specified by the hypothesis such that $V_{2m+1} - 2\delta \leq L_2$. Suppose, to the contrary of the lemma, that $V_3 - 2\delta \geq L_2$. Thus, the law of motion (6) in the case $s = 2$, with $\pi_{\gamma,2} = 1$, implies that M_2 is between L_2 and $V_3 - 2\delta$, hence $V_3 - 2\delta \geq M_2 \geq L_2$. By the law of motion (7) in the case $s = 2m$, M_{2m} is a convex combination among zero, $V_{2m+1} - 2\delta$ and L_2. Thus the hypothesis implies that $M_{2m} \leq L_2$. Consequently, the law of motion (6) in the case $s = 2m - 1$, together with $\pi_{\gamma,2m-1} = 1$ and $M_2 \geq L_2$, implies that $V_{2m-1} \leq M_2$ and hence $V_{2m-1} - 2\delta \leq M_2 - 2\delta$. Then (7) in the case $s = 2m - 2$ implies $M_{2m-2} \leq L_2$. Repeating this reasoning backward, with 3 being odd, we eventually reach state $s = 3$ and obtain $V_3 \leq M_2$. But since $V_3 - 2\delta \geq M_2$, we have a contradiction $V_3 - 2\delta \geq M_2 \geq V_3$. □

Lemma 12 *Denote* $x := \pi_{\gamma,s_*-1}$. *For any integer* m *such that* $1 \leq m \leq s_*/2 - 1$,

$$M_{s_*-(2m-1)} = -\delta \sum_{k=1}^{m-1} 2^{-2k+2} + M_2 \sum_{k=1}^{m-1} 2^{-2k} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)} x \right), \tag{31}$$

$$V_{s_*-2m} = -\delta \sum_{k=1}^{m-1} 2^{-2k+1} + M_2 \sum_{k=1}^{m} 2^{-2k+1} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1} x \right), \tag{32}$$

$$\begin{aligned} V_{s_*-(2m-1)} = & -\delta \sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)}(1-x)v + L_2 \sum_{k=1}^{m-1} 2^{-2k} \\ & + M_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)} x \right), \end{aligned} \tag{33}$$

$$\begin{aligned} M_{s_*-2m} = & -\delta \sum_{k=0}^{m-1} 2^{-2k} + 2^{-2m+1}(1-x)v + L_2 \sum_{k=1}^{m} 2^{-2k+1} \\ & + M_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1} x \right), \end{aligned} \tag{34}$$

$$L_2 = \delta \left(s_* - 4 + 2^{-s_*+3} \right). \tag{35}$$

Proof First, we prove Eqs. (31) and (32). When $m = 1$, Eq. (31), coupled with the summation notation defined in (27), becomes $M_{s_*-1} = xL_2 = \pi_{\gamma,s_*-1} L_2$, which follows from (7) and the fact that $V_s = 2\delta$ and $M_s = 0$ for all $s \geq s_*$, due to Theorem 1. This coupled with Eq. (14) implies that

$$V_{s_*-2} = (M_{s_*-1} + M_2)/2 = M_2/2 + xL_2/2,$$

which is Eq. (32) when $m = 1$ (using again the summation notation in (27)). Suppose, for any integer m' with $1 \leq m' \leq s_*/2 - 2$, that Eqs. (31) and (32) are true with $m = m'$. By the induction hypothesis of (32) and Eq. (14),

$$M_{s_*-(2m'+1)} = \frac{1}{2}\left(V_{s_*-2m'} - 2\delta + L_2\right)$$
$$= -\delta\left(1 + \frac{1}{2}\sum_{k=1}^{m'-1} 2^{-2k+1}\right) + \frac{M_2}{2}\sum_{k=1}^{m'} 2^{-2k+1}$$
$$+ \frac{L_2}{2}\left(1 + \sum_{k=1}^{m'-1} 2^{-2k} + 2^{-2m'+1}x\right),$$

which is Eq. (31) when $m = m' + 1$. By the above calculation of $M_{s_*-(2m'+1)}$ and Eq. (14),

$$V_{s_*-(2m'+2)} = \frac{1}{2}\left(M_{s_*-(2m'+1)} + M_2\right)$$
$$= -\frac{\delta}{2}\left(1 + \frac{1}{2}\sum_{k=1}^{m'-1} 2^{-2k+1}\right) + \frac{M_2}{2}\left(1 + \sum_{k=1}^{m'} 2^{-2k+1}\right)$$
$$+ \frac{L_2}{4}\left(1 + \sum_{k=1}^{m'-1} 2^{-2k} + 2^{-2m'+1}x\right),$$

which is Eq. (32) in the case $m = m' + 1$. Thus Eqs. (31) and (32) are proved.

Next we prove Eqs. (33) and (34). When $m = 1$, Eq. (33), coupled with the notation $\sum_{k=1}^{0} a_k = 0$, becomes $V_{s_*-1} = (1-x)v + xM_2$, which is true by definition of x and the fact $\pi_{\beta,s_*-1} = 0$ (Theorem 1). Then by Eq. (14)

$$M_{s_*-2} = \left(V_{s_*-1} - 2\delta + L_2\right)/2 = \left((1-x)v + xM_2 - 2\delta + L_2\right)/2,$$

which is Eq. (34) when $m = 1$ (again using the notation $\sum_{k=1}^{0} a_k = 0$). Suppose, for any integer m' with $1 \le m' \le s_*/2 - 2$, that Eqs. (33) and (34) are true with $m = m'$. By the induction hypothesis and Eq. (14),

$$V_{s_*-(2m'+1)} = \frac{1}{2}\left(M_{s_*-2m'} + M_2\right)$$
$$= -\frac{\delta}{2}\sum_{k=0}^{m'-1} 2^{-2k} + 2^{-1}2^{-2m'+1}(1-x)v + \frac{L_2}{2}\sum_{k=1}^{m'} 2^{-2k+1}$$
$$+ M_2\left(2^{-1} + 2^{-1}\sum_{k=1}^{m'-1} 2^{-2k} + 2^{-1}2^{-2m'+1}x\right),$$

which is Eq. (33) in the case $m = m' + 1$. By the above calculation and Eq. (14),

$$
\begin{aligned}
M_{s_*-(2m'+2)} &= \frac{1}{2}\left(V_{s_*-(2m'+1)} - 2\delta + L_2\right) \\
&= -\delta\left(1 + \frac{1}{2}\sum_{k=1}^{m'} 2^{-2k+1}\right) + 2^{-1}2^{-2m'}(1-x)v \\
&\quad + L_2\left(\frac{1}{2} + 2^{-1}\sum_{k=1}^{m'} 2^{-2k}\right) + \frac{M_2}{2}\left(\sum_{k=1}^{m'} 2^{-2k+1} + 2^{-2m'}x\right),
\end{aligned}
$$

which is Eq. (34) in the case $m = m' + 1$. Hence Eqs. (33) and (34) are proved.

Finally we prove Eq. (35). Applying Eq. (14) to (8) recursively we obtain, for any integer $s_* \geq 4$, that

$$
\begin{aligned}
L_2 =& \frac{1}{2}\left(V_2 - 3\delta + \frac{1}{2}\left(V_2 - 4\delta + \frac{1}{2}\left(\cdots + \frac{1}{2}\left(V_2 - (s_* - 1)\delta\right)\right)\right)\right) \\
=& \frac{\delta}{2}\left(s_* - 3 + \frac{1}{2}\left(s_* - 4 + \frac{1}{2}\left(\cdots + \frac{1}{2}\cdot 1\right)\right)\right) \\
=& \delta\left(\frac{1}{2}(s_* - 3) + \frac{1}{2^2}(s_* - 4) + \frac{1}{2^3}(s_* - 5) + \cdots + \frac{1}{2^{s_*-3}}\right),
\end{aligned}
$$

which is equal to the right-hand side of (35). In the above multiline calculation, the first and second lines are due to $V_2 = s_*\delta$ (Theorem 1.iv). □

Lemma 13 $V_{s_*-2} - 2\delta \geq L_2 \Longrightarrow \forall m \in \{1, \ldots, s_*/2 - 1\} : V_{s_*-2m} - 2\delta \geq L_2$.

Proof By the law of motion and Eq. (14), Eqs. (31), (32), (33), (34) and (35) hold. Denote

$$
\begin{aligned}
\mu(m) &:= 2^{-2m+1}, \\
\mu_* &:= 2^{-s_*+3}.
\end{aligned}
$$

With the fact $\sum_{k=1}^{m-1} 2^{-2k} = (1 - 2^{-2m+2})/3$, Eq. (32) becomes

$$
\begin{aligned}
V_{s_*-2m} =& -\delta \cdot \frac{2}{3}(1 - 2\mu(m)) + M_2\left(\frac{2}{3}(1 - 2\mu(m)) + \mu(m)\right) \\
& + L_2\left(\frac{1}{3}(1 - 2\mu(m)) + \mu(m)x\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
V_{s_*-2m} - 2\delta - L_2 =& -\delta\left(\frac{2}{3}(1 - 2\mu(m)) + 2\right) + M_2\left(\frac{2}{3}(1 - 2\mu(m)) + \mu(m)\right) \\
& - L_2\left(1 - \frac{1}{3}(1 - 2\mu(m)) - \mu(m)x\right) \\
=& -\frac{4}{3}(2 - \mu(m))\delta + \frac{1}{3}(2 - \mu(m))M_2 \\
& - \left(s_* - 4 + \mu_*\right)\delta\left(\frac{2}{3}(1 + \mu(m)) - \mu(m)x\right),
\end{aligned}
$$

with the second equality due to (35). Thus, $V_{s_*-2m} - 2\delta \geq L_2$ is equivalent to

$$\frac{1}{3}(2-\mu(m))M_2 \geq \delta\Big(\frac{4}{3}(2-\mu(m)) + \big(s_* - 4 + \mu_*\big)\Big(\frac{2}{3}(1+\mu(m)) - \mu(m)x\Big)\Big),$$

i.e.,

$$\frac{M_2}{\delta} \geq 4 + \frac{2(1+\mu(m)) - 3\mu(m)x}{2-\mu(m)}(s_* - 4 + \mu_*). \tag{36}$$

Since $s_* - 4 \geq 0$ by hypothesis, and

$$\begin{aligned}\frac{d}{d\mu(m)}&\left(\frac{2(1+\mu(m)) - 3\mu(m)x}{2-\mu(m)}\right)\\ &= \frac{(2-\mu(m))(2-3x) + 2(1+\mu(m)) - 3\mu(m)x}{(2-\mu(m))^2}\\ &= \frac{6(1-x)}{(2-\mu(m))^2} \;\geq\; 0,\end{aligned}$$

the right-hand side of (36) is weakly increasing in $\mu(m)$, which in turn is strictly decreasing in m. Thus the right-hand side of (36) is weakly decreasing in m. Consequently, $V_{s_*-2m} - 2\delta - L_2 \geq 0$ is satisfied for all m if the inequality holds at the minimum $m = 1$, i.e., if $V_{s_*-2} - 2\delta - L_2 \geq 0$, as claimed. □

Proof of Lemma 3
Let $s \in \{1, 2, \ldots, s_* - 2\}$. If s is even and $V_3 - 2\delta \geq L_2$, then Lemma 11 implies $V_{s+1} - 2\delta > L_2$; thus, by (7) and by the fact that $\pi_{\gamma,s} = 1$ due to the strategy profile specified in Theorem 1, the β player at s gets L_2 if he does not bid, and $\frac{1}{2}(V_{s+1} - 2\delta) + \frac{1}{2}L_2$ if he does. Hence bidding is the unique best response for β at s. If s is odd and $V_{s_*-2} - 2\delta \geq L_2$, then Lemma 13 implies that $V_{s+1} - 2\delta \geq L_2$; thus, by the same token as in the previous case, the β player at s weakly prefers to bid. □

B.6 Theorem 2

Lemma 14 *For any even number $s_* \geq 4$, if Eqs. (14) and (35) hold and $M_2 \geq V_2 = s_*\delta$, then at the initial and second rounds each player strictly prefers to bid.*

Proof First, consider the second round, which means $s = 1$. For each non-α player, becoming the next α player gives him an expected payoff $V_2 - 2\delta = (s_* - 2)\delta$ by the hypothesis $V_2 = s_*\delta$, whereas staying put gives payoff L_2, which is less than $(s_* - 3)\delta$ by Eq. (35). Thus, each non-α player strictly prefers to bid at state one, hence $s = 2$ occurs for sure given $s = 1$. Second, consider the initial state. Based on the analysis of the previous step (from $s = 1$ to $s = 2$), becoming the first α yields the expected payoff $-\delta + M_2$, whereas staying put yields $\frac{1}{2}(V_2 - 2\delta + L_2)$. Since $M_2 \geq V_2$ by hypothesis and $V_2 - 2\delta > L_2$ by the previous analysis, each player strictly prefers to become the first α player. □

Lemma 15 *Any integer* $s_* \geq 3$ *constitutes an equilibrium if* s_* *is an even number and there exists* $(M_2, x, L_2) \in \mathbb{R}_+^3$ *such that—*

a. $(M_2, x, L_2) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+$ *and it solves simultaneously Eq.* (32) *in the case* $m = s_*/2 - 1$, *Eq.* (34) *in the case* $m = s_*/2 - 1$ *such that* $V_2 = s_*\delta$, *and Eq.* (35);
b. $M_2 \geq s_*\delta$ *(which is equivalent to* $V_3 - 2\delta \geq L_2$, *cf.* (12));
c. Ineq. (36) *is satisfied in the case* $m = 1$.

Proof Pick any even number $s_* \geq 4$ and assume Conditions (a)–(c). Consider the strategy profile such that everyone bids in the initial round and, in any future round, acts according to the strategy profile specified in Theorem 1. This strategy profile implies Eq. (14), which allows calculation of the value functions $(V_s, M_s, L_s)_{s=2}^{s_*}$ via the law of motions. By Conditions (a) and (b), $M_2 \geq V_2 = s_*\delta$, hence Lemma 14 implies that bidding at the initial round is a best response for each player, and bidding at second rounds a best response for each non-α player. The incentive for each player to abide by the strategy profile at any state $s \geq s_*$ is the same as in the surplus-dissipating subgame equilibrium. At the state $s_* - 1$, bidding with probability x is a best response for the γ player because he is indifferent about bidding, since $V_2 - s_*\delta = 0 = L_{s_*}$, and not bidding at all is the best response for the β player because $V_{s_*} - 2\delta = 0 < L_2$. At any state s with $2 \leq s \leq s_* - 2$, bidding is the best response for the γ player because $V_2 - (s+1)\delta > L_{s+1}$ (by Eq. (8)); Condition (c) by Lemma 13 suffices the incentive for the β player at every odd state to bid. To incentivize the β player at every even state $s \leq s_* - 2$ to bid, Lemma 11 says that it suffices to have $V_3 - 2\delta \geq L_2$, which is equivalent to $M_2 \geq L_2$ since, by the law of motion and Eq. (14), M_2 is the midpoint between $V_3 - 2\delta$ and L_2. Since $L_2 < s_*\delta$ by Eq. (35), the condition $M_2 \geq L_2$ is guaranteed by Condition (b), $M_2 \geq s_*\delta$. □

Lemma 16 *For any* $s_* \geq 4$, *Condition* (*c*) *in Lemma* 15 *implies Condition* (*b*) *in Lemma* 15.

Proof Condition (c) in Lemma 15 is Ineq. (36) in the case $m = 1$, i.e., when $\mu(m) = 2^{-2m+1} = 1/2$. Hence the condition is equivalent to

$$\frac{M_2}{\delta} \geq 4 + (2 - x)(s_* - 4 + \mu_*). \tag{37}$$

To prove that this inequality implies Condition (b), i.e., $M_2/\delta \geq s_*$, it suffices to show

$$4 + (2 - x)(s_* - 4 + \mu_*) > s_*,$$

i.e.,

$$(1 - x)(s_* - 4) + \mu_*(2 - x) > 0,$$

which is true because $s_* \geq 4$, $\mu_* = 2^{-s_*+3} > 0$ and $x \leq 1$. □

Lemma 17 *Condition* (*a*) *in Lemma* 15 *is equivalent to existence of an* $x \in [0, 1]$ *that solves Eq*. (15).

Proof Condition (a) requires existence of $(M_2, x, L_2) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+$ that satisfies Eqs. (32), (34) and (35) in the case of $m = s_*/2 - 1$ and $V_2 = s_*\delta$. Combine (32) with (35) and use the notation $\mu_* := 2^{-s_*+3}$ and the fact $\sum_{k=1}^{m-1} 2^{-2k} = (1 - 2^{-2m+2})/3$ to obtain

$$s_*\delta = V_2 = -\,\delta \cdot \frac{2}{3}(1 - 2\mu_*) + M_2\Big(\frac{2}{3}(1 - 2\mu_*) + \mu_*\Big) + \underbrace{\delta(s_* - 4 + \mu_*)}_{L_2}\Big(\frac{1}{3}(1 - 2\mu_*) + \mu_* x\Big),$$

i.e.,

$$\frac{M_2}{\delta} = \frac{1}{2 - \mu_*}\big(3s_* + 2(1 - 2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\big). \tag{38}$$

By the same token, (34) coupled with (35) is equivalent to

$$M_2\Big(1 - \frac{1}{3}(1 - 2\mu_*) - \mu_* x\Big) = -\,\delta\Big(1 + \frac{1}{3}(1 - 2\mu_*)\Big) + (1 - x)\mu_* v + \delta(s_* - 4 + \mu_*)\Big(\frac{2}{3}(1 - 2\mu_*) + \mu_*\Big),$$

i.e.,

$$\frac{M_2}{\delta}\big(2(1 + \mu_*) - 3\mu_* x\big) = \frac{3\mu_* v}{\delta}(1 - x) + (2 - \mu_*)(s_* - 6 + \mu_*). \tag{39}$$

Plug (38) into (39) and we obtain Eq. (15). □

Lemma 18 *For any even number* $s_* \geq 4$, *suppose that Eq*. (38) *holds. Then Condition* (*c*) *in Lemma* 15 *is equivalent to Ineq*. (16), *which is implied by* $x \geq 0$ *if and only if* $s_* \leq 6$.

Proof Condition (c) in Lemma 15 has been shown to be equivalent to Ineq. (37). Provided that Eq. (38) is satisfied, Ineq. (37) is equivalent to

$$\begin{aligned} &4 + (2 - x)(s_* - 4 + \mu_*) \\ \leq &\frac{1}{2 - \mu_*}\big(3s_* + 2(1 - 2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\big). \end{aligned}$$

This inequality, given the fact $1 - 2\mu_* \geq 0$, is equivalent to

$$x \geq \frac{1}{2(1 - 2\mu_*)}\Big(5 - 4\mu_* - \frac{3(s_* - 2)}{s_* - 4 + \mu_*}\Big),$$

i.e., Ineq (16). Given Condition (a) in Lemma 15, which implies $x \geq 0$, Ineq. (16) is redundant if and only if the right-hand side of (16) is nonpositive, i.e.,

$$\frac{3(2-\mu_*)}{2(1-2\mu_*)(s_*-4+\mu_*)} \geq 1,$$

i.e.,

$$s_* \leq 4 - 2^{-s_*+3} + \frac{3\left(2-2^{-s_*+3}\right)}{2\left(1-2^{-s_*+4}\right)}.$$

This inequality is satisfied when $s_* \in \{4, 6\}$, as its right-hand side is equal to ∞ when $s_* = 4$, and 61/8 when $s_* = 6$. The inequality does not hold, by contrast, when $s_* \geq 8$, as

$$\begin{aligned} s_* \geq 8 \Rightarrow & 2^{-s_*+2} \leq 2^{-6} \Rightarrow \frac{1-2^{-s_*+2}}{1/4-2^{-s_*+2}} \leq \frac{1-2^{-6}}{1/4-2^{-6}} = \frac{63}{15} \\ & \Rightarrow 4 - 2^{-s_*+3} + \frac{3\left(2-2^{-s_*+3}\right)}{2\left(1-2^{-s_*+4}\right)} < 4 + \frac{3}{2}\cdot\frac{2}{4}\cdot\frac{63}{15} < 8 \leq s_*. \end{aligned}$$

Thus, for all even numbers $s_* \geq 4$, Ineq. (37) follows if and only if $s_* \leq 6$. □

Lemma 19 *Any $s_* \geq 3$ constitutes an equilibrium if s_* is an even number and—*

i. either $s_* \leq 6$ and Eq. (15) admits a solution for $x \in [0, 1]$;
ii. or $s_* \geq 8$ and Eq. (15) admits a solution for $x \in [0, 1]$ such that Ineq. (16).

Proof The lemma follows from Lemma 15, where Condition (a) has been characterized by Lemma 17, Condition (b) by Lemmas 16 can be dispensed with, and Condition (c), by Lemma 18, can be dispensed with when $s_* \leq 6$ (hence Claim (i) of the lemma) and is equivalent to Ineq (16) when $s_* > 6$ (hence Claim (ii) of the lemma). □

Lemma 20 *If $x = 1$, the left-hand side of (15) is less than the right-hand side of (15).*

Proof When $x = 1$, the left-hand side of (15) is equal to $(2-\mu_*)^2(s_*-6+\mu_*)$, and the right-hand side equal to

$$\begin{aligned} & \big(2(1+\mu_*)-3\mu_*\big)\big(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_*)\big) \\ & \quad = (2-\mu_*)\big(2s_*+6-\mu_*-\mu_* s_*-\mu_*^2\big). \end{aligned}$$

Thus, the lemma follows if

$$(2-\mu_*)(s_*-6+\mu_*) < 2s_*+6-\mu_*-\mu_* s_*-\mu_*^2,$$

i.e., $9\mu_* < 18$, which is true because $\mu_* = 2^{-s_*+3}$. □

Lemma 21 $s_* = 4$ *constitutes an equilibrium if and only if* $v/\delta > 35/2$, *and* $s_* = 6$ *constitutes an equilibrium if and only if* $v/\delta > 6801/120$ (= 56.675).

Proof By Lemma 19, with $s_* \le 6$ the necessary and sufficient condition for equilibrium is that Eq. (15) admits a solution for $x \in [0, 1]$. By Lemma 20, the left-hand side of that equation is less than its right-hand side when $x = 1$. Thus, it suffices to show that the left-hand side is greater than the right-hand side when $x = 0$, i.e.,

$$\begin{aligned}&\frac{3\mu_* v}{\delta}(2-\mu_*) + (2-\mu_*)^2(s_* - 6 + \mu_*)\\ >&2(1+\mu_*)\big(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1-2\mu_*)\big),\end{aligned}$$

which is equivalent to

$$\frac{v}{\delta}(2-\mu_*) > s_*(4+\mu_*) + (6-\mu_*)(2/\mu_* - 2 - \mu_*).$$

Since μ_* is equal to 1/2 when $s_* = 4$, and equal to 1/8 when $s_* = 6$, the above inequality is equivalent to $v/\delta > 35/2$ when $s_* = 4$, and $v/\delta > 6801/120$ when $s_* = 6$. □

Proof of Theorem 2

Claims (i) and (ii) of the theorem are just Lemma 21. To prove Claim (iii), pick any even number $s_* \in \{8, 10, 12, \dots\}$. By Lemma 19.ii, s_* constitutes an equilibrium if Eq. (15) admits a solution for $x \in [0, 1]$ that satisfies Ineq. (16). By Lemma 20, the left-hand side of (15) is less than its right-hand side when $x = 1$. Thus, it suffices to show that the left-hand side is greater than the right-hand side when x is equal to some number greater than or equal to the right-hand side of Ineq. (16). To that end, note from $s_* \ge 8$ that $\mu_* = 2^{-s_*+3} \le 1/32$, hence $2 - \mu_* \ge 63/32$ and $1 + \mu_* < 33/32$. Thus, the left-hand side of (15) is greater than

$$\frac{3\mu_* v}{\delta}(1-x)\frac{63}{32} + \left(\frac{63}{32}\right)^2(s_* - 6),$$

and the right-hand side of Ineq. (16)

$$1 - \frac{3(2-\mu_*)}{2(1-2\mu_*)(s_* - 4 + \mu_*)} < 1 - \frac{3 \times 63/32}{2 \times 1 \times (s_* - 3)}.$$

Therefore, it suffices, for s_* to constitute an equilibrium, to have

$$\frac{3\mu_* v}{\delta}(1-x)\frac{63}{32} + \left(\frac{63}{32}\right)^2(s_* - 6)$$

greater than or equal to the right-hand side of (15) when

$$x = x_* := 1 - \frac{3 \times 63}{64(s_* - 3)}.$$

To that end, denote $\phi(s_*, x)$ for the right-hand side of (15), i.e.,

$$\begin{aligned}&\phi(s_*, x)\\ :=&\big(2(1+\mu_*)-3\mu_* x\big)\big(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_* x)\big)\end{aligned}$$

(recall that $\mu_* = 2^{-s_*+3}$). Note, from $0 < x < 1$, that $-1/32 < \mu_*(2-3x) < 1/16$. Hence

$$\begin{aligned}\frac{63}{32}=2-\frac{1}{32} <&2(1+\mu_*)-3\mu_* x < 2+\frac{1}{16}=\frac{33}{16},\\ \frac{15}{16}=1-\frac{1}{16} <&1-2\mu_*+3\mu_* x < 1+\frac{1}{32}=\frac{33}{32}.\end{aligned}$$

Thus, the first factor $2(1+\mu_*)-3\mu_* x$ of $\phi(s_*, x)$ is positive for all $x \in (0,1)$. If the second factor of $\phi(s_*, x)$ is nonpositive when $x = x_*$ then $\phi(s_*, x_*) \le 0$ and we are done, as the left-hand side of (15) is positive. Hence we may assume, without loss of generality, that

$$3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_* x_*) > 0.$$

Consequently, $\phi(s_*, x_*)$ can only get bigger if we replace its first factor by the upper bound 33/16, and the term $1-2\mu_*+3\mu_* x$ in the second factor by its lower bound 15/16 (note that, in the second factor, $s_*-4+\mu_* > 0$ because $s_* \ge 8$). I.e., $\phi(s_*, x_*)$ is less than

$$\begin{aligned}\frac{33}{16}\Big(3s_*+2(1-2\mu_*)-\frac{15}{16}(s_*-4+\mu_*)\Big) =&\frac{33}{16}\Big(\frac{33}{16}s_*+\frac{23}{4}-\frac{79}{16}\mu_*\Big)\\ <&\frac{33}{16}\Big(\frac{33}{16}s_*+\frac{23}{4}\Big)\\ <&5s_*+12.\end{aligned}$$

Therefore, the above observations put together, we are done if

$$\frac{3\mu_* v}{\delta}(1-x_*)\frac{63}{32}+\left(\frac{63}{32}\right)^2(s_*-6) \ge 5s_*+12$$

In other words, it suffices to have

$$\frac{3\mu_* v}{\delta}\cdot\frac{3\times 63}{64(s_*-3)}\cdot\frac{63}{32}+\left(\frac{63}{32}\right)^2(s_*-6) \ge 5s_*+12,$$

i.e.,

$$\frac{3^2\mu_* v}{\delta} \ge -2(s_*-6)(s_*-3)+\frac{32\times 64}{63^2}\big(5s_*+12\big)(s_*-3).$$

With $\frac{32\times 64}{63^2} \approx 0.516$, the above inequality holds if

$$\frac{3^2\mu_* v}{\delta} \geq -2(s_* - 6)(s_* - 3) + (5s_* + 12)(s_* - 3),$$

i.e.,

$$\frac{9\mu_* v}{\delta} \geq 3s_*^2 + 15s_* - 72,$$

which, coupled with $\mu_* = 2^{-s_*+3}$, is equivalent to the hypothesis (17) of Claim (iii).

□

B.7 Theorems 3

First, when $s = s_* = 1$, clearly $M_s = \pi_{\beta,s_*-1}L_2 = \pi_{\beta,s_*-1}\delta/2 > 0$ and $L_s = L_{s_*-1} = 0$. Second, we show that if $s < s_* - 1$ then $M_s > 0$ and $L_s > 0$. By Eq. (8) and Theorem 1, $L_s \geq \frac{1}{2}(V_2 - (s+1)\delta) = \frac{1}{2}(s_*\delta - (s+1)\delta) > 0$, with the last inequality due to $s \leq s_* - 2$. By (7), the follower can secure an expected payoff no less than L_2, through staying put at s (while the underdog bids for sure), hence $M_2 \geq L_2 > 0$. Third, for any state $s \leq s_* - 1$, we show that $V_s > 2\delta$. With $s \leq s_* - 1$, the frontrunner's surplus, by (6), is

i. either $V_s = (M_2 + M_{s+1})/2$ (if $s < s_* - 1$)
ii. or $V_s = (1 - \pi_{\gamma,s_*-1})v + \pi_{\gamma,s_*-1}M_2$ (if $s = s_* - 1$).

In Case (i), his surplus is

$$(M_2 + M_{s+1})/2 > M_2/2 > \big(V_2 + \delta/2\big)/2 \geq (4\delta + \delta/2)/2 > 2\delta,$$

with the second inequality due to Lemma 9, and the third inequality due to $V_2 = s_*\delta$ and $s_* \geq 4$. In Case (ii), his surplus is

$$(1 - \pi_{\gamma,s_*-1})v + \pi_{\gamma,s_*-1}M_2 > (1 - \pi_{\gamma,s_*-1})2\delta + \pi_{\gamma,s_*-1}4\delta > 2\delta,$$

with the first inequality due to $v > 2\delta$ by assumption, the fact $M_2 > V_2$ by Lemma 9, and the fact $V_2 \geq 4\delta$ as in Case (i) □

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