Optimal Splitting among General Distributions

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Abstract

Le Treust and Tomala's (2019) solution for constrained information design is restricted to discrete distributions. This note extends it to general distributions on $\mathbb{R}$.

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The concavification technique in Aumann, Maschler and Stearns [1], due to Kamenica and Gentzkow's [5] introduction, has become the standard workhorse in the large and growing literature on information design and further applied to the design of large markets by Dworczak, Kominers and Akbarpour [4] (henceforth DKA). A recent development of the concavification technique is Le Treust and Tomala's [6] extension that allows for an inequality constraint, which can capture the capacity constraint in information transmission or the budget balancing constraint in market design. However, the choice set they consider is restricted to discrete distributions, while the works that apply their solution (e.g., DKA [4]) would look for an optimum among all distributions.[1] This note proves that Le Treust and Tomala's solution is also optimal among all general distributions defined on $\mathbb{R}$.

Let $\Delta[0,1]$ denote the set of all cumulative distribution functions on $\mathbb{R}$ whose supports are contained in $[0,1]$. Given any $(x,z) \in [0,1] \times \mathbb{R}$ and any integrable functions $g, h : [0,1] \to \mathbb{R}$, the optimal splitting problem is:

$$\begin{aligned} G(x,z) \;:=\; \sup\nolimits_{\mu\in\Delta[0,1]} & \int_0^1 g d\mu \\ \text{s.t.} \quad & \int_0^1 s d\mu(s) = x \\ & \int_0^1 h d\mu \geq z. \end{aligned} \tag{1}$$

In DKA [4], the x corresponds to the average quantity for the market under consideration that the designer wants to split into a distribution, and the inequality constraint is the part of the budget balancing condition that this market needs to satisfy. Note that the choice set $\Delta[0,1]$ in (1) includes all distributions on $\mathbb{R}$ with any support in $[0,1]$. However, Le Treust and Tomala restrict the choice set to the distributions whose supports are discrete. To state the problem they have solved, denote for any $n \in \{1, 2, \ldots\}$

$$\mathscr{F}_n := \left\{ (\alpha_m, x_m)_{m=1}^n \mid \forall m \, [x_m \in [0,1]] \, ; (\alpha_m)_{m=1}^n \in \Delta\{x_m \mid m = 1, \ldots, n\} \right\},$$

where ΔS denotes the set of probability measures on S. That is, an element $(\alpha_m, x_m)_{m=1}^n$ of $\mathscr{F}_n$ is a discrete probability measure that assigns a probability α_m to x_m ($m = 1, \ldots, n$).

[1] Citing Le Treust and Tomala's solution, Doval and Skreta [3] restrict the choice set to distributions with finite supports. See their proofs of Props. A.3 and 5.1 for the role played by the finite-support restriction.

Le Treust and Tomala have solved the following discrete counterpart to (1):

$$\begin{aligned} \mathscr{G}(x,z) \; := \; \sup_{(\alpha_m,x_m)_{m=1}^n \in \bigcup_{n=1}^\infty \mathscr{F}_n} \quad & \sum_m^n \alpha_m g(x_m) \\ \text{s.t.} \quad & \sum_m^n \alpha_m x_m = x \\ & \sum_m^n \alpha_m h(x_m) \geq z. \end{aligned} \tag{2}$$

The question is to what extent Le Treust and Tomala's solution applies to (1). While it seems intuitive that general distributions can be approximated by the discrete distributions in (2), the approximations need to satisfy the equality constraint and in doing so might violate the inequality constraint. Thus the implication requires a proof. With the theorem presented next, Le Treust and Tomala's solution does apply to (1) when g and h are continuous.

Theorem *If $g, h : [0,1] \to \mathbb{R}$ are continuous, then $G(x,z) = \mathscr{G}(x,z)$ for any $(x,z) \in [0,1] \times \mathbb{R}$.*

Lemma *If $g, h : [0,1] \to \mathbb{R}$ are continuous, then for any $(x,z) \in [0,1] \times \mathbb{R}$,*

$$\lim_{\epsilon \downarrow 0} \mathscr{G}(x, z - \epsilon) = \mathscr{G}(x,z). \tag{3}$$

Proof For any $(x,z) \in [0,1] \times \mathbb{R}$, denote

$$\Gamma(z) := \left\{ (\alpha_m, x_m)_{m=1}^n \in \bigcup_{n=1}^\infty \mathscr{F}_n \;\middle|\; \sum_m \alpha_m x_m = x; \sum_m \alpha_m h(x_m) \geq z \right\}.$$

Pick any $z \in \mathbb{R}$ and let $(z^k)_{k=1}^\infty$ converge to z such that $z^k \leq z$ for all k.[2] (The hypothesis $z^k \leq z$ for all k is justified because the limit in (3) is taken when ϵ is converges to zero.)

Note: If $\Gamma(z) \neq \varnothing$ then $\Gamma(z^k) \neq \varnothing$ for all k. That is because $z^k \leq z$ and hence $\Gamma(z^k) \supseteq \Gamma(z)$ for all k. Consequently, if $\Gamma(z^k) = \varnothing$ for all sufficiently large k then $\Gamma(z) = \varnothing$, so by (2) both sides of (3) are equal to $-\infty$.[3]

[2] The theorem of maximum is not readily available for the lemma, because one needs to topologize the space $\bigcup_{n=1}^\infty \mathscr{F}_n$ appropriately even before establishing the desired properties of Γ (cf. Ausubel and Deneckere [2]).

[3] We adopt the convention that $\sup \varnothing := -\infty$.

Thus, assume without loss of generality that for any k there exist an n_k and an $(\alpha_m^k, x_m^k)_{m=1}^{n_k} \in \Gamma(z^k)$. For each k, letting $\alpha_m^k := x_m^k := 0$ for all $m > n_k$, we can write $(\alpha_m^k, x_m^k)_{m=1}^{n_k}$ equivalently as $(\alpha_m^k, x_m^k)_{m=1}^{\infty}$. Thus we have a sequence $((\alpha_m^k, x_m^k)_{m=1}^{\infty})_{k=1}^{\infty}$ such that, for each k, $\alpha_m^k = x_m^k = 0$ for all sufficiently large m. Since $(\alpha_m^k, x_m^k) \in [0,1]^2$ for all m and k, we can extract by the diagonal trick an infinite subsequence $((\alpha_m^{k_j}, x_m^{k_j})_{m=1}^{\infty})_{j=1}^{\infty}$ that converges to some $(\alpha_m, x_m)_{m=1}^{\infty}$ such that $\alpha_m = x_m = 0$ for all $m > n$ for some n. That is, $((\alpha_m^{k_j}, x_m^{k_j})_{m=1}^{\infty})_{j=1}^{\infty}$ converges to some $(\alpha_m, x_m)_{m=1}^{n} \in \mathscr{F}_n$ for some n. For each j, since $(\alpha_m^{k_j}, x_m^{k_j})_{m=1}^{\infty} \in \Gamma(z^{k_j})$, we have $\sum_j \alpha_m^{k_j} x_m^{k_j} = x$ and $\sum_j \alpha_m^{k_j} h(x_m^{k_j}) \geq z_{k_j}$. Take the limit to $j \to \infty$ and use the continuity of h to obtain $\sum_m \alpha_m x_m = x$ and $\sum_m \alpha_m h(x_m) \geq z$. Thus $(\alpha_m, x_m)_{m=1}^{n} \in \Gamma(z)$.

If $\mathscr{G}(x, z^k) = \infty$ for all sufficiently large k, there exists a sequence $\left((\alpha_m^k, x_m^k)_{m=1}^{n_k}\right)_{k=1}^{\infty}$ such that $(\alpha_m^k, x_m^k)_{m=1}^{n_k} \in \Gamma(z^k)$ for any k and $\lim_{k\to\infty} \sum_m \alpha_m^k g(x_m^k) = \infty$. By the previous paragraph, this sequence has an infinite subsequence that converges to some $(\alpha_m, x_m)_{m=1}^{n} \in \Gamma(z)$ and, by continuity of g, $\mathscr{G}(x, z) \geq \sum_m \alpha_m g(x_m) = \lim_{k\to\infty} \sum_m \alpha_m^k g(x_m^k) = \infty$. Thus (3) holds in this case.

Thus, assume without loss of generality that, for any k, $\mathscr{G}(x, z^k) < \infty$ and hence the maximization problem corresponding to (2) is solved by some $(\alpha_m^k, x_m^k)_{m=1}^{n_k} \in \Gamma(z^k)$. As shown above, we can extract an infinite subsequence $((\alpha_m^{k_j}, x_m^{k_j})_{m=1}^{\infty})_{j=1}^{\infty}$ that converges to some $(\alpha_m, x_m)_{m=1}^{n} \in \Gamma(z)$. We claim that $(\alpha_m, x_m)_{m=1}^{n}$ solves Problem (2) given z. To show that, consider any $(\hat{\alpha}_m, \hat{x}_m)_{m=1}^{n} \in \Gamma(z)$. For all j, since $z^{k_j} \leq z$ by hypothesis, $(\hat{\alpha}_m, \hat{x}_m)_{m=1}^{n} \in \Gamma(z^{k_j})$. Then the optimality of $(\alpha_m^{k_j}, x_m^{k_j})_{m=1}^{n_{k_j}}$ given z^{k_j} implies $\sum_m \alpha_m^{k_j} g(x_m^{k_j}) \geq \sum_m \hat{\alpha}_m g(\hat{x}_m)$ for all j. Take the limit to $j \to \infty$ and use the continuity of g to obtain $\sum_m \alpha_m g(x_m) \geq \sum_m \hat{\alpha}_m g(\hat{x}_m)$. Thus the claim is true. It follows that

$$\mathscr{G}(x, z) = \sum_m \alpha_m g(x_m) = \lim_{j\to\infty} \mathscr{G}(x, z^{k_j}).$$

For any infinite subsequence that attains $\limsup_k \mathscr{G}(x, z^k)$, we apply the above reasoning to further extract an infinite subsequence for which the above-displayed equation holds. The same is true for any infinite subsequence that attains $\liminf_k \mathscr{G}(x, z^k)$. Thus we obtain

$$\limsup_k \mathscr{G}(x, z^k) = \mathscr{G}(x, z) = \liminf_k \mathscr{G}(x, z^k),$$

namely, (3). ■

Proof of the Theorem Clearly $G(x,z) \geq \mathscr{G}(x,z)$. Conversely, to prove $G(x,z) \leq \mathscr{G}(x,z)$, pick any $\epsilon > 0$ and any $\mu \in \Delta[0,1]$ feasible to (1), so $\int s d\mu(s) = x$ and $\int h d\mu \geq z$. According to Miller and Rice [7], the distribution μ can be approximated by a subset $\{(\alpha_m, x_m)_{m=1}^n : n\}$ of $\bigcup_n \mathscr{F}_n$ that preserves the mean x of μ and, with g and h assumed continuous on $[0,1]$ and hence each approximated uniformly by polynomials (Stone-Weierstrass Theorem), the sums $\sum_m \alpha_m g(x_m)$ and $\sum_m \alpha_m h(x_m)$ approximate the integrals $\int g d\mu$ and $\int h d\mu$. Thus, there exists $(\alpha_m, x_m)_{m=1}^n \in \mathscr{F}_n$ for some n such that

$$\begin{aligned} \sum_m \alpha_m x_m &= \int_0^b s d\mu(s) \;=\; x, \\ \sum_m \alpha_m g(x_m) &\geq \int_0^b g d\mu - \epsilon, \\ \sum_m \alpha_m h(x_m) &\geq \int_0^b h d\mu - \epsilon \;\geq\; z - \epsilon. \end{aligned}$$

Consequently, $(\alpha_m, x_m)_{m=1}^n \in \Gamma(z-\epsilon)$ and hence

$$\mathscr{G}(x, z-\epsilon) \geq \sum_m \alpha_m g(x_m) \geq \int g d\mu - \epsilon.$$

This being true for all μ feasible to the problem that defines $G(x,z)$, we have

$$\mathscr{G}(x, z-\epsilon) \geq G(x,z) - \epsilon.$$

Taking the limit to $\epsilon \to 0$ and applying the lemma, we obtain $\mathscr{G}(x,z) = G(x,z)$. ■

References

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