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Necessary and sufficient conditions for peace: Implementability versus security

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Abstract

This paper investigates the conditions for full preemption of conflicts in the form of all-pay auctions. I define two notions of conflict preemption: to *implement* peace on path with commonly expected continuation plays should one veto a peace proposal, or to *secure* that each player accepts a peace proposal no matter what continuation play he might expect to occur should he veto it. For each notion I prove a necessary and sufficient condition in terms of the primitives. The conditions imply that peace cannot be secured when the infimum of a player's type support is sufficiently low, regardless of the distribution functions of the players' types. The conditions also imply that peace can be implemented even when each player forecasts that should he veto peace the cost he incurs in the ensuing conflict is infinitesimal. The findings are obtained through a distributional method on two-player all-pay auctions that unifies the methods previously separated by discrete versus continuous distributions.

JEL classification: D44; D74; D82

Keywords: Conflict mediation; All-pay auction; Outside option; Endogenous conflict; Asymmetric auction; Distributional strategy

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1. Introduction

Under what conditions can two rivals avoid conflicts through a mediated settlement? While the conditions must be that the outcome of conflict be sufficiently unattractive to each player, the question is What does such unattractiveness mean in terms of the primitives? The answer implied by the conflict mediation literature, such as Bester and Wärneryd (2006), Compte and Jehiel (2009), Fey and Ramsay (2011), Hörner et al. (2015) and Spier (1994), is that each player has a sufficiently bad exogenous outside option as the alternative to peace. This exogenous outside option is either the player's nonparticipation payoff as his type, or his expected payoff from an exogenous lottery determined by the two players' types. Thus, peace is guaranteed if each player's nonparticipation-payoff type is distributed on a sufficiently low support, or if each player's lottery-winning type is so stochastically dominant that the opponent's expected payoff from triggering the conflict lottery is sufficiently small.

The answer is more complicated, and different to a large extent, than the above if the outcome of the conflict is determined not by an exogenous payoff or lottery but rather by an endogenous continuation play during the conflict. That means a player's assessment of his outside option depends on how he thinks his opponent would do should conflict ensue. Specifically, this paper considers conflict as an all-pay auction for the contested prize, with each player's type equal to the reciprocal of his marginal cost of bids, interpreted as his strength level in the conflict. To see how our answer may differ from the above, suppose that player 1's type is drawn from a distribution supported by $[0, \epsilon]$. Consider his decision, given some type $t \in (0, \epsilon)$, on whether to veto (unilaterally reject) a peaceful split of the prize proposed by the mediator. Suppose, for the moment, that player 1 thinks that, should he veto peace, his opponent would believe that player 1's type is zero. Driven by this belief, player 2 would bid arbitrarily close to zero; then player 1 would think that he can easily win the auction by bidding slightly above zero. Consequently, in contrast to the above literature, player 1 would reject any peace proposal that offers him less than the full prize, no matter how small the supremum ϵ of his type support is, and no matter how stochastically dominant his opponent's type distribution is.

Thus, the prospect of conflict preemption depends on what each player forecasts as the continuation play should conflict ensue. With conflict off path in any equilibrium that fully preempts conflict, such forecasts are arbitrary, not subject to Bayes's rule. That leads to different notions of conflict preemption, depending on the degree to which the mediator can coordinate the players into having the same forecast about continuation plays in off-path events. For a mediator with such coordination power is the notion *implementability* of peace, meaning that a peace proposal admits a perfect Bayesian equilibrium (PBE) on the path of which conflict occurs with zero probability. By contrast, for a mediator without such coordination power is the notion *security* of peace, meaning that every type of each player is willing to accept the peace proposal no matter what continuation play he forecasts to occur in the event where he vetoes the proposal. For each of the two notions, this paper delivers a necessary and sufficient condition, in terms of the prior distributions of the players' types, for full preemption of conflict (Theorem 1).

The condition for peace security has an unprecedented implication: When the infimum of a player's type support is sufficiently lower than its supremum, peace is not securable, regardless of any other aspect of the players' type distributions (Corollaries 1 and 2.b and Theorem 3). Even if there is a conflict-preempting PBE, a player may forecast a different off-path continuation play than what the PBE prescribes, so he would find it profitable to deviate. Such disagreements in forecasting off-path plays have been justified by the self-confirming equilibrium literature such as Fudenberg and Levine (1993). In particular, the player may predict that, in the off-path event where he vetoes peace, his opponent will be complacent in the conflict because of her posterior belief that his type is the infimum; then he would rather engage her in the conflict and take advantage of her complacency, however stochastically dominant is her type distribution. This result does not require enlarging the spread or riskiness of a player's type distribution. In the previous example, the result obtains even when the support $[0, \epsilon]$ of player 1's type is arbitrarily narrow. Here is a discontinuity of security between ϵ-likelihoods and zero-likelihoods: Start with a case where peace is secured and slightly perturb each player i's type distribution from its support $[a_i, z_i]$ so that the type belongs to $[a_i, z_i]$ with probability $1 - \epsilon$ and otherwise belongs to $[a_i/2, a_i/2 + \epsilon]$, then Theorem 3 implies that peace is not securable, however small is the positive ϵ.

The condition for peace implementability, however, has precedents in the conflict mediation literature, but there is an important difference. The similarity is in the implication that the prospect of satisfying the condition is improved when a player's type distribution becomes more stochastically dominant than before (Theorem 2). An important difference is that to implement peace the literature also relies on an assumption that conflict reduces the value of the contested prize by a sufficiently large exogenous cost, whereas this paper assumes no exogenous cost of conflict. Furthermore, in the PBE that this paper constructs to implement peace, all but one type of each player expects to incur zero or arbitrarily small cost in the off-path event that he vetoes peace (Theorem 7.a.ii). Thus, from the perspective that the cost of conflicts is endogenous, peace is implementable even when each player expects that conflict is not costly should he trigger it. This finding indicates that, regarding the question Why peace is implementable, the exogenous cost assumption would overstate the importance of the destructiveness of conflict, though the assumption may be justifiable with respect to the question Why conflict happens despite its costs, which concerned much of the conflict mediation literature.

To obtain a condition both necessary and sufficient for all types of each player to accept a peace proposal, one would need to characterize the entire set of endogenous outside options for all of his types, including the deviating types that are not expected in, and may (due to possibility of ties) have no best response to, the continuation play in the conflict. This task becomes tractable because the supremum among a player's expected payoffs in responding to a continuation play—when his bid ranges in $\mathbb{R}_+$—is monotone and continuous in his type (Theorem 5) and hence it suffices to characterize the set of all outside options only for the strongest type of each player. This set corresponds to all the Bayesian Nash equilibria (BNE) of the all-pay auction in the off-path event where the player vetoes peace, with each BNE rationalized by an off-path posterior belief about the vetoer. Since off-path posteriors are arbitrary, we need to characterize the BNEs of the all-pay auction given arbitrary type distributions, allowing for gaps and atoms.

In solving the all-pay auction game with arbitrary type distributions, this paper develops a distributional method generalized from Vickrey (1961, Section II) and Milgrom and Weber (1985). The method, encapsulated by Eqs. (9) and (19), unifies the previously separate approaches to two-player all-pay auctions in the literature, one based on discrete or degenerate distributions, and the other, continuous, strictly increasing, and often identical distributions. The first approach is not conducive to a general formula for equilibria, which we need in order to compare their performances; the second one provides general formulas but it relies on the pure strategy of an equilibrium and the invertibility thereof to map one's bid to the other's type submitting the same bid, whereas we need to handle mixed and non-invertible strategies due to type distributions with atoms and gaps. Given such general settings, my method obtains new properties of the bid-to-type correspondence despite its possible discontinuities (Sections 5 and B.5) and characterizes the equilibrium in terms of its distributions of bids (Appendix B.6). The result generalizes the second approach (e.g., Amann and Leininger, 1996 and Kirkegaard, 2008) and includes all cases handled by the first one, except when types are correlated across bidders (Krishna and Morgan, 1997; Siegel, 2014 and Lu and Parreiras, 2017).[1]

The all-pay auction game solved given arbitrary type distributions, this paper finds for each player's strongest type the posterior belief that rationalizes the best BNE, and the posterior belief that rationalizes the worst BNE, among the BNEs in the off-path event where he vetoes peace (Theorem 7). What the best BNE provides for this type of the player is the minimum payoff that a peace proposal needs to offer the player in order to secure his acceptance whichever off-path BNE he might anticipate; what the worst BNE provides is the minimum payoff to make his acceptance a best response to some off-path BNE. These minimum peaceful payoffs are derived from the parameters explicitly (Eqs. (7) and (8)). Thus come the necessary and sufficient condition for peace to be securable, and that for peace to be implementable (Theorem 1).

Balzer and Schneider (2018) have independently considered conflict mediation with endogenous conflict. They provide characterization of conflict-probability-minimizing mechanisms in terms of the on-path posterior belief system in the associated equilibria given the assumption that both players are drawn from an identical discrete distribution and that conflict cannot be fully preempted. Celik and Peters (2011) have considered endogenous outside options in an oligopoly environment of cartel formation. Their focus is the possible loss of generality due to the full participation condition on mechanisms.

After presenting the primitives, Section 2 defines the two notions of conflict preemption. The conditions for conflict preemption according to these notions are derived in Section 3, which assumes a minimum set of properties of the conflict stage so that the derivation could shed a light on conflicts that are not necessarily all-pay auctions. The two conditions for conflict preemption, implementability versus security, are contrasted in Section 4, as well as in the examples after Theorem 1. Then Section 5 presents a general, distributional approach to two-player all-pay auctions that delivers the properties of the conflict stage assumed in Section 3. All formally stated claims are proved in the Appendix, in their order of appearance.

2. The preliminaries

The primitives Two players, indexed by $i \in \{1, 2\}$, compete for a prize. First, each player i's type t_i is independently drawn according to a commonly known cumulative distribution function (c.d.f.) F_i and becomes i's private information. Assume $F_i(0) = 0$, let $[a_i, z_i]$ be the convex hull of supp F_i, the support of F_i, and assume $0 \le a_i < z_i$. Second, a neutral mediator proposes a peaceful split in the form of $v \in [0, 1]$. Third, each player independently announces whether to accept or reject the proposed split. If both accept it, the game ends with player 1 getting

[1] Krishna and Morgan, Siegel, and Lu and Parreiras allow for correlation between players but restrict the extent of correlation to retain monotonicity of the equilibrium strategy (cf. Footnote 3). Lu and Parreiras also use a technique, dating back to Milgrom and Weber (1985), that transforms any continuous and strictly increasing distribution into the uniform distribution on $[0, 1]$. This technique does not simplify our task of comparing the BNEs rationalized by different type distributions, not even among continuous and strictly increasing ones, because such transformation from different distributions generates different transforms of each bidder's valuation functions, hence the task of comparing distributions would become comparing such valuation transforms. For references to other all-pay auction literature, see Kaplan and Zamir (2015).

a payoff equal to v, and player 2 getting $1 - v$.[2] If at least one of them rejects the proposed split, the game enters its final, conflict stage, where each player i, after observing each other's response to the proposed split, submits a sealed bid $b_i \in \mathbb{R}_+$. The player who submits the higher bid wins the prize, with ties broken randomly with equal probabilities. Then player i's payoff is equal to $1 - b_i/t_i$ if he wins the prize, and $-b_i/t_i$ if otherwise. When $t_i = 0$, the notation b_i/t_i means ∞ if $b_i > 0$, and zero if $b_i = 0$. Both players are assumed risk neutral.

Implementability versus security of peace Given any proposed split, a multistage game is defined. Denote $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ for the continuation game in the conflict stage such that $\tilde{F}_1$ is the posterior distribution of players 1's type, and $\tilde{F}_2$ that of player 2's. Since this paper considers only equilibria where conflict is off path, let us focus on the off-path event where player i has deviated through vetoing (or unilaterally rejecting) the peaceful split. In that event, Bayes's rule implies $\tilde{F}_{-i} = F_{-i}$, as the other player $-i$ accepts the split almost surely. Whereas, $\tilde{F}_i$ is off-path and hence arbitrary. In the spirit of the "no signaling what you don't know" condition in Fudenberg and Tirole (1991), I impose an *independence condition* on $\tilde{F}_i$: If player i makes a unilateral deviation in responding to a proposed split, then $\tilde{F}_i$ is independent of the realized type of $-i$. This condition rules out scenarios where the players' types, assumed stochastically independent at the outset, suddenly become correlated without the two having had any communication.[3] Given any such $\tilde{F}_i$, a continuation equilibrium in the event that results from player i's unilateral deviation is an element of

$$\mathscr{E}_i(\tilde{F}_i) := \text{the set of BNEs of } \mathscr{G}(\tilde{F}_i, F_{-i}), \tag{1}$$

where BNE stands for Bayesian Nash equilibrium. Thus, I restrict the notion *perfect Bayesian equilibrium* (PBE) by the condition that, for each $i \in \{1, 2\}$, the continuation play conditional on player i's unilateral deviation is an element of

$$\mathscr{E}_i := \bigcup \left\{ \mathscr{E}_i(\tilde{F}_i) : \operatorname{supp} \tilde{F}_i \subseteq \operatorname{supp} F_i;\ \tilde{F}_i \text{ and } F_{-i} \text{ are independent} \right\}. \tag{2}$$

A PBE is said *peaceful* if and only if on its path conflict occurs with zero probability relative to the prior distributions. Peace is said *implementable* if and only if a peaceful PBE exists in the multistage game given some proposed split.

However, it may be unrealistic that players have the same forecast on the continuation equilibria in off-path events. To preempt conflict through a peaceful PBE, a mediator needs to be able to tell (implicitly) the players which continuation equilibrium to play in the off-path event of conflict; should it be off path, she would be unable to check empirically whether they abide by her coordination.[4] Thus a stronger concept than implementability is germane. A peaceful PBE is said *secure* if and only if, for each player $i \in \{1, 2\}$, for almost every type of i, and for any

[2] Restriction to such fixed splits, which each player can only accept or reject, causes no loss of generality in this paper, which concerns the existence of equilibria that induce zero occurrence of conflict and assumes that a player's type affects his payoff only when conflict occurs. See Appendix A for more explanation.

[3] The independence condition ensures monotonicity of any equilibrium strategy in the continuation game (Lemma 6), needed for Lemma 2. Without the independence condition, if the correlation is sufficiently small, monotonicity can still be guaranteed (cf. Footnote 1). But if the correlation is strong then monotonicity cannot be guaranteed (Rentschler and Turocy, 2016).

[4] For further explanations see the self-confirming equilibrium literature.

$\sigma \in \mathscr{E}_i$,[5] if σ is the continuation play conditional on his deviation of rejecting the proposed split, it is a best response for i to accept the peace proposal. Peace is said *securable* if and only if a secure, peaceful PBE exists in the multistage game given some proposed split.

3. The conditions for peace

For any $i \in \{1, 2\}$, any $\sigma \in \mathscr{E}_i$ and any $t_i \in \operatorname{supp} F_i$, denote $U_i(t_i|\sigma)$ for the supremum among the expected payoffs during the conflict stage for the type t_i of player i when his bid ranges in $\mathbb{R}_+$, expecting the other player, $-i$, to abide by σ. It is easy to show that $U_i(\cdot|\sigma)$ is weakly increasing on $[a_i, z_i]$ (Lemma 3, Appendix B.1). Consequently, for each player i the type most tempted to veto a proposed split is z_i: If σ is the continuation play that i anticipates for the event where i vetoes the proposal, then accepting the proposed split is a best response for all types of player i if and only if the split would give i at least $U_i(z_i|\sigma)$. Thus, among the class of peaceful PBEs, the lowest possible peaceful payoff that can induce acceptance from all types of player i is

$$\underline{u}_i := \inf\{U_i\,(z_i|\sigma) : \sigma \in \mathscr{E}_i\}. \tag{3}$$

For a peaceful PBE to be secure, however, acceptance needs to be a best response for almost all types of player i no matter which $\sigma \in \mathscr{E}_i$ player i expects as the continuation play. Thus, for a proposed split to have a securely peaceful PBE, it needs to offer player i at least

$$\overline{u}_i := \sup\{U_i\,(z_i|\sigma) : \sigma \in \mathscr{E}_i\}. \tag{4}$$

Lemma 1. *(a) Peace is securable if $\overline{u}_1 + \overline{u}_2 \le 1$; (b) if $\underline{u}_i$ is attained in Eq. (3) for each $i \in \{1, 2\}$, then peace is implementable if $\underline{u}_1 + \underline{u}_2 \le 1$; (c) if, for any $i \in \{1, 2\}$ and any $\sigma \in \mathscr{E}_i$, $U_i\,(\cdot|\sigma)$ is continuous at z_i, then peace is implementable only if $\underline{u}_1 + \underline{u}_2 \le 1$, and securable only if $\overline{u}_1 + \overline{u}_2 \le 1$.*

Our task is mainly to turn the endogenous conditions $\underline{u}_1 + \underline{u}_2 \le 1$ and $\overline{u}_1 + \overline{u}_2 \le 1$ into conditions purely about the primitives through finding the formulas that map the primitives to $\underline{u}_i$ and $\overline{u}_i$. Derivation of the formulas requires careful analysis of the conflict continuation game and will be presented in Section 5. Here I shall state the outcome of the derivation thereby presenting the main results. The upshot of the derivation is that, for each player i, $\underline{u}_i$ is attained by the unique element $\underline{\sigma}^i$ of $\mathscr{E}_i(\delta_{z_i})$, and $\overline{u}_i$ by the unique element $\overline{\sigma}^i$ of $\mathscr{E}_i(\delta_{a_i})$ (Theorem 7).[6] In other words, to the type z_i of player i, the worst off-path posterior belief about him upon his vetoing a peaceful split is that his type is z_i, and the best off-path posterior is that his type is a_i. The rough intuition is that the opponent would be most aggressive in fighting against the strongest possible player i, and most complacent in fighting against the weakest possible player i.[7]

[5] Note that "for any $\sigma \in \mathscr{E}_i$" is just a short way to say "for any posterior $\tilde{F}_i$ satisfying the independence condition with respect to the on-path actions of the PBE, and for any $\sigma \in \mathscr{E}_i(\tilde{F}_i)$." Thus, the notion of security can be applied to general multistage games, with "accepting the proposed split" generalized to on-path actions with respect to the PBE.

[6] For any $x \in \mathbb{R}$, δ_x denotes the Dirac measure, as well as the distribution, whose support is $\{x\}$.

[7] This intuition about complacency, however, would have difficulty when a_i is larger than the type supremum z_{-i} of the opponent. Neither can the intuition about aggressiveness explain why the opponent would not suppress her bids out of the fear that player i's type is too strong for her small winning chance to be worth her sunk bidding cost (cf. Footnote 8).

Our finding of the best and worst posteriors for type z_i of player i should not be confused with observations that a bidder would rather have his type independently drawn from the same distribution as his rival's than have his type, say t_i, be commonly known (e.g., Kovenock et al., 2015). Such observations are binary comparisons between the game $\mathscr{G}(F, F)$

With $\underline{u}_i$ attained by $\underline{\sigma}^i$, and $\overline{u}_i$ by $\overline{\sigma}^i$, calculation of $\underline{\sigma}^i$ and $\overline{\sigma}^i$ gives us

$$\underline{u}_i = c^*_{-i}, \tag{5}$$

$$\overline{u}_i = 1 - \frac{a_i}{z_i}\left(1 - c^o_{-i}\right), \tag{6}$$

where c^*_{-i} is the probability with which the opponent player $-i$ bids zero in $\underline{\sigma}^i$, and c^o_{-i} player $-i$'s probability of bidding zero in $\overline{\sigma}^i$. And, as properties of $\underline{\sigma}^i$ and $\overline{\sigma}^i$, these probabilities are determined by the primitives according to the next formulas for each $i \in \{1, 2\}$:

$$c^*_i := \inf\left\{ c_i \in [0,1] : z_{-i} \int_{c_i}^{1} \frac{1}{F_i^{-1}(s)} ds \leq 1 \right\}, \tag{7}$$

$$c^o_i := \inf\left\{ c_i \in [0,1] : a_{-i} \int_{c_i}^{1} \frac{1}{F_i^{-1}(s)} ds \leq 1 \right\}, \tag{8}$$

where, for any $s \in [0, 1]$,

$$F_i^{-1}(s) := \inf\{t \in \operatorname{supp} F_i : F_i(t) \geq s\}. \tag{9}$$

The last paragraph of Section 5 will explain how we obtain Eqs. (5)–(8).

Eqs. (5) and (6), combined with Lemma 1, imply the main result, where the conditions for peace are purely about the prior distributions (F_1, F_2) according to Eqs. (7) and (8):

Theorem 1. *(a) Peace is implementable if and only if*

$$c^*_1 + c^*_2 \leq 1. \tag{10}$$

(b) Peace is securable if and only if

$$\sum_{i=1}^{2} \frac{a_i}{z_i}\left(1 - c^o_{-i}\right) \geq 1. \tag{11}$$

Corollary 1. *If $a_i = 0$ for some $i \in \{1, 2\}$, then peace is not securable.*

Example 1. Suppose that player 1's type distribution is supported by $[0, \epsilon]$, and player 2's by $[a_2, z_2]$, with $a_2 > \epsilon$. Then $c^*_2 = 0$ by Eq. (7), because $z_1/F_2^{-1}(s) < z_1/a_2 < 1$ for all $s \in [0, 1]$. Since $c^*_1 \leq 1$ by definition, Ineq. (10) is satisfied. Thus, peace is implementable: The mediator can propose to give the entire prize to player 2; player 1 is so much stochastically weaker than player 2 that he expects zero surplus from vetoing the proposal as long as he expects the continuation play $\underline{\sigma}^1$. By contrast, because $a_1 = 0$, the left-hand side of Ineq. (11) is equal to $a_2/z_2(1 - c^o_1) < 1$, hence (11) is not satisfied, and peace is not securable: Unless the mediator

with an identical type distribution F versus the game $\mathscr{G}(\delta_{t_i}, F)$ with the distribution pair (δ_{t_i}, F), and the identical distribution F in the former, facilitating an explicit solution, is crucial to such observations. The comparisons in this paper, by contrast, are not binary, but rather are between a game $\mathscr{G}(\delta_{t_i^*}, F_{-i})$—where t_i^* need not be the player's true type—and a continuum of other games $\mathscr{G}(\tilde{F}_i, F_{-i})$, with $\tilde{F}_i$ ranging among all posteriors of i. The finding does not rely on identical distributions.

proposes to give the entire prize to the stochastically inferior player 1, which player 2 would surely reject because $\overline{u}_2 = 1 - a_2/z_2 > 0$ by Eqs. (6) and (8), the mediator cannot ensure that player 1 would not reject the proposal out of an optimistic forecast that, should he reject it, player 2 would believe that player 1's type is zero and so would bid so complacently that player 1 can win with an arbitrarily small bid.

The rest of this section focuses on symmetric players. Corollary 2.b makes an interesting observation that the condition for peace security depends only on the infimum–supremum ratio of the type support and not at all on any other aspect of the type distributions.

Corollary 2. *Suppose that* $[a, z] =$ *convex hull* $(\operatorname{supp} F_i)$ *for all* $i \in \{1, 2\}$.

a. *If* $F_1 = F_2 = F$ *for some c.d.f.* F*, then there exists a unique* $c_* \in [0, 1)$ *such that*

$$z \int_{c_*}^{1} \frac{1}{F^{-1}(s)} ds = 1, \tag{12}$$

and peace is implementable if and only if $c_* \le 1/2$.

b. *Peace is securable if and only if* $2a \ge z$.

Example 2. Suppose $F_1 = F_2 = F$ and F is the uniform distribution on $[a, z]$. Then peace is securable if and only if $2a \ge z$. By contrast, peace is always implementable. To see that, note $F^{-1}(s) = a + (z - a)s$ for any $s \in [0, 1]$. Hence the left-hand side of Eq. (12) is equal to

$$z \int_{c_*}^{1} (a + (z - a)s)^{-1} ds = \frac{z}{z - a} \ln \frac{z}{a + (z - a)c_*}.$$

Thus Eq. (12) implies

$$c_* = \frac{e^{-1+a/z} - a/z}{1 - a/z}.$$

We claim that $c_* \le 1/2$, which, by the above equation and the fact $a \le z$, is equivalent to

$$2e^{-1+r} - r \le 1$$

for all $r \in [0, 1]$. Since the left-hand side of this inequality is convex in r, it attains its maximum at either $r = 0$ or $r = 1$. When $r = 0$, $2e^{-1+r} - r = 2/e < 1$; when $r = 1$, $2e^{-1+r} - r = 1$. Thus, $2e^{-1+r} - r \le 1$ for all $r \in [0, 1]$, as claimed.

Example 3. The peace implementability condition $c_* \le 1/2$ is satisfied when $F(t) = \sqrt{t}$ for all $t \in [0, 1]$, as $c_* = 1/2$ by Eq. (12). By contrast, if $F(t) = t^{1/3}$ for all $t \in [0, 1]$, Eq. (12) becomes $\int_{c_*}^{1} s^{-3} ds = \left(c_*^{-2} - 1\right)/2 = 1$, i.e., $c_* = 1/\sqrt{3} > 1/2$, violating the implementability condition. The peace security condition, by Corollary 1, can never be satisfied.

Example 4. To underscore the applicability of our result to both continuous and discrete distributions, suppose that the type of each player is independently drawn from the same binary distribution F, supported by $\{a, z\}$ with $a < z$, such that $F(a) = \theta$ for some $\theta \in (0, 1)$. The condition for security of peace is again $2a \geq z$. To calculate the condition for implementability of peace, note from Eq. (9) that

$$F^{-1}(s) = \begin{cases} z & \text{if } s \in (\theta, 1] \\ a & \text{if } s \in [0, \theta]. \end{cases}$$

If $c > \theta$ then $z \int_c^1 \frac{1}{F^{-1}(s)} ds = (z/z)(1 - c) < 1$; if $c \in [0, \theta]$,

$$z \int_c^1 \frac{1}{F^{-1}(s)} ds \leq 1 \iff \frac{z}{z}(1 - \theta) + \frac{z}{a}(\theta - c) \leq 1 \iff c \geq \left(1 - \frac{a}{z}\right)\theta.$$

Thus, $c_* = (1 - a/z)\,\theta$ by Eq. (12), and peace is implementable if and only if $(1 - a/z)\,\theta \leq 1/2$, requiring that the probability θ of being the weak type be sufficiently small.

4. Insufficient security of deterrence by strength

By Theorem 1, improving the prospect of peace boils down to shrinking the left-hand side of Ineq. (10) for peace implementability, and enlarging the left-hand side of Ineq. (11) for peace security. Obviously, one of such desirable changes is to reduce c_i^* for Ineq. (10), and c_i^o for (11). To reduce c_i^* and c_i^o, one readily sees from their definitions that it suffices to make a prior distribution F_i rank higher in stochastic dominance. For any two distributions F and $\widehat{F}$, write $\widehat{F} \rhd F$ if and only if $\widehat{F}$ first-order stochastically dominates F and $\operatorname{supp} \widehat{F} = \operatorname{supp} F$. Replacing the prior F_i of player i's type by an $\widehat{F_i}$ with $\widehat{F_i} \rhd F_i$ amounts to transferring some weight from low types to high types thereby making player i stochastically stronger ex ante. The next theorem observes that such operations never undermine peace implementability and, furthermore, there exists such an operation that improves it.

Theorem 2. *Given any prior distributions (F_1, F_2) of the players' types:*

a. *if $\widehat{F_i} \rhd F_i$ for each $i \in \{1, 2\}$ then peace implementability (resp. security) given $(F_i)_{i=1}^2$ implies peace implementability (resp. security) given $(\widehat{F_i})_{i=1}^2$;*
b. *for any $(F_i)_{i=1}^2$ given which peace is not implementable, there exists $(\widehat{F_i})_{i=1}^2$ such that $\widehat{F_i} \rhd F_i$ for each $i \in \{1, 2\}$ and, if $(F_i)_{i=1}^2$ is replaced by $(\widehat{F_i})_{i=1}^2$, peace is implementable.*

It is worth noting that improving peace-implementability does not require making one player ex ante weaker than the other. Rather, implementability of peace never gets hurt when both players become ex ante strong including when they become equally so. One can easily prove a corollary of Theorem 2: For any prior distributions (F_1, F_2) such that $\operatorname{supp} F_1 = \operatorname{supp} F_2$, there exists $\widehat{F}$ such that $\widehat{F} \rhd F_i$ for each $i \in \{1, 2\}$ and, if (F_1, F_2) is replaced by $(\widehat{F}, \widehat{F})$, peace is implementable.

In contrast to implementability of peace, which can be improved simply by strengthening some prior distributions (Theorem 2.b), it is impossible to improve security of peace through strengthening distributions when each player's infimum type is sufficiently apart from his supremum type:

Theorem 3. *If $a_i/z_i < 1/2$ for each $i \in \{1, 2\}$, then security of peace is impossible.*

Regardless of the functional forms of the prior distributions, peace security is never undermined by the following modification of the infimum–supremum ratios of the prior distributions. It enlarges the type support for one player and shrinks that for the other.

Theorem 4. *For any* $(\widehat{F}_1, \widehat{F}_2)$ *such that* $\inf \operatorname{supp} \widehat{F}_i = \hat{a}$ *and* $\sup \operatorname{supp} \widehat{F}_i = \hat{z}$ *for each* $i \in \{1, 2\}$, *if* $\hat{a}/\hat{z} = \frac{1}{2}(a_1/z_1 + a_2/z_2)$, *then peace security given* $(F_i)_{i=1}^2$ *implies peace security given* $(\widehat{F}_i)_{i=1}^2$.

To understand the crucial role of the infimum–supremum ratio of each player's type support, manifested in Theorems 3 and 4 as well as Corollary 2.b, recall that security requires the compliance of the strongest type z_i of each player i for whatever forecast that he may have regarding what his deviation would entail. The most optimistic forecast is that the opponent $-i$ will bid based on the posterior belief that the vetoer i's type is the infimum a_i (Theorem 7.b.i) because such a belief would drive $-i$ to bid merely in the order of a_i no matter how stochastically strong she is, the auction being all-pay. Then the vetoer i's bid would also be in the order of a_i and hence his cost, given type z_i, would merely be in the order of a_i/z_i. Hence when a_i/z_i is sufficiently small, the type z_i of player i would rather veto peace if he has such an optimistic forecast. Ex ante strength of the opponent $-i$ is insufficient to deter z_i given i's optimistic outlook of conflict.

It is interesting to note that ex ante disparity between the players' strength levels is neither necessary to implement peace, as explained immediately after Theorem 2, nor sufficient to secure peace, as shown in Example 1, Corollary 2.b and Theorem 3.

5. Details: general analysis of two-player contests

Let us derive the formulas, Eqs. (5)–(8), that turn the peace conditions from those about the endogenous variables $\underline{u}_i$ and $\overline{u}_i$ (Lemma 1) into those about the parameters (Theorem 1). By the definition of $\underline{u}_i$ and $\overline{u}_i$, to derive these formulas we need to evaluate all the possible continuation plays in the event that a player vetoes an otherwise mutually acceptable peaceful split. Such events off-path, the posterior belief about the vetoer is arbitrary. Thus following is the analysis of two-player all-pay auctions given arbitrary distributions.

Distributional strategies Consider the all-pay auction game $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ such that each player i's type is independently drawn from a distribution $\tilde{F}_i$ whose support is contained in $[a_i, z_i]$. A player i's distributional strategy $\boldsymbol{\sigma}_i$ in $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ is a probability measure on (Borel) subsets of the product space of the possible types and bids of player i such that the marginal distribution of i's type is $\tilde{F}_i$. Note that $\boldsymbol{\sigma}_i$ corresponds to the equivalence class of behavioral strategies, each being a mapping σ_i that associates to any realized type t_i of player i a c.d.f. $\sigma_i(\cdot, t_i)$ of his bid, such that coupled with $\tilde{F}_i$ they generate the same probability measure $\boldsymbol{\sigma}_i$ (Milgrom and Weber, 1985). Thus, I shall identify a distributional strategy $\boldsymbol{\sigma}_i$ with any behavioral strategy that belongs to its equivalence class and, unless more clarity is required, call both *strategy* and denote both by σ_i. An equilibrium (BNE) of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ means a Nash equilibrium where each player chooses a distributional strategy.

For any strategy pair $\sigma := (\sigma_1, \sigma_2)$ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any $i \in \{1, 2\}$, define the induced distribution $H_{i,\sigma}$ of player i's bids and the supremum x_σ of bids by, for any $b \in \mathbb{R}$,

$$H_{i,\sigma}(b) := \int_{\mathbb{R}} \int_{-\infty}^{b} \sigma_i(dr, t_i) d\tilde{F}_i(t_i), \tag{13}$$

$$x_\sigma := \max_{i \in \{1,2\}} \sup \operatorname{supp} H_{i,\sigma}. \tag{14}$$

In bidding b against the rival who abides by σ, player i's probability of winning the auction, incorporating the possibility of ties and the uniform tie-breaking rule, is equal to

$$H^*_{-i,\sigma}(b) := \begin{cases} H_{-i,\sigma}(b) & \text{if } b \text{ is not an atom of } H_{-i,\sigma} \\ \frac{1}{2}\left(H_{-i,\sigma}(b) + \lim_{b' \uparrow b} H_{-i,\sigma}(b')\right) & \text{if } b \text{ is an atom of } H_{-i,\sigma}. \end{cases} \tag{15}$$

Surplus from an equilibrium For any $i \in \{1, 2\}$, any realized type t_i of player i and any strategy pair σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, by the definition of $U_i(t_i|\sigma)$ at the start of Section 3,

$$U_i(t_i|\sigma) = \sup_{b \in \mathbb{R}_+} H^*_{-i,\sigma}(b) - b/t_i. \tag{16}$$

The operator in Eq. (16) is sup instead of max because a maximum need not exist when $H_{-i,\sigma}$ has an atom, at which the equal-probability tie-breaking rule renders the objective function discontinuous. However, if σ is an equilibrium of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, the induced bid distribution $H_{-i,\sigma}$ has no atom except possibly at the zero bid (Lemma 6). Thus, unless $b = 0$, player i's probability of winning by bidding b is equal to $H_{-i,\sigma}(b)$, as if the tie-breaking rule were altered to always pick him the winner in the (zero-probability) event that the opponent also bids b. That is also true when $b = 0$ unless zero is an atom of $H_{-i,\sigma}$. When zero is an atom of $H_{-i,\sigma}$, given the uniform tie-breaking rule, player i of any positive type would rather bid slightly above zero to secure an expected payoff approximately $H_{-i,\sigma}(0)$ than bid exactly zero to get only $H_{-i,\sigma}(0)/2$; if, in addition, he cannot do better than $H_{-i,\sigma}(0)$, the supremum among his expected payoffs, when his bid ranges in $\mathbb{R}_+$, is equal to $H_{-i,\sigma}(0)$, again as if he were bidding exactly zero and the tie-breaking rule were altered to always favor him. Thus the continuity condition for Lemma 1.c is ensured:

Theorem 5. *For any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, any BNE σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any $i \in \{1, 2\}$, $U_i(\cdot|\sigma)$ is continuous on $[a_i, z_i] \setminus \{0\}$ and, for any $t_i \in [a_i, z_i] \setminus \{0\}$,*

$$U_i(t_i|\sigma) = \max_{b \in \mathbb{R}_+} H_{-i,\sigma}(b) - b/t_i. \tag{17}$$

Solving for equilibria By Eq. (17), the decision for player i with any type $t_i > 0$ is equivalent to maximizing $t_i H_{-i,\sigma}(b) - b$ over all $b \in \mathbb{R}_+$. With $H_{-i,\sigma}$ weakly increasing, this objective is differentiable almost everywhere in $[0, x_\sigma]$, which by equilibrium condition is the support of both $H_{1,\sigma}$ and $H_{2,\sigma}$. Any such differentiable point $b \in (0, x_\sigma)$ satisfies the first-order necessary condition for b to be a bid prescribed by σ_i to t_i:

$$\frac{d}{db} H_{-i,\sigma}(b) = \frac{1}{t_i}. \tag{18}$$

To characterize σ based on this equation, we need to map the bid b to a type t_i for which b is a bid prescribed by σ_i. If $\tilde{F}_i$ has neither atom nor gap, then naturally the mapping is

$$\gamma_{i,\sigma}(b) = \tilde{F}_i^{-1}\left(H_{i,\sigma}(b)\right) \tag{19}$$

for all b, so that $\gamma_{i,\sigma}(b)$ is the type whose cumulative mass is equal to the cumulative mass of the bid b. However, with more general $\tilde{F}_i$, the two cumulative masses may be impossible to be the same. Hence we generalize $\gamma_{i,\sigma}(b)$ to be the infimum among i's types whose cumulative masses are not below that of b. More precisely, define the generalized inverse $\tilde{F}_i^{-1}$ of $\tilde{F}_i$ by

Eq. (9), where the role of F_i is played by $\tilde{F}_i$ here. Now define $\gamma_{i,\sigma}$ by Eq. (19) for each $b \in \mathbb{R}$. Then we prove that, for almost all $b \in [0, x_\sigma]$, Eq. (18) holds with $t_i = \gamma_{i,\sigma}(b)$ for each $i \in \{1, 2\}$ (Lemma 11). That gives us a differential equation system for $(H_{1,\sigma}, H_{2,\sigma})$.

Solving $(H_{i,\sigma})_{i=1}^2$ through integrating this differential system, however, is nontrivial. The differential system of $H_{i,\sigma}$ is available only for almost every bid in $[0, x_\sigma]$ rather than for every bid there. Hence it is not immediate that $H_{i,\sigma}$ is an antiderivative. A technique in the literature to obtain uniqueness of the solution for a differential system, if applicable here, would be to show that $H_{i,\sigma}$ is Lipschitz on $[0, x_\sigma]$ via a revealed-preference argument, which would rely on continuity of $\gamma_{-i,\sigma}$ (Griesmer et al., 1967, Lemma 3.6). But that technique does not work here, because $\gamma_{-i,\sigma}$ need not be continuous, as $\tilde{F}_{-i}$ may have gaps.

My approach is to decompose $H_{i,\sigma}$ into the sum of two distributions on $\mathbb{R}$, one absolutely continuous and the other singular with respect to Lebesuge measure. The second part, with $H_{i,\sigma}$ having no atom except at zero (Lemma 6), can be discontinuous only at zero. Furthermore, it is constantly equal to $H_{i,\sigma}(0)$ on $[0, x_\sigma]$, otherwise the distribution would be so steep at some point that the equilibrium condition for player $-i$ is violated (Appendix B.6). Then integration of the differential equation yields the absolutely continuous part, to which we add the second part, which is just the mass $H_{i,\sigma}(0)$, denoted by $c_{i,\sigma}$, at zero. Thus, despite arbitrary type distributions, we obtain the pair of equilibrium bid distributions, which in turn pins down the pair of equilibrium strategies uniquely (Corollary 4, Appendix B.7).

Theorem 6. *For any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any BNE σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ there exists a unique triple $(x_\sigma, c_{1,\sigma}, c_{2,\sigma}) \in \mathbb{R}_{++} \times [0, 1]^2$ such that $c_{1,\sigma} c_{2,\sigma} = 0$ and, for each $i \in \{1, 2\}$ and all $b \in [0, x_\sigma]$, $H_{i,\sigma}(x_\sigma) = 1$ and*

$$H_{i,\sigma}(b) = c_{i,\sigma} + \int_0^b \frac{1}{\tilde{F}_{-i}^{-1}\left(H_{-i,\sigma}(y)\right)} dy. \tag{20}$$

When one of the bidders say i has a degenerate type distribution, Eq. (20) gives us explicitly the probability $c_{-i,\sigma}$ with which player $-i$ bids zero:

Corollary 3. *If σ is a BNE of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, $i \in \{1, 2\}$, $t_i^* > 0$ and* $\operatorname{supp} \tilde{F}_i = \{t_i^*\}$, *then*

$$c_{-i,\sigma} = \inf \left\{ c \in [0, 1] : t_i^* \int_c^1 \frac{1}{\tilde{F}_{-i}^{-1}(s)} ds \le 1 \right\}. \tag{21}$$

Comparing equilibria Given any BNE σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, since the strategy is monotone (Eq. (22)) and has no atom at the bid supremum x_σ (Lemma 6), bidding x_σ guarantees a win and is a best response for the type z_i of each player i. Hence

$$U_i(z_i|\sigma) = 1 - \frac{x_\sigma}{z_i}.$$

Thus, the search for the best and the worst plays in the auction for the type z_i of each player i boils down to finding the lowest and the highest x_σ among all $\sigma \in \mathscr{E}_i$. The upshot, through a nontrivial application of Eq. (20), is that x_σ is highest when player i is believed to be type z_i, and lowest when believed to be type a_i[8]:

Lemma 2. *For any $i \in \{1,2\}$ and any posterior distribution $\tilde{F}_i$ of player i's type, if $\underline{\sigma}^i \in \mathscr{E}_i(\delta_{z_i})$, $\overline{\sigma}^i \in \mathscr{E}_i(\delta_{a_i})$ and $\sigma \in \mathscr{E}_i(\tilde{F}_i)$, then $x_{\underline{\sigma}^i} \geq x_\sigma \geq x_{\overline{\sigma}^i}$.*

The lemma results from a subtle linkage between the two players' marginal costs of bids. Here is the intuition for $x_{\underline{\sigma}^i} \geq x_\sigma$, and that for the other part is analogous. Note that a player i's marginal cost of bids is equal to $1/t_i$ when t_i is supposed to be his type that submits the bid. At the equilibrium $\underline{\sigma}^i$, with i's type degenerate to the type supremum z_i, his marginal cost $1/z_i$ is less than his marginal cost $1/t_i$ at any equilibrium say σ given other posteriors. Thus, his marginal revenue of bids at equilibrium $\underline{\sigma}^i$ is less than that at equilibrium σ. Since player i's marginal revenue of bids is the slope of his opponent $-i$'s bid distribution function, player $-i$'s bid distribution $H_{-i,\underline{\sigma}^i}$ at $\underline{\sigma}^i$ is less steep than her $H_{-i,\sigma}$ at σ. Thus, unless $x_{\underline{\sigma}^i} \geq x_\sigma$, $H_{-i,\sigma}$ first-order stochastically dominates $H_{-i,\underline{\sigma}^i}$. Hence for any bid the type of $-i$ that submits the bid in equilibrium $\underline{\sigma}^i$ is higher than the type of $-i$ that submits it in σ. In other words, player $-i$'s marginal cost of bids, and hence her marginal revenue, are lower in $\underline{\sigma}^i$ than in σ. Thus, her marginal revenue being the slope of her opponent i's bid distribution, player i's bid distribution $H_{i,\underline{\sigma}^i}$ rises at a lower rate in equilibrium $\underline{\sigma}^i$ than the bid distribution $H_{i,\sigma}$ does in σ. Since $H_{i,\underline{\sigma}^i}(0) = 0$ (due to the above derivation and the fact that the zero bid cannot be an atom for both players), $H_{i,\underline{\sigma}^i}$ stochastically dominates $H_{i,\sigma}$, which implies that at the supremums of their supports, $x_{\underline{\sigma}^i} \geq x_\sigma$.

The best and the worst plays for type z_i Lemma 2, coupled with an existence proof of equilibria given degenerate beliefs (Appendix B.8.2), results in the next theorem, which locates the best and the worst plays in the auction game for the type z_i of each player i and describes what the other types of player i get in response to either play, especially those types of i that have empty best response to the play. Denote

$$\mathrm{BR}_i(t_i, \epsilon \mid \sigma) := \left\{b \in \mathbb{R}_+ : \forall b' \in \mathbb{R}_+ \left[H^*_{-i,\sigma}(b) - b/t_i \geq H^*_{-i,\sigma}(b') - b'/t_i - \epsilon\right]\right\}$$

for any $t_i \in [a_i, z_i]$, any $\epsilon \geq 0$ and any strategy pair σ.

Theorem 7. *For any $i \in \{1,2\}$:*

a. $\mathscr{E}_i(\delta_{z_i}) = \left\{\underline{\sigma}^i\right\}$ *and:*
   i. $\underline{u}_i = c_{-i,\underline{\sigma}^i} = U_i(t_i|\underline{\sigma}^i)$ *for all $t_i \in [a_i, z_i] \setminus \{0\}$;*
   ii. *for any $t_i \in [a_i, z_i)$, $\lim_{\epsilon\downarrow 0} \sup \mathrm{BR}_i\left(t_i, \epsilon \mid \underline{\sigma}^i\right) = 0$, and if $H_{-i,\underline{\sigma}^i}(0) = 0$ then $\mathrm{BR}_i\left(t_i, 0 \mid \underline{\sigma}^i\right) = \{0\}$;*

b. *if $a_i > 0$, $\mathscr{E}_i(\delta_{a_i}) = \left\{\overline{\sigma}^i\right\}$ and:*
   i. $\overline{u}_i = 1 - \frac{a_i}{z_i}\left(1 - c_{-i,\overline{\sigma}^i}\right)$;
   ii. *for any $t_i \in (a_i, z_i]$, $\mathrm{BR}_i(t_i, 0 \mid \overline{\sigma}^i) = \{x_{\overline{\sigma}^i}\}$;*

c. *if $a_i = 0$ then $\overline{u}_i = 1$.*

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8 This observation is not obvious even if we restricted the search within the plays rationalized by degenerate posteriors $\delta_{t_i^*}$ about i. By Eq. (21), higher t_i^* means larger $c_{-i,\sigma}$: the stronger is the type that the opponent $-i$ believes player i to be, the more likely is $-i$ to bid zero, hence the larger is i's payoff from submitting an infinitesimal bid.

Part (c) of the theorem handles the special case $a_i = 0$, where the belief δ_{a_i}, that i's type is zero, renders player $-i$'s best response empty due to the uniform tie-breaking rule. Part (a.ii) describes behaviors of the non-z_i types of i in the play $\underline{\sigma}^i$: Expecting none but type z_i of player i, the opponent $-i$ bids according to $\underline{\sigma}^i$, which may have an atom at zero; hence player i with any type but z_i would respond by bidding just slightly above the atom zero (Lemma 14.c). Part (a.ii) is interesting because it implies that almost all types of each player i expect to bear only an infinitesimal sunk cost in the auction, which in the context of Section 2 is the off-path conflict that keeps him from vetoing the peace proposal despite the infinitesimal cost of vetoing it. Part (b.ii) describes player i's best response, which is more straightforward, to the play $\overline{\sigma}^i$.

Finally we obtain the equations that deliver Theorem 1. Apply Eq. (21), with the roles of i and $-i$ switched, to the cases where the role of t_i^* is played by z_{-i} or a_{-i}, and $\tilde{F}_{-i}$ played by F_i. That shows c_i^* and c_i^o, defined in Eqs. (7) and (8), are equal to the masses $c_{i,\underline{\sigma}^{-i}}$ and $c_{i,\overline{\sigma}^{-i}}$ in the plays $\underline{\sigma}^{-i}$ and $\overline{\sigma}^{-i}$. Thus, Part (a.i) of Theorem 7 gives Eq. (5) and implies that $\underline{u}_i$ is attained; and Parts (b.i) and (c) together give Eq. (6).

6. Conclusion

Fundamental to humanity is the question whether conflicts can be preempted by peace settlements. The received literature on conflict mediation is based on an assumption that should conflict ensue the outcome is exogenously determined. Replacing this exogenous outside option by endogenous continuation plays, this paper enriches both the question and the answer. Because such continuation plays are off-path were the conflict fully preempted, endogenization leads to new questions: Can the mediator be sure, when she proposes a peace settlement, that both players forecast the off-path outside option in the same way as she desires, and if not how can she secure that the peace settlement is accepted by both? This paper handles the new questions by proposing two notions of conflict preemption. One, for the mediator with full coordination power, is implementability of peace. The other, for the mediator with no coordination power, is security of peace. For each notion this paper delivers a necessary and sufficient condition purely in terms of the primitives. Moreover, these conditions have implications unprecedented in the literature: On one hand, peace cannot be secured when one of the players has a sufficiently low infimum of his type support, no matter how stochastically disparate the two players are ex ante. On the other hand, peace can be implemented even when all but one type of each player forecast that should he veto the peace proposal the cost he has to incur in the ensuing conflict is infinitesimal.

Motivated by the all-pay aspect of conflicts, this paper focuses on all-pay auctions. But the distributional method it develops can handle cases where the all-pay auction is replaced by other formats of auctions, hence applicable to the study of bidding collusion. Extension is trivial for second-price auctions, but nontrivial for first-price auctions (cf. Zheng, 2018). In a similar spirit, though relevant to different contexts, is to investigate how various contest mechanisms in the conflict phase may affect the prospect of conflict preemption. This problem is germane, and anticipated by Spier (1994), when the conflict is litigation, where the fraction of the winner's fees that the loser needs to pay varies with the litigation system. While Klemperer (2003) argues that such fee-shifting rules are irrelevant when the revenue equivalence theorem applies, the theorem is inapplicable to our continuation game because the posteriors, being endogenous, need not be identical between contestants.

A related question is how the peace condition may be affected by refinement conditions. For instance, consider the condition for peace implementation. One can prove that it remains true when the off-path posteriors are restricted by the intuitive and divinity criteria. If the mediator can propose peaceful allocations that do not exhaust the value of the prize, one can show that when Condition (10) is satisfied the peaceful allocation $(\underline{u}_1, \underline{u}_2)$ is ratifiable in the sense of Cramton and Palfrey (1995), albeit causing a deadweight loss $1 - \underline{u}_1 - \underline{u}_2$ from the players' viewpoint. Without such flexibility, however, ratifiability of peaceful allocations other than $(\underline{u}_1, \underline{u}_2)$ is an open question.

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Appendix A. General mechanisms for mediation

Instead of proposing a type-independent split which the players can only accept or reject, the mechanism in the mediation stage can be any communication mechanism à la Myerson (1986), which solicits a confidential message from each player and then computes a recommendation, which is either conflict or a peaceful split, and finally delivers to each player a confidential message that contains this recommendation, possibly accompanied with some truthful information about the message submitted by the other player. Once such a mechanism is announced, each player, already privately informed of his own type, announces independently and publicly whether to participate. If both participate, the mechanism is operated; if the recommendation thereof is to settle via a split, each player independently announces, publicly, whether to accept or reject it; if it is accepted by both then the game ends with the peaceful split. In any other case, conflict ensues.

Correspondingly, we generalize the independence condition for a PBE: If player i's unilateral deviation from the PBE is made before receiving any message from $-i$ (via the mechanism), then the posterior belief about i is independent of the realized type of $-i$.[9] Then the notion of peaceful PBE, as well as that of securely peaceful PBE, can be trivially extended to this general setup, with "Accept a proposed split" extended to "Participate in the mechanism and accept its recommendation."

By assumption, a player's type affects his payoff only in case of conflict. Thus, in any peaceful PBE, expecting zero probability of conflict, he would send to the mechanism whatever message that maximizes his peaceful share. Consequently, one can easily prove that, given any mechanism coupled with a peaceful PBE that it admits, for any $i \in \{1, 2\}$ there exists a unique $k_i \in \mathbb{R}_+$ such that player i's on-path expected payoff is equal to k_i for almost all types of i and $k_1 + k_2 = 1$.

9 This is less restrictive than Fudenberg and Tirole's (1991) "no signaling what you don't know" condition because it regulates the off-path posteriors only in the events where the deviating player has had no communication with the other player and hence can signal nothing new about the latter.

Then the original mechanism can be replaced by the type-independent split (k_1, k_2) without first soliciting messages from the players, whose participation means to accept, and nonparticipation means to reject, the proposal. The peaceful PBE given (k_1, k_2) is essentially the same as the original one, with the event where player i vetoes (k_1, k_2) identified with the event where player i does not participate in the original mechanism while the other player does.

Thus, there is no loss of generality to restrict mechanisms to the kind of splits assumed in the main text. Given such proposed splits, the players have no chance to communicate before they independently choose their responses; hence the independence condition applies, requiring that the off-path posterior $\tilde{F}_i$ in Eq. (1) be independent of the realized type of $-i$. Thus, Eq. (2) applies to this general setup, and so does the rest.

Appendix B. Proofs

Definition A distributional strategy $\boldsymbol{\sigma}_i$ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ is said *monotone* if and only if its corresponding equivalence class contains a behavioral strategy σ_i such that

$$t_i'' > t_i' \Longrightarrow \inf \operatorname{supp} \sigma_i(\cdot, t_i'') \geq \sup \operatorname{supp} \sigma_i(\cdot, t_i'). \tag{22}$$

Since elements of the equivalence class of $\boldsymbol{\sigma}_i$ differ only by a subset of zero measure with respect to $\tilde{F}_i$, I shall identify a monotone distributional strategy $\boldsymbol{\sigma}_i$ with the element σ_i of its equivalence class that satisfies (22) for all elements t_i' and t_i'' of $\mathbb{R}_+$.

Definition Given any c.d.f. $\tilde{F}_i$ of player i's type, a strategy σ_i is said to *generate* a c.d.f. H_i of i's bid if and only if $H_i(b) = \int_{\mathbb{R}} \int_{-\infty}^{b} \sigma_i(dr, t_i) d\tilde{F}_i(t_i)$ for all $b \in \mathbb{R}$.

B.1. Lemma 1

Lemma 3. *For any $i \in \{1, 2\}$ and any strategy pair σ, $U_i(\cdot|\sigma)$ is weakly increasing on $[a_i, z_i]$.*

Proof. Let $t_i' > t_i$. If $t_i = 0$ then bidding zero is the best response for player i, as the cost of any positive bid is infinite to the zero type (Section 2), hence $U_i(t_i|\sigma) = 0$ and so $U_i(t_i'|\sigma) \geq U_i(t_i|\sigma)$ because type t_i' can alway ensure zero payoff by bidding zero. Hence assume that $t_i > 0$. Then $H^*_{-i,\sigma}(b) - b/t_i' \geq H^*_{-i,\sigma}(b) - b/t_i$ for any $b \in \mathbb{R}_+$. Consequently,

$$U_i(t_i'|\sigma) = \sup_{b \in \mathbb{R}_+} H^*_{-i,\sigma}(b) - b/t_i' \geq \sup_{b \in \mathbb{R}_+} H^*_{-i,\sigma}(b) - b/t_i = U_i(t_i|\sigma). \quad \square$$

Proof of Lemma 1. Part (a): Suppose $\overline{u}_1 + \overline{u}_2 \leq 1$. Then there exists a split $k_1 + k_2 = 1$ such that $k_i \geq \overline{u}_i$ for each $i \in \{1, 2\}$. For each i, by definition of $\overline{u}_i$ and Lemma 3, $k_i \geq U_i(z_i|\sigma) \geq U_i(t_i|\sigma)$ for any $\sigma \in \mathscr{E}_i$ and any $t_i \in [a_i, z_i]$. Thus, if (k_1, k_2) is the proposed split, accepting the proposal is a best response for all types of player i to whichever continuation play σ that $-i$ will abide by in the event where i vetoes the split, hence mutual acceptance of the split is a secure, peaceful PBE.

Part (b): Suppose for each player i that $\underline{u}_i$ is attained, hence $\underline{u}_i = U_i(z_i|\underline{\sigma}^i)$ for some $\underline{\sigma}^i \in \mathscr{E}_i(\tilde{F}_i)$ and some posterior $\tilde{F}_i$ independent of the prior F_{-i} about $-i$. Suppose, in addition, that $\underline{u}_1 + \underline{u}_2 \leq 1$. Then there exists a split $k_1 + k_2 = 1$ such that $k_i \geq \underline{u}_i$ for each $i \in \{1, 2\}$. Given (k_1, k_2) as the proposed split, mutual acceptance of the split, coupled with the provision that, for each i, if player i vetoes the split then $\underline{\sigma}^i$ is the continuation play rationalized by the posteriors $(\tilde{F}_i, F_{-i})$, constitutes a peaceful PBE: for all $t_i \in [a_i, z_i]$, $U_i(t_i|\underline{\sigma}^i) \le U_i(z_i|\underline{\sigma}^i) = \underline{u}_i \le k_i$ by Lemma 3.

Part (c): First, we claim that, for any $k \in \mathbb{R}$, if $k < U_i(z_i|\sigma)$ then $k < U_i(t_i|\sigma)$ for all t_i in a set whose measure relative to F_i is positive: either z_i is an atom of F_i or, by continuity of $U_i(\cdot|\sigma)$ at z_i and z_i being the supremum of $\operatorname{supp} F_i$, a neighborhood of z_i satisfies this strict inequality and is of positive F_i-measure. Second, if peace is implementable, then some split (k_1, k_2) admits a peaceful PBE, supported by the provision that if i vetoes the split then some $\sigma_i \in \mathscr{E}_i$ is the continuation play; by the previously established claim, $k_i \ge U_i(z_i|\sigma) \ge \underline{u}_i$ for both $i \in \{1, 2\}$, with the last inequality due to the definition of $\underline{u}_i$; thus $1 = k_1 + k_2 \ge \underline{u}_1 + \underline{u}_2$. Third, if peace is securable, then some split (k_1, k_2) admits a secure, peaceful PBE; thus, by definition of security, for each player i, any $\sigma \in \mathscr{E}_i$ and almost all types t_i of i, accepting the split to get payoff k_i is no worse than vetoing it to get $U_i(t_i|\sigma)$, hence the claim established in the first step implies $k_i \ge U_i(z_i|\sigma)$. This true for all $\sigma \in \mathscr{E}_i$, we have $k_i \ge \overline{u}_i$ by definition of $\overline{u}_i$. Sum the last inequalities across i to obtain $1 = k_1 + k_2 \ge \overline{u}_1 + \overline{u}_2$. □

B.2. Theorem 1 and Corollaries 1 and 2

Theorem 1 First, Lemma 1 is applicable because its conditions are satisfied: $U_i(\cdot|\sigma)$ is continuous by Theorem 5; $\underline{u}_i$ is attained by Theorem 7.a.i. Second, by Theorem 7, Eqs. (5)–(8) are valid (proved in the last paragraph of Section 5). Therefore, Part (a) of the theorem follows directly from plugging Eq. (5) into Lemma 1. To prove part (b), plug Eq. (6) into Lemma 1 to obtain that peace is securable if and only if

$$1 - \frac{a_1}{z_1}\left(1 - c_2^o\right) + 1 - \frac{a_2}{z_2}\left(1 - c_1^o\right) \le 1,$$

which is simplified to Ineq. (11). □

Corollary 1 This follows from Theorem 1.b. □

Corollary 2 Claim (a): With $F_1 = F_2 = F$, Eq. (7) becomes $c_i^* := \inf\{c_i \in [0, 1] : \phi(c_i) \le 1\}$ such that $\phi(c) = z \int_c^1 \frac{1}{F^{-1}(s)} ds$. Thus, $c_1^* = c_2^*$. Note that ϕ is continuous and strictly decreasing on $[0, 1]$, $\phi(1) = 0$ and $\phi(0) \ge 1$, as $\phi(0) = z/F^{-1}(\xi)$ for some $\xi \in [0, 1]$, with $F^{-1}(\xi) \le z$. Thus, Eq. (12), $\phi(c_*) = 1$, admits a unique solution c_* in $[0, 1]$. By continuity of ϕ, $c_* = c_1^* = c_2^*$. Thus Claim (a) follows from Theorem 1.a.

Claim (b): With $(a_i, z_i) = (a, z)$, Eq. (8) becomes $c_i^o := \inf\{c \in [0, 1] : \varphi(c) \le 1\}$ such that $\varphi(c) = a \int_c^1 \frac{1}{F_i^{-1}(s)} ds$. Since the integrand $1/F_i^{-1}(s) \le 1/a$ for all $s \in [0, 1]$, $\varphi(c) \le 1$ for all $c \in [0, 1]$, hence $c_i^o = 0$ for each $i \in \{1, 2\}$. Plug this into Ineq. (11) to obtain $2a/z \ge 1$. Thus Claim (b) follows from Theorem 1.b. □

B.3. Theorems 2, 3 and 4

Lemma 4. *If $\widehat{F}$ and F are each a c.d.f. and $\widehat{F}(t) \le F(t)$ for all $t \in \mathbb{R}$, then $\widehat{F}^{-1}(s) \ge F^{-1}(s)$ for all $s \in [0, 1]$.*

Proof. For any $s \in [0,1]$, the hypothesis $F \geq \widehat{F}$ implies $F\left(\widehat{F}^{-1}(s)\right) \geq \widehat{F}\left(\widehat{F}^{-1}(s)\right) \geq s$, with the second inequality due to the definition of $\widehat{F}^{-1}(s)$—Eq. (9)—and upper semicontinuity of any c.d.f. Now that $F\left(\widehat{F}^{-1}(s)\right) \geq s$, Eq. (9) applied to F implies $\widehat{F}^{-1}(s) \geq F^{-1}(s)$. □

Lemma 5. *For each $i \in \{1,2\}$, if F_i becomes more stochastically dominant while a_{-i} and z_{-i} are either unchanged or lowered, then c_i^* and c_i^o become weakly smaller than before.*

Proof. Let the prior F_i be replaced by another $\widehat{F}_i$ that dominates F_i, and the supremum z_{-i} replaced by a $\hat{z}_{-i} \leq z_{-i}$. By definition of dominance and Lemma 4, $\widehat{F}_i^{-1}(s) \geq F_i^{-1}(s)$ for all $s \in [0,1]$. Thus, for any $c_i \in [0,1]$,

$$\int_{c_i}^{1} \frac{1}{\widehat{F}_i^{-1}(s)} ds \leq \int_{c_i}^{1} \frac{1}{F_i^{-1}(s)} ds$$

and hence, since $\hat{z}_{-i} \leq z_{-i}$,

$$\hat{z}_{-i} \int_{c_i}^{1} \frac{1}{\widehat{F}_i^{-1}(s)} ds \leq z_{-i} \int_{c_i}^{1} \frac{1}{F_i^{-1}(s)} ds.$$

Thus, by Eq. (7), the c_i^* given $\left(\widehat{F}_i, \hat{z}_{-i}\right)$ is weakly smaller than the c_i^* given (F_i, z_{-i}). Similarly, if $\hat{a}_{-i} \leq a_{-i}$, the c_i^o given $\left(\widehat{F}_i, \hat{a}_{-i}\right)$ is weakly smaller than the c_i^o given (F_i, a_{-i}). □

Theorem 2 Claim (a): By Theorem 1, the necessary and sufficient condition for peace implementation is $c_1^* + c_2^* \leq 1$, and that for peace security is $\sum_{i=1}^{2}(a_i/z_i)(1 - c_{-i}^o) \geq 1$. Thus, the claim follows from Lemma 5, which implies that, when $\widehat{F}_i \rhd F_i$ for both $i \in \{1,2\}$, c_i^* and c_i^o decrease weakly for each i when $(F_i)_{i=1}^2$ is replaced by $(\widehat{F}_i)_{i=1}^2$.

Claim (b): Pick the $i \in \{1,2\}$ for whom $z_i \geq z_{-i}$. Note from Eq. (7) that $c_{-i}^* < 1$. To satisfy $c_1^* + c_2^* \leq 1$, given the definition of c_i^*, it suffices to replace F_i by some $\widehat{F}_i$ such that

$$z_{-i} \int_{1-c_{-i}^*}^{1} \frac{1}{\widehat{F}_i^{-1}(s)} ds \leq 1 \tag{23}$$

and $\widehat{F}_i \rhd F_i$. To satisfy Ineq. (23), note from $c_{-i}^* < 1$ that there exists $\epsilon > 0$ for which

$$\frac{z_{-i}}{z_{-i} - \epsilon} c_{-i}^* < 1.$$

Pick any c.d.f. F_i^* with $\operatorname{supp} F_i^* = \operatorname{supp} F_i$ such that $1 - F_i^*(z_{-i} - \epsilon) > 1 - c_{-i}^*$, which is compatible with $\operatorname{supp} F_i^* = \operatorname{supp} F_i$ because $z_i \geq z_{-i}$ by the choice of i. Let $\widehat{F}_i := \min\left\{F_i, F_i^*\right\}$ pointwise. Then $\widehat{F}_i \rhd F_i$ and, by the definition of the generalized inverse $\widehat{F}_i^{-1}$, the left-hand side of (23) is less than or equal to

$$z_{-i}(1 - 1 + c_{-i}^*) \cdot \frac{1}{z_{-i} - \epsilon} < 1.$$

Hence (23) is satisfied. Thus, with $\hat{c}_i^*$ defined by Eq. (7) where F_i is replaced by $\widehat{F}_i$ here, $\hat{c}_i^* < 1 - c_{-i}^*$. Since $F_{-i} \rhd F_{-i}$, the pair $(\widehat{F}_i, \widehat{F}_{-i})$, with $\widehat{F}_{-i} := F_{-i}$, is what Claim (b) needs. □

Theorem 3 By Eq. (8) the definition of c_i^o, $c_i^o \in [0,1]$ for each $i \in \{1,2\}$. Thus, if $a_i/z_i < 1/2$ for each i, then $\sum_{i=1}^2 (a_i/z_i)(1-c_{-i}^o) \le \sum_{i=1}^2 (a_i/z_i) < 1$, and hence Ineq. (11), the condition for security of peace, can never be satisfied. □

Theorem 4 Let $\hat{c}_i^o$ be the counterpart of c_i^o in the case where the priors are $(\widehat{F}_i)_{i=1}^2$ instead of $(F_i)_{i=1}^2$. Since $\hat{a}_1 = \hat{a}_2 = \hat{a}$, $\widehat{F}_i^{-1}(s) \ge \hat{a}_i = \hat{a}_{-i}$ for all $s \in [0,1]$. Thus $\hat{a}_{-i} \int_c^1 (1/\widehat{F}_i^{-1}(s))ds \le 1$ for all $c \in [0,1]$ and hence Eq. (8) implies $\hat{c}_i^o = 0$ for each $i \in \{1,2\}$. Then

$$\sum_{i=1}^{2} \frac{\hat{a}_i}{\hat{z}_i}\left(1-\hat{c}_{-i}^o\right) = \sum_{i=1}^{2} \frac{\hat{a}_i}{\hat{z}_i} = \sum_{i=1}^{2} \frac{a_i}{z_i} \ge \sum_{i=1}^{2} \frac{a_i}{z_i}\left(1-c_{-i}^o\right),$$

with the second equality due to the hypothesis of the theorem, and the inequality due to the fact that $c_{-i}^o \in [0,1]$ for each $-i$. Thus, the conclusion follows from Theorem 1.b. □

B.4. Theorem 5

Lemma 6. *For any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, any BNE $\sigma := (\sigma_1, \sigma_2)$ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any $i \in \{1,2\}$:*

a. *the support of $H_{i,\sigma}$ is $[0, x_\sigma]$ and $H_{i,\sigma}$ has neither gap nor atom in $(0, x_\sigma]$;*
b. *σ_i is monotone.*

Proof. Claim (a): The supremum of the support of $H_{i,\sigma}$ exists by individual rationality, with z_i finite. By the payment rule of an all-pay auction and the equilibrium condition, this supremum is the same between the two players, and $H_{i,\sigma}$ has no gap in $[0, x_\sigma]$. To prove the no-atom claim, pick any $b \in (0, x_\sigma]$. We have noted that $H_{-i,\sigma}$ has no gap, hence for any $\epsilon > 0$ there exists a strictly positive mass of player $-i$'s equilibrium bids belonging to $(b-\epsilon, b)$. Thus, if b is an atom of $H_{i,\sigma}$, those types t_{-i} of $-i$ that submit such bids would deviate from such bids to a bid slightly above b when ϵ is sufficiently small, as the incremental revenue $H_{i,\sigma}(b+\epsilon) - H_{i,\sigma}(b-\epsilon)$ outweighs the incremental cost $2\epsilon/t_{-i}$. This contradiction to the equilibrium condition implies that b is not an atom of $H_{i,\sigma}$.

Claim (b): Consider any behavioral strategy σ_i in the equivalence class of the equilibrium distributional strategy. There is an $S \subseteq \operatorname{supp} \tilde{F}_i$, with $\mathbb{R}_+ \setminus S$ of zero measure with respect to $\tilde{F}_i$, such that $\sigma_i(\cdot, t_i)$ best replies $H_{-i,\sigma}$ for any $t_i \in S$. Pick any $t_i', t_i'' \in S$. There are $B' \subseteq \operatorname{supp} \sigma_i(\cdot, t_i')$ and $B'' \subseteq \operatorname{supp} \sigma_i(\cdot, t_i'')$, with B' of full measure with respect to $\sigma_i(\cdot, t_i')$ and B'' of full measure with respect to $\sigma_i(\cdot, t_i'')$, such that any element of B' best replies $H_{-i,\sigma}$ for type t_i' of player i, and any element of B'' best replies for type t_i''. Let $b' \in B'$ and $b'' \in B''$. By revealed preference and Eq. (16),

$$H^*_{-i,\sigma}(b') - b'/t_i' \ge H^*_{-i,\sigma}(b'') - b''/t_i',$$
$$H^*_{-i,\sigma}(b'') - b''/t_i'' \ge H^*_{-i,\sigma}(b') - b'/t_i''.$$

Sum the two inequalities to obtain $\left(b'' - b'\right)/t_i' \ge \left(b'' - b'\right)/t_i''$. Thus, $t_i' < t_i'' \Rightarrow b' \le b''$. With b' and b'' being any elements of B' and B'' respectively, (22) is satisfied on S. Since $\mathbb{R}_+ \setminus S$ is of zero measure with respect to $\tilde{F}_i$, modify σ_i so that (22) is satisfied throughout $\mathbb{R}_+$ and thus obtain another element in the same equivalence class of the distributional strategy. Since this element satisfies (22) on $\mathbb{R}_+$, the distributional strategy is monotone. □

Theorem 5 With σ an equilibrium of the all-pay auction, Lemma 6.a says that the bid distribution $H_{-i,\sigma}$ has no atom except possibly at the zero bid. Thus, by Eq. (15), $H^*_{-i,\sigma}(b) = H_{-i,\sigma}(b)$ for all $b \in \mathbb{R}_+ \setminus \{0\}$, and by the uniform tie-breaking rule $H^*_{-i,\sigma}(0) = H_{-i,\sigma}(0)/2$. Thus, for any $t_i \in [a_i, z_i] \setminus \{0\}$,

$$\begin{aligned} U_i(t_i|\sigma) &\overset{(16)}{=} \sup_{b \in \{0\} \cup \mathbb{R}_{++}} H^*_{-i,\sigma}(b) - b/t_i \\ &= \max\left\{ H_{-i,\sigma}(0)/2, \lim_{b \downarrow 0} H_{-i,\sigma}(b) - b/t_i, \sup_{b \in \mathbb{R}_{++}} H_{-i,\sigma}(b) - b/t_i \right\} \\ &= \max\left\{ H_{-i,\sigma}(0), \sup_{b \in \mathbb{R}_{++}} H_{-i,\sigma}(b) - b/t_i \right\} \\ &= \sup_{b \in \mathbb{R}_+} H_{-i,\sigma}(b) - b/t_i. \end{aligned} \tag{24}$$

Since $H_{-i,\sigma}$, a c.d.f., is upper semicontinuous and its only possible discontinuous point is zero (Lemma 6.a), $H_{-i,\sigma}$ restricted to $\mathbb{R}_+$ is continuous. This, combined with the fact that the domain for b in the problem (24) can be bounded without loss by $[0, z_i]$, implies that the maximum in (24) is attained. Thus $U_i(t_i|\sigma) = \max_{b \in \mathbb{R}_+} H_{-i,\sigma}(b) - b/t_i$ for all $t_i \in [a_i, z_i] \setminus \{0\}$. Since $\max_{b \in \mathbb{R}_+} H_{-i,\sigma}(b) - b/t_i$ is continuous in t_i for all $t_i \in \mathbb{R}_{++}$ by the theorem of maximum, $U_i(t_i|\sigma)$ is continuous in t_i for all $t_i \in [a_i, z_i] \setminus \{0\}$. □

B.5. Properties of $\gamma_{i,\sigma}$

Lemma 7. *Suppose that H is a c.d.f. that has neither gap nor atom in $(0, x]$, with $[0, x]$ being its support. For any c.d.f. F let $\gamma(b) := F^{-1}(H(b))$ for all $b \in \mathbb{R}$. Then—*

a. *for any $b \in [0, x]$, $F(\gamma(b)) \geq H(b)$;*
b. *γ is weakly increasing on $[0, x]$;*
c. *$[\gamma(b) = t = \gamma(b')$ and $b \neq b'] \iff [t$ is an atom of $F]$;*
d. *if there is a unique $b \in [0, x]$ such that $\gamma(b) = t$, then $F(t) = H(b)$;*
e. *if $t \in \operatorname{supp} F \setminus \operatorname{range} \gamma$ then either (i) $F(t) < H(0)$ and $t < \gamma(0)$, or (ii) there exists a unique $b \in [0, x]$ such that $F(t) = H(b)$ and $F(\gamma(b)) = F(t)$.*

Proof. Claim (a): By Eq. (9) and the definition of $\gamma(b)$,

$$\gamma(b) = \inf\{\tau \in \operatorname{supp} F : F(\tau) \geq H(b)\} \tag{25}$$

for all b. Thus Claim (a) follows from upper semicontinuity of any distribution.

Claim (b): Let $b' > b$. By Claim (a), $F(\gamma(b')) \geq H(b')$; hence $F(\gamma(b')) \geq H(b)$ as H is increasing. Then Eq. (25) implies $\gamma(b') \geq \gamma(b)$.

Claim (c): Let $b' \geq b$ and $\gamma(b) = t = \gamma(b')$. By Claim (a), $F(t) \geq H(b')$. For any $t' < t$ such that $t' \in \operatorname{supp} F$, Eq. (25) implies $F(t') < H(b)$. Hence $\lim_{t' \uparrow t} F(t') \leq H(b)$. Thus,

$$F(t) - \lim_{t' \uparrow t} F(t') \geq H(b') - H(b).$$

Since H has no gap, $H(b') - H(b) > 0 \iff b' > b$. Thus, $b' > b \iff F(t) - \lim_{t' \uparrow t} F(t') > 0 \iff t$ is an atom of F.

Claim (d): Since $\gamma(b) = t$, $F(t) \geq H(b)$ by Claim (a). Suppose $F(t) > H(b)$. Then there exists $b' > b$ for which $F(t) \geq H(b')$, as H has no gap. Thus, for any $t' \in \operatorname{supp} F$ such that $t' < t$, $F(t') < H(b')$, otherwise $F(t') \geq H(b') \geq H(b)$ and hence by Eq. (25) $\gamma(b) \neq t$, contradiction. Now that $F(t') < H(b')$ for all such t', by $F(t) \geq H(b')$ and (25) we have $\gamma(b') = t$, contradicting the uniqueness of b. Hence $F(t) \leq H(b)$, as desired.

Claim (e): Let $t \in \operatorname{supp} F \setminus \operatorname{range} \gamma$. By hypothesis of this lemma, H is a continuous bijection from $[0, x]$ to $[H(0), 1]$; thus either (i) $F(t) < H(0)$ or (ii) $F(t) = H(b)$ for a unique $b \in [0, x]$. In Case (i), $t < \gamma(0)$ because $F(\gamma(0)) \geq H(0)$ by Claim (a). In Case (ii), the fact $t \neq \gamma(b)$ implies, by Eq. (25), that $t > \gamma(b)$. Then $F(\gamma(b)) \leq F(t) = H(b) \leq F(\gamma(b))$, with the last inequality due to Claim (a). Hence $F(\gamma(b)) = F(t)$. □

Lemma 8. *Given any c.d.f. F and any strategy σ, let H be the distribution generated by σ given F, and $\gamma(b) := F^{-1}(H(b))$ for all $b \in \mathbb{R}$. If H has neither gap nor atom in $(0, x]$, with $[0, x]$ being its support, and if σ is monotone, then for any $b \in [0, x]$ and any $t, t' \in \operatorname{supp} F$ such that $t < \gamma(b) < t'$:*

a. $\sup \operatorname{supp} \sigma(\cdot, t) \leq b \leq \inf \operatorname{supp} \sigma(\cdot, t')$*;*
b. *if $b \in \operatorname{supp} \sigma(\cdot, t')$, then $\left(\gamma(b), t'\right)$ is a gap of F;*
c. $b \in \operatorname{supp} \sigma(\cdot, \gamma(b))$.

Proof. Claim (a): By Eq. (25) and $t < \gamma(b)$, $F(t) < H(b)$. If $\sup \operatorname{supp} \sigma(\cdot, t) > b$, then by monotonicity of σ no type above t would bid b, hence $H(b) \leq F(t)$, contradiction. To prove the second inequality of Claim (a), suppose, to the contrary, that $b > \inf \operatorname{supp} \sigma(\cdot, t') =: b'$. By monotonicity of σ, no type below t' bids above b', hence $H(b') \geq \lim_{\tau \uparrow t'} F(\tau)$. Thus

$$H(b) > H(b') \geq \lim_{\tau \uparrow t'} F(\tau) \geq F(\gamma(b)) \geq H(b),$$

with the strict inequality due to H having no gap, the second last inequality due to $\gamma(b) < t'$, and the last, Lemma 7.a. The contradiction displayed above implies Claim (a).

Claim (b): Pick any $\tau \in (\gamma(b), t')$. Applying the second inequality in Claim (a) to the case where τ plays the role of t', we have $b \leq \inf \operatorname{supp} \sigma(\cdot, \tau)$. Hence

$$b \leq \inf \operatorname{supp} \sigma(\cdot, \tau) \leq \sup \operatorname{supp} \sigma(\cdot, \tau) \leq \inf \operatorname{supp} \sigma(\cdot, t') \leq b,$$

with the second last inequality due to monotonicity of σ, and the last due to the hypothesis $b \in \operatorname{supp} \sigma(\cdot, t')$. Thus $\operatorname{supp} \sigma(\cdot, \tau) = \{b\}$ and hence $F(\tau) \leq H(b)$. Consequently,

$$\lim_{\tau \uparrow t'} F(\tau) \leq H(b) \overset{(25)}{\leq} F(\gamma(b)) \leq \lim_{\tau \uparrow t'} F(\tau).$$

Thus $\lim_{\tau \uparrow t'} F(\tau) = F(\gamma(b))$ for all $\tau \in \left(\gamma(b), t'\right)$. I.e., $\left(\gamma(b), t'\right)$ is a gap of F.

Claim (c): Since σ is monotone and the H that it generates has no gap,

$$\sup \cup_{\tau < \gamma(b)} \operatorname{supp} \sigma(\cdot, \tau) = \inf \operatorname{supp} \sigma(\cdot, \gamma(b)).$$

Thus, by the fact $\sup \cup_{\tau < \gamma(b)} \operatorname{supp} \sigma(\cdot, \tau) = \sup_{\tau < \gamma(b)} \sup \operatorname{supp} \sigma(\cdot, \tau)$, we have

$$\sup_{\tau < \gamma(b)} \sup \operatorname{supp} \sigma(\cdot, \tau) = \inf \operatorname{supp} \sigma(\cdot, \gamma(b)).$$

Analogously, $\inf_{\tau' > \gamma(b)} \inf \operatorname{supp} \sigma(\cdot, \tau') = \sup \operatorname{supp} \sigma(\cdot, \gamma(b))$. Then Claim (a), applied to all $\tau < \gamma(b)$ and $\tau' > \gamma(b)$, implies

$$\inf \operatorname{supp} \sigma(\cdot, \gamma(b)) = \sup_{\tau < \gamma(b)} \sup \operatorname{supp} \sigma(\cdot, \tau) \leq b \leq \inf_{\tau' > \gamma(b)} \inf \operatorname{supp} \sigma(\cdot, \tau')$$
$$= \sup \operatorname{supp} \sigma(\cdot, \gamma(b)).$$

Thus, $\operatorname{supp} \sigma(\cdot, \gamma(b))$, convex because H has no gap and σ is monotone, contains b. □

B.6. Theorem 6: the equilibrium bid distributions

Consider any $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$ and any BNE σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$. The associated $(H_{i,\sigma})_{i=1}^2$, x_σ and $(\gamma_{i,\sigma})_{i=1}^2$ are each uniquely defined by Eqs. (13), (14), and (19). Since each player's distributional strategy in the equilibrium is monotone (Lemma 6.b), I shall identify it with the corresponding behavioral strategy that satisfies the monotonicity condition (22) throughout $\mathbb{R}_+$. All the claims in this subsection refer to the same tuple $\left(\mathscr{G}(\tilde{F}_1, \tilde{F}_2), \sigma, x_\sigma, (H_{i,\sigma}, \gamma_{i,\sigma})_{i=1}^2\right)$ specified above.

Lemma 9. *For any $i \in \{1, 2\}$, if $b > 0$ then $b \notin \operatorname{supp} \sigma_i(\cdot, 0)$.*

Proof. Suppose, to the contrary, that $b \in \operatorname{supp} \sigma_i(\cdot, 0)$. With σ_i monotone, no type above zero bids in $[0, b)$. Thus, since $H_{i,\sigma}$ has no gap (Lemma 6.a), $\operatorname{supp} \sigma_i(\cdot, 0)$ contains $(0, b]$ and 0 is an atom of $\tilde{F}_i$. Hence $\{0\} \times (0, b]$ is a set of positive measure with respect to the distributional strategy corresponding to σ_i, whose marginal distribution of player i's type is $\tilde{F}_i$. But no element of $\{0\} \times (0, b]$ satisfies the best response condition, because to the zero type the cost of any positive bid is infinity (Section 2). That contradicts σ_i being a best response. □

Lemma 10. *For any $i \in \{1, 2\}$, if $b > 0$, $t_i \in \operatorname{supp} \tilde{F}_i$ and $b \in \operatorname{supp} \sigma_i(\cdot, t_i)$, then b is a best response to $H_{-i,\sigma}$ for the type t_i of player i.*

Proof. Let $b > 0$ and $b \in \operatorname{supp} \sigma_i(\cdot, t_i)$. By Lemma 9, $t_i \neq 0$ and hence $t_i > 0$. With $b > 0$, b is not an atom of $H_{-i,\sigma}$ (Lemma 6.a), hence it yields an expected payoff $H_{-i,\sigma}(b) - b/t_i$ for player i of type t_i. Suppose, to the contrary of the lemma, that b is not a best response for type t_i. Then there exists $b' \in \mathbb{R}_+ \setminus \{b\}$ for which

$$H_{-i,\sigma}(b) - b/t_i < H_{-i,\sigma}(b') - b'/t_i,$$

where the right-hand side is equal to the expected payoff that b' yields because $H_{-i,\sigma}(b')$ is i's winning probability with b' unless $b' = 0$ and $H_{-i,\sigma}(0) > 0$, in which case we can choose b' to be a bid slightly above zero instead of zero. The above-displayed strict inequality, combined with $H_{-i,\sigma}$ being continuous on $\mathbb{R}_+$ (as it is right-continuous and, by Lemma 6.a, can be discontinuous only at zero), $b > 0$ and $t_i > 0$, implies that there exists $\delta > 0$ such that

$$H_{-i,\sigma}(y) - y/\tau < H_{-i,\sigma}(b') - b'/\tau \tag{26}$$

for any $y \in (b - \delta, b + \delta)$ and any $\tau \in (t_i - \delta, t_i + \delta)$. We claim that

$$\int_{t_i-\delta}^{t_i+\delta} \int_{b-\delta}^{b+\delta} \sigma_i(dy, \tau) d\tilde{F}_i(\tau) > 0. \tag{27}$$

This is true if t_i is an atom of $\tilde{F}_i$, as in that case $\operatorname{supp} \sigma_i(\cdot, t_i)$ is a nondegenerate interval containing b, since b cannot be an atom of $H_{i,\sigma}$. Thus, suppose t_i is not an atom of $\tilde{F}_i$. Then $\operatorname{supp} \sigma_i(\cdot, t_i)$

is singleton since $H_{i,\sigma}$ has no gap (Lemma 6.a) and σ_i is monotone. Hence $\{b\} = \operatorname{supp}\sigma_i(\cdot, t_i)$. Now suppose that (27) does not hold. Then, with $H_{i,\sigma}$ gapless and σ_i monotone,

$$\sup \bigcup_{\tau \le t_i - \delta} \operatorname{supp}\sigma_i(\cdot, \tau) = b = \inf \bigcup_{\tau' \ge t_i + \delta} \operatorname{supp}\sigma_i(\cdot, \tau').$$

Consequently, if $t_i - \delta < \tau < t_i + \delta$ then monotonicity of σ_i requires $\{b\} = \operatorname{supp}\sigma_i(\cdot, \tau)$. Then, as b is not atom of $H_{i,\sigma}$, the measure of $(t_i - \delta, t_i + \delta)$ with respect to $\tilde{F}_i$ is zero, which contradicts the hypothesis $t_i \in \operatorname{supp}\tilde{F}_i$. That proves (27), which coupled with (26) contradicts the fact that the strategy σ_i is a best response to $H_{-i,\sigma}$. □

Lemma 11. *For any $i \in \{1, 2\}$ and almost every $b \in [0, x_\sigma]$, $H_{i,\sigma}$ is differentiable at b and*

$$\frac{d}{db} H_{i,\sigma}(b) = \frac{1}{\gamma_{-i,\sigma}(b)}. \tag{28}$$

Proof. Denote $\Gamma_{i,\sigma}(b) := \{t_i \in \operatorname{supp}\tilde{F}_i : b \in \operatorname{supp}\sigma_i(\cdot, t_i)\}$. By Lemma 9, $b > 0 \Rightarrow 0 \notin \Gamma_{i,\sigma}(b)$. A monotone function, $H_{-i,\sigma}$ is differentiable at almost every b in $[0, x_\sigma]$. For any such b with $b > 0$, let $t_i \in \Gamma_{i,\sigma}(b)$, hence $t_i > 0$. By Lemma 10, b is a best reply for the type t_i of player i; thus, with $t_i > 0$, the derivative of his expected payoff at b is $H'_{-i,\sigma}(b) - 1/t_i \ge 0$, which in turn implies, for any $t'_i > t_i$, that $H'_{-i,\sigma}(b) - 1/t'_i > 0$ and hence b cannot be a best reply for the type t'_i. Thus $\Gamma_{i,\sigma}(b)$ is singleton. Note $\gamma_{i,\sigma}(b) \in \Gamma_{i,\sigma}(b)$ because Lemma 8.c says $b \in \operatorname{supp}\sigma_i(\cdot, \gamma_{i,\sigma}(b))$ and Eq. (25) implies $\gamma_{i,\sigma}(b) \in \operatorname{supp}\tilde{F}_i$. Thus, $\{\gamma_{i,\sigma}(b)\} = \Gamma_{i,\sigma}(b)$. Switch the roles between i and $-i$ to obtain $\{\gamma_{-i,\sigma}(b)\} = \Gamma_{-i,\sigma}(b)$ for almost all such b at which $H_{i,\sigma}$ is differentiable. Thus, any such b satisfies the first-order necessary condition to best reply $H_{i,\sigma}$ for the type $\gamma_{-i,\sigma}(b)$ of player $-i$, and hence satisfies (28). □

Lemma 12. *For any $i \in \{1, 2\}$, there exist functions $H^{ac}_{i,\sigma}$, $H^*_{i,\sigma} : \mathbb{R} \to [0, 1]$ such that: (i) for all $b \in \mathbb{R}$,*

$$H_{i,\sigma}(b) = H^*_{i,\sigma}(b) + H^{ac}_{i,\sigma}(b); \tag{29}$$

(ii) for all $b \in [0, x_\sigma]$,

$$H^{ac}_{i,\sigma}(b) = \int_0^b \frac{1}{\gamma_{-i,\sigma}(y)} dy; \tag{30}$$

(iii) $H^*_{i,\sigma}$ is constant almost everywhere, and weakly increasing and right-continuous on $\mathbb{R}$; and (iv) if $H^*_{i,\sigma}$ is discontinuous at b then $b = 0$.

Proof. By the Lebesuge decomposition theorem, the distribution $H_{i,\sigma}$ can be uniquely decomposed into the sum of two functions $H^{ac}_{i,\sigma}, H^*_{i,\sigma} : \mathbb{R} \to [0, 1]$ such that each is weakly increasing and right-continuous, $H^{ac}_{i,\sigma}(b) = H^*_{i,\sigma}(b) = 0$ for all $b < 0$, $H^{ac}_{i,\sigma}$ is absolutely continuous on $\mathbb{R}$, $H^*_{i,\sigma}$ is constant almost everywhere (with respect to Lebesuge measure) and (29) holds for all $b \in \mathbb{R}$. Hence (i) and (iii) are immediate. For (iv), note that $H^*_{i,\sigma}$ is discontinuous at b if and only if $H_{i,\sigma}$ is discontinuous at b, since $H^{ac}_{i,\sigma}$ is absolutely continuous. To $H_{i,\sigma}$, a weakly increasing function, any discontinuity is a jump discontinuity. Thus, if $H^*_{i,\sigma}$ is discontinuous at b, then b is an atom of $H_{i,\sigma}$ and hence, since the only possible atom of $H_{i,\sigma}$ is $\{0\}$ (Lemma 6.a), $b = 0$. That proves (iv). To show (ii), note from the monotonicity of $H_{i,\sigma}$, $H^{\mathrm{ac}}_{i,\sigma}$ and $H^*_{i,\sigma}$ that, at almost every $b \in \mathbb{R}$, all three functions are differentiable and

$$\frac{d}{db} H_{i,\sigma}(b) \overset{(29)}{=} \frac{d}{db} H^{\mathrm{ac}}_{i,\sigma}(b) + \frac{d}{db} H^*_{i,\sigma}(b) = \frac{d}{db} H^{\mathrm{ac}}_{i,\sigma}(b),$$

with the second equality due to $H^*_{i,\sigma}$ being constant almost everywhere. The above equation, coupled with (28) for almost all b in $[0, x_\sigma]$ (Lemma 11), implies $\frac{d}{db} H^{\mathrm{ac}}_{i,\sigma} = 1/\gamma_{-i,\sigma}$ almost everywhere in $[0, x_\sigma]$. Thus, since $H^{\mathrm{ac}}_{i,\sigma}$ is absolutely continuous, for all $b \in [0, x_\sigma]$ we have

$$H^{\mathrm{ac}}_{i,\sigma}(b) = H^{\mathrm{ac}}_{i,\sigma}(0) + \int_0^b \frac{1}{\gamma_{-i,\sigma}(y)} dy = \int_0^b \frac{1}{\gamma_{-i,\sigma}(y)} dy,$$

with the last equality due to $H^{\mathrm{ac}}_{i,\sigma}$ being continuous at zero and $H^{\mathrm{ac}}_{i,\sigma}(b') = 0$ for all $b' < 0$. Thus we obtain (30), hence (ii) follows. □

Proof of Theorem 6. Given any equilibrium σ of $\mathscr{G}(\tilde{F}_1, \tilde{F}_2)$, the associated $(H_{1,\sigma}, H_{2,\sigma})$, x_σ and $(\gamma_{1,\sigma}, \gamma_{2,\sigma})$ are each uniquely defined by Eqs. (13), (14) and (19). If Eq. (20) holds then, for each player i, $H_{i,\sigma}(0) = c_{i,\sigma}$ and then $c_{1,\sigma} c_{2,\sigma} = 0$ because $\{0\}$ cannot be an atom of both players' equilibrium bid distributions, otherwise such nonzero measure of either player's zero-bidding types would deviate to a bid slightly above zero.

To prove (20), given (19), (29) and (30), it suffices to show $H^*_{i,\sigma}(b) = H^*_{i,\sigma}(0)$ for all $b \in (0, x_\sigma]$. Suppose, to the contrary, that $H^*_{i,\sigma}(b) \neq H^*_{i,\sigma}(0)$ for some $b \in (0, x_\sigma]$. Then, since $\frac{d}{db} H^*_{i,\sigma} = 0$ almost everywhere (Lemma 12), $H^*_{i,\sigma}$ is not absolutely continuous on $[0, x_\sigma]$. Note that $H^*_{i,\sigma}$ is continuous on $[0, x_\sigma]$ because it is right-continuous and can be discontinuous only at zero (Lemma 12). This, combined with the fact that $H^*_{i,\sigma}$ is weakly increasing (Lemma 12) and is not absolutely continuous on $[0, x_\sigma]$, implies that there exists $\epsilon > 0$ for which $H^*_{i,\sigma}$ is not absolutely continuous on $[\epsilon, x_\sigma]$ (Royden and Fitzpatrick, 2010, Problem 37.ii, p123).

By Lemma 9, $\epsilon \notin \operatorname{supp} \sigma_{-i,\sigma}(\cdot, 0)$. That implies $\gamma_{-i,\sigma}(\epsilon) \neq 0$ (Lemma 8.c) and hence $\gamma_{-i,\sigma}(\epsilon) > 0$. Thus, there exists $K \in \mathbb{R}$ such that

$$K > \frac{1}{\gamma_{-i,\sigma}(\epsilon)} - \frac{1}{z_{-i}}.$$

Since $H^*_{i,\sigma}$ is not absolutely continuous on $[\epsilon, x_\sigma]$, it is not Lipschitz on $[\epsilon, x_\sigma]$, hence there exist b and b' such that $\epsilon \leq b < b' \leq x_\sigma$ and

$$\frac{H^*_{i,\sigma}(b') - H^*_{i,\sigma}(b)}{b' - b} > K.$$

By Lemma 8.c, $b \in \operatorname{supp} \sigma_{-i}(\cdot, \gamma_{-i,\sigma}(b))$. Also note that $\gamma_{-i,\sigma}(b) > 0$ because $b \geq \epsilon$ implies $\gamma_{-i,\sigma}(b) \geq \gamma_{-i,\sigma}(\epsilon)$ (Lemma 7.b), and that $\gamma_{-i,\sigma}(b) \in \operatorname{supp} \tilde{F}_{-i}$ because of Eq. (25). Thus, Lemma 10 implies that b is a best response to $H_{i,\sigma}$ for player $-i$ of type $\gamma_{-i,\sigma}(b)$. However, the gain for player $-i$ of type $\gamma_{-i,\sigma}(b)$ to deviate to the bid b' instead of b is equal to

$$\left(H_{i,\sigma}(b') - \frac{b'}{\gamma_{-i,\sigma}(b)} \right) - \left(H_{i,\sigma}(b) - \frac{b}{\gamma_{-i,\sigma}(b)} \right)$$
$$\overset{(29)}{=} (b' - b) \left(\frac{H^{\mathrm{ac}}_{i,\sigma}(b') - H^{\mathrm{ac}}_{i,\sigma}(b)}{b' - b} - \frac{1}{\gamma_{-i,\sigma}(b)} + \frac{H^*_{i,\sigma}(b') - H^*_{i,\sigma}(b)}{b' - b} \right)$$

$$\overset{(30)}{=} (b' - b)\left(\frac{1}{b'-b}\int_b^{b'} \frac{1}{\gamma_{-i,\sigma}(y)}dy - \frac{1}{\gamma_{-i,\sigma}(b)} + \frac{H^*_{i,\sigma}(b') - H^*_{i,\sigma}(b)}{b'-b}\right)$$

$$\geq (b' - b)\left(\frac{1}{z_{-i}} - \frac{1}{\gamma_{-i,\sigma}(\epsilon)} + \frac{H^*_{i,\sigma}(b') - H^*_{i,\sigma}(b)}{b'-b}\right) \tag{31}$$

$$> (b' - b)\left(-K + \frac{H^*_{i,\sigma}(b') - H^*_{i,\sigma}(b)}{b'-b}\right),$$

which is positive by the choice of b and b'. Here the line (31) uses the aforementioned fact $\gamma_{-i,\sigma}(b) \geq \gamma_{-i,\sigma}(\epsilon)$. The desired contradiction established, we have $H^*_{i,\sigma}(b) = H^*_{i,\sigma}(0)$ for all $b \in (0, x_\sigma]$, as asserted. □

B.7. Recovering equilibrium strategies from bid distributions

Corollary 4. *If H is a c.d.f. that has neither gap nor atom in $(0, x]$, with $[0, x]$ being its support, then given any c.d.f. F there is at most one monotone distributional strategy that generates H.*

Proof. Define $\gamma(b) := F^{-1}(H(b))$ for all $b \in \mathbb{R}$. By (9), Eq. (25) holds. Let σ generate H, i.e., $H(b) = \int_{\mathbb{R}} \int_{-\infty}^{b} \sigma(dr, t)dF(t)$ for all b. Suppose that σ is monotone. Then $H(b) = \int_{\mathbb{R}} \int_{-\infty}^{b} \sigma(dr, t)dF(t)$ is equivalent to

$$H(b) = \int_{\mathbb{R}} \left(\mathbf{1}_{t<\gamma(b)} + \mathbf{1}_{t=\gamma(b)}\sigma(b,t)\right) dF(t)$$

$$= \lim_{t\uparrow\gamma(b)} F(t) + \sigma(b, \gamma(b))\left(F(\gamma(b)) - \lim_{t\uparrow\gamma(b)} F(t)\right). \tag{32}$$

By monotonicity of σ, $\lim_{t\uparrow\gamma(b)} F(t) = H(\beta_*(b))$ where $\beta_*(b) := \inf \operatorname{supp} \sigma(\cdot, \gamma(b))$, hence

$$H(b) = H(\beta_*(b)) + \sigma(b, \gamma(b))(F(\gamma(b)) - H((\beta_*(b))).$$

If $\gamma(b)$ is an atom of F, its mass is equal to $F(\gamma(b)) - H(\beta_*(b))$ and $\sigma(b, \gamma(b))$ is uniquely determined by Eq. (32); else Eq. (32) is reduced to $H(b) = H(\beta_*(b))$, which by strict monotonicity of H means $\beta_*(b) = b$ and $\operatorname{supp} \sigma(\cdot, \gamma(b)) = \{\beta_*(b)\} = \{b\}$.

To pin down σ completely, consider any $t \in \operatorname{supp} F \setminus \operatorname{range} \gamma$. By Lemma 7.e, either (i) $F(t) < H(0)$ and $t < \gamma(0)$ or (ii) $t > \gamma(b)$ and $F(t) = F(\gamma(b))$ for a unique b. In Case (i), with $t < \gamma(0)$, monotonicity of σ implies that $\sigma(\cdot, t)$ is the Dirac measure at zero. In Case (ii), the facts $t \neq \gamma(b)$ and $F(t) = F(\gamma(b))$ together imply that t is not an atom of F. Thus, with H gapless, $\operatorname{supp} \sigma(\cdot, t)$ is singleton. Consequently, σ cannot prescribe to t a bid $b' < b$; otherwise, a positive measure of types above t would be prescribed to bid in (b', b) since H has no gap and σ is monotone, but then $H(b) > F(t)$, contradiction. By the same token, σ cannot prescribe to t a bid above b. Thus, $\sigma(\cdot, t)$ is the Dirac measure at b. All t in $\operatorname{supp} F$ considered, the behavioral strategy σ is thus uniquely determined up to a set of zero measure with respect to F. Thus the corresponding distributional strategy is unique. □

Corollary 5. *Given the same hypothesis and notations of Lemma 7, we have:*

a. *there exists a monotone strategy* σ *that generates* H *given* F*;*
b. *if* H *is generated by a monotone strategy* σ *given* F*, then—*
   i. $\big[\operatorname{supp}\sigma(\cdot,t)$ *is not singleton*$\big] \Longleftrightarrow t$ *is an atom of* F*;*
   ii. *for any* $t \in \operatorname{supp} F$*, if* $t \geq \gamma(0)$ *then either* $\operatorname{supp}\sigma(\cdot,t) \subseteq \{b : \gamma(b) = t\}$ *or* $\operatorname{supp}\sigma(\cdot,t) = \{b\}$ *for which* $F(t) = H(b)$ *(and* $(\gamma(b), t)$ *is a gap of* F*); if* $t < \gamma(0)$ *then* $\operatorname{supp}\sigma(\cdot,t) = \{0\}$*;*

Proof. Claim (a): First, construct a strategy σ: For any $t \in \operatorname{supp} F$ such that $t = \gamma(b)$ for some $b \in \mathbb{R}_+$, if t is an atom of F then define a c.d.f. $\sigma(\cdot,t)$ according to Eq. (32); else define $\sigma(\cdot,t)$ to be the Dirac measure at b. For any $t \in \operatorname{supp} F$ that does not belong to the range of γ, if $t < \gamma(0)$ then let $\sigma(\cdot,t)$ be the Dirac measure at 0; else there exists a unique $b \in [0,x]$ for which $F(t) = H(b)$, and we let $\sigma(\cdot,t)$ be the Dirac measure at b.

We show that the σ constructed above is monotone. For any $t \in \operatorname{range}\gamma$, $\operatorname{supp}\sigma(\cdot,t)$ by construction is contained in the γ-inverse image of $\{t\}$. Thus, σ restricted to the range of γ is monotone, because γ is weakly increasing (Lemma 7.b). To show that monotonicity is preserved when types $t \in \operatorname{supp} F \setminus \operatorname{range}\gamma$ are also included, pick any such t. By Lemma 7.e, either (i) $F(t) < H(0)$ and $t < \gamma(0)$, in which case our σ prescribes to t the zero bid, or (ii) $t > \gamma(b)$ and $F(t) = F(\gamma(b))$ for a unique b, in which case σ prescribes the bid b. In Case (i), as zero is the lowest possible bid and $t < \gamma(0)$, $\sigma(\cdot,t)$ does not violate monotonicity. In Case (ii), σ prescribes to t the same bid b as it does to $\gamma(b)$ and, since $(\gamma(b),t)$ is a gap of F, we can simply set σ to prescribe the same b to those in the gap; any type below $\gamma(b)$ that belongs to $\operatorname{range}\gamma$ is prescribed by σ a bid no higher than b, by monotonicity of σ restricted to $\operatorname{range}\gamma$; likewise, any type above t belonging to $\operatorname{range}\gamma$ is prescribed by σ a bid higher than or equal to $\sup\operatorname{supp}\sigma(\cdot,\gamma(b))$, which is at least as high as b. Thus, monotonicity of σ is preserved when such t is included. This being true for all $t \in \operatorname{supp} F \setminus \operatorname{range}\gamma$, we have extended monotonicity of σ to the entire $\operatorname{supp} F$.

By its construction, σ satisfies Eq. (32) for all b. And Eq. (32), according to the proof of Corollary 4, which applies since σ is monotone, is equivalent to $H(b) = \int_{\mathbb{R}} \int_{-\infty}^{b} \sigma(dr,t)dF(t)$, i.e., the σ constructed above generates H. Thus Claim (a) follows.

Claim (b): By Corollary 4, any monotone strategy that generates H is the same as the σ constructed above. Thus, parts (i)–(ii) in the claim follow by construction of σ. □

B.8. The equilibrium given a degenerate type distribution

For any $i \in \{1,2\}$ and any $t_i^* \in [a_i, z_i]$, recall that $\mathscr{E}_i(\delta_{t_i^*})$ denotes the set of all BNEs of the all-pay auction where the distribution of i's type is the Dirac measure $\delta_{t_i^*}$ at t_i^*, and that of $-i$'s type is the prior F_{-i}.

Lemma 13. *For any* $i \in \{1,2\}$ *and any* $t_i^* \in [a_i, z_i] \setminus \{0\}$, $\mathscr{E}_i(\delta_{t_i^*}) = \{\sigma^*\}$ *such that*

$$\forall b \in [0, x_{\sigma^*}]: \quad H_{i,\sigma^*}(b) = c_{i,\sigma^*} + \int_0^b \left(F_{-i}^{-1}\left(\frac{s}{t_i^*} + c_{-i,\sigma^*} \right) \right)^{-1} ds, \tag{33}$$

$$\forall b \in [0, x_{\sigma^*}]: \quad H_{-i,\sigma^*}(b) = \frac{b}{t_i^*} + c_{-i,\sigma^*}, \tag{34}$$

$$c_{i,\sigma^*} c_{-i,\sigma^*} = 0, \tag{35}$$

$$x_{\sigma^*}/t_i^* = 1 - c_{-i,\sigma^*}, \tag{36}$$

$$1 - c_{i,\sigma^*} = t_i^* \int_{c_{-i,\sigma^*}}^{1} \frac{1}{F_{-i}^{-1}(s)} ds. \tag{37}$$

The lemma is proved in two steps. Section B.8.1 proves the uniqueness of the equilibrium in $\mathscr{E}_i(\delta_{t_i^*})$, and Section B.8.2, its existence.

B.8.1. The uniqueness proof for Lemma 13

Pick any $\sigma \in \mathscr{E}_i(\delta_{t_i^*})$ and denote $(H_{i,\sigma}, H_{-i,\sigma}, c_{i,\sigma}, c_{-i,\sigma}, x_\sigma)$ for the associated tuple of bid distributions, masses of $\{0\}$ and bid supremum. We shall show that σ is unique. By definition of $\delta_{t_i^*}$, supp $\tilde{F}_i = \{t_i^*\}$ if $\tilde{F}_i$ denotes the c.d.f. corresponding to $\delta_{t_i^*}$. Hence $\gamma_{i,\sigma} = t_i^*$ on $[0, x_\sigma]$ by Eq. (19). Thus, Eq. (20), where the role of i is played by $-i$ here, implies that

$$H_{-i,\sigma}(b) = \frac{b}{t_i^*} + c_{-i,\sigma}$$

for all $b \in [0, x_\sigma]$, i.e., Eq. (34) is satisfied. By definition of $\gamma_{-i,\sigma}$, for all $b \in [0, x_\sigma]$,

$$\gamma_{-i,\sigma}(b) = F_{-i}^{-1}\left(H_{-i,\sigma}(b)\right) = F_{-i}^{-1}\left(\frac{b}{t_i^*} + c_{-i,\sigma}\right).$$

Then again Eq. (20) implies that, for all $b \in [0, x_\sigma]$,

$$H_{i,\sigma}(b) = c_{i,\sigma} + \int_0^b \left(F_{-i}^{-1}\left(\frac{s}{t_i^*} + c_{-i,\sigma}\right)\right)^{-1} ds.$$

Hence Eq. (33) follows. Eq. (35) is also satisfied due to Lemma 12. Apply Eq. (34) to the supremum x_σ of the bid distribution $H_{-i,\sigma}$ to obtain $1 = x_\sigma/(t_i^*) + c_{-i,\sigma}$, i.e., Eq. (36). And apply Eq. (33) to the supremum x_σ to get

$$1 - c_{i,\sigma} = \int_0^{x_\sigma} \left(F_{-i}^{-1}\left(\frac{y}{t_i^*} + c_{-i,\sigma}\right)\right)^{-1} dy = t_i^* \int_{c_{-i,\sigma}}^{x_\sigma/t_i^* + c_{-i,\sigma}} \left(F_{-i}^{-1}(s)\right)^{-1} ds,$$

with the second equality due to the change of variables $s := y/t_i^* + c_{-i,\sigma}$. This equation coupled with Eq. (36) gives Eq. (37), which for any $c_{i,\sigma}$ admits at most one solution for $c_{-i,\sigma}$, with the right-hand side strictly decreasing in $c_{-i,\sigma}$. Consequently, Eqs. (35) and (37) together determine uniquely $(c_{i,\sigma}, c_{-i,\sigma})$, hence Eq. (36) determines x_σ uniquely, and so $H_{i,\sigma}$ and $H_{-i,\sigma}$ are each uniquely determined by Eqs. (33) and (34). Note from Eqs. (33)–(35) that both bid distributions $H_{i,\sigma}$ and $H_{-i,\sigma}$ are gapless and atomless on $(0, x_\sigma]$. This coupled with the monotonicity of any BNE of the auction game (Lemma 6.b) implies that Corollary 4 is applicable. Hence σ is unique as $(H_{i,\sigma}, H_{-i,\sigma})$ is unique. □

B.8.2. The existence proof for Lemma 13

Step 1: construction and preparation Clearly, Eqs. (35)–(37) together admit a unique solution for $(c_{i,\sigma^*}, c_{-i,\sigma^*}, x_{\sigma^*}) \in [0,1]^2 \times \mathbb{R}_+$. Plug this solution into Eq. (33)–(34) to obtain a pair $(H_{i,\sigma^*}, H_{-i,\sigma^*})$, each a c.d.f. with support $[0, x_{\sigma^*}]$ due to Eqs. (36) and (37). Suppress the symbol σ^* in the subscripts and write the tuple as $(H_i, H_{-i}, c_i, c_{-i}, x)$, which we shall prove constitutes an equilibrium in $\mathscr{E}_i(\delta_{t_i^*})$. We shall refer to σ^* and (H_i, H_{-i}) interchangeably. Let $\gamma_{-i}(b) := F_{-i}^{-1}(H_{-i}(b))$ for all b, hence Eq. (25) applies.

Lemma 14. *Given the above-constructed H_{-i} based on t_i^*, the following is true for any type $t_i \in [a_i, z_i] \setminus \{0\}$ of player i:*

a. *if $t_i = t_i^*$, then any bid in $(0, x]$ is a best reply to H_{-i};*
b. *if $c_{-i} = 0$ and $t_i \le t_i^*$, then bidding zero is a best reply to H_{-i};*
c. *if $c_{-i} > 0$ and $t_i < t_i^*$, then the best reply to H_{-i} is null, and*

$$\lim_{\epsilon \downarrow 0} \sup \mathrm{BR}_i \left(t_i, \epsilon \mid \sigma^*\right) = 0;$$

d. *if $t > t_i^*$ then the best reply to H_{-i} is uniquely x;*
e. *if $t_i \ge t_i^*$, $U_i(t_i|\sigma^*) = 1 - x/t_i = 1 - (t_i^*/t_i)(1 - c_{-i})$;*
f. *if $t_i \le t_i^*$, $U_i(t_i|\sigma^*) = c_{-i}$.*

Proof. We need only to consider bids in the support $[0, x]$ of H_{-i}, as player i has no incentive to bid outside it. Given type $t_i \in [a_i, z_i] \setminus \{0\}$, player i's expected payoff from submitting any bid $b \in (0, x]$ is equal to, by Eq. (34),

$$\frac{b}{t_i^*} + c_{-i} - \frac{b}{t_i} = c_{-i} + b\left(\frac{1}{t_i^*} - \frac{1}{t_i}\right);$$

and his expected payoff from bidding zero is equal to $c_{-i}/2$ due to the equal-probability tie-breaking rule. Thus, Claims (a), (b), (c) and (d) follow. (The equation in (c) is due to the fact that $\mathrm{BR}_i(t_i, \epsilon|\sigma^*) = \left(0, \epsilon/\left(1/t_i - 1/t_i^*\right)\right]$ for any $t_i < t_i^*$ and any $\epsilon > 0$.) Claims (a) and (d), combined with the fact that x is not an atom of H_{-i} and hence bidding x wins for sure, imply the first equality in Claim (e); then the second equality of (e) follows from Eq. (36). Taking the limit of the above-displayed expected payoff when $b \downarrow 0$, coupled with the definition of $U_i(t_i|\sigma^*)$ in Eq. (16), we obtain Claim (f). □

Lemma 15. *For any $t_{-i} \in \operatorname{supp} F_{-i}$, define for any $b \in [0, x]$*

$$u_{-i}(b, t_{-i}) := \begin{cases} t_{-i} H_i(b) - b & \text{if } b > 0 \\ t_{-i} H_i(0)/2 & \text{if } b = 0. \end{cases} \tag{38}$$

Then $u_{-i}(\cdot, t_{-i})$ is concave on $(0, x]$ and, if $H_i(0) = 0$, also on $[0, x]$; furthermore, if $u_{-i}(\cdot, t_{-i})$ is differentiable at b, then $\frac{\partial}{\partial b} u_{-i}(b, t_{-i})$, denoted by $D_1 u_{-i}(b, t_{-i})$, satisfies

$$D_1 u_{-i}(b, t_{-i}) = \frac{t_{-i}}{\gamma_{-i}(b)} - 1. \tag{39}$$

Proof. By the definition of u_{-i}, to prove concavity of $u_{-i}(\cdot, t_{-i})$ on $(0, x]$ it suffices to show that H_i is concave on $[0, x]$. By Eq. (33), H_i restricted to $[0, x]$ is absolutely continuous; thus, we need only to show that the left-derivative of H_i is never smaller than the right-derivative and that, whenever the two coincide, the derivative is weakly decreasing. To that end, pick any $b \in [0, x)$. By Eq. (33), the right-derivative of H_i at b is

$$\begin{aligned} D_+ H_i(b) &= \lim_{b'' \downarrow b} \frac{1}{b'' - b} \int_b^{b''} \left(F_{-i}^{-1}\left(\frac{s}{t_i^*} + c_{-i}\right)\right)^{-1} ds \\ &= \lim_{b'' \downarrow b} \left(F_{-i}^{-1}\left(\frac{\xi(b, b'')}{t_i^*} + c_{-i}\right)\right)^{-1}, \end{aligned}$$

with $b \le \xi(b, b'') \le b''$ by the intermediate-value theorem. Thus, with F_{-i}^{-1} weakly increasing,

$$D_+ H_i(b) \le \left(F_{-i}^{-1} \left(\frac{b}{t_i^*} + c_{-i} \right) \right)^{-1}.$$

Analogously, the left-derivative at any $b \in (0, x]$ is

$$D_- H_i(b) = \lim_{b'' \uparrow b} \frac{1}{b - b''} \int_{b''}^{b} \left(F_{-i}^{-1} \left(\frac{s}{t_i^*} + c_{-i} \right) \right)^{-1} ds \ge \left(F_{-i}^{-1} \left(\frac{b}{t_i^*} + c_{-i} \right) \right)^{-1}.$$

Thus, for any $b \in (0, x)$, $D_- H_i(b) \ge D_+ H_i(b)$ and, when they coincide,

$$\frac{d}{db} H_i(b) = D_+ H_i(b) = D_- H_i(b) = \left(F_{-i}^{-1} \left(\frac{b}{t_i^*} + c_{-i} \right) \right)^{-1} = \frac{1}{\gamma_{-i}(b)},$$

with the last equality due to the definition of γ_{-i} and Eq. (34). Thus, by Lemma 7.b, $\frac{d}{db} H_i(b)$ is weakly decreasing. Hence $u_{-i}(\cdot, t_{-i})$ is concave on $(0, x]$, and Eq. (39) holds. If $H_i(0) = 0$ then $u_{-i}(\cdot, t_{-i})$ by Eq. (38) is continuous at zero, hence also concave on $[0, x]$. □

Step 2: verification To verify the equilibrium condition from player i's standpoint, since $\{t_i^*\} = \operatorname{supp} \delta_{t_i^*}$, it suffices to show that H_i best responds to H_{-i} for the type t_i^* of player i. By Lemma 14.a, every nonzero element of the support $[0, x]$ of H_i is a best response for t_i^*. We need to verify that the zero bid is also a best response only when $H_i(0) > 0$; in that case, $c_{-i} = 0$ by Eq. (35), hence Lemma 14.b implies that the zero bid is a best response for t_i^*.

Thus the rest of the proof concerns player $-i$. By Corollaries 4 and 5.a, there is a unique monotone strategy σ_{-i} that generates H_{-i} given F_{-i}. We shall show that σ_{-i} is player $-i$'s best response to H_i for all types but a set of zero F_{-i}-measure. Since $F_{-i}(0) = 0$ by assumption, we may assume without loss that $t_{-i} > 0$. Thus, player $-i$'s decision in the auction is equivalent to maximizing $u_{-i}(\cdot, t_{-i})$, defined in Eq. (38).

Pick any (b, t_{-i}) such that $t_{-i} \in \operatorname{supp} F_{-i}$, $t_{-i} > 0$ and $b \in \operatorname{supp} \sigma_{-i}(\cdot, t_{-i})$. To verify the best response condition for player $-i$, there is no loss of generality to assume that either $b > 0$ or "$b = 0$ and $H_i(0) = 0$" holds. That is because if $H_i(0) > 0$ then by the construction in Step 1 we have $H_{-i}(0) = 0$ (Eq. (35)), which means that either the type t_{-i} belongs to the zero-measure set of types that bid zero according to σ_{-i} and hence can be omitted, or the bid zero is assigned zero weight according to $\sigma_{-i}(\cdot, t_{-i})$ and hence can be omitted.

First, consider the case where $b > 0$. With $b \in \operatorname{supp} \sigma_{-i}(\cdot, t_{-i})$ and σ_{-i} monotone, $F_{-i}(t_{-i}) \ge H_{-i}(b) > H_{-i}(0)$. Thus by Eq. (25) $t_{-i} \ge \gamma_{-i}(0)$. By Corollary 5.b.ii, either (A) $\gamma_{-i}(b) = t_{-i}$ or (B) $F_{-i}(t_{-i}) = H_{-i}(b)$ and $(\gamma_{-i}(b), t_{-i})$ is a gap of F_{-i}. Pick any $b'' > b$. In (A), monotonicity of γ_{-i} (Lemma 7.b) implies $\gamma_{-i}(b'') \ge \gamma_{-i}(b) = t_{-i}$. In (B), by Lemma 7.a and H_{-i} having no gap,

$$F_{-i}(\gamma_{-i}(b'')) \ge H_{-i}(b'') > H_{-i}(b) = F_{-i}(t_{-i}),$$

hence $\gamma_{-i}(b'') \ge t_{-i}$. Thus $\gamma_{-i}(b'') \ge t_{-i} \ge \gamma_{-i}(b)$ in each case. This, again coupled with the monotonicity of γ_{-i}, implies that in each case

$$b' < b < b'' \Longrightarrow \gamma_{-i}(b') \le t_{-i} \le \gamma_{-i}(b'').$$

Thus, for any $b' < b < b''$ such that $u_{-i}(\cdot, t_{-i})$ is differentiable at b' and b'', Eq. (39) implies

$$D_1 u_{-i}(b', t_{-i}) = \frac{t_{-i}}{\gamma_{-i}(b')} - 1 \geq 0 \geq \frac{t_{-i}}{\gamma_{-i}(b'')} - 1 = D_1 u_{-i}(b'', t_{-i}).$$

Since $u_{-i}(\cdot, t_{-i})$ is concave on $(0, x]$ by Lemma 15, and $u_{-i}(0, t_{-i}) \leq \lim_{b' \downarrow 0} u_{-i}(b', t_{-i})$ by Eq. (38), we have shown that b is a global maximum of $u_{-i}(\cdot, t_{-i})$.

Second, consider the case $b = 0$, hence $0 \in \operatorname{supp} \sigma_{-i}(\cdot, t_{-i})$. As explained above, $H_i(0) = 0$. Hence $u_{-i}(\cdot, t_{-i})$ is concave on $[0, x]$ by Lemma 15. For any $b'' > 0$ at which $u_{-i}(\cdot, t_{-i})$ is differentiable, we have $b'' \in \operatorname{supp} \sigma_{-i}(\cdot, \gamma_{-i}(b''))$ (Lemma 8.c) and hence, by monotonicity of σ_{-i} (Eq. (22)), $t_{-i} \leq \gamma_{-i}(b'')$; thus Eq. (39) implies

$$D_1 u_{-i}\left(b'', t_{-i}\right) = \frac{t_{-i}}{\gamma_{-i}(b'')} - 1 \leq 0.$$

This, combined with the fact that zero is the left corner of the domain of bids, and $u_{-i}(\cdot, t_{-i})$ concave on $[0, x]$, implies that zero $(= b)$ is a best response for type t_{-i}. All cases considered, σ_{-i} best responds H_i for player $-i$. □

B.9. *Corollary 3, Lemma 2 and Theorem 7*

B.9.1. *Proof of Corollary 3*

Let $t_i^* \in [a_i, z_i] \setminus \{0\}$. The σ and $\tilde{F}_{-i}$ referred to in this corollary corresponds to the σ^* and F_{-i} in Lemma 13. Consider first the case $c_{-i,\sigma^*} = 0$. Eq. (37) in Lemma 13 implies

$$1 \geq 1 - c_{i,\sigma^*} = t_i^* \int_{c_{-i,\sigma^*}}^{1} \left(1/F_{-i}^{-1}(s)\right) ds = t_i^* \int_{0}^{1} \left(1/F_{-i}^{-1}(s)\right) ds.$$

Hence the right-hand side of Eq. (21) is equal to zero, so Eq. (21) holds. Next consider the other case, $c_{-i,\sigma^*} > 0$. Then $c_{i,\sigma^*} = 0$ by Eq. (35) in Lemma 13. Hence Eq. (37) implies

$$1 = t_i^* \int_{c_{-i,\sigma^*}}^{1} \left(1/F_{-i}^{-1}(s)\right) ds.$$

Thus, this equation admits one solution for c_{-i,σ^*}, and it is the only one as $t_i^* \int_c^1 \left(1/F_{-i}^{-1}(s)\right) ds$ is strictly decreasing in c. Since $t_i^* \int_c^1 \left(1/F_{-i}^{-1}(s)\right) ds$ is continuous in c, when the equation admits a solution for c_{-i,σ^*}, it must also admit the right-hand side of Eq. (21) as its solution. Thus, the right-hand side of Eq. (21) is again equal to c_{-i,σ^*}. □

B.9.2. *Proof of Lemma 2: linkage between the players' marginal costs of bids*

Given any c.d.f. $\tilde{F}_i$ with support contained in $[a_i, z_i]$, pick any $\sigma \in \mathscr{E}(\tilde{F}_i)$. First, we prove that $x_{\underline{\sigma}^i} \geq x_\sigma$. Suppose, to the contrary, that $x_\sigma > x_{\underline{\sigma}^i}$. Define $\gamma_{i,\sigma}$ by Eq. (19), and likewise for $\gamma_{i,\underline{\sigma}^i}$, $\gamma_{-i,\sigma}$ and $\gamma_{-i,\underline{\sigma}^i}$. Since i's type in the equilibrium $\underline{\sigma}^i$ is degenerate to z_i, $\gamma_{i,\underline{\sigma}^i} = z_i$. Eq. (20), with the roles of i and $-i$ switched, implies

$$\frac{d}{db} H_{-i,\sigma}(b) = \frac{1}{\gamma_{i,\sigma}(b)} \geq \frac{1}{z_i} = \frac{1}{\gamma_{i,\underline{\sigma}^i}(b)} = \frac{d}{db} H_{-i,\underline{\sigma}^i}(b) \tag{40}$$

for almost every $b \in [0, x_{\underline{\sigma}^i}]$. This, combined with the fact $1 = H_{-i,\sigma}(x_\sigma) > H_{-i,\sigma}(x_{\underline{\sigma}^i})$ (because $x_\sigma > x_{\underline{\sigma}^i}$ by supposition and $H_{-i,\sigma}$ has no gap) and the continuity of H_{-i} on $\mathbb{R}_+$, implies

$H_{-i,\underline{\sigma}^i} > H_{-i,\sigma}$ on $[0, x_{\underline{\sigma}^i}]$. Thus, $H_{-i,\underline{\sigma}^i}(0) > H_{-i,\sigma}(0) \geq 0$. Then $H_{i,\underline{\sigma}^i}(0) = c_{i,\underline{\sigma}^i} = 0$ by Eq. (35). Thus, $H_{-i,\underline{\sigma}^i} \geq H_{-i,\sigma}$ throughout $[0, x_\sigma]$. Consequently, since F_{-i}^{-1} is weakly increasing according to its definition Eq. (9),

$$\gamma_{-i,\underline{\sigma}^i}(b) = F_{-i}^{-1}\left(H_{-i,\underline{\sigma}^i}(b)\right) \geq F_{-i}^{-1}\left(H_{-i,\sigma}(b)\right) = \gamma_{-i,\sigma}(b)$$

for all $b \in [0, x_\sigma]$. Thus, Eq. (20) implies that, for almost every $b \in [0, x_\sigma]$,

$$\frac{d}{db}H_{i,\sigma}(b) = \frac{1}{\gamma_{-i,\sigma}(b)} \geq \frac{1}{\gamma_{-i,\underline{\sigma}^i}(b)} = \frac{d}{db}H_{i,\underline{\sigma}^i}(b).$$

This, coupled with the fact $H_{i,\sigma}(0) \geq H_{i,\underline{\sigma}^i}(0)$ (due to $H_{i,\underline{\sigma}^i}(0) = 0$) and the supposition $x_\sigma > x_{\underline{\sigma}^i}$, implies $H_{i,\sigma}(x_{\underline{\sigma}^i}) \geq H_{i,\underline{\sigma}^i}(x_{\underline{\sigma}^i}) = 1$. Thus, $x_{\underline{\sigma}^i} \geq x_\sigma$, contradicting the supposition $x_\sigma > x_{\underline{\sigma}^i}$. Thus we have proved $x_{\underline{\sigma}^i} \geq x_\sigma$.

Second, we prove $x_\sigma \geq x_{\overline{\sigma}^i}$. By hypothesis of the lemma, $\overline{\sigma}^i$ exists as a BNE of the game $\mathscr{G}(\delta_{a_i}, F_{-i})$; thus $a_i > 0$, otherwise $\mathscr{G}(\delta_{a_i}, F_{-i})$ admits no BNE (because bidder i, of type $a_i = 0$, would bid zero for sure, rendering bidder $-i$'s best response empty). Now suppose that $x_{\overline{\sigma}^i} > x_\sigma$. Repeat the previous reasoning with σ there replaced by $\overline{\sigma}^i$ here, $\underline{\sigma}^i$ there replaced by σ here, and (40) there replaced by

$$\frac{d}{db}H_{-i,\overline{\sigma}^i}(b) = \frac{1}{\gamma_{i,\overline{\sigma}^i}(b)} = \frac{1}{a_i} \geq \frac{1}{\gamma_{i,\sigma}(b)} = \frac{d}{db}H_{-i,\sigma}(b),$$

with $1/a_i$ meaningful because $a_i > 0$. Thus, following the reasoning in the previous paragraph, we have $H_{-i,\sigma} > H_{-i,\overline{\sigma}^i}$ on $[0, x_\sigma]$. Then $H_{i,\sigma}(0) = 0$ and $\frac{d}{db}H_{i,\overline{\sigma}^i} \geq \frac{d}{db}H_{i,\sigma}$. Consequently, $x_\sigma \geq x_{\overline{\sigma}^i}$, contradicting the supposition $x_{\overline{\sigma}^i} > x_\sigma$. $\square$

B.9.3. Proof of Theorem 7

Lemma 13, applied to the cases of $t_i^* = z_i$ and $t_i^* = a_i > 0$, implies existence of $\underline{\sigma}^i \in \mathscr{E}_i(\delta_{z_i})$ and $\overline{\sigma}^i \in \mathscr{E}_i(\delta_{a_i})$ asserted by Parts (a) and (b) of the theorem. Pick any $\sigma \in \mathscr{E}_i$. By Lemma 2, $x_{\overline{\sigma}^i} \leq x_\sigma \leq x_{\underline{\sigma}^i}$. Note, to any BNE $\sigma' \in \mathscr{E}_i$, bidding its bid supremum $x_{\sigma'}$ is a best response for the type z_i of player i (Lemma 6.b); thus, since $H_{-i,\sigma'}$ is atomless at $x_{\sigma'}$ (Lemma 6.a), $U_i(z_i|\sigma') = 1 - x_{\sigma'}/z_i$ for any $\sigma' \in \mathscr{E}_i$. Thus,

$$U_i(z_i|\overline{\sigma}^i) = 1 - x_{\overline{\sigma}^i}/z_i \geq 1 - x_\sigma/z_i = U_i(z_i|\sigma) \geq 1 - x_{\underline{\sigma}^i}/z_i = U_i(z_i|\underline{\sigma}^i).$$

Thus, $\overline{u}_i = U_i(z_i|\overline{\sigma}^i)$ and $\underline{u}_i = U_i(z_i|\underline{\sigma}^i)$. This, combined with Parts (e) and (f) of Lemma 14, implies Claims (a.i) and (b.i) of the theorem. Claim (a.ii) of the theorem follows from (b) and (c) of Lemma 14. Claim (b.ii) of the theorem follows from (d) of the same lemma. To prove Claim (c) of the theorem, let $a_i = 0$ and pick any $t_i^* \in (0, z_i)$. By Lemma 13, a BNE $\sigma^* \in \mathscr{E}_i(\delta_{t_i^*})$ exists; by Lemma 14.e,

$$U_i(z_i|\sigma^*) = 1 - (t_i^*/z_i)(1 - H_{-i,\sigma^*}(0)),$$

which converges to one as t_i^* goes to zero. Thus, by definition of $\overline{u}_i$, $\overline{u}_i \geq 1$; because one is the highest possible payoff a player can get in this environment, $\overline{u}_i = 1$, as asserted. $\square$

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