B Supplement to "A Method to Characterize Reduced-Form Auctions"

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B.1 Reduced-Form Characterization Given Partial Assignment

In the *partial assignment* model, the set X of feasible allocation outcomes is constant to all $t \in T$ and is defined to be the set of all $((x_{kj})_{k=1}^{2})_{j=1}^{N} \in \{0,1\}^{2N}$ that satisfy (25) and

$$\forall k \in I_1 : \sum_{j \in I_2} x_{kj} \leq 1.$$

That is, each bidder gets at most one object and may get none. A feasible allocation outcome corresponds to a subset of I that satisfies (26). In other words, X is equivalent to the set $\mathscr{M}_{\mathrm{P}}$ of the subsets M of I $(= \{1,2\} \times \{1,\ldots,N\})$ that satisfy (26). Different from the $\mathscr{M}$ in the assignment model in Section 6, the cardinality of an element of $\mathscr{M}_{\mathrm{P}}$ may be less than two.

Theorem 5 *In the partial assignment model such that* $|I_1| = 2$, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.

Social Planner's Solution To prove the theorem, following the road map in Section 3, pick any $\alpha \in \mathbb{R}^{\mathscr{X}}$ and consider the social planner's problem for any $t := (t_1, t_2) \in T$:

$$\begin{aligned} \max_{x \in \mathrm{cv} X} \sum_{i \in I} x_i \alpha(i, t_{i_1}) &= \max_{x \in X} \sum_{i \in I} x_i \alpha(i, t_{i_1}) = \max_{M \in \mathscr{M}_{\mathrm{P}}} \sum_{i \in M} \alpha(i, t_{i_1}) \\ &= \max_{j \in I_2} \max_{j' \in I_2 \setminus \{j\}} \left(\max\{0, \alpha(1, j, t_1)\} + \max\{0, \alpha(2, j', t_2)\} \right), \end{aligned}$$

where the first two "=" are the same as in the previous model, and the last "=" highlights the difference, that there is no loss for a solution to assign zero quantity to any interim state whose α-value is nonpositive. Thus, the social planner's problem is solved by coupling the first- or second-highest *positive* $\alpha(1, j, t_1)$ among $j \in I_2$ with the first- or second-highest *positive* $\alpha(2, j, t_2)$ among $j \in I_2$ such that the couple have different j-coordinates. To illustrate, consider the following table that displays the α-values given ex post state (t_1, t_2), with rows corresponding to objects, and columns bidder-types:

	$(1,t_1)$	$(2,t_2)$
1	2	3
2	-4	0
3	-1	$1/2$

The solution in the previous model would have been $\{(1,1),(2,3)\}$ (giving good 1 to bidder 1, and good 3 to bidder 2). That would produce a total α-value $2+1/2$, while the second best would be $\{(1,3),(2,1)\}$, producing a total α-value $-1+3=2$. In the partial assignment model, by contrast, the solution is $\{(2,1)\}$ (giving good 1 to bidder 2 and none to bidder 1). That produces a total positive α-value $\max\{0,-1\}+3=3$, while the second best, $2+1/2$.

To define a solution to the social planner's problem in general, let

$$j^1(k,t_k) := \begin{cases} \min\left(\arg\max_{j\in I_2}\alpha(k,j,t_k)\right) & \text{if } \max_{j\in I_2}\alpha(k,j,t_k)>0 \\ 0 & \text{else} \end{cases}$$

$$j^2(k,t_k) := \begin{cases} \min\left(\arg\max_{j\in I_2\setminus\{j^1(k,t_k)\}}\alpha(k,j,t_k)\right) & \text{if } \max_{j\in I_2\setminus\{j^1(k,t_k)\}}\alpha(k,j,t_k)>0 \\ 0 & \text{else} \end{cases}$$

for any $k\in I_2$ and any $t_k\in T_k$. For any $j\in I_2$ $(=\{1,\ldots,N\})$, let

$$\delta_{\mathrm{P}}(k,j,t_k) := \max\{0,\alpha(k,j,t_k)\} - \max_{j'\in I_2\setminus\{j\}}\max\{0,\alpha(k,j',t_k)\}.$$

For any $t:=(t_1,t_2)\in T$, define $M_*(t_1,t_2)\in\mathscr{M}_{\mathrm{P}}$ by:

a. if $j^1(1,t_1)\neq j^1(2,t_2)$, let $M_*(t_1,t_2):=\{(1,j^1(1,t_1)),(2,j^1(2,t_2))\}\setminus\{(1,0),(2,0)\}$;

b. else $(j^1(1,t_1)=j^1(2,t_2))$ then:

   i. if $\delta_{\mathrm{P}}(1,j^1(1,t_1),t_1)\geq\delta_{\mathrm{P}}(2,j^1(1,t_1),t_2)$ $(=\delta_{\mathrm{P}}(2,j^1(2,t_2),t_2))$, let

$$M_*(t_1,t_2) := \left\{(1,j^1(1,t_1)),(2,j^2(2,t_2))\right\}\setminus\{(1,0),(2,0)\};$$

   ii. else $(\delta_{\mathrm{P}}(1,j^1(1,t_1),t_1)<\delta_{\mathrm{P}}(2,j^1(1,t_1),t_2))$, let

$$M_*(t_1,t_2) := \left\{(1,j^2(1,t_1)),(2,j^1(2,t_2))\right\}\setminus\{(1,0),(2,0)\}.$$

This definition of M_* parallels that of M_* in the previous model except that $M_*(t)$ here excludes any bidder-object pair with nonpositive α-value at state t (through "$\setminus\{(1,0),(2,0)\}$"). Clearly, $M_*(t)$ is a solution to the social planner's problem for every $t\in T$, hence M_* corresponds to the q^* that Step 1 of our method needs.

Partial Revealed Preferences Step 2, as in the previous model, is to construct partial orders that rationalize M_* partially, one for every "column" of interim states referring to a common bidder-type (k, t_k) and every "row" of interim states referring to a common object j.

For any $j \in I_2$, define the "row"

$$\mathscr{Z}_j := \left\{(k, j, t_k) \mid k \in I_1; t_k \in T_k; j = j^1(k, t_k) \neq 0\right\}.$$

The definition is the same as its counterpart in the previous model except for the nonzero condition on the right-hand side, which excludes those "top" contenders whose α-values are nonpositive. The binary relation $\succ_j$ on $\mathscr{Z}_j$ is defined in the same way as that in Section 6, where δ is replaced by δ_{P} here. Correspondingly, list the elements of $\mathscr{Z}_j$ in descending order of $\succ_j$ as in (29), and define its upper contour sets U_j^n as there. By the same proof of Lemma 9 (Appendix A.6), one verifies that U_j^n is upward universally binding for any $n = 1, \ldots, |\mathscr{Z}_j|$.

The partial order $\succeq_{k,t_k}$ within a "column" $\{k\} \times I_2 \times \{t_k\}$ is needed only when both $j^1(k, t_k)$ and $j^2(k, t_k)$ are nonzero; otherwise the tradeoff within the column is trivial. Thus, consider any $k \in I_1$ and any $t_k \in T_k$ for which $j^2(k, t_k) \neq 0$ (which implies $\alpha(k, j^2(k, t_k), t_k) > 0$ and hence $j^1(k, t_k) \neq 0$). Define $\succeq_{k,t_k}$ by:

a. let $j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k)$;

b. for any $\{j, j'\} \neq \{j^2(k, t_k), j^1(k, t_k)\}$:

   i. if $\alpha(k, j, t_k) > 0$ and $\alpha(k, j', t_k) > 0$, let $j \succ_{k,t_k} j'$ iff

$$\alpha(k, j, t_k) > \alpha(k, j', t_k) \text{ or } [\alpha(k, j, t_k) = \alpha(k, j', t_k) \text{ and } j < j'];$$

   ii. if $\alpha(k, j, t_k) > 0 \geq \alpha(k, j', t_k)$, let $j \succ_{k,t_k} j'$.

Let there be N_{k,t_k} distinct elements $j \in I_2$ for which $\alpha(k, j, t_k) > 0$. Note $N_{k,t_k} \geq 2$ by the supposition $j^2(k, t_k) > 0$. List all these elements (whose α-values are positive) in descending order of $\succeq_{k,t_k}$, just like (28):

$$j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k) \succ_{k,t_k} j^3 \succ_{k,t_k} j^4 \succ_{k,t_k} \cdots \succ_{k,t_k} j^{N_{k,t_k}}. \tag{49}$$

Correspondingly, let $(k, j, t_k) \succ_{k,t_k} (k, j', t_k) \iff j \succ_{k,t_k} j'$, and likewise for $\sim_{k,t_k}$. Define the upper contour sets $(V_{k,t_k}^n)_{n=2}^{N_{k,t_k}}$ with respect to $\succeq_{k,t_k}$ just as in Section 6. For any $z \in \mathscr{Z}$ such that $\alpha(z) \leq 0$, define (a lower contour set)

$$L(z) := \{z\}.$$

Lemma 10 *In the partial assignment model, for any $k \in I_1$ and any $t_k \in T_k$ such that $j^2(k, t_k) > 0$, V^n_{k,t_k} is upward universally binding for any $n \in \{2, \ldots, N_{k,t_k}\}$, and $L(k, j, t_k)$ downward universally binding whenever $\alpha(k, j, t_k) \leq 0$.*

Proof The definition of V^n_{k,t_k} implies $\varnothing =: V^1_{k,t_k} \subsetneq V^2_{k,t_k} \subsetneq V^3_{k,t_k} \subsetneq \cdots \subsetneq V^{N_{k,t_k}}_{k,t_k}$, as in the (10) in Lemma 2. To verify (12) in that lemma, pick any $t' := (t'_k, t'_{-k}) \in T$. If $t'_k \neq t_k$ then $I(V^n_{k,t_k}, t') = \varnothing$ for all n and the proof is trivial. Suppose $t'_k = t_k$. Then for any $n \in \{2, \ldots, N_{k,t_k}\}$, $I(V^n_{k,t_k}, t') = \{k\} \times \{j^1(k, t_k), j^2(k, t_k)\} \cup \{j^m \mid 2 \leq m \leq n\}$. This, combined with (3) and (24), implies $f(V^n_{k,t_k}, t') = 1$. Thus,

$$\begin{aligned} f(V^2_{k,t_k}, t') - f(V^1_{k,t_k}, t') &= 1 - 0 = 1 \\ f(V^n_{k,t_k}, t') - f(V^{n-1}_{k,t_k}, t') &= 1 - 1 = 0 \quad \forall n \geq 3. \end{aligned}$$

Meanwhile, since both $j^1(k, t_k)$ and $j^2(k, t_k)$ are nonzero (hypothesis of the lemma), exactly one of them gets one object according to $M_*(t_1, t_2)$, while none of the other interim states ranked behind them get any. Thus,

$$\begin{aligned} \sum_{(k,j,t_k) \in V^2_{k,t_k} \setminus V^1_{k,t_k}} q^*_{k,j}(t') &= q^*_{k,j^1(k,t_k)}(t') + q^*_{k,j^2(k,t_k)}(t') = 1 \\ \sum_{(k,j,t_k) \in V^n_{k,t_k} \setminus V^{n-1}_{k,t_k}} q^*_{k,j}(t') &= 0 \quad \forall n \geq 3. \end{aligned}$$

Thus (12) is satisfied, and hence V^n_{k,t_k} is upward universally binding.

Let $\alpha(k, j, t_k) \leq 0$. Then $L(k, j, t_k) =: L^1 \supsetneq L^0 := \varnothing$, hence the (11) in Lemma 2 holds. To verify (13), note from (4), (24) and (25) that $g(S, t') = 0$ for all $S \subseteq \mathscr{Z}$. Thus $g(L^1, t') - g(L^0, t') = 0$. Meanwhile, $\alpha(k, j, t_k) \leq 0$ implies $q^*_{k,j}(t_k, t_{-k}) = 0$ for any t_{-k} by the definition of M_*. Thus (13) is true, and $L(k, j, t_k)$ downward universally binding. ■

Existence of the Price Function Let

$$\begin{aligned} \mathscr{S}_+ &:= \left\{U^n_j \mid j \in I_2; \mathscr{Z}_j \neq \varnothing; n = 1, \ldots, |\mathscr{Z}_j|\right\} \\ &\quad \bigcup \left\{V^n_{k,t_k} \mid k \in I_1; t_k \in T_k; j^2(k, t_k) \neq 0; n = 2, \ldots, N_{k,t_k}\right\}, \\ \mathscr{S}_- &:= \{L(z) \mid z \in \mathscr{Z}; \alpha(z) \leq 0\}. \end{aligned}$$

Different from its counterpart in the previous model, $\mathscr{S}_-$ consists of the singletons of interim states whose α-values are nonpositive (and hence excluded by M_*).

Following Lemma 3, define the matrix $[\mathbf{M}_+, \mathbf{M}_-]$ with respect to the $\mathscr{S}_+$ and $\mathscr{S}_-$ here. For any $z \in \mathscr{Z}$, let $[z]$ denote the row of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ corresponding to z. Let $[0]$ denote the zero vector in the space spanned by the row vectors of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$. That is, $[0](S) = [0](-\boldsymbol{\alpha}) = 0$ for all $S \in \mathscr{S}_+ \sqcup \mathscr{S}_-$.

To prove Theorem 5 by Lemma 3, it suffices to prove that, for any nonempty subset Z of $\mathscr{Z}$ and any $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$, (30) and (31) cannot hold simultaneously. To that end, the next lemma says that we can exclude from Z all the interim states z for which $\alpha(z) \leq 0$. The reason is intuitive. By the construction of the upper and lower contour sets, if $\alpha(z) \leq 0$ then the only member in $\mathscr{S}_+ \sqcup \mathscr{S}_-$ that contains z is the $\{z\}$ $(= L(z))$ in $\mathscr{S}_-$. Thus the row $[z]$ has zero for all its entries except $[z](L(z)) = -1$ and $[z](-\boldsymbol{\alpha}) = -\alpha(z) \geq 0$. Then adding $[z]$ to any Gaussian elimination violates the nonnegativity condition (30), as no other row can cancel out $[z](L(z)) = -1$. Neither can subtracting $[z]$ from the Gaussian operation improve the prospect for (31), as $-[z]$ adds a nonnegative increment $-\alpha(z)$ to the left-hand side of (31).

Lemma 11 *If (30) implies $\sum_{z\in Z} \beta_z \alpha(z) \geq 0$ for any $Z \subseteq \mathscr{Z}$ such that $\alpha(z) > 0$ for all $z \in Z$ and $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$, then (30) implies $\sum_{z\in Z} \beta_z \alpha(z) \geq 0$ for any $Z \subseteq \mathscr{Z}$ and any $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$.*

Proof For $Z \subseteq \mathscr{Z}$ and $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$ that satisfy (30), define

$$\begin{aligned} Z^+ &:= \{z \in Z \mid \beta_z > 0\}, \\ Z^- &:= Z \setminus Z^+ \; (= \{z \in Z \mid \beta_z < 0\}). \end{aligned}$$

Observe that $\alpha(z) \leq 0 \Rightarrow z \notin Z^+$. That is because $\alpha(z) \leq 0$ implies $[z](L(z)) = -1$. Since $L(z) = \{z\}$, $[z'](L(z)) = 0$ for all $z' \neq z$. Thus, $z \in Z^+$, combined with the fact $Z^+ \cap Z^- = \varnothing$ and the hypothesis $\beta_z > 0$, contradicts (30).

Thus, if $z \in Z$ and $\alpha(z) \leq 0$, then $z \in Z^-$. Let

$$Z' := Z \setminus \{z \in Z^- \mid \alpha(z) \leq 0\}.$$

Note: Z' satisfies (30) where the role of Z is replaced by Z'. That is because $\alpha(z) \leq 0$ implies that $[z](S) = 0$ for all $S \in \mathscr{S}_+ \sqcup S_- \setminus \{L(z)\}$, $[z](L(z)) = -1$, and $[z](-\boldsymbol{\alpha}) = -\alpha(z) \geq 0$. Thus, removing $\beta_z[z]$ from $\sum_{z'\in Z} \beta_{z'}[z']$ has no effect on $\sum_{z'\in Z} \beta_{z'}[z'](S)$ if $S \neq L(z)$ and, when $S = L(z)$, merely turns

$$\sum_{z' \in Z} \beta_{z'}[z'](L(z)) = \beta_z(-1) = -\beta_z = |\beta_z|$$

into $\sum_{z' \in Z \setminus \{z\}} \beta_{z'}[z'](L(z)) = \beta_z(-1) = 0$ because $[z'](L(z) = 0$ for all $z' \neq z$. Consequently, Z' with all such z removed preserves (30). Then, by the hypothesis of the lemma, $\sum_{z' \in Z'} \beta_{z'} \alpha(z') \geq 0$. Thus the desired conclusion follows:

$$\sum_{z \in Z} \beta_z \alpha(z) = \sum_{z' \in Z'} \beta_{z'} \alpha(z') + \sum_{z \in Z^- : \alpha(z) \leq 0} \beta_z \alpha(z) \geq 0. \quad \blacksquare$$

Thus, we can assume, without loss, that $\alpha(z) > 0$ for any $z \in Z$ in (30). Also assume $Z^- \neq \varnothing$, which is without loss because $Z^- = \varnothing \Rightarrow \sum_{z \in Z} \beta_z \alpha(z) \geq 0$, as $\alpha(z) > 0$ for all $z \in Z$. Then follows the analogous observation to the previous Lemma 5:

Lemma 12 *For any subset $Z \subseteq \mathscr{Z}$ such that $\alpha(z) > 0$ for all $z \in Z$ and $Z^- \neq \varnothing$, and for any $(\beta_z)_{z \in Z} \in (\mathbb{R} \setminus \{0\})^Z$, if (30) is true then there exist a set H and a positive $(\tilde{\beta}_h)_{h \in H} \in \mathbb{R}^H_{++}$ for which*

$$\sum_{z \in Z} \beta_z [z] = \sum_{h \in H} \tilde{\beta}_h \left([z'_h] - [z_h] \right) \tag{50}$$

such that for every $h \in H$ there exist $k \in I_1$ and $t_k \in T_k$ that satisfy one of the following:

i. $z_h = (k, j, t_k)$ and $z'_h = (k, j', t_k)$ for some $j, j' \in I_2$ such that $j \neq j'$, $\alpha(z_h) > 0$, $\alpha(z'_h) > 0$, and $j' \succeq_{k,t_k} j$ ($\succ_{k,t_k}$ or $\sim_{k,t_k}$);

ii. or $[z_h] = [0]$ and $z'_h = (k, j', t_k)$ such that $j' = j^1(k, t_k) > 0 = j^2(k, t_k)$;

iii. or $[z'_h] = [0]$ and $z_h = (k, j, t_k)$ such that $j = j^1(k, t_k) > 0 = j^2(k, t_k)$.

Proof As in the case of Lemma 5, let us assume, without loss of generality, that $Z \subseteq \{k\} \times I_2 \times \{t_k\}$ for some $k \in I_1$ and some $t_k \in T_k$.

First suppose $j^2(k, t_k) = 0$. Then $\alpha(k, j, t_k) \leq 0$ for all $j \in I_2 \setminus \{j^1(k, t_k)\}$. Consequently, if $z \in Z$ then $\alpha(z) > 0$ (by hypothesis of the lemma) and hence $z = (k, j^1(k, t_k), t_k)$ and $j^1(k, t_k) > 0$. Thus, $Z = \{z\}$ ($Z \subseteq \{k\} \times I_2 \times \{t_k\}$ by the previous paragraph). Since $Z = Z^+ \sqcup Z^-$, it follows that z cannot be in both Z^+ and Z^-. If $z \in Z^+$ then (50) holds trivially, with $H := \{1\}$ and $z'_1 := z$, which is case (ii) in the conclusion. Else, $z \in Z^-$, then (50) holds trivially, with $H := \{1\}$ and $z_1 := z$, which is case (iii) of the conclusion.

Next suppose $j^2(k,t_k) \neq 0$. Then the binary relation $\succeq_{k,t_k}$ is defined on the column $\{k\} \times I_2 \times \{t_k\}$, and $V^n_{k,t_k} \in \mathscr{S}_+$ for all $n \in \{2,\ldots,N_{k,t_k}\}$. Thus, if n is the rank of z in the list (49) (with rank of both $j^1(k,t_k)$ and $j^2(k,t_k)$ be 2), then $[z](V^m_{k,t_k})$ is equal to one for all $m \geq n$, and equal to zero for all $m < n$. Consequently, the proof of Eq. (40) (Appendix A.7) applies. Now that (40) obtains, if $Z_* \neq \varnothing$, simply rewrite the $\sum_{z' \in Z_*} \tilde{\beta}_{z'}[z']$ on its right-hand side as $\sum_{z' \in Z_*} \tilde{\beta}_{z'}([z'] - [0])$ and modify the definition of H there by $H := H \sqcup Z_*$. Then (50) obtains, with the elements of Z_* belonging to case (ii), and all the other elements of H belonging to case (i). ■

In Lemma 12, case (i) is the same as the last clause in the statement of Lemma 5 (except for the positive-α property thanks to Lemma 11 in the current model). Cases (ii) and (iii) are the special (and trivial) cases that arise because the planner's solution in the current model assigns zero quantity to any interim state with a nonpositive α-value. When a row $[z]$ enters a Gaussian elimination, z may be the only interim state in $\{k\} \times I_2 \times \{t_k\}$ that has a positive α-value. The rows corresponding to the other interim states in $\{k\} \times I_2 \times \{t_k\}$ have zero for all their entries, because none of them, already precluded by the planner's solution, belong to any upper contour set, and any interim state that belongs to any lower contour set is already excluded from the Gaussian procedure, thanks to Lemma 11. That is why the zero vector $[0]$ appears in Cases (ii) and (iii).

Mimicking the proof in the previous model that derives Lemma 6 from Lemma 5, we obtain a consequence of Lemma 12 in the current model.[17] Then Theorem 5 is proved in the same manner as the ending paragraph of Section 6.

B.2 An Example to Apply the Necessity Part of Theorem 1

Let us apply the necessity observation in Theorem 1 to an example in Che et al. [7]. Multiple units of a homogeneous object are to be allocated to three bidders, named 1, 2 and 3. Now

[17] Case (i) in Lemma 12 is the same as Lemma 5. Neither cases (ii) nor (iii) cause any complication. That is because the proof of Lemma 6 is to balance every negatively weighted difference $[z'_h] - [z_h]$ by some positively weighted differences, where the former means that $([z'_h] - [z_h])(U^n_j) = -[z_h](U^n_j) = -1$ for some upper contour set U^n_j among the top contenders that refer to the same object j as z_h refers to. Thus case (ii) is a nonissue because $[z_h] = [0]$ in (ii) and thus $[z'_h] - [z_h]$, equal to $[z'_h]$, is not negatively weighted. In case (iii), z_h is a top contender and $[z'_h] = [0]$; thus $([z'_h] - [z_h])(U^n_j) = -[z_h](U^n_j) = -1$ for any upper contour set U^n_j that contains z_h. Thus $[z'_h] - [z_h]$ behaves exactly like a negatively weighted difference described above.

that the set I_2 of objects is singleton, without loss denote $I := I_1 := \{1, 2, 3\}$. Each bidder's type is drawn from the same set $\{\underline{\theta}, \overline{\theta}\}$, so $T = \{\underline{\theta}, \overline{\theta}\}^3$. The set X of feasible allocation outcomes is defined by (16) such that $f\{1\} = f\{2\} = f\{3\} = 3$, $f\{1, 2\} = f\{2, 3\} = f\{3, 1\} = 4$, $f\{1, 2, 3\} = 6$, and $f(\varnothing) = g(E) = 0$ for all $E \subseteq I$. One readily sees that the f in this example is not submodular on 2^I. Thus Theorem 2 does not apply. Che et al. note $\mathscr{Q}_{\mathrm{B}} \not\subseteq \mathscr{Q}$ in this example based on unpublished computations. Here I prove $\mathscr{Q}_{\mathrm{B}} \not\subseteq \mathscr{Q}$ through showing that the universal binding condition, necessary for $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ by Theorem 1, cannot be satisfied in this example.

In this example, the set $\mathscr{X}$ of interim states is $\{(i, t_i) \mid \in \{1, 2, 3\}, t_i \in \{\underline{\theta}, \overline{\theta}\}\}$. For all $i \in \{1, 2, 3\}$, let

$$\alpha(i, \underline{\theta}) := 1, \quad \alpha(i, \overline{\theta}) := 3.$$

For any $t \in T$ $(= \{\underline{\theta}, \overline{\theta}\}^3)$, let

$$q^*(t) \in \arg\max_{x \in \mathrm{cv} X} \sum_{i \in I} x_i \alpha(i, t_i).$$

One readily sees that $q^*(t) = (2, 2, 2)$ if $t = (\underline{\theta}, \underline{\theta}, \underline{\theta})$ or $t = (\overline{\theta}, \overline{\theta}, \overline{\theta})$, and for any other t, $q^*(t) \in \{(3, 1, 1), (1, 3, 1), (1, 1, 3)\}$ such that one of the bidders whose types are $\overline{\theta}$ is assigned 3 units, and one of the bidders whose types are $\underline{\theta}$ is assigned 1 unit.

By the necessity assertion in Theorem 1, it suffices $\mathscr{Q}_{\mathrm{B}} \not\subseteq \mathscr{Q}$ to prove that that there exists no $(p_+, p_-) : 2^{\mathscr{X}} \to \mathbb{R}^2_+$ that satisfies both (7) and (9) given any q^* with the above-stated property. Suppose, to the contrary, that there is such a (p_+, p_-).

Claim: For any $S \subseteq \mathscr{X}$, $(i, \underline{\theta}) \in S$ implies $p_+(S) = 0$. To prove the claim, without loss, let $(1, \underline{\theta}) \in S$. First, suppose that $(i', \underline{\theta}) \in S$ for some $i' \neq 1$. Without loss, let $i' := 2$. Let $t := (\underline{\theta}, \underline{\theta}, \overline{\theta})$. Then $q^*(t) = (1, 1, 3)$. Meanwhile, $I(S, t) \supseteq \{1, 2\}$ and hence $f(S, t)$ is equal to either 4 (when $I(S, t) = \{1, 2\}$) or 6 (when $I(S, t) = \{1, 2, 3\}$). In either case, $f(S, t) > \sum_{i \in I(S,t)} q_i^*(t)$, because $\sum_{i \in I(S,t)} q_i^*(t) = q_1^*(t) + q_2^*(t) = 2$ when $I(S, t) = \{1, 2\}$, and $\sum_{i \in I(S,t)} q_i^*(t) = \sum_{i=1}^3 q_i^*(t) = 5$ when $I(S, t) = \{1, 2, 3\}$. Thus (9) is violated unless $p_+(S) = 0$.

Thus, if $p_+(S) > 0$ then $(i', \underline{\theta}) \notin S$ for any $i' \neq 1$. Let $t' := (\underline{\theta}, \underline{\theta}, \underline{\theta})$. Then $q^*(t') = (2, 2, 2)$ and $I(S, t') = \{1\}$. Thus $f(S, t') = 3$ whereas $\sum_{i \in I(S,t')} q_i^*(t') = q_1^*(t') = 2$. But then $p_+(S) > 0$ violates (9): contradiction. Thus $p_+(S) = 0$, and the claim is proved.

By the claim just proved, Eq. (7) for any $i \in \{1,2,3\}$ implies

$$\alpha(i,\underline{\theta}) = -\sum_{S \subseteq \mathscr{X}} p_-(S)\chi_S(i,\underline{\theta}) \leq 0,$$

which contradicts the fact that $\alpha(i,\underline{\theta}) = 1$. Thus, there exists no $(p_+,p_-) : 2^{\mathscr{X}} \to \mathbb{R}^2_+$ that satisfies both (7) and (9) given any social planner's solution q^*, as asserted. ∎

B.3 Paramodularity Implies Decomposability

In contrast to Lang and Yang's [20] total unimodularity assumption, for which they do not offer any example where a player may have more than two types, the assumptions of Theorem 3 allow for arbitrary numbers of types per player. That is because paramodularity implies decomposability as well as the other assumptions in the theorem, as observed next.

Remark 2 *If $|T| < \infty$ and if the set X of feasible allocation outcomes is paramodular, then h (defined in (23)) is linear, and X satisfies (18) for some $(\mathscr{F},\mathscr{G},\hat{f},\hat{g})$ that is decomposable and satisfies (22).*

Proof Let X be paramodular and defined by $(f,g) : 2^I \to \mathbb{R}^2_+$ via (16). To prove linearity of h, pick any $F, G \subseteq I$ with $F \cap G = \varnothing$. By (23) the definition of h,

$$h(F,G) = \max_{x \in \mathrm{cv}X} \sum_{i \in I} (\chi_F(i) - \chi_G(i))\, x_i.$$

This problem is solved by the greedy-generous algorithm (due to paramodularity). Thus,

$$h(F,G) = f(F) - g(G) = \max_{x \in X} \sum_{i \in F} x_i - \min_{x \in X} \sum_{i \in G} x_i = h(F,\varnothing) + h(\varnothing,G),$$

with the second equality due to (14) and (15). Thus h is linear.

To prove the other parts of the remark, define $(\mathscr{F},\mathscr{G},\hat{f},\hat{g})$ by $\mathscr{F} := \mathscr{G} := 2^I$, $\hat{f} := f$, and $\hat{g} := g$. Then (18) holds, and $(\mathscr{F},\mathscr{G},\hat{f},\hat{g})$ is a constraint structure considered in Section 5. To prove that (22) is satisfied, note that cv$X$ is equal to the $\mathscr{Q}$ in the special case where everyone's type is common knowledge, i.e., T_{i_1} is singleton for all $i_1 \in I_1$. Thus, by paramodularity, Theorem 2 implies

$$\mathrm{cv}X = \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \;\middle|\; \forall E \subseteq I \left[g(E) \leq \sum_{i \in E} x_i \leq f(E) \right] \right\},$$

which becomes (22) because $\mathscr{F} = \mathscr{G} = 2^I$, $\hat{f} = f$ and $\hat{g} = g$.

To prove that $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ is decomposable, pick any $\alpha \in \mathbb{R}^{\mathscr{X}}$. Since X is assumed nonempty and compact, so is cv$X$. Thus the problem on the right-hand side of (6) has a finite optimum, and hence so does its dual. Since $\mathscr{F} = \mathscr{G} = 2^I$, $\hat{f} = f$ and $\hat{g} = g$, this dual is problem (20). Then, by Frank et al. [10, Prop. 2] due to the paramodular assumption, there exists a solution (φ^*, γ^*) to problem (20) such that, for any $t \in T$, the supports

$$\begin{aligned}\operatorname{supp} \varphi^*(\cdot, t) &:= \{F \subseteq I \mid \varphi^*(F, t) > 0\} \quad \text{and} \\ \operatorname{supp} \gamma^*(\cdot, t) &:= \{G \subseteq I \mid \gamma^*(G, t) > 0\}\end{aligned}$$

are both laminar families on disjoint ground sets. Denote

$$\begin{aligned}\alpha^+ &:= (\max\{0, \alpha(i, t_{i_1})\})_{(i, t_{i_1}) \in \mathscr{X}}, \\ \alpha^- &:= (\min\{0, \alpha(i, t_{i_1})\})_{(i, t_{i_1}) \in \mathscr{X}}.\end{aligned}$$

Apply Lang and Yang [20, Lemmas 6 & 7] to see that $(\alpha^+, \varphi^*, \mathbf{0})$ belongs to

$$\mathcal{P}_1 := \{(\alpha, \varphi, \gamma) \in \mathcal{P} \mid \alpha \geqq \mathbf{0}, \gamma = \mathbf{0}, [F \notin \operatorname{supp} \varphi^*(\cdot, t) \Rightarrow \varphi(F, t) = 0]\},$$

$(\alpha^-, \mathbf{0}, \gamma^*)$ belongs to

$$\mathcal{P}_2 := \{(\alpha, \varphi, \gamma) \in \mathcal{P} \mid \alpha \leqq \mathbf{0}, \varphi = \mathbf{0}, [G \notin \operatorname{supp} \gamma^*(\cdot, t) \Rightarrow \gamma(G, t) = 0]\},$$

and $\mathcal{P}_1 \cup \mathcal{P}_2$ is contained in the cone generated by some finite subset $\{(\alpha_k, \varphi_k, \gamma_k) \mid k \in \mathcal{K}\} \subset \mathcal{P}$ that satisfies (21). Thus, there exist finite sets K_1 and K_2, $(\beta_k^1)_{k \in K_1} \in \mathbb{R}_{++}^{K_1}$, $(\beta_k^2)_{k \in K_2} \in \mathbb{R}_{++}^{K_2}$, $(\alpha_k^1, \varphi_k^1, \gamma_k^1)_{k \in K_1} \in \mathcal{P}^{K_1}$, and $(\alpha_k^2, \varphi_k^2, \gamma_k^2)_{k \in K_2} \in \mathcal{P}^{K_2}$, such that every α_k^j satisfies (21), and

$$\begin{aligned}\left(\alpha^+, \varphi^*, \mathbf{0}\right) &= \sum_{k \in K_1} \beta_k^1 \left(\alpha_k^1, \varphi_k^1, \gamma_k^1\right), \\ \left(\alpha^-, \mathbf{0}, \gamma^*\right) &= \sum_{k \in K_2} \beta_k^2 \left(\alpha_k^2, \varphi_k^2, \gamma_k^2\right).\end{aligned}$$

Sum the two equations and note $\alpha = \alpha^+ + \alpha^-$ to obtain

$$(\alpha, \varphi^*, \gamma^*) = \sum_{k \in K_1} \beta_k^1 \left(\alpha_k^1, \varphi_k^1, \gamma_k^1\right) + \sum_{k \in K_2} \beta_k^2 \left(\alpha_k^2, \varphi_k^2, \gamma_k^2\right).$$

That is, $(\alpha, \varphi^*, \gamma^*)$ is a conic combination of $\bigcup_{j=1}^2 \{(\alpha_k^j, \varphi_k^j, \gamma_k^j) \mid k \in K_j\}$, and each α_k^j is $\{0, 1, -1\}$-valued. Thus the constraint structure is decomposable. ■

Different from decomposability, the total unimodularity assumption has not been observed to include paramdoularity as a special case. The reason is that total unimodularity requires that the entire set $\mathcal{P}$ be generated by extreme rays whose α-components are $\{0, 1, -1\}$-valued, and φ- and γ-components are $\{0, 1\}$-valued, whereas decomposability requires only a subset of $\mathcal{P}$ be generated by extreme rays whose α-components are $\{0, 1, -1\}$-valued, and hence decomposability is broad enough to include paramodularity. Total unimodularity may be an unnecessary condition for reduced-form characterization.[18]

B.4 Non-Paramodularity of the Assignment Models

In both assignment models, the set X of feasible allocation outcomes satisfies (16) such that

$$\forall i_2 \in I_2: \quad f(I_1 \times \{i_2\}) = 1, \tag{51}$$

$$\forall i_1 \in I_1\ \forall E \subsetneq I_2: \quad g(\{i_1\} \times E) = 0, \tag{52}$$

$$\forall i \in I: \quad f\{i\} = 1, \tag{53}$$

$$\forall M \subseteq I: \quad [(i_1, i_2), (i_1', i_2') \in M, i_1 \neq i_1', i_2 \neq i_2'] \Rightarrow f(M) = 2, \tag{54}$$

and either

$$\forall i_1 \in I_1 : g(\{i_1\} \times I_2) = 1 \tag{55}$$

in the assignment model (Section 6), or

$$\forall i_1 \in I_1 : f(\{i_1\} \times I_2) = 1 \quad \text{and} \quad \forall E \subseteq I : g(E) = 0 \tag{56}$$

in the partial assignment model (Online Supplement B.1).

The assignment model in Section 6 violates the compliance condition:

$$f\{(1,1),(2,1)\} - f(\{(1,1),(2,1)\} \setminus (\{1\} \times I_2)) = 1 - f\{(2,1)\} = 1 - 1 = 0,$$

with the first equality due to (51) and the second due to (53). Whereas, by (55),

$$g(\{1\} \times I_2) - g((\{1\} \times I_2) \setminus \{(1,1),(2,1)\}) = 1 - g(\{1\} \times (I_2 \setminus \{1\})) = 1 - 0 = 1,$$

with the second last "=" due to (52). Thus (f, g) violates compliance.

[18] Lang and Yang [20] cite an example from Che et al. [7] for which neither the reduced-form characterization nor the total unimodularity assumption are valid. Needless to say, existence of such an example does not imply that total unimodularity is a necessary condition for the reduced-form characterization to be valid.

The partial assignment model (Online Supplement B.1) violates the submodular condition for f: $f\{(1,1),(2,1)\} + f\{(2,1),(2,2)\} = 1 + 1 = 2$ by (51) and (56), whereas

$$\begin{aligned} & f(\{(1,1),(2,1)\} \cup \{(2,1),(2,2)\}) + f(\{(1,1),(2,1)\} \cap \{(2,1),(2,2)\}) \\ = \;& f\{(1,1),(2,1),(2,2)\} + f\{(2,1)\} = 2 + 1 = 3, \end{aligned}$$

where $f\{(1,1),(2,1),(2,2)\} = 2$ by (54). Thus f is not submodular. ■

B.5 Infinite Type Spaces

This section removes the assumption $|T| < \infty$ in the main text.

Theorem 6 *For any $(f, g) : 2^I \to \mathscr{R}_+^2$, if there exists $\epsilon > 0$ such that for any integer $m > 1/\epsilon$, $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$ holds given any $|T| < \infty$ and any constraint structure defined by (16) where $(f_t, g_t) = (f, g^m)$ for all t and, for any $m \in \{1, 2, \ldots\}$ and any $E \subseteq I$,*

$$g^m(E) = \max\{0, g(E) - 1/m\}, \tag{57}$$

then $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$ given the constraint structure defined by (16) where $(f_t, g_t) = (f, g)$ for all t, whether $|T|$ is finite or not.

Proof It is easy to adapt Lemma 7 to obtain $\mathscr{Q}_{\mathrm{B}} \supseteq \mathscr{Q}$ for infinite type spaces.[19] The proof of $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ is a passing-to-limit argument: Given any type space T and any constraint structure X defined by (f, g), pick any $Q \in \mathscr{Q}_{\mathrm{B}}$. Construct a sequence $(Q^m)_{m=1}^{\infty}$ of finite-type interim allocations converging to Q so that, for all sufficiently large m, $Q^m \in \mathscr{Q}_{\mathrm{B}}$ given any constraint structure defined by (f, g^m). Then the hypothesis of the theorem implies $Q^m \in \mathscr{Q}$ with respect to constraint structure (f, g^m). Consequently, the convergence of $Q^m \to Q$ and $g^m \to g$ implies $Q \in \mathscr{Q}$ with respect to constraint structure (f, g). Since Q can be any element of $\mathscr{Q}_{\mathrm{B}}$ given constraint structure (f, g), we have $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ given T and (f, g). Next are the details of this argument.

[19]Since X_t is assumed compact for all $t \in T$, every element of $\mathscr{Q}$ is μ-essentially bounded due to (1). Thus, following Border [2], treat both $\mathscr{Q}$ and $\mathscr{Q}_{\mathrm{B}}$ as subsets of the $L_\infty(\mu)$-space of functions $\mathscr{X} \to \mathbb{R}$. For any L_∞-function $Q := (Q_i)_{i \in I} \in \mathbb{R}^{\mathscr{X}}$ and any $L_1(\mu)$-function $\alpha \in \mathbb{R}^{\mathscr{X}}$, define the inner product $\langle Q, \alpha \rangle := \int_T \sum_{i \in I} Q_i(t_{i_1}) \alpha(i, t_{i_1}) d\mu(t)$. Then $\langle \cdot, \alpha \rangle$ is a continuous linear functional on the L_∞-space of interim allocations. The rest is the same as the proof of Lemma 7.

Let $Q := (Q_{i_1,i_2})_{(i_1,i_2)\in I_1\times I_2} \in \mathscr{Q}_{\mathrm{B}}$ given any type space T and constraint structure (f,g). Write Q as $(Q_{i_1})_{i_1\in I_1}$ such that $Q_{i_1} := (Q_{i_1,i_2})_{i_2\in I_2}$ for each $i_1 \in I_1$. For any $m = 1,2,3,\ldots$, partition $\mathbb{R}^{I_2}$ into a collection $\mathscr{C}_m$ of cells each of which has diameter at most $1/m$.[20] For each bidder $i_1 \in I_1$ and each cell $C \in \mathscr{C}_m$ that has nonempty intersection with the range of Q_{i_1}, denote $([\min C]_{i_2})_{i_2\in I_2}$ for the coordinate-wise minimum among all elements of C, with $[\min C]_{i_2}$ being its coordinate in the i_2th dimension, and define

$$Q^m_{i_1,i_2}(t_{i_1}) := \max\{0, [\min C]_{i_2}\}$$

for all $i_2 \in I_2$ and all t_{i_1} in the inverse image $Q^{-1}_{i_1}(C)$ of C. Thus,

$$\max\{0, Q_{i_1,i_2}(t_{i_1}) - 1/m\} \le Q^m_{i_1,i_2}(t_{i_1}) \le Q_{i_1,i_2}(t_{i_1})$$

for each m, each $(i_1,i_2) \in I$ $(= I_1 \times I_2)$ and each $t_{i_1} \in T_{i_1}$. (The second inequality follows from the definition of $Q^m_{i_1,i_2}(t_{i_1})$ and the fact that there is no loss to restrict the range of Q_{i_1} to $\mathbb{R}^{I_2}_+$, as f and g are both nonnegative-valued.) Since $Q \in \mathscr{Q}_{\mathrm{B}}$ with respect to (f,g), Q satisfies (5) with respect to (f,g). Thus, by the above-displayed inequalities, $Q^m := (Q^m_{i_1,i_2})_{(i_1,i_2)\in I}$ satisfies the (5) with respect to (f, g^m), where g^m is defined by (57).

Since there is no loss to restrict the range of Q_{i_1} to a bounded set (as f and g are each finite-valued), for each m there are only finitely many cells in $\mathscr{C}_m$ that intersect with the range of Q_{i_1}. Thus, $Q^m_{i_1}$ is equivalent to a function defined on the finite type space

$$T^m_{i_1} := \left\{Q^{-1}_{i_1}(C) \mid Q^{-1}_{i_1}(C) \neq \varnothing; C \in \mathscr{C}_m\right\}.$$

It follows that for any m, $Q^m \in \mathscr{Q}_{\mathrm{B}}$ given type space $T^m := \prod_{i_1\in I_1} T^m_{i_1}$ and constraint structure (f, g^m). Thus, by the hypothesis in the theorem, for all sufficiently large m, Q^m belongs to the $\mathscr{Q}$ given T^m and (f, g^m). For any such m, by the definition of $\mathscr{Q}$, there exists an ex post allocation q^m given T^m and (f, g^m). Consequently, one can extract a subsequence $(q^{m_k})_{k=1}^{\infty}$ converging to some ex post allocation q given the original type space T and original constraint structure (f,g). Furthermore, following the reasoning (and topologies) in Border [2], $\lim_{k\to\infty} Q^{m_k}$ is the reduced form of q, and $Q = \lim_{k\to\infty} Q^{m_k}$. That is, $Q \in \mathscr{Q}$ given T and (f,g), as desired. ■

The proof of Theorem 6 is an extension of Che et al.'s [7, Online Appendix B.2] passing-to-limit argument. The main assumption around (57) is to ensure that, when any given

[20] A *cell* in $\mathbb{R}^{I_2}$ is any set $\prod_{i_2\in I_2}[y_{i_2}, y'_{i_2})$ for some real numbers $y_{i_2} < y'_{i_2}$ $(\forall i_2 \in I_2)$.

interim allocation is being approximated from below by the nearest nonnegative grid points, the floor constraints in the Border condition within the discretized model is satisfied. This assumption is true given paramodularity or partial assignment. Thus follow the next two corollaries. By contrast, the assumption is not satisfied in the full assignment model, because its floor constraint involves positive integers and hence cannot be perturbed downward. By the same token, neither is the assumption satisfied by Lang and Yang's total unimodular model, which allows for positive integer floor constraints.

Corollary 1 *If (f, g) is constant across ex post states $t \in T$ and is paramodular on 2^I, and if $\mathscr{R} = \mathbb{R}$, then $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$.*

Proof We shall prove that, for any sufficiently large integer m, $-g^m$ is submodular and (f, g^m) is compliant. (Since $\mathscr{R} = \mathbb{R}$ by the assumption of the corollary, the g^m defined in (57) is a legitimate constraint function.) Submodularity of $-g^m$ means

$$g^m(E) + g^m(E') \leq g^m(E \cup E') + g^m(E \cap E') \tag{58}$$

for all $E, E' \subseteq I$. Since 2^I is finite, it suffices to show, given any $E, E' \subseteq I$, that (58) holds for all sufficiently large m. If $g(E) > 0$ and $g(E') > 0$, then (57) implies that, for any large enough m, $g^m(E) = g(E) - 1/m$ and $g^m(E') = g(E') - 1/m$; meanwhile, the right-hand side of (58) is never less than $g(E \cup E') + g(E \cap E') - 2/m$ (by (57)). Thus (58) follows from $g(E) + g(E') \leq g(E \cup E') + g(E \cap E')$ (submodularity of $-g$) for all large m. If $g(E) = 0$ and $g(E') = 0$, then $g^m(E) = g^m(E') = 0$ by the definition of g^m, and (58) follows trivially because its right-hand side is always nonnegative (by the definition of g^m). Else, one of $g(E)$ and $g(E')$ is zero, and the other positive. Then $g(E \cup E') > 0$ and $g(E \cap E') = 0$ (monotonicity of g, due to submodularity of $-g$). Without loss of generality, say $g(E) > 0 = g(E')$. Then for any m sufficiently large, (58) becomes $g(E) - 1/m \leq g(E \cup E') - 1/m$, which is true by $g(E') \leq g(E \cup E')$ (submodularity of $-g$). Thus, (58) is true for any sufficiently large m.

Compliance of (f, g^m) means

$$f(E') - f(E' \setminus E) \geq g^m(E) - g^m(E \setminus E') \tag{59}$$

for all $E, E' \subseteq I$. Suppose that (59) does not hold no matter how large m is. Then, it follows from the fact $f(E') - f(E' \setminus E) \geq g(E) - g(E \setminus E')$ (compliance of (f, g)) that $g^m(E \setminus E') = g(E \setminus E') - 1/m$ and $g^m(E) = 0$ for any m. Then by the definition of g^m we have $g(E) = 0 < g(E \setminus E')$, contradicting the monotonicity of g noted previously. Thus, (59) holds for all sufficiently large m. Since there are only finitely many subsets of I, (59) holds for all subsets of I when m is sufficiently large. Thus, for any sufficiently large m, (f, g^m) is paramodular. Then the conclusion follows from Theorems 2 and 6. ■

Corollary 2 *In the partial assignment model,* $\mathcal{Q}_{\mathrm{B}} = \mathcal{Q}$.

Proof In the partial assignment model, the constraint structure X satisfies (16) wherein $f(E)$ is equal to the $\sum_{i \in E} x_i$ in Example 2 (Section 2), and $g(E) = 0$ for all $E \subseteq I$. Then $g^m = 0 = g$ for all m. The conclusion therefore follows from Theorems 5 and 6. ■

Corollary 1 also gives the characterization in the two-player bargaining model, a special case of paramodularity.

Corollary 3 *In the two-player bargaining model defined in Section 4,* $\mathcal{Q}_{\mathrm{B}} = \mathcal{Q}$.

Proof In the two-player bargaining model, $I_1 = \{1, 2\}$ and I_2 is singleton. Thus we can let $I := \{1, 2\}$. The constraint structure is defined by (16) where $f\{1, 2\} = f\{1\} = f\{2\} = 1$, $g\{1\} = g\{2\} = 0$ and $g\{1, 2\} = 1$, as well as the standard $f(\varnothing) = g(\varnothing) = 0$. By Corollary 1, it suffices to prove that (f, g) is paramodular on 2^I. Submodularity of f follows directly from $f\{1, 2\} = f\{1\} = f\{2\} = 1$, and submodularity of $-g$ directly from $g\{1\} = g\{2\} = 0$ and $g\{1, 2\} = 1$. To prove compliance, pick any $E, E' \subseteq \{1, 2\}$ to prove

$$f(E') - f(E' \setminus E) \geq g(E) - g(E \setminus E'). \tag{60}$$

By the g defined above, $g(E) - g(E \setminus E') \in \{0, 1\}$ and $f(E') - f(E' \setminus E) \geq 0$. Thus, if $g(E) - g(E \setminus E') = 0$ then (60) follows trivially. Suppose that $g(E) - g(E \setminus E') = 1$. Then by the definition of g, $E \setminus E'$ is either singleton or empty, and $E = \{1, 2\}$. Thus, $E' \neq \varnothing$ and $E' \setminus E = \varnothing$. Consequently, $f(E') - f(E' \setminus E) = f(E') - f(\varnothing) = 1$, so (60) holds. Thus, (f, g) is compliant, as desired. ■