A Method to Characterize Reduced-Form Auctions*

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Abstract

This paper proposes a method to characterize the interim allocations that are ex post feasible given multiple objects, possibly unit-demand preferences, and arbitrary distributions of types. To obtain such characterizations it suffices to take any allocation that has binding ex post feasibility constraints and replace them with some binding interim constraints, each associated with a set of interim states. The method considers any allocation with binding ex post feasibility constraints as a choice function among interim states. From this choice function I derive a family of partial revealed preferences, each rationalizing the choice within a subset of interim states. The upper or lower contour sets of interim states with respect to these partial revealed preferences correspond to the binding interim constraints. The method applies easily to generalize the received result in the mainstream model (paramodularity) and establish a counterpart to a contemporary result (total unimodularity), and it applies nontrivially to the assignment problems between $N \geq 2$ objects and two bidders each of whom can have any number of types.

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1 Introduction

Like the prisoners in Plato's Cave who see only the shadows projected on the wall from objects of higher dimensions, a player in a Bayesian game often sees only a projection from the actual game. For instance, in an assignment situation where a player cannot get more than one object, if player k gets object j then any different object $j' \neq j$ can only be allocated to a different player $k' \neq k$. This constraint has to be respected by any equilibrium outcome say $((q_{kj}(t_1, \ldots, t_m))_{j=1}^n)_{k=1}^m$ conditional on the realized type profile $(t_1, \ldots, t_m)$ across all players, with q_{kj} associating to each type profile a probability with which objet j goes to player k. However, at the interim stage of decision making, any player k is uncertain about the types t_{-k} of the other players and hence the variable that affects his decision on behalf of $((q_{kj})_{j=1}^n)_{k=1}^m$ is only the reduced form $(Q_{kj}(t_k))_{j=1}^n$, each $Q_{kj}(t_k)$ equal to the expected value of $q_{kj}(t_k, t_{-k})$ with t_{-k} being the random variable. The design of a Bayesian game therefore often amounts to designing a profile $((Q_{kj})_{j=1}^n)_{k=1}^m$ of such marginals, called *interim allocation*, as opposed to the underlying $((q_{kj})_{j=1}^n)_{k=1}^m$, called *ex post allocation*. The question is how to tell whether a profile $((Q_{kj})_{j=1}^n)_{k=1}^m$, each Q_{kj} being a function of only player k's type, is indeed the reduced form of some feasible ex post allocation $((q_{kj})_{j=1}^n)_{k=1}^m$, each q_{kj} being a function of the type profile across all players.

Characterizing such feasible reduced forms has long been an important area in the auction theory literature.[1] A characterization is first obtained by Border [2] for single-unit symmetric auctions, extended to asymmetric single-unit models by Border [3], Manelli and Vincent [22], Mierendorff [25], Goeree and Kushnir [13][2] and Yang [29], and generalized to multiple units of a single object by Che et al. [7]. Border's characterization is also obtained through majorization techniques by Hart and Reny [14] and generalized by Kleiner et al. [18] and Kolesnikov et al. [19] in a symmetric model.[3] Toikka et al. [27] obtain a characterization

[1]Examples include auctions with risk aversion such as Maskin and Riley [23] and Matthews [24], endogenous valuation such as Gershkov et al. [12], information acquisition such as Li [21], and budget constraints such as Boulatov and Severinov [4]. While the reduced form issue could be partially avoided through restricting attention to dominance- instead of Bayesian incentive compatible mechanisms, the restriction has loss of generality because the equivalence between the two requires other conditions such as single-dimensionality for each bidder's type (cf. Manelli and Vincent [22] and Gershkov et al. [11]).

[2]Goeree and Kushnir also consider social choice models.

[3]In the single-object multiunit case of [18], Gershkov et al. [12] find a connection between the majorization method and Border's characterization. He et al. [16] suggest a connection between feasible reduced forms

in a Bayesian persuasion problem by direct calculations of the elimination cone associated with the reduced forms.

The frontier of the area is on multiple heterogeneous objects with combinatorial complications such as the aforementioned assignment problems. The mainstream model such as Che et al. [7] relies on a paramodularity assumption, which does not incorporate assignment problems even if it is extended to have multiple objects. Cai et al. [6] consider multiple objects but assume additive utilities, so assignment problems are not included either. Recently, the total unimodularity model of Lang and Yang [20], as well as their complete-information precedent Budish et al. [5], includes all assignment models, but their application to assignments is restricted to cases where a bidder can have at most two types. Valenzuela-Stookey's [28] assignment model allows for arbitrary numbers of types per bidder, though his characterization is "approximate," namely, sufficient but not necessary for reduced forms.

This paper proposes a method to obtain an exact characterization with multiple objects and arbitrary numbers of types. It applies easily to extend Che et al.'s [7] paramodularity result for multiple objects (Theorem 2) and to establish a counterpart (Theorem 3) to Lang and Yang's [20] contemporary total unimodularity result. The method applies to an assignment model between $N \geq 2$ objects and two bidders, with an arbitrary number of types per bidder (Theorem 4). The crucial constraint here is that each bidder is to be assigned exactly one object. While the same characterization also obtains when the constraint is weakened to allow for a bidder getting no object (Online Supplement B.1), this paper focuses on the aforementioned constraint as it is not considered in Valenzuela-Stookey's [28] model.

The method starts with a basic idea that characterizing feasible reduced forms is simply to replace the ex post feasibility constraints by their interim counterparts. An ex post constraint is an inequality in terms of the ex post allocation at an instance of the *ex post state*, a profile of the realized types across all players. An interim constraint is an inequality of the interim allocation regarding a set of players and a set of their realized types. This pair of sets is succinctly represented by a set of *interim states*, each being a player-object pair together with a realized type of the player. Any ex post constraint whose ex post state is compatible with some element in the set of interim states implies an inequality, a *projection* on the set of interim states. An interim constraint is equal to the marginal of all these projections on the associated set of interim states. Thus, interim constraints are

and feasible private information structures.

only implications of ex post constraints, and replacing ex post constraints with interim ones risks the danger of replacing the feasible set with something too broad. To validate the replacement, we need to show that any allocation on the boundary of the feasible set is just about to violate some interim constraint. The problem boils down to finding a method to locate such binding interim constraints.

To that end, Theorem 1 suggests that we need only to search among the interim constraints whose associated sets of interim states satisfy a *universal binding* condition. A set of interim states is said to be upward (resp. downward) universally binding if every projection on this set from the ex post constraints is binding upward (resp. downward). An interim constraint can be binding only if the associated set of interim states is either upward or downward universally binding. Intuitively, the sum of a set of weak inequalities in the same direction cannot be an equality if some of the inequalities is strict.

Then how to find such universally binding sets of interim states? The idea is to think of a boundary point allocation mentioned before as a stochastic choice function among interim states. Say bidder L's type is a and the allocation assigns bidder L to task 1 when bidder R's type is b, and assigns bidder L to no task when bidder R's type is b'. That means the allocation chooses the interim state "task 1 for type a of bidder L" over another interim state "task 2 for type a of bidder L" when the ex post state is (a, b), and neither of them is chosen over the other when the ex post state is (a, b'). If there is a preference relation among the interim states that is consistent with the choices, then any upper or lower contour set with respect to the preferences satisfies the universal binding condition (Lemma 2).

Due to combinatorial complications, a preference relation on all interim states need not exist or come directly from the supporting hyperplane of the feasible set at the boundary point allocation. In such situations, we construct multiple partial orders, each consistent with the choice function within a subset of interim states. In an assignment problem, for example, an interim state has two kinds of rivals under the choice function. Because an object cannot go to different bidders, an interim state say "task 1 for type a of bidder L" competes with all the interim states associated with the other bidder for the same object such as "task 1 for type b of bidder R" and "task 1 for type c of bidder R." Think of these rivals in a row. Meanwhile, since no bidder can have more than one object, the interim state also competes with those referring to the same bidder-type for different objects such as "task 2 for type a of bidder L" and "task 3 for type a of bidder L." Think of these rivals in a column. Within every row or column, I construct a partial order among its elements that is consistent with the choice function therein. Then I derive a family of upper or lower contour sets, each with respect to some of such partial orders.

The final step is to prove that the derived family of upper or lower contour sets corresponds to the family of binding interim constraints at the boundary point allocation. This amounts to proving existence of a nonnegative solution (the shadow prices for the interim constraints) in a system of linear equations. In simple cases, the system can be directly solved. In general, the existence proof is to verify that no Gaussian elimination on the linear system can produce a contradictory equation (Lemma 3). This suffices the existence thanks to Chu et al.'s [8] hyper-rectangle cover theory. Once the shadow prices are proved to exist, some of them strictly positive due to the equation system, the nonempty set of binding interim constraints is located, and so the reduced form characterization obtains.

The next section introduces the basic definitions including the characterization statement. Then Section 3 introduces the method, which is applied to the paramodularity model in Section 4 and to a counterpart to the total unimodularity model in Section 5. Section 6 then explains at length the application to the assignment model. Appendix A contains all delayed details.

2 Basic Definitions

Let I_1 be the set of bidders, I_2 the set of objects, both assumed to be finite. Let

$$I := I_1 \times I_2$$

denote the set of all possible bidder-object pairs. For any $i_1 \in I_1$, let T_{i_1} be the set of the possible types of bidder i_1. Any (i_1, i_2, t_{i_1}), with $(i_1, i_2) \in I_1 \times I_2$ and $t_{i_1} \in T_{i_1}$, is called *interim state*, often denoted by (i, t_{i_1}) with $i := (i_1, i_2) \in I$. Let $T := \prod_{i_1 \in I_1} T_{i_1}$ be the type space. Any element of T is a possible profile of realized types across all bidders, called *ex post state*, is in the form

$$t := (t_{i_1})_{i_1 \in I_1}.$$

Assume that T is a finite set,[4] and μ the probability measure on T that has T as the support (so $\mu(t) > 0$ for any $t \in T$, where $\mu(t) := \mu(\{t\})$ denoting the measure of the singleton $\{t\}$).

[4]Some results in this paper are extended to infinite type spaces in Online Supplement B.5.

Let $\mathscr{R}$ denote either the set $\mathbb{R}$ of real numbers or the set $\mathbb{Z}$ of integers, and let $\mathscr{R}_+$ be the set of nonnegative elements of $\mathscr{R}$. For any $t \in T$, let $X_t \subseteq \mathscr{R}^I$ be the set of allocation outcomes that are feasible when t is the ex post state. An element of X_t is in the form of $x := (x_i)_{i \in I}$, or $(x_{i_1,i_2})_{(i_1,i_2) \in I}$, with x_{i_1,i_2} being the quantity of object i_2 allocated to bidder i_1. (Thus, quantities are divisible if $\mathscr{R} = \mathbb{R}$, and indivisible if $\mathscr{R} = \mathbb{Z}$.) The mapping $t \mapsto X_t$ is the entire structure of *ex post constraints*. For each $t \in T$, denote $\mathrm{cv}X_t$ for the convex hull of X_t. Assume that X_t is nonempty and compact for all $t \in T$.

An *ex post allocation* is a profile $(q_i)_{i \in I}$ of functions $q_i : T \to \mathbb{R}$ ($\forall i \in I$) such that $(q_i(t))_{i \in I} \in \mathrm{cv}X_t$ for any $t \in T$. Thus, an ex post allocation can randomize on the feasible allocation outcomes, and its range is a subset of $\mathbb{R}^I$ even when $X_t \subseteq \mathbb{Z}^I$. An *interim allocation* is a profile $Q := (Q_i)_{i \in I}$ of functions $Q_i : T_{i_1} \to \mathbb{R}$ ($\forall i := (i_1, i_2) \in I$).

For each $i_1 \in I_1$, let μ_{i_1} be the marginal measure of μ onto T_{i_1}; let $T_{-i_1} := \prod_{j \in I_1 \setminus \{i_1\}} T_j$, and denote $\mu_{-i_1}(\cdot|t_{i_1})$ for the conditional measure on T_{-i_1} according to μ conditional on t_{i_1}. An interim allocation $(Q_i)_{i \in I}$ is said to be the *reduced form* of some ex post allocation $(q_i)_{i \in I}$ if and only if Q_i is the marginal of q_i onto T_{i_1} for all $i := (i_1, i_2) \in I$, namely,

$$Q_i(t_{i_1}) = \sum_{t_{-i_1} \in T_{-i_1}} q_i(t_{i_1}, t_{-i_1}) \mu_{-i_1}(t_{-i_1}|t_{i_1}) \tag{1}$$

for any $i := (i_1, i_2) \in I$ and any $t_{i_1} \in T_{i_1}$. Let $\mathscr{Q}$ denote the set of interim allocations that are reduced forms of some ex post allocations.

Denote the set of interim states by

$$\mathscr{L} := \bigcup_{(i_1,i_2) \in I} \left(\{(i_1, i_2)\} \times T_{i_1} \right).$$

For any $S \subseteq \mathscr{L}$ and any $t := (t_{i_1})_{i_1 \in I_1} \in T$, define

$$I(S, t) := \{(i_1, i_2) \in I \mid (i_1, i_2, t_{i_1}) \in S\}. \tag{2}$$

In other words, $I(S, t)$ is the set of bidder-object pairs due to which S is subject to some ex post constraint when the ex post state is t: As long as the i_1-th component of the ex post sate t matches the realized type t_{i_1} in an interim state (i_1, i_2, t_{i_1}) within S, the bidder-object pair (i_1, i_2) is subject to the ex post constraints X_t at t. To capture the implications of the ex post constraints on a set of interim states, define

$$f(S,t) \;:=\; \max_{x \in X_t} \sum_{i \in I(S,t)} x_i \tag{3}$$

$$g(S,t) \;:=\; \min_{x \in X_t} \sum_{i \in I(S,t)} x_i \tag{4}$$

for any $S \subseteq \mathscr{X}$ and any $t \in T$. Thus, $f(S,t)$ and $g(S,t)$ are the ceiling and floor of the total quantity that S can get for its members that are ex post constrained when the ex post state is t. Obviously, satisfaction of such upper and lower bounds in expectation, as in the Border condition defined next, is necessary for an interim allocation to be feasible. What is nontrivial, however, is whether satisfaction thereof is also sufficient for an interim allocation to be feasible.

An interim allocation $(Q_i)_{i \in I}$ is said to satisfy the *Border condition* if and only if

$$\sum_{t \in T} g(S,t)\mu(t) \leq \sum_{i \in I} \sum_{t \in T} Q_i(t_{i_1}) \chi_S(i, t_{i_1}))\mu(t) \leq \sum_{t \in T} f(S,t)\mu(t) \tag{5}$$

for all $S \subseteq \mathscr{X}$ (χ_S being the characteristic function of S).

Examples To illustrate the Border condition in specific settings, define $S_i := \{t_{i_1} \in T_{i_1} \mid (i, t_{i_1}) \in S\}$ for any $S \subseteq \mathscr{X}$ and any $i \in I$. Then $I(S,t) = \{i \in I \mid t_{i_1} \in S_i\}$, and the Border condition is equivalent to satisfaction of (5) for all profiles $(S_i)_{i \in I}$ such that S_i is a measurable subset of T_{i_1} for each $i \in I$, with the S in (5) standing for the profile $(S_i)_{i \in I}$, and $\chi_S(i, t_{i_1})$ replaced by $\chi_{S_i}(t_{i_1})$. Expressing the Border condition explicitly amounts to computing the left- and right-hand sides of (5) for any such profile S.

1. *Single-Unit Auction*: Here I_2 is singleton and so we can write $I := I_1$, and S becomes a profile $(S_{i_1})_{i_1 \in I_1}$ across bidders. With only a single unit available, the maximum $\sum_{i \in E} x_i$ among all feasible x is equal to one if $E \neq \varnothing$, and zero otherwise. Assume that types are independent across bidders. Then the right-hand side of (5)

$$\sum_{t \in T} f(S,t)\mu(t) = \sum_{t \in T} \left(1 - \prod_{i_1 \in I_1} \chi_{\neg S_{i_1}}(t_{i_1})\right) \mu(t) = 1 - \prod_{i_1 \in I_1} \left(1 - \mu_{i_1}(S_{i_1})\right),$$

where $\neg S_k$ denotes the complement of S_k. The left-hand side of (5) is equal to zero because the minimum of $\sum_{i \in E} x_i$ among all feasible x is zero, as no-sale is an option. If we further restrict attention to the symmetric model, namely, $Q_{i_1} = Q_{i'_1} =: Q$,

$T_{i_1} = T_{i_1'} =: \mathcal{T}$ and $\mu_{i_1} = \mu_{i_1'} =: \nu$ for all $i_1, i_1' \in I_1$, then the Border condition is reduced to satisfaction of (5) for all measurable subsets S of $\mathcal{T}$, the formula displayed above reduced to $1 - (1 - \nu(S))^{|I_1|}$, and (5) reduced to the well-known condition

$$\sum_{\tau \in \mathcal{T}} Q(\tau)\chi_S(\tau)\nu(\tau) \leq \frac{1}{|I_1|}\left(1 - (1 - \nu(S))^{|I_1|}\right).$$

2. *Assignment*: Suppose that there are two bidders and $N \geq 2$ objects subject to a unit-demand constraint: Each bidder is to be assigned exactly one object. Let $I_1 := \{1, 2\}$ denote the set of bidders. With I_2 non-singleton, the S in (5) becomes a profile $(S_{kj})_{(k,j) \in I}$ across all bidder-object pairs (rather than merely across all bidders) such that $S_{kj} \subseteq T_k$ for all $k \in \{1, 2\}$ and $j \in I_2$. To calculate the right-hand side of (5), note for any $E \subseteq I$ that the maximum $\sum_{i \in E} x_i$ among all $(x_i)_{i \in I}$ subject to the unit-demand constraint is equal to two if there exist $j, j' \in I_2$ for which $j \neq j'$ and $\{(1, j), (2, j')\} \subseteq E$, equal to one if there exist no such j and j' while $E \neq \varnothing$, and equal to zero if $E = \varnothing$. Thus, for any profile $S := ((S_{kj})_{k=1}^2)_{j=I_2}$ and any $t := (t_k)_{k=1}^2 \in T$, the maximum $\sum_{(k,j) \in I(S,t)} x_{kj}$ among all feasible $(x_i)_{i \in I}$ is equal to two if and only if

$$\prod_{j \in I_2} \left(\chi_{\neg S_{1j}}(t_1) + \chi_{S_{1j}}(t_1) \prod_{j' \in I_2 \setminus \{j\}} \chi_{\neg S_{2j'}}(t_2) \right) = 0,$$

and the maximum is equal to one if and only if

$$\prod_{j \in I_2} \left(\chi_{\neg S_{1j}}(t_1) + \chi_{S_{1j}}(t_1) \prod_{j' \in I_2 \setminus \{j\}} \chi_{\neg S_{2j'}}(t_2) \right) - \prod_{(k,j) \in I_1 \times I_2} \chi_{\neg S_{kj}}(t_k) = 1.$$

Then, with types assumed independent, the right-hand side of (5) is

$$\sum_{t \in T} f(S,t)\mu(t) = 2 - \prod_{(k,j) \in I_1 \times I_2} (1 - \mu_k(S_{kj})) - \prod_{j \in I_2} \left(1 - \mu_1(S_{1j}) + \mu_1(S_{1j}) \prod_{j' \neq j} (1 - \mu_2(S_{2j'})) \right).$$

To calculate the left-hand side, note for any $E \subseteq I$ that the minimum $\sum_{i \in E} x_i$ among all x subject to both constraints is now equal to two if $E = I$, equal to one if there exists exactly one $k \in \{1, 2\}$ for which $\{k\} \times I_2 \subseteq E$, and zero if none of such k exists. Then the left-hand side of (5) is

$$\sum_{t \in T} g(S,t)\mu(t) = \prod_{j \in I_2} \mu_1(S_{1j}) + \prod_{j \in I_2} \mu_2(S_{2j}).$$

The characterization in Example 2 is new, valid due to Theorem 4.

Let $\mathscr{Q}_{\mathrm{B}}$ denote the set of interim allocations that satisfy the Border condition. The characterization of reduced forms is the claim $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$. Proving $\mathscr{Q}_{\mathrm{B}} \supseteq \mathscr{Q}$ is easy (Appendix A.1). It is the converse that we shall investigate: When is $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ true?

3 The Universal Binding Condition

Since T is assumed finite, the set of interim allocations is a Euclidean space $\mathbb{R}^{\mathscr{Z}}$, and hence any continuous linear operator on $\mathscr{Q}$ corresponds to a vector $\alpha := (\alpha(z))_{z\in\mathscr{Z}} \in \mathbb{R}^{\mathscr{Z}}$. The next lemma (proved in Appendix A.2), due to the separating hyperplane theorem, provides a starting point to establish $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.

Lemma 1 *$\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if, for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$,*

$$\max_{Q\in\mathscr{Q}_{\mathrm{B}}} \sum_{i\in I} \sum_{t_{i_1}\in T_{i_1}} \alpha(i, t_{i_1}) Q_i(t_{i_1}) \mu_{i_1}(t_{i_1}) \le \sum_{t\in T} \mu(t) \max_{(q_i(t))_{i\in I}\in \mathrm{cv} X_t} \sum_{i\in I} q_i(t) \alpha(i, t_{i_1}). \tag{6}$$

Given any linear valuation α of the interim states, the maximization problem on the right-hand side of (6) is to choose, for every ex post state t, an allocation outcome $(q_i(t))_{i\in I}$ to maximize the total α-value subject to all the ex post constraints that X_t represents. I call it *social planner's problem* for its resemblance to a social planner's (micromanaging) plan specific to every ex post state. By contrast, the problem on the left-hand-side of (6) is subject to only $Q \in \mathscr{Q}_{\mathrm{B}}$ (the Border condition (5)), uniform to all ex post states. Thus the constraint $Q \in \mathscr{Q}_{\mathrm{B}}$ could be too broad for Q to satisfy the ex post constraints on the right-hand side, upsetting (6).

Theorem 1 observes the exact condition for (6) to hold, thereby suggesting a general method to validate the characterizations of reduced forms. The proof (Appendix A.3) is based on comparing the dual of the left-hand side in (6) with the right-hand side.

Theorem 1 *$\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$ there exist $(p_+, p_-) : 2^{\mathscr{Z}} \to \mathbb{R}^2_+$ and $q^* := (q^*_i)_{i\in I} : t \mapsto (q^*_i(t))_{i\in I} \in \mathrm{cv} X_t$ such that*

$$\alpha(z) = \sum_{S\subseteq\mathscr{Z}} p_+(S)\chi_S(z) - \sum_{S\subseteq\mathscr{Z}} p_-(S)\chi_S(z) \tag{7}$$

for all $z \in \mathscr{Z}$,

$$q^*(t) \in \arg\max_{(x_i)_{i\in I}\in \mathrm{cv} X_t} \sum_{i\in I} x_i \alpha(i, t_{i_1}) \tag{8}$$

for all $t := (t_{i_1})_{i_1 \in I_1} \in T$, *and, for all* $S \subseteq \mathscr{X}$,

$$\begin{aligned} p_+(S) > 0 \quad &\Rightarrow \quad \forall t \in T \left[f(S,t) = \textstyle\sum_{i \in I} q_i^*(t) \chi_S(i, t_{i_1}) \right] \\ p_-(S) > 0 \quad &\Rightarrow \quad \forall t \in T \left[g(S,t) = \textstyle\sum_{i \in I} q_i^*(t) \chi_S(i, t_{i_1}) \right]. \end{aligned} \tag{9}$$

The reduced form of the solution q^* in (8) is a boundary point between the feasible set $\mathscr{Q}$ and the supporting hyperplane normal to the vector α. Thus q^* is just about to violate some ex post constraints. To validate the reduced-form characterization, we need to replace such ex post constraints by some interim constraints. Theorem 1 tells us where to find such binding interim constraints. They are those whose shadow prices $p_+(S)$ or $p_-(S)$ are positive, S being the set of interim states associated with the interim constraint. Only those S that satisfy (9) can be associated with a binding interim constraint. That is, according to the boundary point q^*, the total quantity that S gets for its members needs to be either maxed out to the ceiling $f(S,t)$ for the price $p_+(S)$ to be positive, or reduced to the floor $g(S,t)$ for the price $p_-(S)$ to be positive, for every ex post state t. This binding condition of S is universal in the sense of being required for all ex post states t.

The Method The universal binding condition gives a clue on how to construct a price vector (p_+, p_-) thereby obtaining a characterization of reduced forms.[5] Note that the condition is defined with respect to q^*. Thus, for any set S of interim states that is *upward* universally binding in the sense of $f(S,t) = \sum_{i \in I} q_i^*(t) \chi_S(i, t_{i_1})$ for all $t \in T$ (so that $p_+(S)$ can be positive), our fictitious social planner acts as if she strictly prefers any element in S to any element outside S, raising the quantity for S up to the ceiling $f(S,t)$. Analogously, for any S that is *downward* universally binding in the sense of $g(S,t) = \sum_{i \in I} q_i^*(t) \chi_S(i, t_{i_1})$ for all $t \in T$ (so that $p_-(S)$ can be positive), the planner acts as if she strictly prefers any element outside S to any element in S, reducing the quantity for S down to the floor $g(S,t)$. Therefore, the support of p_+ consists only of the upper contour sets, and the support of p_- only the lower contour sets, with respect to such revealed preferences of the planner's solution q^*. The next lemma formalizes this idea.

Lemma 2 *Let* $q^* := (q_i^*)_{i \in I}$ *map every* $t \in T$ *to a* $(q_i^*(t))_{i \in I} \in \mathrm{cv} X_t$, *and let* $\{U^n \mid n =$

[5] That is to exploit the sufficiency observation of Theorem 1. Its necessity observation can be useful to establish impossibility results (e.g., Online Supplement B.2).

$1, \ldots, n_*\}$ *and* $\{L^n \mid n = 1, \ldots, n^*\}$ *be collections of some subsets of* $\mathscr{Z}$ *such that*

$$\varnothing = U^0 \subsetneq U^1 \subsetneq \cdots \subsetneq U^{n_*-1} \subsetneq U^{n_*}, \tag{10}$$

$$L^{n^*} \supsetneq L^{n^*-1} \supsetneq \cdots \supsetneq L^1 \supsetneq L^0 = \varnothing. \tag{11}$$

i. If for any $t := (t_{i_1})_{i_1 \in I_1} \in T$ *and any* $n = 1, \ldots, n_*$,

$$\sum_{(i,t_{i_1}) \in U^n \setminus U^{n-1}} q_i^*(t) = f(U^n, t) - f(U^{n-1}, t), \tag{12}$$

then U^n *is upward universally binding for any* $n = 1, \ldots, n_*$.

ii. If for any $t := (t_{i_1})_{i_1 \in I_1} \in T$ *and any* $n = 1, \ldots, n^*$,

$$\sum_{(i,t_{i_1}) \in L^n \setminus L^{n-1}} q_i^*(t) = g(L^n, t) - g(L^{n-1}, t), \tag{13}$$

then L^n *is downward universally binding for any* $n = 1, \ldots, n^*$.

Proof To prove claim (i), pick any $n = 1, \ldots, n_*$. By (10), $U^n = \bigsqcup_{k=1}^{n} \left(U^k \setminus U^{k-1}\right)$. Thus, for any $t := (t_{i_1})_{i_1 \in I_1} \in T$,

$$\begin{aligned} \sum_{i \in I} q_i^*(t) \chi_{U^n}(i, t_{i_1}) &= \sum_{k=1}^{n} \sum_{i \in I} q_i^*(t) \chi_{U^k \setminus U^{k-1}}(i, t_{i_1}) \\ &= \sum_{k=1}^{n} \sum_{(i,t_{i_1}) \in U^k \setminus U^{k-1}} q_i^*(t) \\ &= \sum_{k=1}^{n} \left(f(U^k, t) - f(U^{k-1}, t)\right) \\ &= f(U^n, t), \end{aligned}$$

with the third line due to (12), and the last line due to cancelation and $f(\varnothing, t) = 0$. Thus, U^n is upward universally binding. The proof for claim (ii) is analogous. ■

To unpack the implications of Lemma 2, note from (10) and (12) that the choice q^*, when restricted within U^{n_*}, behaves as if following a greedy algorithm according to a preference relation $\succeq_U$ that strictly prefers any element of U^k to any element of $U^n \setminus U^k$ for all $n > k$, and is indifferent among the elements in $U^n \setminus U^{n-1}$ for each n.[6] That is, q^* maxes

[6] To see why (12) corresponds to a greedy algorithm, plug $n = 1$ into (12) to see $\sum_{U^1} q_i^*(t) = f(U^1, t)$, and then plug $n = 2$ into (12) to see $\sum_{U^2 \setminus U^1} q_i^*(t) = f(U^2, t) - f(U^1, t)$, and so on. See Schrijver [26, Ch. 40] for an extensive coverage of greedy algorithms. The hierarchical allocation familiar in the optimal auction literature is a special case thereof, with the said preference relation being the ranking of virtual utilities.

out the feasible allocation to any preferred interim state before allocating any quantity at all to any less preferred ones. Analogously, from (11) and (13) one can see that q^*, when restricted within L^{n^*}, acts as if following a generous algorithm—the mirror image of the greedy algorithm—according to a preference relation $\succeq_L$ that strictly prefers any element of L^k to any element of $L^n \setminus L^k$ for all $n > k$, and is indifferent among the elements in $L^n \setminus L^{n-1}$ for each n. That is, $q^*|_{L^{n^*}}$ follows the greedy algorithm on L^{n^*} such that the role of f is played by $-g$.[7] Thus, the sets U^n are the upper contour sets with respect to $\succeq_U$, and the sets L^n the lower contour sets with respect to $\succeq_L$. By Lemma 2, these sets satisfy the universal binding condition. They suit our need to support the price vector (p_+, p_-). In other words, to construct the support for the (p_+, p_-) required in Theorem 1, it suffices to look for such partially revealed preferences thereby to derive the upper or lower contour sets.

Such revealed preferences, in general, does not come directly from the linear valuation α parametric to the social planner's problem. For example, consider an assignment problem between two bidders and two objects such that each bidder is to be allocated exactly one object, and vice versa. There are four bidder-object pairs, $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$. Suppose when the ex post state is (t_1, t_2), the α-values are $\alpha(1, 1, t_1) = 10$, $\alpha(1, 2, t_1) = 4$, $\alpha(2, 1, t_2) = 9$ and $\alpha(2, 2, t_2) = 2$. The ranking according to the α-values is then $(1, 1, t_1) \succ (2, 1, t_2) \succ (1, 2, t_1) \succ (2, 2, t_2)$. Had the greedy algorithm been valid according to this ranking, the social planner would have given a full quantity 1 to $(1, 1, t_1)$, namely, $q^*_{1,1}(t_1, t_2) = 1$, and hence zero quantity to $(2, 1, t_2)$ (since good 1 cannot be assigned to both bidders). But the planner is to maximize the total α-value among all feasible allocation outcomes, which can only be either $\{(1, 1), (2, 2)\}$ (giving good 1 to bidder 1, and good 2 to bidder 2) or $\{(2, 1), (1, 2)\}$ (giving good 1 to bidder 2, and good 2 to bidder 1). The maximum is $\{(2, 1), (1, 2)\}$, yielding a total value equal to $9 + 4 = 13$ instead of a total value $10 + 2 = 12$ from the alternative. Thus, the correct solution has $q^*_{1,1}(t_1, t_2) = 0$ and $q^*_{2,1}(t_1, t_2) = 1$. So the revealed preference is $(2, 1, t_2) \succ (1, 1, t_1)$, opposite to the ordinal ranking of the α-values.

Due to such combinatorial complications, in general the arbitrarily given α does not yield directly a total order on the entire set $\mathscr{X}$ of interim states that rationalizes the social planner's solution q^*. Rather, we need multiple partial orders on $\mathscr{X}$ such that each rationalizes q^* restricted within a subset of $\mathscr{X}$. Then a collection of upper or lower contour sets is obtained, each with respect to a partial order within the corresponding subset.

[7]See Hassin [15] for a definition of generous algorithms.

With such upper or lower contour sets, to satisfy the condition required in Theorem 1 it suffices to prove that the (p_+, p_-) as a function of these sets satisfies (7). That is, we need to prove that the linear system (7) has a nonnegative solution for (p_+, p_-). In simple cases, this can be done through directly solving the equation system. To handle general cases, the following lemma provides a method based on the hyper-rectangle cover theory.

Let $\mathscr{S}_+$ and $\mathscr{S}_-$ be two collections of subsets in $\mathscr{Z}$. (In applications, the elements of $\mathscr{S}_+$ are the upward universally binding sets to support p_+, and the elements of $\mathscr{S}_-$ the downward universally binding sets to support p_-.) Let $\mathbf{M}_+$ be a matrix with $|\mathscr{Z}|$ rows and $|\mathscr{S}_+|$ columns, so that rows are indexed by $\mathscr{Z}$, and columns by $\mathscr{S}_+$. For each $z \in \mathscr{Z}$ and each $S \in \mathscr{S}_+$, let the entry at the intersection between row z and column S be equal to $\chi_S(z)$. Analogously, let $\mathbf{M}_-$ be a $|\mathscr{Z}|$-by-$|\mathscr{S}_-|$ matrix whose rows are indexed by $\mathscr{Z}$ and columns by $\mathscr{S}_-$, and whose entry at the intersection between row z and column S be equal to $-\chi_S(z)$ ($\forall z \in \mathscr{Z} \forall S \in \mathscr{S}_-$). Thus $[\mathbf{M}_+, \mathbf{M}_-]$ is a matrix with $|\mathscr{Z}|$ rows and $|\mathscr{S}_+| + |\mathscr{S}_-|$ columns. Denote $\boldsymbol{p}$ for the column vector

$$\boldsymbol{p} := \left[(p_+(S))_{S \in \mathscr{S}_+}, (p_-(S))_{S \in \mathscr{S}_-} \right]^\intercal,$$

and $\boldsymbol{\alpha}$ for the column vector

$$\boldsymbol{\alpha} := \left[(\alpha(z))_{z \in \mathscr{Z}} \right]^\intercal.$$

Then (7) is equivalent to $[\mathbf{M}_+, \mathbf{M}_-]\, \boldsymbol{p} = \boldsymbol{\alpha}$, namely, with $\boldsymbol{\alpha}$ moved to the left-hand side,

$$[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]\, \boldsymbol{p} = \mathbf{0}.$$

Lemma 3 *For any $\alpha \in \mathbb{R}^{\mathscr{Z}}$, and any $\mathscr{S}_+, \mathscr{S}_- \subseteq 2^{\mathscr{Z}}$, with the associated matrices $\mathbf{M}_+$ and $\mathbf{M}_-$, if no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ can produce any nonnegative row whose entry at the $-\boldsymbol{\alpha}$ position is (strictly) positive, then there exist $p_+ : \mathscr{S}_+ \to \mathbb{R}_+$ and $p_- : \mathscr{S}_- \to \mathbb{R}_+$ that satisfy (7).*[8]

Proof As explained previously, the existence of the (p_+, p_-) specified thereof is equivalent to the existence of a nonnegative solution of $\boldsymbol{p}$ for $[\mathbf{M}_+, \mathbf{M}_-]\, \boldsymbol{p} = \boldsymbol{\alpha}$. By Theorem 2 of Chu et al. [8], such a nonnegative solution exists if the cover order of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ is less than or equal to the cover order of $[\mathbf{M}_+, \mathbf{M}_-]$. According to the procedure in their Section 4, the cover order of any matrix say $\mathbf{A}$ is equal to the maximum number of (strictly) positive

[8] Extend (p_+, p_-) to $2^{\mathscr{Z}}$ trivially by setting $p_+(S) := 0$ and $p_-(S') := 0$ for all $S \notin \mathscr{S}_+$ and $S' \notin \mathscr{S}_-$.

entries among all the nonnegative rows that any Gaussian elimination on $\mathbf{A}$ can produce. Since the only difference between $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ and $[\mathbf{M}_+, \mathbf{M}_-]$ is the $-\boldsymbol{\alpha}$ column, any nonnegative row produced by a Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ that has zero at the entry for $-\boldsymbol{\alpha}$ can be produced by the same Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-]$. Thus, the desired inequality between their cover orders follows from the hypothesis in the lemma. ■

To understand the intuition behind Lemma 3, consider two instances of (7) for interim states $z', z'' \in \mathscr{Z}$ such that $\alpha(z') < \alpha(z'')$. Subtract the instance of (7) when $z = z'$ by the instance of (7) when $z = z''$, so the left-hand side is a negative number. Then a contradiction occurs if the right-hand side is nonnegative, which occurs if $\chi_S(z') \geq \chi_S(z'')$ whenever $p_+(S) > 0$ ($S \in \mathscr{S}_+$) and $-\chi_S(z') \geq -\chi_S(z'')$ whenever $p_-(S) > 0$ ($S \in \mathscr{S}_-$). That is, the subtraction between the two instances of (7) produces a nonnegative row $\big[(\chi_S(z') - \chi_S(z''))_{S\in\mathscr{S}_+}, (-\chi_S(z') + \chi_S(z''))_{S\in\mathscr{S}_-}, -\alpha(z') - (-\alpha(z''))\big]$, with the last positive component signifying the contradictorily negative left-hand side $\alpha(z') - \alpha(z'')$. This contradictory case is ruled out by the hypothesis in the lemma, which also rules out any linear combination of similar subtractions that produces a contradiction to (7).

Thus comes a road map to obtain reduced-form characterizations:

1. For any linear valuation $\alpha \in \mathbb{R}^{\mathscr{Z}}$, find a solution q^* to the social planner's problem (8).

2. Derive from q^* some partial orders $\succeq_Z$ on $\mathscr{Z}$ that rationalizes $q^*|_Z$: For each $\succeq_Z$, q^* restricted to $Z \subseteq \mathscr{Z}$ follows a greedy-generous algorithm according to $\succeq_Z$. The upper or lower contour sets with respect to $\succeq_Z$ satisfy the universal binding condition.

3. Prove that (7) has a nonnegative solution for (p_+, p_-) such that p_+ is supported by the upper contour sets, and p_- supported by the lower contour sets.

4 Paramodularity

The method applies easily to the paramodularity model, where the social planner's solution reflects the ranking of the linear valuation α directly. For any $t \in T$ and any $E \subseteq I$, define

$$f_t(E) := \max_{x\in X_t} \sum_{i\in E} x_i, \tag{14}$$

$$g_t(E) := \min_{x\in X_t} \sum_{i\in E} x_i. \tag{15}$$

$$\begin{tikzcd} E & E \cup E' \\ E \cap E' \arrow[u] \arrow[r, "f : \geq", "g : \leq"'] & E \arrow[u] \end{tikzcd}$$

Figure 1: Submodularity of f and $-g$

A constraint structure $(X_t)_{t\in T}$ is said to be *paramodular* iff two conditions hold for any $t \in T$: (i) the pair (f_t, g_t) defined above completely characterizes X_t in the sense that

$$X_t = \left\{ (x_i)_{i\in I} \in \mathscr{R}^I \;\middle|\; \forall E \subseteq I \left[g_t(E) \leq \sum_{i\in E} x_i \leq f_t(E) \right] \right\}, \tag{16}$$

and (ii) (f_t, g_t) is *paramodular* on 2^I: first, f_t and $-g_t$ are each *submodular* in the sense that

$$\begin{aligned} f_t(E) - f_t(E \cap E') &\geq f_t(E \cup E') - f_t(E') \\ g_t(E) - g_t(E \cap E') &\leq g_t(E \cup E') - g_t(E') \end{aligned}$$

for all $E, E' \subseteq I$; second, (f_t, g_t) is *compliant* in the sense that, for all $E, E' \subseteq I$,

$$f_t(E') - f_t(E' \setminus E) \geq g_t(E) - g_t(E \setminus E').$$

Thus, the feasible allocation outcomes are defined by the upper and lower bounds of the total quantities for various sets of bidder-object pairs. The paramodularity assumption regulates these bounds so that their marginal changes are the binding constraints when we add an element to, or remove it from, a set of bidder-object pairs. Submodularity of f_t is a notion of diminishing marginal upper bounds, and submodularity of $-g_t$ a notion of increasing marginal lower bounds, with respect to set inclusion (Figure 1). Compliance of (f_t, g_t) is a notion of marginal upper bounds never falling below marginal lower bounds.[9]

The paramodular model includes as special cases single-unit auctions (I_2 is singleton, $f_t(E) = 1$ for all $\varnothing \neq E \subseteq I$ and $g_t \equiv 0$), multiunit auctions (I_2 is singleton, as in Che et al. [7]), and two-player bargainings (I_2 is singleton and, with I simplified to I_1, $g_t\{1,2\} = 1$, $g_t\{1\} = g\{2\} = 0$, and $f_t\{1,2\} = f_t\{1\} = f_t\{2\} = 1$; see Corollary 3, Online Supplement B.5). The following theorem includes these cases as well as the case of multiple heterogeneous objects and state-dependent allocation constraints.

[9] One can prove that compliance is equivalent to $e \in E \subseteq I \Rightarrow f_t((\neg E) \cup \{e\}) - f_t(\neg E) \geq g_t(E) - g_t(E \setminus \{e\})$.

Theorem 2 *If a constraint structure is paramodular, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.*

Proof For any $t \in T$ and any $S \subseteq \mathscr{Z}$ we have $f(S,t) = f_t(I(S,t))$ by (3) and (14), and $g(S,t) = g_t(I(S,t))$ by (4) and (15). From the definition of $I(S,t)$, it is easy to prove that $I(S \cup S', t) = I(S,t) \cup I(S',t)$ and $I(S \cap S', t) = I(S,t) \cap I(S',t)$ for all $S, S' \subseteq \mathscr{Z}$ and all $t \in T$. It then follows from the paramodularity of (f_t, g_t) on 2^I that the pair $(f(\cdot,t), g(\cdot,t))$ is paramodular on $2^{\mathscr{Z}}$. It also follows that the X_t in (16) is isomorphic to

$$\tilde{X}_t := \left\{ (x_z)_{z\in\mathscr{Z}} \in \mathscr{R}^{\mathscr{Z}} \;\middle|\; \forall S \subseteq \mathscr{Z} \left[g(S,t) \le \sum_{z\in S} x_z \le f(S,t) \right] \right\},$$

because for any $z = (i, t'_{i_1}) \in \mathscr{Z}$ such that $t'_{i_1} \neq t_{i_1}$, $I(\{z\},t) = \varnothing$ and hence $f(\{z\},t) = g(\{z\},t) = 0$, forcing $x_z = 0$.

Thus, for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$ and any $t := (t_{i_1})_{i_1\in I_1} \in T$, the social planner's problem is equivalent to

$$\max_{(x_z)_{z\in\mathscr{Z}} \in \mathrm{cv}\tilde{X}_t} \sum_{z\in\mathscr{Z}} x_z \alpha(z). \tag{17}$$

List all interim states as $\left(z^1, z^2, \cdots, z^{n_*}, \cdots, z^{|\mathscr{Z}|}\right)$ in descending order of $\alpha(z)$, so that

$$\alpha(z^1) \ge \alpha(z^2) \ge \cdots \ge \alpha(z^{n_*}) \ge 0 > \alpha(z^{n_*+1}) \ge \cdots \ge z(\alpha^{|\mathscr{Z}|}).$$

For any $n = 1, \ldots, |\mathscr{Z}|$, define

$$S^n := \begin{cases} \{z^k \mid 1 \le k \le n\} & \text{if } n \le n_* \\ \{z^k \mid n \le k \le |\mathscr{Z}|\} & \text{if } n \ge n_* + 1. \end{cases}$$

Since $\tilde{X}_t$ is defined by the paramodular pair $(f(\cdot,t), g(\cdot,t))$, problem (17) is solved by the greedy-generous solution (Hassin [15, Theorems 4 & 5])[10] $(x^*_z)_{z\in\mathscr{Z}} := (q_z(t))_{z\in\mathscr{Z}}$ such that

$$q^*_{z^n}(t) = \begin{cases} f\left(S^n, t\right) - f\left(S^{n-1}, t\right) & \text{if } n \le n_* \\ g\left(S^n, t\right) - g\left(S^{n+1}, t\right) & \text{if } n > n_* \end{cases}$$

for all $n \in \{1, \ldots, |\mathscr{Z}|\}$. For any $i \in I$ there is a unique n such that $(i, t_{i_1}) = z^n$. Apply the equation displayed above to $z^n = (i, t_{i_1})$ to obtain

$$q^*_{i,t_{i_1}}(t) = \begin{cases} f\left(S^n, t\right) - f\left(S^{n-1}, t\right) & \text{if } n \le n_* \\ g\left(S^n, t\right) - g\left(S^{n+1}, t\right) & \text{if } n > n_*, \end{cases}$$

[10] The compliance condition in Hassin's Theorem 4 is meant to be assumed for all subsets rather than only those related by set inclusion.

where $q^*_{i,t_{i_1}}(t)$ can be succinctly denoted by $q^*_i(t)$. This equation of $q^*_i(t)$, applied to all $t \in T$ and all $i \in I$, is the social planner's solution that Step 1 in our method needs.

By the definition of S^n, $\varnothing = S^0 \subsetneq S^1 \subsetneq S^2 \subsetneq \cdots \subsetneq S^{n_*}$, as the hierarchy $(U^n)_{n=1}^{n^*}$ in (10); and $S^{n_*+1} \supsetneq \cdots \supsetneq S^{|\mathscr{X}|-1} \supsetneq S^{|\mathscr{X}|} \supsetneq S^{|\mathscr{X}|+1} = \varnothing$, as the hierarchy $(L^n)_n$ in (11). Then the q^* obtained above satisfies (12) and (13), because $S^n \setminus S^{n-1} = \{z^n\}$ for all $n \leq n_*$, and $S^n \setminus S^{n+1} = \{z^n\}$ for all $n > n_*$. It then follows from Lemma 2 that S^n is upward universally binding for all $n \leq n_*$, and downward universally binding for all $n > n_*$.

Thus, by Theorem 1, the proof completes because a nonnegative solution of (7) exists: $p_+(S^n) = \alpha(z^n) - \alpha(z^{n+1})$ for all $n < n_*$, $p_+(S^{n_*}) = \alpha(z^{n_*})$, $p_-(S^{n_*+1}) = -\alpha(z^{n_*+1})$, and $p_-(S^n) = \alpha(z^{n-1}) - \alpha(z^n)$ for all $n > n_* + 1$. ■

We see from the proof that the greedy-generous algorithm is to raise the quantity for a positive-α interim state up to its marginal upper bound, and to reduce the quantity for a negative-α one down to its marginal lower bound. The paramodularity assumption is just to ensure that such a hierarchical solution is feasible and optimal. Thus the revealed preference of the planner's choice q^* presents itself as the ordinal ranking of the α-values. Consequently, it is trivial to solve (7) for a price system that Theorem 1 requires.

5 Decomposability

Lang and Yang [20] adopt a total unimodularity assumption in the combinatorial optimization literature (e.g., Edmonds [9], Frank et al. [10], Hoffman [17]) to handle some cases outside the paramodular model. From the perspective of our method, total unimodularity implies that any linear valuation α parametric to the social planner's problem can be decomposed into a conic combination of $\{-1, 0, 1\}$-valued extreme rays. That turns out to be a main drive to rationalize the social planner's solution with multiple preference relations, one for each of the $\{-1, 0, 1\}$-valued extreme rays, thereby achieving Step 2 in our method, the other steps trivial in this case. The outcome is a counterpart to Lang and Yang's characterization.

Consider constraint structures in the form of $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ where $\mathscr{F}, \mathscr{G} \subseteq 2^I$, $\hat{f} : \mathscr{F} \to \mathbb{Z}_+$, $\hat{g} : \mathscr{G} \to \mathbb{Z}_+$, $\hat{f} \geqq \hat{g}$ on $\mathscr{F} \cap \mathscr{G}$, $\varnothing \in \mathscr{F} \Rightarrow \hat{f}(\varnothing) = 0$, $\varnothing \in \mathscr{G} \Rightarrow \hat{g}(\varnothing) = 0$ and X_t is a nonempty compact set X constant to all $t \in T$:

$$X = \left\{ (x_i)_{i \in I} \in \mathbb{Z}^I \;\middle|\; \forall E \in \mathscr{F} \left[\sum_{i \in E} x_i \leq \hat{f}(E) \right]; \forall E \in \mathscr{G} \left[\sum_{i \in E} x_i \geq \hat{g}(E) \right] \right\}. \tag{18}$$

Given $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$, let $\mathcal{P}$ be the set of vectors $(\alpha, \varphi, \gamma) \in \mathbb{R}^{\mathscr{X}} \times \mathbb{R}_+^{\mathscr{F} \times T} \times \mathbb{R}_+^{\mathscr{G} \times T}$ such that

$$\forall i \in I \, \forall t \in T : \alpha(i, t_{i_1}) - \sum_{F \in \mathscr{F}} \varphi(F, t) \chi_F(i) + \sum_{G \in \mathscr{G}} \gamma(G, t) \chi_G(i) = 0. \tag{19}$$

We can think of $\varphi(F, t)$ as the Lagrange multiplier for the ceiling constraint $\sum_{i \in F} x_i \leq \hat{f}(F)$, and $\gamma(G, t)$ the Lagrange multiplier for the floor constraint $\sum_{i \in G} x_i \geq \hat{g}(G)$, in the social planner's problem specifically for the ex post state t. Note that these Lagrange multipliers are tailored individually for each specific ex post state t. By contrast, the price vector (p_+, p_-) sought after in our method needs to be uniform across all ex post states t.

Lang and Yang's *total unimodular* assumption stipulates that the matrix $[\mathbf{M}_1, \mathbf{M}_2]^\intercal$ for which $\mathcal{P} = \{\boldsymbol{v} \mid \mathbf{M}_1 \boldsymbol{v} = \mathbf{0}; \mathbf{M}_2 \boldsymbol{v} \geqq \mathbf{0}\}$ is totally unimodular, namely, the determinant of every square submatrix of every order is 0, 1, or -1. The main implication of this assumption is that every $(\alpha, \varphi, \gamma) \in \mathcal{P}$ is a conic combination of some $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ for which each α_k is $\{-1, 0, 1\}$-valued, and all φ_k and γ_k are $\{0, 1\}$-valued. Instead, the decomposability assumption I propose next requires only the elements of a subset of $\mathcal{P}$ to be conic combinations of some $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ for which only every α_k is required to be $\{-1, 0, 1\}$-valued.

A constraint structure $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ is said to be *decomposable* iff for any $\alpha \in \mathbb{R}^{\mathscr{X}}$ there exists a solution (φ^*, γ^*) to the problem

$$\begin{aligned} \min_{(\varphi, \gamma) \in \mathbb{R}_+^{\mathscr{F} \times T} \times \mathbb{R}_+^{\mathscr{G} \times T}} \quad & \textstyle\sum_{t \in T} \mu(t) \left(\sum_{F \in \mathscr{F}} \hat{f}(F) \varphi(F, t) - \sum_{G \in \mathscr{G}} \hat{g}(G) \gamma(G, t) \right) \\ & \text{subject to (19)} \end{aligned} \tag{20}$$

such that $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ is a conic combination among some $(\alpha_1, \varphi_1, \gamma_1), \ldots, (\alpha_K, \varphi_K, \gamma_K) \in \mathcal{P}$ (for some integer K) and

$$\forall k \in \{1, \ldots, K\} \, \forall i \in I \, \forall t_{i_1} \in T_{i_1} : \alpha_k(i, t_{i_1}) \in \{0, 1, -1\}. \tag{21}$$

One can verify that total unimodularity implies decomposability (Appendix A.5).

I also assume another implication of total unimodularity noted by Lang and Yang, that

$$\text{cv}X = \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \;\middle|\; \forall E \in \mathscr{F} \left[\sum_{i \in E} x_i \leq \hat{f}(E) \right]; \forall E \in \mathscr{G} \left[\sum_{i \in E} x_i \geq \hat{g}(E) \right] \right\}. \tag{22}$$

That is, the set of randomized ex post allocation outcomes is characterized by $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$.

The other assumption I adopt from Lang and Yang is linearity of a function defined by

$$h(F, G) := \max_{x \in \text{cv}X} \left(\sum_{i \in F} x_i - \sum_{i \in G} x_i \right) \tag{23}$$

for any $F, G \subseteq I$ such that $F \cap G = \varnothing$. The function h is said to be *linear* iff

$$h(F, G) = h(F, \varnothing) + h(\varnothing, G)$$

for all $F, G \subseteq I$ such that $F \cap G = \varnothing$.[11]

Theorem 3 *If the construct structure $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ is decomposable and satisfies (22), and if h is linear, then $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.*

Proof To apply Theorem 1, pick any $\alpha \in \mathbb{R}^{\mathscr{Z}}$. By the decomposability assumption, there exist $(\varphi^*, \gamma^*) \in \mathbb{R}_+^{\mathscr{F} \times T} \times \mathbb{R}_+^{\mathscr{G} \times T}$, $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ and $(\beta_k)_{k=1}^K \in \mathbb{R}_{++}^K$ for some integer K such that (φ^*, γ^*) is a solution to problem (20), $(\alpha_k, \varphi_k, \gamma_k) \in \mathcal{P}$ and satisfies (21) for any $k = 1, \ldots, K$, and $\alpha = \sum_{k=1}^K \beta_k \alpha_k$, $\varphi^* = \sum_{k=1}^K \beta_k \varphi_k$, and $\gamma^* = \sum_{k=1}^K \beta_k \gamma_k$. Now that (20) has a solution, its dual has a solution $q^* := (q_i^*)_{i \in I}$. By (22) and $\mu(t) > 0$ for all $t \in T$ ($|T| < \infty$), the dual is equivalent to the social planner's problem for all $t \in T$:

$$\max_{(q(t))_{t \in T} \in \prod_{t \in T} \mathrm{cv} X} \sum_{i \in I} q_i(t) \alpha(i, t_{i_1}).$$

Thus, q^* satisfies (8), which is what Step 1 in our method needs.

For any $k = 1, \ldots, K$, define

$$\begin{aligned} S^{k,+} &:= \{z \in \mathscr{Z} \mid \alpha_k(z) = 1\}, \\ S^{k,-} &:= \{z \in \mathscr{Z} \mid \alpha_k(z) = -1\}. \end{aligned}$$

Due to the decomposability and linearity assumptions, one can prove (Appendix A.4):

Lemma 4 *For every $k \in \{1, \ldots, K\}$ and every $t := (t_{i_1})_{i_1 \in I_1} \in T$,*

$$\begin{aligned} \sum_{i \in I} q_i^*(t) \chi_{S^{k,+}}(i, t_{i_1}) &= f\left(S^{k,+}, t\right), \\ \sum_{i \in I} q_i^*(t) \chi_{S^{k,-}}(i, t_{i_1}) &= g\left(S^{k,-}, t\right). \end{aligned}$$

[11] The linearity assumption is introduced by Lang and Yang [20]. Without the assumption, their main result is a condition that needs to be checked for any pair $(S_+, S_-) \in \left(2^{\mathscr{Z}}\right)^2$ of sets of interim states such that $S_+ \cap S_- = \varnothing$. By contrast, my characterization, the Border condition $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$, needs only to be checked for any set $S \in 2^{\mathscr{Z}}$ of interim states. As noted in their Corollary 1, their condition reduces to mine if the linearity is assumed. Thus the next Theorem 3 is slightly stronger than their main result in conclusions, and corresponds to a slight generalization of their Corollary 1, which also assumes linearity. Online Supplement B.3 has more on the difference between decomposability and total unimodularity.

For every $k \in \{1, \ldots, K\}$, let $\varnothing =: U^0 \subsetneq U^1 := S^{k,+}$, as the hierarchy (10), and $S^{k,-} =: L^1 \supsetneq L^0 := \varnothing$, as the hierarchy (11). Since $U^1 \setminus U^0 = U^1 = S^{k,+}$ and $L^1 \setminus L^0 = L^1 = S^{k,-}$, the two equations in Lemma 4 are the same as the (12) and (13) in Lemma 2. It then follows from Lemma 2 that $S^{k,+}$ is upward universally binding, and $S^{k,-}$ downward universally binding, for all $k = 1, \ldots, K$. Thus we obtain the supports for p_+ and p_- that Step 2 in our method needs. To achieve Step 3 in the method, simply use the conic combination condition

$$\forall i \in I \, \forall t_{i_1} \in T_{i_1} : \alpha(i, t_{i_1}) \quad = \quad \sum_{k=1}^{K} \beta_k \left(\chi_{S^{k,+}}(i, t_{i_1}) - \chi_{S^{k,-}}(i, t_{i_1}) \right)$$

to obtain a solution for (7): $p_+(S^{k,+}) = p_-(S^{k,-}) = \beta_k$ for all k and $p_+(S) = p_-(S) = 0$ for all other subsets S of $\mathscr{X}$. Thus, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ follows from Theorem 1. ∎

As said earlier, the drive in the decomposability assumption is to decompose any linear valuation for the social planner's problem into a conic combination of $\{-1, 0, 1\}$-valued valuations. That means the social planner's choice can be rationalized by multiple preference relations among the interim states, each partitioning the interim states into only three indifference sets, the good (those z with $\alpha_k(z) = 1$), the bad (those with $\alpha_k(z) = -1$), and the neutral ($\alpha_k(z) = 0$). They give the upper or lower contour sets that our method needs.

6 Assignments

It is known that the paramodular model cannot cover assignment problems, due to the unit-demand assignment constraint (Online Supplement B.4). While the total unimodular model allows for such constraint structures, its application is restricted to at most two possible types per bidder.[12] This section considers an assignment model between $N \geq 2$ objects and two bidders such that each bidder can have any number of types, Our method applies nontrivially. The support for the price function is derived from multiple partial orders, each rationalizing the social planner's solution locally. The existence proof of the price function relies on the hyper-rectangle cover theory as in Lemma 3.

[12] See Lang and Yang's [20] Theorem 2 for the application of total unimodularity to assignment models, which allows for any number of bidders. Although one can show that the decomposability assumption, an implication of and substitute for total unimodularity, includes paramodularity as a special case and hence allows for more than two types per bidder (Online Supplement B.3), it is unknown whether its application beyond paramodularity can do so.

Let the set of bidders be $I_1 := \{1, 2\}$, and the set of objects $I_2 := \{1, \ldots, N\}$, $N \geq 2$. There is exactly one unit for each object, indivisible. An allocation outcome is in the form of $((x_{kj})_{k=1}^{2})_{j=1}^{N} \in \{0, 1\}^{2N}$, signifying bidder k getting object j.[13] The ex post constraint consists of the unit-demand constraint[14]

$$\forall k \in I_1 : \sum_{j \in I_2} x_{kj} = 1 \tag{24}$$

and the resource-feasibility constraint

$$\forall j \in I_2 : \sum_{k \in I_1} x_{kj} \leq 1. \tag{25}$$

It is convenient to represent an allocation outcome $((x_{kj})_{k=1}^{2})_{j=1}^{N}$ equivalently as a set $M \subseteq I$ $(= I_1 \times I_2)$ such that $(k, j) \in M \iff x_{k,j} = 1$ (note $x_{k,j} \neq 1 \Rightarrow x_{k,j} = 0$). Then the ex post feasibility constraint is equivalent to the condition that M has exactly two elements and

$$M \supseteq \{(k, j), (k', j')\} \Longrightarrow [k \neq k', j \neq j']\,. \tag{26}$$

And the set X of feasible allocation outcomes is equivalent to the set $\mathscr{M}$ of two-element subsets M of I that satisfy (26).

Theorem 4 *In the assignment model such that* $|I_1| = 2$, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.

Social Planner's Solution The proof of Theorem 4 follows the road map in Section 3. For Step 1, pick any $\alpha \in \mathbb{R}^{\mathscr{X}}$ and consider the social planner's problem for any ex post state $t := (t_1, t_2) \in T$:

$$\max_{x \in \mathrm{cv} X} \sum_{(k,j) \in I} x_{k,j} \alpha(k, j, t_k) = \max_{x \in X} \sum_{(k,j) \in I} x_{k,j} \alpha(k, j, t_k) = \max_{M \in \mathscr{M}} \sum_{(k,j) \in M} \alpha(k, j, t_k),$$

where the first equality is due to the linear programming with X finite, and the second equality due to the definition of $\mathscr{M}$. Obviously the problem is solved by coupling the first- or second-highest $\alpha(1, j, t_1)$ among $j \in I_2$ with the first- or second-highest $\alpha(2, j, t_2)$ among

[13] In what follows the index for bidders and that for objects will play distinct roles. Thus bidders will often be indexed by k, and objects by j.

[14] When (24) is weakened to $\sum_{j \in I_2} x_{kj} \leq 1$ for each bidder $k \in I_1$, namely, that each bidder can get at most one object and may get none, Online Supplement B.1 obtains the same result as Theorem 4 in this section by a similar proof.

$j \in I_2$ such that the two have different j-coordinates. In the next table (with three objects), the entry at row j and column (k, t_k) signifies $\alpha(k, j, t_k)$. The social planner's solution is $\{(1, 2), (2, 1)\}$ (giving good 2 to bidder 1, and good 1 to bidder 2) thereby generating total α-value $4 + 3 = 7$ when the ex post state is (t_1, t_2), and $\{(1, 3), (2, 2)\}$ (giving good 3 to bidder 1, and good 2 to bidder 2) generating total α-value $2+3 = 5$ when the state is (t_1, t'_2).

	$(1, t_1)$	$(2, t_2)$	$(1, t_1)$	$(2, t'_2)$
1	-1	3	-1	$1/2$
2	4	0	4	3
3	2	$1/2$	2	0

To define a solution to the social planner's problem in general, let

$$\begin{aligned} j^1(k, t_k) &:= \min\left(\arg\max_{j \in I_2} \alpha(k, j, t_k)\right) \\ j^2(k, t_k) &:= \min\left(\arg\max_{j \in I_2 \setminus \{j^1(k,t_k)\}} \alpha(k, j, t_k)\right) \end{aligned}$$

for any $t := (t_1, t_2) \in T$ and any $k \in \{1, 2\}$. For any $j \in I_2$ $(= \{1, \ldots, N\})$, let

$$\delta(k, j, t_k) := \alpha(k, j, t_k) - \max_{j' \in I_2 \setminus \{j\}} \alpha(k, j', t_k). \tag{27}$$

Define $M_*(t_1, t_2) \in \mathscr{M}$ by:

a. if $j^1(1, t_1) \neq j^1(2, t_2)$, let $M_*(t_1, t_2) := \{(1, j^1(1, t_1)), (2, j^1(2, t_2))\}$;

b. else $(j^1(1, t_1) = j^1(2, t_2))$ then:

   i. if $\delta(1, j^1(1, t_1), t_1) \geq \delta(2, j^1(1, t_1), t_2)$ $(= \delta(2, j^1(2, t_2), t_2))$, let

$$M_*(t_1, t_2) := \left\{(1, j^1(1, t_1)), (2, j^2(2, t_2))\right\};$$

   ii. else $(\delta(1, j^1(1, t_1), t_1) < \delta(2, j^1(1, t_1), t_2))$, let

$$M_*(t_1, t_2) := \left\{(1, j^2(1, t_1)), (2, j^1(2, t_2))\right\}.$$

In other words, if the highest $\alpha(1, j, t_1)$ among all j and the highest $\alpha(2, j, t_2)$ among all j are of different objects, couple them as the solution (Case (a)). Otherwise (Case (b)), they refer to the same object and so coupling them is infeasible with respect to (26); thus the solution is either coupling the highest $\alpha(1, j, t_1)$ with the second highest $\alpha(2, j, t_2)$, or coupling the second highest $\alpha(1, j, t_1)$ with the highest $\alpha(2, j, t_2)$, whichever is larger in total α-values, and break the tie by favoring bidder 1. To see that case (b) is exactly what is said, note from (27) and the definition of j^2 that $\delta(1, j^1(1, t_1), t_1) \geq \delta(2, j^1(2, t_2), t_2)$ is equivalent to

$$\alpha(1, j^1(1, t_1), t_1) + \alpha(2, j^2(2, t_2), t_2) \geq \alpha(2, j^1(2, t_2), t_2) + \alpha(1, j^2(1, t_1), t_1).$$

It is clear that $M_*(t)$ is a solution to the social planner's problem for every $t \in T$ and hence M_* corresponds to the q^* that Step 1 in our method needs.

Partial Revealed Preferences Step 2 is to construct a set of partial revealed preferences of the social planner's solution that is rich enough to cover every interim state. Every interim state (k, j, t_k) competes with the other interim states that refer to the same object j (due to the feasibility condition (25)) and with those that refer to the same bidder-type (k, t_k) (condition (24)). We construct a partial order for each "row" consisting of the interim states referring to a same object j, and each "column" referring to a same bidder-type (k, t_k).

For any $k \in I_1$ and any $t_k \in T_k$, both the top contender $j^1(k, t_k)$ and the second highest contender $j^2(k, t_k)$ in the column $\{k\} \times I_2 \times \{t_k\}$ may get coupled with some $(-k, j')$ to be the solution in $M_*(t_k, t_{-k})$ for some t_{-k}, while the other elements of the column have no chance by the definition of M_*. Thus define a binary relation $\succeq_{k,t_k}$ by:

i. $j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k)$;

ii. for any $\{j, j'\} \neq \{j^2(k, t_k), j^1(k, t_k)\}$, let $j \succ_{k,t_k} j'$ if and only if

$$\alpha(k, j, t_k) > \alpha(k, j', t_k) \text{ or } [\alpha(k, j, t_k) = \alpha(k, j', t_k) \text{ and } j < j'].$$

List the elements of I_2 ($|I_2| = N$) as

$$j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k) \succ_{k,t_k} j^3 \succ_{k,t_k} j^4 \succ_{k,t_k} \cdots \succ_{k,t_k} j^N. \tag{28}$$

Correspondingly, let $(k, j, t_k) \succ_{k,t_k} (k, j', t_k) \iff j \succ_{k,t_k} j'$, and likewise for $\sim_{k,t_k}$. Define:

$$\begin{aligned} V^n_{k,t_k} &:= \{k\} \times \left(\{j^1(k, t_k), j^2(k, t_k)\} \cup \{j^m \mid 3 \leq m \leq n\}\right) \times \{t_k\} \quad (\forall n = 2, \ldots, N) \\ L_{k,t_k} &:= \{k\} \times I_2 \times \{t_k\}. \end{aligned}$$

One can verify that V^n_{k,t_k} is upward universally binding, and L_{k,t_k} downward universally binding, for any $k \in I_1$, any $t_k \in T_k$ and any $n \in \{2, \ldots, N\}$ (Lemma 8, Appendix A.6).

For any $j \in I_2$, among the interim states that refer to object j, the rivalry is only between top contenders, namely, between elements of

$$\mathscr{Z}_j := \{(k, j, t_k) \mid k \in I_1; t_k \in T_k; j = j^1(k, t_k)\},$$

because for any ex post state (t_k, t_{-k}), if (k, j, t_k) is the top contender from its column while the other $(-k, j, t_{-k})$ is not, the top contender is chosen by $M_*(t_k, t_{-k})$ to be paired with a top contender that refers to a different object than j. Define a binary relation $\succ_j$ on $\mathscr{Z}_j$ by: for any $(k, j, t_k), (k', j, t'_{k'}) \in \mathscr{Z}_j$, let $(k, j, t_k) \succ_j (k', j, t'_{k'})$ if and only if:

i. either $\delta(k, j, t_k) > \delta(k', j, t'_{k'})$

ii. or $\delta(k, j, t_k) = \delta(k', j, t'_{k'})$ and $k < k'$ (i.e., $k = 1$ and $k' = 2$)

iii. or $\delta(k, j, t_k) = \delta(k', j, t'_{k'})$ and $k = k'$ and $t_k \triangleright t'_k$ ($\triangleright$ being a strict total order on T_k).

Conditions (i) and (ii), together with the definition of M_*, guarantee that $(k, j, t_k) \succ_j (-k, j, t_{-k})$ is equivalent to $(k, j) \in M_*(t_k, t_{-k})$ for any $(k, j, t_k), (-k, j, t_{-k}) \in \mathscr{Z}_j$. Condition (iii) is to break the tie between two interim states that refer to the same bidder k. Although (k, j, t_k) and (k, j, t'_k) are not in rivalry, as any ex post state t'' is compatible with at most one of them, we rank them so that $\succ_j$ is a total order on $\mathscr{Z}_j$.

Enumerate the elements of $\mathscr{Z}_j$ in descending order of $\succ_j$ so that

$$z^1 \succ_j z^2 \succ_j \cdots \succ_j z^{|\mathscr{Z}_j|}. \tag{29}$$

Correspondingly, for every $n = 1, \ldots, |\mathscr{Z}_j|$ define the upper contour set of z^n as

$$U_j^n := \{z^m \in \mathscr{Z}_j \mid 1 \leq m \leq n\}.$$

One can verify that U_j^n is upward universally binding for any $j \in I_2$ and any $n = 1, \ldots, |\mathscr{Z}_j|$ (Lemma 9, Appendix A.6).

In Figure 2, the interim state (k, j^1, t_k) is the top contender within its column. Thus it is contained in all the upper contour sets V_{k,t_k}^m in the column. As a top contender, (k, j^1, t_k) belongs to the row $\mathscr{Z}_j$ for which $j = j^1$. Within $\mathscr{Z}_j$, it ranks third, labeled as z^3. Thus it is not contained in the upper contour sets U_j^1 or U_j^2, but it is in the other U_j^n's in that row. In Figure 3, by contrast, (k, j^3, t_k) is not a top contender, but rather ranks third in its column. Thus it is contained only by the V_{k,t_k}^n for which $n \geq 3$. Regardless of their ranks in the column, however, all the members (k, j^n, t_k) in the column are contained by L_{k,t_k}, as it is defined to be the entire column.

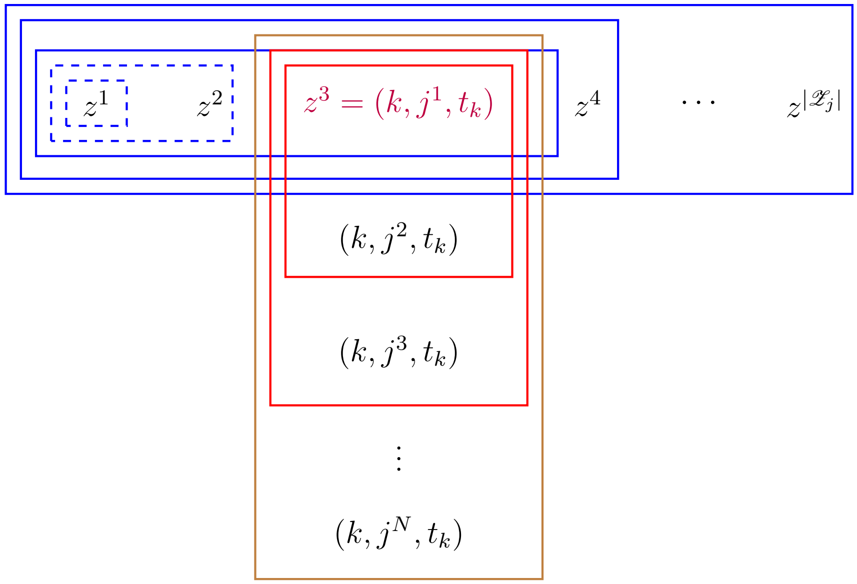

Figure 2: Solid-line boxes: the U_j^n (blue) or V_{k,t_k}^m (red or brown) that contain z^3; dotted-line boxes: upper contour sets that do not contain z^3; the brown box is also L_{k,t_k}

Existence of the Price Function Following Step 3 in our road map, let

$$\begin{aligned}\mathscr{S}_+ &:= \left\{U_j^n \mid j \in I_2; n = 1, \ldots, |\mathscr{Z}_j|\right\} \bigcup \left\{V_{k,t_k}^n \mid k \in I_1; t_k \in T_k; n = 2, \ldots, N\right\} \\ \mathscr{S}_- &:= \left\{L_{k,t_k} \mid k \in I_1; t_k \in T_k\right\}.\end{aligned}$$

We prove existence of price vectors p_+ and p_- supported by $\mathscr{S}_+$ and $\mathscr{S}_-$ respectively. According to Lemma 3, let $[\mathbf{M}_+, \mathbf{M}_-]$ be the matrix defined there with respect to the $\mathscr{S}_+$ and $\mathscr{S}_-$ here. For any $z \in \mathscr{Z}$, denote $[z]$ for the row of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ corresponding to z, so $[z](S)$ is the entry in the row at the intersection with column S, and $[z](-\boldsymbol{\alpha}) := -\alpha(z)$.[15]

By Lemma 3, to verify the condition required in Theorem 1 it suffices to prove that no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ can produce any nonnegative row whose entry at the $-\boldsymbol{\alpha}$ position is positive. Since any Gaussian elimination on a matrix corresponds to a linear combination of its rows, it suffices to prove that there exist no $Z \subseteq \mathscr{Z}$ and $(\beta_z)_{z \in Z} \in (\mathbb{R} \setminus \{0\})^Z$ for which

$$\sum_{z \in Z} \beta_z [z](S) \geq 0 \quad \forall S \in \mathscr{S}_+ \sqcup \mathscr{S}_- \quad \text{and} \tag{30}$$

$$\sum_{z \in Z} \beta_z \alpha(z) < 0. \tag{31}$$

[15] The rows and columns in the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ are not to be confused with the "rows" $\mathscr{Z}_j$ and "columns" $\{k\} \times I_2 \times \{t_k\}$ such as those in Figures 2 and 3.

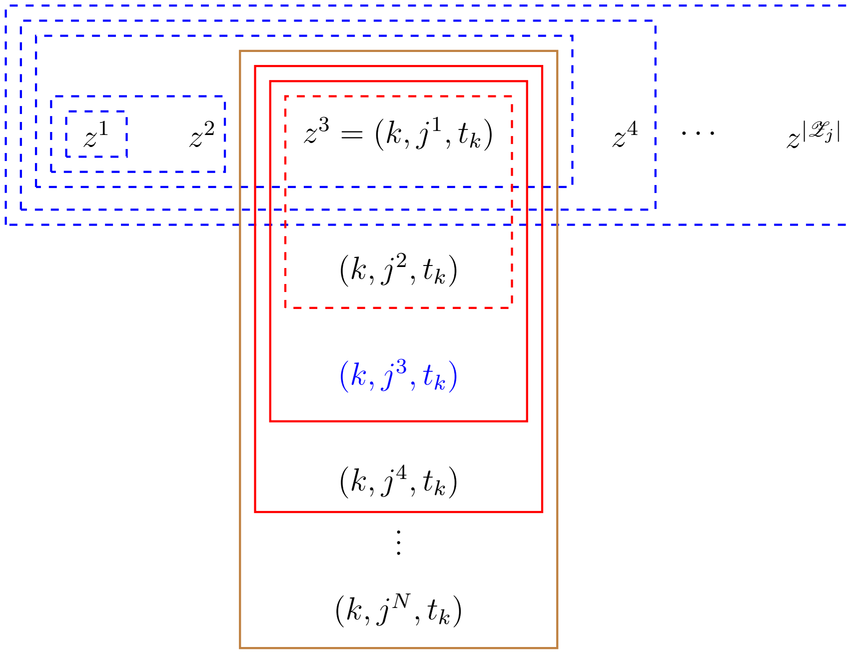

Figure 3: Solid-line boxes: the upper or lower contour sets that contain (k, j^3, t_k)

Intuitively speaking, suppose for now that the negative outcome (31) could be achieved in one single operation. That would involve subtracting a row $[z]$ in $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ from another row $[z']$ for which $\alpha(z) > \alpha(z')$. Meanwhile, for the operation to satisfy the non-negativity condition (30), for every $S \in \mathscr{S}_+ \sqcup \mathscr{S}_-$ the entry $[z](S)$ in row $[z]$ has to be no greater than the corresponding entry $[z'](S)$ in row $[z']$. By the definitions of $\mathbf{M}_-$, that requires z and z' to be in the same "column" $\{k\} \times I_2 \times \{t_k\}$. Otherwise, the negative entry $[z'](L_{k',t'_{k'}})$, for whatever $\{k'\} \times I_2 \times \{t'_{k'}\}$ that contains z', cannot be balanced by $[z]$, because $[z](L_{k',t'_{k'}}) = 0$ by the definition of $L_{k',t'_{k'}}$. Now that they belong to the same $\{k\} \times I_2 \times \{t_k\}$, within which the upper contour sets are nested (cf. Figures 2 and 3), either $z' \succ_{k,t_k} z$ (so that any V^m_{k,t_k} that contains z also contains z') or $z \sim_{k,t_k} z'$ (so z and z' are contained by the same family of V^m_{k,t_k}'s). Since $\alpha(z) > \alpha(z')$, $z' \succ_{k,t_k} z$ is impossible. Thus the only possibility is $z \sim_{k,t_k} z'$. That is, z is the top contender in $\{k\} \times I_2 \times \{t_k\}$, and z' the second-highest one therein. Being on the top (like the z^3 in Figure 2), z belongs to some upper contour set U^n_j of the top rivals in $\mathscr{Z}_j$. But since the operation is $[z'] - [z]$, it produces a row whose entry at the U^n_j position is negative, $-[z](U^n_j) = -1$, as $[z'](U^n_j) = 0$. Therefore, to maintain the nonnegativity condition (30), we need to add to $[z'] - [z]$ another row $[z'_*]$ in $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ so that $[z'_*](U^n_j) \geq [z](U^n_j)$. That means $z'_* \succ_j z$, since the upper contour sets in $\mathscr{Z}_j$ are nested. Then the definition of $\succ_j$ implies that $\delta(z'_*) \geq \delta(z)$. In other words, the α-value differential of z'_*, compared to its highest rival say z_*, is no less than the α-value differential of z compared to its highest rival, which is z'. Thus, even if we minimize the positive contribution of z'_* by subtracting from $[z'_*]$ its highest rival $[z_*]$, the "difference of differences" $[z'_*] - [z_*] - ([z] - [z'])$ would still have a nonnegative net value of α, to the opposite of (31).

Rearranging Gaussian Eliminations The formal argument of the above is just to decompose the sum in (30) into a conic combination of such "differences of differences" quadruples so that (31) cannot hold. The first step is to rearrange the sum in (30) into a conic combination of the differences $[z'] - [z]$ such that z and z' refer to the same (k, t_k).

Lemma 5 *For any $\varnothing \neq Z \subseteq \mathscr{Z}$ and any $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$ that satisfy (30), there exist a finite set H and a positive vector $(\tilde{\beta}_h)_{h\in H} \in \mathbb{R}^H_{++}$ for which*

$$\sum_{z\in Z} \beta_z[z] = \sum_{h\in H} \tilde{\beta}_h \left([z'_h] - [z_h]\right) \tag{32}$$

such that for every $h \in H$ there exist $k \in I_1$ and $t_k \in T_k$ for which $z_h = (k, j, t_k)$ and $z'_h = (k, j', t_k)$ for some $j, j' \in I_2$ such that $j' \neq j$ and $j' \succeq_{k,t_k} j$ ($\succeq_{k,t_k}$: $\succ_{k,t_k}$ or $\sim_{k,t_k}$).

The nonnegativity condition (30) guarantees that for any $[z]$ that enters a Gaussian operation negatively (i.e., $\beta_z < 0$ in the sum $\sum_{r\in Z} \beta_r[r]$) there is some $[z']$ that enters the operation positively. The proof of Lemma 5 (Appendix A.7) is to show that such z and z' can be matched in pairs as in the right-hand side of (32). It is obvious that any matched pair needs to refer to the same bidder-type say (k, t_k). Otherwise, if z' does not belong to the $\{k\} \times I_2 \times \{t_k\}$ that contains z then $[z']$ cannot balance the negative entry $-[z](V^n_{k,t_k})$ in $-[z]$ for any V^n_{k,t_k} that contains z. The question is how to match the z and z' within the same $\{k\} \times I_2 \times \{t_k\}$. To sketch the idea, suppose for now that $|\beta_{z''}| = 1$ for all the rows $[z'']$ that refer to the same (k, t_k). Then

$$\sum_{z\in Z\cap(\{k\}\times I_2\times\{t_k\})} \beta_z[z] = [z'_1] + [z'_2] + \cdots + [z'_r] - [z_1] - [z_2] - \cdots - [z_r]$$

for some integer r, where the rows are labeled according to the ranking (28) so that $z_1 \succeq_{k,t_k} z_2 \succeq_{k,t_k} \cdots \succeq_{k,t_k} z_r$ and $z'_1 \succeq_{k,t_k} z'_2 \succeq_{k,t_k} \cdots \succeq_{k,t_k} z'_r$. Since $V^2_{k,t_k} \subsetneq V^3_{k,t_k} \subsetneq \cdots \subsetneq V^N_{k,t_k}$,

$$\begin{aligned} \psi_+(n) &:= [z'_1(V^n_{k,t_k})] + [z'_2](V^n_{k,t_k}) + \cdots + [z'_r](V^n_{k,t_k}) \quad \text{and} \\ \psi_-(n) &:= [z_1(V^n_{k,t_k})] + [z_2](V^n_{k,t_k}) + \cdots + [z_r](V^n_{k,t_k}) \end{aligned}$$

are each a nondecreasing step function of n. By the nonnegativity condition, $\psi_+ \geqq \psi_-$. Consequently, $z'_r \succeq_{k,t_k} z_r$. Otherwise there exists V^n_{k,t_k} that contains z_r and not z'_r; then $[z_m](V^n_{k,t_k}) = 1$ for all $m = 1, \ldots, r$ while $[z'_r](V^n_{k,t_k}) = 0$; thus

$$\psi_-(n) = r > r - 1 \geq [z'_1(V^n_{k,t_k})] + [z'_2](V^n_{k,t_k}) + \cdots + [z'_{r-1}](V^n_{k,t_k}) = \psi_+(n),$$

contradiction. Furthermore, $[z'_r]$ should be paired with $[z_r]$. To see why, consider the case

$$z'_{r-1} \succ_{k,t_k} z_{r-1} \succ_{k,t_k} z'_r \succ_{k,t_k} z_r.$$

If $[z'_{r-1}]$ instead of $[z'_r]$ is paired with $[z_r]$, we need to find a match with $[z_{r-1}]$. it cannot be $[z'_r]$, as it cannot balance $[z_{r-1}]$ since $z_{r-1} \succ_{k,t_k} z'_r$ (which means $z_{r-1} \in V^m_{k,t_k} \not\ni z'_r$ for some m and so $([z'_r] - [z_{r-1}])(V^m_{k,t_k}) = 0 - 1 = -1$). If we match $[z_{r-1}]$ with some element ranked before $[z'_{r-1}]$, we may have problem matching $[z_{r-2}]$ and so on. Thus, $[z_r]$ should be paired with $[z'_r]$. Note that $\psi_+(n) - [z'_r](V^n_{k,t_k})$ and $\psi_-(n) - [z_r](V^n_{k,t_k})$, analogous to $\psi_+(n)$ and $\psi_-(n)$, are each nondecreasing in n and $\psi_+(n) - [z'_r](V^n_{k,t_k}) \geq \psi_-(n) - [z_r](V^n_{k,t_k})$. Thus we can recursively pair $[z_{r-1}]$ with $[z'_{r-1}]$, pair $[z_{r-2}]$ with $[z'_{r-2}]$, and so forth.

In the general case where the weights β_z in the sum $\sum_{z \in Z} \beta_z[z]$ are not identical in absolute value, divide any term $\beta_z[z]$ with a larger weight into smaller portions. For example, suppose z_r remains to be the $\succeq_{k,t_k}$-minimum among those z that enter the sum negatively, and let z'_s be the $\succeq_{k,t_k}$-minimum among those z' that enter the sum positively and $z' \succeq_{k,t_k} z_r$. Suppose $\beta_{z'_s} = 2$ and $\beta_{z_r} = -5$. Then split the term $-5[z_r]$ in the sum into $-2[z_r]$ and $-3[z_r]$. Pair $-2[z_r]$ with $2[z'_s]$ (so 2 corresponds to a $\tilde{\beta}_h$ on the right-hand side of (32)). Then repeat the process recursively on the remainder of the sum after $2([z'_s] - [z_r])$ is removed, treating the residual $-3[z_r]$ as a separate term in the remainder. Eventually all the weight of $[z_r]$ is eliminated in this manner, then the set of rows that enter the remainder negatively is reduced by one element. Continue the process until all such "negative" rows are eliminated.

The second step of showing the incompatibility between (30) and (31) is to rearrange the right-hand side of (32), a conic combination of differences, into a conic combination of "differences of differences." For any $j \in I_2$ and given the set H obtained by Lemma 5, define

$$H_j := \{h \in H \mid z_h \in \mathscr{Z}_j \text{ or } z'_h \in \mathscr{Z}_j\}.$$

Note: $H = \bigsqcup_{j \in I_2} H_j$. That is because for any (k, t_k) there is at most one $j \in I_2$ for which $j = j^1(k, t_k)$ (by the definition of j^1). Since z_h and z'_h refer to the same (k, t_k) (Lemma 5), it is impossible for $z_h \in \mathscr{Z}_j$ and $z'_h \in \mathscr{Z}_{j'}$ for some $j, j' \in I_2$. Thus, the sum in (32) is equal to $\sum_{j\in I_2} \sum_{h\in H_j} \tilde{\beta}_h ([z'_h] - [z_h])$. The next lemma is to reorganize each $\sum_{h\in H_j} \tilde{\beta}_h ([z'_h] - [z_h])$.

Lemma 6 *Assume the conclusion of Lemma 5. Then for any $j \in I_2$ there exist $W \subset (H_j)^2$ and $H_* \subseteq H_j$ and a nonnegative vector $(\widehat{\beta}_w)_{w\in W\sqcup H_*} \in \mathbb{R}_+^{W\sqcup H_*}$ for which*

$$\sum_{h\in H_j} \tilde{\beta}_h ([z'_h] - [z_h]) = \sum_{(w,w_*)\in W} \widehat{\beta}_w \left([z'_{w_*}] - [z_{w_*}] - ([z_w] - [z'_w])\right) + \sum_{h\in H_*} \widehat{\beta}_h ([z'_h] - [z_h]) \quad (33)$$

such that $\{z'_{w_*}, z_w\} \subseteq \mathscr{Z}_j$ and $z'_{w_*} \succeq_j z_w$ for any $(w, w_*) \in W$, and $z_h \notin \mathscr{Z}_j$ for any $h \in H_*$.

The proof of Lemma 6 (Appendix A.7) parallels that of Lemma 5 (Appendix A.7), where the role of $\succ_{k,t_k}$ is played by the $\succ_j$ here. In the conic combination $\sum_{h\in H_j} \tilde{\beta}_h ([z'_h] - [z_h])$ of differences, a difference $[z'_h]-[z_h]$ has a negative weight iff $([z'_h]-[z_h])(U_j^m) = -[z_h](U_j^m) = -1$ for some upper contour set U_j^n among the top contenders that refer to the same object j (which means $z_h \in U_j^n$ is the top contender among the interim states that refer to the same bidder-type as z_h does). Lemma 6 is to balance this negative weight by some other differences say $[z'_{h_*}] - [z_{h_*}]$ that enter the conic combination with positive weights in the sense that $([z'_{h_*}] - [z_{h_*}])(U_j^m) = [z'_{h_*}](U_j^m) = 1$ (which means that z'_{h_*} is a top contender referring to the same object j as z_h does and ranks before z_h). The nonnegativity condition (30) guarantees existence of such positively weighted differences in the conic combination. Just like Lemma 5, Lemma 6 pairs every negatively weighted difference with an appropriately chosen positively weighted difference so that the negative entry $([z'_h]-[z_h])(U_j^m) = -1$ in the former is canceled out by the positive entry $([z'_{h_*}] - [z_{h_*}])(U_j^m) = 1$ in the latter. These "differences of differences" are collected in the first sum on the right-hand side of (33). Those positively weighted differences that do not get to be used in the pairing procedure are collected in the second sum on the right-hand side. Eq. (33) parallels the Eq. (40) that constitutes the gist of the proof of Lemma 5.

Proof of Theorem 4 Sum (32) across all $j \in I_2$ and then use (33) to have

$$\sum_{z\in Z} \beta_z [z] = \sum_{(w,w_*)\in W} \widehat{\beta}_w \left([z'_{w_*}] - [z_{w_*}] + [z'_w] - [z_w]\right) + \sum_{h\in H_*} \widehat{\beta}_h ([z'_h] - [z_h]).$$

Applied to the $-\boldsymbol{\alpha}$ component, this equation becomes

$$\sum_{z\in Z} \beta_z \alpha(z) = \sum_{(w,w_*)\in W} \widehat{\beta}_w \left(\alpha(z'_{w_*}) - \alpha(z_{w_*}) + \alpha(z'_w) - \alpha(z_w)\right) + \sum_{h\in H_*} \widehat{\beta}_h (\alpha(z'_h) - \alpha(z_h)).$$

On the right-hand side, for each pair (z'_s, z_s) (s being w_*, w or h), we know by Lemma 5 that z'_s and z_s are distinct and both refer to the same bidder-type say (k, t_k), and that $z'_s \succeq_{k,t_k} z_s$. Thus, unless $z'_s \sim_{k,t_k} z_s$ and hence z_s is the top contender among those referring to (k, t_k) (namely, unless $z_s \in \mathscr{Z}_j$ for some $j \in I_2$), we have $\alpha(z'_s) - \alpha(z_s) \geq 0$. If this inequality holds for every s on the right-hand side of the above-displayed equation, then (31) cannot hold, and we are done. Thus suppose that $z_s \in \mathscr{Z}_j$ for some s and some j. This s can only be the w in some $(w, w_*) \in W$, because Lemma 6 says that $z_h \notin \mathscr{Z}_j$ for all $h \in H_*$, and that $z_{w_*} \notin \mathscr{Z}_j$ (as $z'_{w_*} \in \mathscr{Z}_j$) for the w_* in any $(w, w_*) \in W$. Consequently, it suffices to consider any $(w, w_*) \in W$ for which $z_w \in \mathscr{Z}_j$ and prove

$$\alpha(z'_{w_*}) - \alpha(z_{w_*}) + \alpha(z'_w) - \alpha(z_w) \geq 0.$$

To prove that, note from $z'_{w_*} \succeq_j z_w$ (Lemma 6) and the definition of $\succeq_j$ that $\delta(z_w) \leq \delta(z'_{w_*})$. Then the definition of δ implies

$$\alpha(z_w) - \max_{j' \in I_2 \setminus \{j\}} \alpha(k, j', t_k) \leq \alpha(z'_{w_*}) - \max_{j' \in I_2 \setminus \{j\}} \alpha(k^*, j', t^*_{k^*}) \leq \alpha(z'_{w_*}) - \alpha(z_{w_*}),$$

where (k, t_k) denotes the bidder-type that z_w refers to, $(k^*, t^*_{k_*})$ the bidder-type that z'_{w_*} refers to, and the second inequality follows from the fact $z_{w_*} \notin \mathscr{Z}_j$ noted previously. Meanwhile,

$$\alpha(z'_w) = \max_{j' \in I_2 \setminus \{j\}} \alpha(k, j', t_k)$$

because z'_w is the second highest contender in $\{k\} \times I_2 \times \{t_k\}$, as $z_w \in \mathscr{Z}_j$ in the current case under consideration while $z'_w \succeq_{k,t_k} z_w$ and $z'_w \neq z_w$ by Lemma 5. Then the desired inequality $\alpha(z_w) - \alpha(z'_w) \leq \alpha(z'_{w_*}) - \alpha(z_{w_*})$ follows, which completes the proof of Theorem 4.

7 Conclusion

Characterizing reduced-form allocations has been an important area in mechanism design. The frontier of the literature calls for a method to tackle the combinatorial complications among multiple objects, exemplified by the assignment problems between bidders with non-trivial type spaces. This paper contributes to this frontier with a novel, unifying method. The main received result, based on the paramodularity model, and the counterpart to a contemporary result that goes beyond that model, turn out to be easy applications of the method. Following the method I obtain a new result, the exact characterization of the reduced forms in the assignment problems between $N \geq 2$ objects and two bidders with an arbitrary number of types per bidder, whether or not full assignment is required as part of the feasibility constraint. With the exact characterization of reduced forms available, the approach of optimal mechanism design, which was largely unavailable to the assignment literature until now, is at hand for a nontrivial set of these problems.

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A Delayed Details

A.1 Necessity of the Border Condition

Lemma 7 $\mathscr{Q}_{\mathrm{B}} \supseteq \mathscr{Q}$.

Proof For any $Q := (Q_i(t_{i_1}))_{(i,t_{i_1})\in\mathscr{X}} \in \mathbb{R}^{\mathscr{X}}$ and $\alpha := (\alpha(i,t_{i_1}))_{(i,t_{i_1})\in\mathscr{X}} \in \mathbb{R}^{\mathscr{X}}$, denote

$$\langle Q, \alpha\rangle := \sum_{i\in I}\sum_{t_{i_1}\in T_{i_1}} Q_i(t_{i_1})\alpha(i,t_{i_1})\mu_{i_1}(t_{i_1}). \tag{34}$$

Then $\langle Q, \alpha\rangle \leq \sup_{Q'\in\mathscr{Q}}\langle Q', \alpha\rangle$ for any $Q \in \mathscr{Q}$. This inequality implies $Q \in \mathscr{Q}_{\mathrm{B}}$, namely, Q satisfies the Border condition (5): Pick any $S \subseteq \mathscr{X}$. To obtain the second inequality in (5), apply $\langle Q, \alpha\rangle \leq \sup_{Q'\in\mathscr{Q}}\langle Q', \varphi\rangle$ to the case where $\alpha = \chi_S$. For any $Q' \in \mathscr{Q}$, Q' satisfies (1) with respect to some ex post allocation q'. Thus

$$\begin{aligned}
\sup_{Q'\in\mathscr{Q}}\langle Q', \chi_S\rangle &= \sup_{(q'(t))_{t\in T}\in\prod_{t\in T}\mathrm{cv}X_t}\sum_{i\in I}\sum_{t_{i_1}\in T_{i_1}}\sum_{t_{-i_1}\in T_{-i_1}} q_i'(t_{i_1},t_{-i_1})\chi_S(i,t_{i_1})\mu_{-i_1}(t_{-i_1}|t_{i_1})\mu_{i_1}(t_{i_1}) \\
&= \sum_{t\in T}\sup_{q'(t)\in\mathrm{cv}X_t}\sum_{i\in I} q_i'(t))\chi_S(i,t_{i_1})\mu(t) \\
&= \sum_{t\in T}\sup_{q'(t)\in X_t}\sum_{i\in I} q_i'(t))\chi_S(i,t_{i_1})\mu(t) \\
&\leq \sum_{t\in T} f(S,t)\mu(t),
\end{aligned}$$

with the third line due to the fact that X_t contains all extremal points of its convex hull, and the last line due to (3). Thus the second inequality in (5) follows. The first inequality in (5) is analogous via $\alpha := -\chi_S$. Thus $Q \in \mathscr{Q}_{\mathrm{B}}$. ■

A.2 Proof of Lemma 1

By the definition of $\mathscr{Q}$ and (1), the right-hand side of (6) is

$$\begin{aligned}
\sum_{t\in T}\mu(t)\max_{(q_i(t))_{i\in I}\in\mathrm{cv}X_t}\sum_{i\in I} q_i(t)\alpha(i,t_{i_1}) &= \max_{((q_i(t))_{i\in I})_{t\in T}\in\prod_{t\in T}(\mathrm{cv}X_t)}\sum_{t\in T}\sum_{i\in I} q_i(t)\alpha(i,t_{i_1})\mu(t) \\
&= \max_{Q\in\mathscr{Q}}\sum_{i\in I}\sum_{t_{i_1}\in T_{i_1}}\alpha(i,t_{i_1})Q_i(t_{i_1})\mu_{i_1}(t_{i_1}).
\end{aligned}$$

Then, with the notation defined in (34), (6) is equivalent to

$$\max_{Q\in\mathscr{Q}_{\mathrm{B}}}\langle Q, \alpha\rangle \leq \max_{Q\in\mathscr{Q}}\langle Q, \alpha\rangle.$$

Note that the set $\mathscr{Q}$ of reduced forms is convex and compact, since the set of ex post allocation is convex and compact (each assigning to each state t an element of the convex hull of the compact X_t) and the mapping from ex post allocations to their reduced forms, Eq. (1), is linear and continuous. It then follows from a finite-dimensional application of the separating hyperplane theorem (Theorem 7.51 of Aliprantis and Border [1, p288]) that

$$\mathscr{Q} = \left\{ \bar{Q} \in \mathbb{R}^{\mathscr{Z}} \,\middle|\, \forall \alpha \in \mathbb{R}^{\mathscr{Z}} \left[\langle \bar{Q}, \alpha \rangle \leq \max_{Q \in \mathscr{Q}} \langle Q, \alpha \rangle \right] \right\}.$$

Claim: $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if (6) holds for all $\alpha \in \mathbb{R}^{\mathscr{Z}}$. Clearly $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ implies (6) for all $\alpha \in \mathbb{R}^{\mathscr{Z}}$. To prove the converse, suppose that (6) is true for all $\alpha \in \mathbb{R}^{\mathscr{Z}}$. If $\bar{Q} \notin \mathscr{Q}$ then, by the equation displayed above, there exists an α given which $\langle \bar{Q}, \alpha \rangle > \max_{Q \in \mathscr{Q}} \langle Q, \alpha \rangle$. Then $\langle \bar{Q}, \alpha \rangle > \max_{Q \in \mathscr{Q}_{\mathrm{B}}} \langle Q, \alpha \rangle$ by (6). Thus $\bar{Q} \notin \mathscr{Q}_{\mathrm{B}}$. ■

A.3 Proof of Theorem 1

By the definition of $\mathscr{Q}_{\mathrm{B}}$ (condition (5)), the left-hand side of (6) is equivalent to

$$\begin{aligned}
\max_{(Q_i)_{i \in I} \in \mathbb{R}^I} \quad & \sum_{i \in I} \sum_{t_{i_1} \in T_{i_1}} \mu_{i_1}(t_{i_1}) Q_i(t_{i_1}) \alpha(i, t_{i_1}) \\
\forall S \subseteq \mathscr{Z} : \quad & \sum_{i \in I} \sum_{i_1 \in T_{i_1}} \mu_{i_1}(t_{i_1}) Q_i(t_{i_1}) \chi_S(i, t_{i_1}) \leq \sum_{t \in T} \mu(t) f(S, t) \\
& \sum_{i \in I} \sum_{i_1 \in T_{i_1}} \mu_{i_1}(t_{i_1}) Q_i(t_{i_1}) \chi_S(i, t_{i_1}) \geq \sum_{t \in T} \mu(t) g(S, t).
\end{aligned}$$

Treat the $Q_i(t_{i_1})\mu_{i_1}(t_{i_1})$ in this problem as a choice variable to obtain its dual:

$$\begin{aligned}
\min_{(p_+, p_-): 2^{\mathscr{Z}} \to \mathbb{R}^2_+} \quad & \textstyle\sum_{t \in T} \mu(t) \sum_{S \subseteq \mathscr{Z}} \left(p_+(S) f(S, t) - p_-(S) g(S, t) \right) \\
\forall z \in \mathscr{Z} \quad & \alpha(z) = \textstyle\sum_{S \subseteq \mathscr{Z}} \left(p_+(S) - p_-(S) \right) \chi_S(z).
\end{aligned}$$

Thus, it follows from Lemma 1 that $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$ there exists $(p_+, p_-) : 2^{\mathscr{Z}} \to \mathbb{R}^2_+$ for which (7) holds for all $z \in \mathscr{Z}$, and

$$\sum_{t \in T} \mu(t) \sum_{S \subseteq \mathscr{Z}} \left(p_+(S) f(S, t) - p_-(S) g(S, t) \right) \leq \sum_{t \in T} \mu(t) \max_{(q_i(t))_{i \in I} \in \mathrm{cv} X_t} \sum_{i \in I} q_i(t) \alpha(i, t_{i_1}). \tag{35}$$

The right-hand side of this inequality is equal to

$$\sum_{t \in T} \mu(t) \sum_{i \in I} q_i^*(t) \alpha(i, t_{i_1})$$

for any q^* that satisfies (8). Thus, (35) is equivalent to

$$\sum_{t\in T}\mu(t)\sum_{S\subseteq\mathscr{X}}(p_+(S)f(S,t)-p_-(S)g(S,t))\leq\sum_{t\in T}\mu(t)\sum_{i\in I}q_i^*(t)\alpha(i,t_{i_1}).$$

Plug (7) into the right-hand side of this inequality to rewrite the inequality as

$$\begin{aligned}&\sum_{t\in T}\mu(t)\sum_{S\subseteq\mathscr{X}}(p_+(S)f(S,t)-p_-(S)g(S,t))\\ \leq\;&\sum_{t\in T}\mu(t)\sum_{i\in I}q_i^*(t)\left(\sum_{S\subseteq\mathscr{X}}p_+(S)\chi_S(i,t_{i_1})-\sum_{S\subseteq\mathscr{X}}p_-(S)\chi_S(i,t_{i_1})\right).\end{aligned}$$

Rearrange terms to rewrite this inequality as

$$\begin{array}{r}\sum_{S\subseteq\mathscr{X}}p_+(S)\sum_{t\in T}\mu(t)\left(f(S,t)-\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1})\right)\\ +\sum_{S\subseteq\mathscr{X}}p_-(S)\sum_{t\in T}\mu(t)\left(-g(S,t)+\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1})\right)\end{array}\leq 0. \tag{36}$$

Thus, the existence of (p_+,p_-) that satisfies (7) and (35) is equivalent to the existence of (p_+,p_-) that satisfies (7) and (36) for some q^* that satisfies (8). For any $S\subseteq\mathscr{X}$ and $t\in T$,

$$f(S,t)=\max_{x\in X_t}\sum_{i\in I(S,t)}x_i=\max_{x\in\mathrm{cv}X_t}\sum_{i\in I(S,t)}x_i\geq\sum_{i\in I(S,t)}q_i^*(t)=\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1}),$$

with the first "=" due to (3), the second "=" due to X_t containing all extremal points of $\mathrm{cv}X_t$, the inequality due to $q_i^*(t)\in\mathrm{cv}X_t$, and the last "=" due to the definition of $I(S,t)$. By the same token,

$$-g(S,t)+\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1})\geq 0$$

for all $S\subseteq\mathscr{X}$ and all $t\in T$. This, coupled with the assumption that $\mu(t)>0$ for all $t\in T$, implies that (36) holds if and only if (9) holds for all $S\subseteq\mathscr{X}$. ∎

A.4 Proof of Lemma 4 in Theorem 3

Since (φ^*,γ^*) is a solution to (20) and q^* a solution to the dual thereof, (q^*,φ^*,γ^*) is a saddle point of the Lagrangian

$$\begin{aligned}L(q,\varphi,\gamma)\;:=\;&\sum_{i\in I}\sum_{t\in T}\alpha(i,t_{i_1})q_i(t)\mu(t)+\sum_{F\in\mathscr{F}}\sum_{t\in T}\varphi(F,t)\left(\hat{f}(F)-\sum_{i\in F}q_i(t)\right)\mu(t)\\ &+\sum_{G\in\mathscr{G}}\sum_{t\in T}\gamma(G,t)\left(\sum_{i\in F}q_i(t)-\hat{g}(G)\right)\mu(t).\end{aligned}$$

For any $t \in T$, $k \in \{1, \ldots, K\}$, $q(t) \in \mathrm{cv}X$, $\varphi_k(\cdot, t) \in \mathbb{R}_+^{\mathscr{F}}$ and any $\gamma(\cdot, t) \in \mathbb{R}_+^{\mathscr{G}}$, define

$$\begin{aligned} L_k^t(q(t), \varphi(\cdot, t), \gamma(\cdot, t)) \;&:=\; \sum_{i \in I} \alpha_k(i, t_{i_1}) q_i(t) \mu(t) + \sum_{F \in \mathscr{F}} \varphi_k(F, t) \left(\hat{f}(F) - \sum_{i \in F} q_i(t) \right) \mu(t) \\ &\quad + \sum_{G \in \mathscr{G}} \gamma_k(G, t) \left(\sum_{i \in G} q_i(t) - \hat{g}(G) \right) \mu(t). \end{aligned}$$

Claim: For any $k = 1, \ldots, K$ and any $t \in T$, $(q^*(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ is a saddle point of $L_k^t(q(t), \varphi(\cdot, t), \gamma(\cdot, t))$. First, since $(\alpha_k, \varphi_k^*, \gamma_k^*)$ belongs to $\mathcal{P}$ and hence satisfies (19), $q^*(t)$ maximizes $L_k(q(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ over all $q(t) \in \mathbb{R}^I$. Second, note that

$$L\left(q, \sum_k \beta_k \varphi_k, \sum_k \beta_k \gamma_k\right) = \sum_{k=1}^{K} \beta_k \sum_{t \in T} L_k^t(q(t), \varphi_k(\cdot, t), \gamma_k(\cdot, t)).$$

Since $(q^*, \varphi^*, \gamma^*)$ is a saddle point of L, L is minimized by $(\sum_k \beta_k \varphi_k^*, \sum_k \beta_k \gamma_k^*)$ given $q = q^*$, as $\varphi^* = \sum_k \beta_k \varphi_k^*$ and $\gamma^* = \sum_k \beta_k \gamma_k^*$. Consequently, since $\beta_k > 0$ for all k and $(\varphi_k(\cdot, t), \gamma_k(\cdot, t))$ does not enter $L_{k'}^{t'}$ for any $(k', t') \neq (k, t)$, $(\varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ minimizes L_k^t given $q(t) = q^*(t)$, for any k and any t. Thus, $(q^*(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ is a saddle point of L_k^t, as claimed.

By the claim proved above, for any $k = 1, \ldots, K$ and any $t \in T$, $q^*(t)$ solves

$$\begin{aligned} \max_{(q_i(t))_{i \in I} \in \mathbb{R}^I} \quad & \sum_{i \in I} \alpha_k(i, t_{i_1}) q_i(t) \mu(t) \\ \text{s.t.} \quad & \left(\hat{f}(F) - \sum_{i \in F} q_i(t) \right) \mu(t) \geq 0 \quad (\forall F \in \mathscr{F}) \\ & \left(\sum_{i \in G} q_i(t) - \hat{g}(G) \right) \mu(t) \geq 0 \quad (\forall G \in \mathscr{G}), \end{aligned}$$

because L_k^t is the Lagrangian associated to this problem. This, combined with $\mu(t) > 0$ and (22), implies that for any $k = 1, \ldots, K$ and any $t \in T$, $q^*(t)$ solves $\max_{x \in \mathrm{cv}X} \sum_{i \in I} \alpha_k(i, t_{i_1}) x_i$.

For any $k = 1, \ldots, K$ and any $t := (t_{i_1})_{i_1 \in I_1} \in T$, since α_k is $\{0, 1, -1\}$-valued by (21),

$$\max_{x \in \mathrm{cv}X_t} \sum_{i \in I} \alpha_k(i, t_{i_1}) x_i = \max_{x \in \mathrm{cv}X_t} \left(\sum_{i:\, \alpha_k(i, t_{i_1}) = 1} x_i - \sum_{i:\, \alpha_k(i, t_{i_1}) = -1} x_i \right) = h\left(I(S^{k,+}, t), I(S^{k,-}, t) \right),$$

where the last "=" uses (23), which applies because $I(S^{k,+}, t) \cap I(S^{k,-}, t) = \varnothing$ by the definitions of $S^{k,+}$ and $S^{k,-}$. Since h is assumed linear,

$$\begin{aligned} h\left(I(S^{k,+}, t), I(S^{k,-}, t) \right) \;&=\; \max_{x \in \mathrm{cv}X} \sum_{i \in I(S^{k,+}, t)} x_i - \min_{x \in \mathrm{cv}X} \sum_{i \in I(S^{k,-}, t)} x_i \\ &=\; \max_{x \in X} \sum_{i \in I(S^{k,+}, t)} x_i - \min_{x \in X} \sum_{i \in I(S^{k,-}, t)} x_i, \end{aligned}$$

where the first line is due to linearity of h and (23), and the second line due to the fact that X contains all extremal points of its convex hull. Let $(x_i^+)_{i\in I}$ be a solution to $\max_{x\in X}\sum_{i\in I(S^{k,+},t)} x_i$, and $(x_i^-)_{i\in I}$ a solution to $\min_{x\in X}\sum_{i\in I(S^{k,-},t)} x_i$. Since $(q_i^*(t))_{i\in I}$ is a solution to $\max_{x\in \mathrm{cv}X_t}\sum_{i\in I}\alpha_k(i,t_{i_1})x_i$, the two formulas displayed above together yield

$$\sum_{i\in I(S^{k,+},t)} q_i^*(t) - \sum_{i\in I(S^{k,-},t)} q_i^*(t) = \sum_{i\in I(S^{k,+},t)} x_i^+ - \sum_{i\in I(S^{k,-},t)} x_i^-.$$

Since $I(S^{k,+},t)\cap I(S^{k,-},t) = \varnothing$, the above equation implies that $q_i^*(t) = x_i^+$ for all $i \in I(S^{k,+},t)$, and $q_i^*(t) = x_i^-$ for all $i\in I(S^{k,-},t)$. Thus,

$$\begin{aligned}\sum_{i\in I} q_i^*(t)\chi_{S^{k,+}}(i,t_{i_1}) &= \max_{x\in X_t}\textstyle\sum_{i\in I\left(S^{k,+},t\right)} x_i &= f\left(S^{k,+},t\right),\\ \sum_{i\in I} q_i^*(t)\chi_{S^{k,-}}(i,t_{i_1}) &= \min_{x\in X_t}\textstyle\sum_{i\in I\left(S^{k,-},t\right)} x_i &= g\left(S^{k,-},t\right),\end{aligned}$$

where the last "=" on each line uses the notation defined in (3) and (4). ■

A.5 Total Unimodularity Implies Decomposability

Remark 1 *For any constraint structure $(\mathscr{F},\mathscr{G},\hat f,\hat g)$ such that $|T|<\infty$, if it satisfies total unimodularity then it is decomposable.*

Proof Let $\alpha\in\mathbb{R}^{\mathscr{L}}$. Since X is assumed nonempty and compact, so is $\mathrm{cv}X$. Thus the social planner's problem has a finite optimum, and hence its dual, problem (20), has a solution (φ^*,γ^*). It follows that $(\alpha,\varphi^*,\gamma^*)\in\mathcal{P}$. By total unimodularity, every element of $\mathcal{P}$ is a conic combination of some $\{(\alpha_k,\varphi_k,\gamma_k)\mid k=1,\ldots,K\}\subset\mathcal{P}$ for some integer K such that (21) is true and the ranges of both φ_k and γ_k are $\{0,1\}$ (Lang and Yang's [20] Lemma 13, or Hoffman's [17] Lemma 3.1). Thus, $(\alpha,\varphi^*,\gamma^*)$ is the conic combination of $\{(\alpha_k,\varphi_k,\gamma_k)\mid k=1,\ldots,K\}\subset\mathcal{P}$ satisfying (21), and hence decomposability is satisfied. ■

A.6 The Universal Binding Property of U_j^n, V_{k,t_k}^n and L_{k,t_k}

Lemma 8 *For any $k\in I_1$, $t_k\in T_k$ and $n\in\{2,\ldots,N\}$, V_{k,t_k}^n is upward universally binding, and L_{k,t_k} downward universally binding, in the full assignment model.*

Proof By the definition of V_{k,t_k}^n and L_{k,t_k}, we have $\varnothing =: V_{k,t_k}^1 \subsetneq V_{k,t_k}^2 \subsetneq V_{k,t_k}^3 \subsetneq \cdots \subsetneq V_{k,t_k}^N$ and $L_{k,t_k}\supsetneq L_{k,t_k}^0 := \varnothing$, as in the (10) and (11) in Lemma 2. To verify (12) and (13) there, combine (3), (4) with (24) to obtain for any $t' \in T$ and any $t_k \in T_k$:

$$[\varnothing \neq S \subseteq \{k\} \times I_2 \times \{t_k\}, t'_k = t_k] \Longrightarrow f(S, t') = 1 \tag{37}$$

$$S \subsetneq \{k\} \times I_2 \times \{t_k\} \Longrightarrow g(S, t') = 0 \tag{38}$$

$$t'_k = t_k \Longrightarrow g(\{k\} \times I_2 \times \{t_k\}, t') = 1. \tag{39}$$

It suffices (12) and (13) to consider all $t' \in T$ such that $t'_k = t_k$, because if $t'_k \neq t_k$ then $I(\cdot, t') = \varnothing$ and (12) and (13) are trivial. Thus, pick any $t' \in T$ for which $t'_k = t_k$. For any $n > 2$, $V^n_{k,t_k} \setminus V^{n-1}_{k,t_k} = \{(k, j^n, t_k)\}$. The social planner's solution M_* always assigns zero quantity to (k, j^n) because $n > 2$. That is, $q^*_{k,j^n}(t') = 0$. Thus

$$\sum_{(k,j,t'_k) \in V^n_{k,t_k} \setminus V^{n-1}_{k,t_k}} q^*_{k,j}(t') = q^*_{k,j^n}(t') = 0 = f(V^n_{k,t_k}, t') - f(V^{n-1}_{k,t_k}, t'),$$

with the third "=" due to (37). For $n = 2$, $V^2_{k,t_k} \setminus V^1_{k,t_k} = \{(k, j^1(k, t_k), t_k), (k, j^2(k, t_k), t_k)\}$, and hence by the definition of M_*

$$\sum_{(k,j,t'_k) \in V^2_{k,t_k} \setminus V^1_{k,t_k}} q^*_{k,j}(t') = \sum_{j \in \{j^1(k,t_k), j^2(k,t_k)\}} q^*_{k,j}(t') = 1,$$

which is equal to $f(V^2_{k,t_k}, t') - f(V^1_{k,t_k}, t')$ by (37) and $V^1_{k,t_k} = \varnothing$. Thus (12) is verified. To verify (13), note that $L_{k,t_k} \setminus L^0_{k,t_k} = \{k\} \times I_2 \times \{t_k\}$. Thus

$$\sum_{(k,j,t'_k) \in L_{k,t_k} \setminus L^0_{k,t_k}} q^*_{k,j}(t') = \sum_{j \in I_2} q^*_{k,j}(t') = 1$$

by the definition of M_*. Meanwhile,

$$g(L_{k,t_k}, t') - g(L^0_{k,t_k}, t') = 1 - 0 = 1,$$

with the first "=" due to (38) and (39). Thus (13) is satisfied. ■

Lemma 9 *For any $j \in I_2$ and any $n = 1, \ldots, |\mathscr{Z}_j|$, U^n_j is upward universally binding in the full assignment model.*

Proof The definition of U^n_j implies $\varnothing =: U^0 \subsetneq U^1_j \subsetneq \cdots \subsetneq U^{|\mathscr{Z}_j|}_j$, as the (10) in Lemma 2. To verify (12) there, pick any $n \in \{1, \ldots, |\mathscr{Z}_j|\}$, so $\{z^n\} = U^n \setminus U^{n-1}$. Pick any $t := (t_1, t_2) \in T$. To avoid triviality, suppose that $z^n = (k, j, t_k)$ (so $(k, j) \in I(\{z^n\}, t)$). First consider the case where there exists $k < n$ for which $z^k = (k', j, t_{k'})$ (so $(k', j) \in I(U^k, t)$). Since $z^n \neq z^k$, we have $k' \neq k$ and hence $k' = -k$. The premise $k < n$ (namely $z^k \succ_j z^n$), together with $k' = -k$, implies that $(k', j) \in M_*(t_1, t_2)$ (and hence $(k, j) \notin M_*(t_1, t_2)$). Thus, $q^*_{k,j}(t_1, t_2) = 0$. Meanwhile, since both $I(U^{n-1}, t)$ and $I(U^n, t)$ are nonempty (each containing $(-k, j)$),

$$f(U^n, t) - f(U^{n-1}, t) = 1 - 1 = 0,$$

with the first "=" due to (25) coupled with (3). Thus (12) is satisfied in this case. In the other case, there is no $k < n$ for which $z^k = (k', j, t_{k'})$, then $I(U^{n-1}, t) = \varnothing$ and hence $f(U^{n-1}, t) = 0$. Whereas, $I(U^n, t) = \{(k, j)\}$ since $z^n \in U^n$, hence $f(U^n, t) = 1$ by (3) and (25). Thus

$$f(U^n, t) - f(U^{n-1}, t) = 1 - 0 = 1.$$

Meanwhile, since there is no $(-k, j, t_{-k}) \succ_j (k, j, t_k)$ in this case, and $j = j^1(k, t_k)$ by the definition of $\mathscr{Z}_j$, we have $(k, j) \in M_*(t_1, t_2)$ by the definition of M_*. That is, $q^*_{k,j}(t_1, t_2) = 1$. Thus (12) is satisfied in both cases. ■

A.7 Proofs of Lemmas 5 and 6 for Theorem 4

Lemma 5 It suffices to prove the existence of a $Z_* \subseteq Z$ and a $(\tilde{\beta}_{z'})_{z' \in Z_*} \in \mathbb{R}_+^{Z_*}$ for which

$$\sum_{r \in Z} \beta_r [r] = \sum_{h \in H} \tilde{\beta}_h \left([z'_h] - [z_h] \right) + \sum_{z' \in Z_*} \tilde{\beta}_{z'} [z'] \tag{40}$$

such that H, $\tilde{\beta}_h$, z'_h and z_h are specified in the lemma. That suffices because every $z' \in Z_* \subseteq Z$ belongs to $\{k\} \times I_2 \times \{t_k\}$, and hence L_{k,t_k}, for some bidder-type (k, t_k). Thus $[z'](L_{k,t_k}) = -1$ by the definition of the matrix $\mathbf{M}_-$. Since $\tilde{\beta}_{z'} \geq 0$, this negative entry cannot be eliminated in the sum $\sum_{z' \in Z_*} \tilde{\beta}_{z'} [z']$. Neither can it be eliminated in the other sum $\sum_{h \in H} \tilde{\beta}_h \left([z'_h] - [z_h] \right)$, because as specified by the lemma, for every $h \in H$, z'_h and z_h refer to the same bidder-type and thus $([z'_h] - [z_h])(L_{k,t_k}) = 0$. Consequently, (30) implies that $Z_* = \varnothing$, and thus (40) reduces to (32).

The rest of the proof is to prove (40). For any $k \in I_1$ and any $t_k \in T_k$, define

$$Z_{k,t_k} := \{ z \in Z \mid z = (k, j, t_k) \text{ for some } j \in I_2 \} \quad (= Z \cap (\{k\} \times I_2 \times \{t_k\})).$$

Note that Z can be partitioned into the family of disjoint subsets Z_{k,t_k} across k and across t_k. Thus, it suffices to prove (32) in the case where Z is replaced by Z_{k,t_k} for any $k \in I_1$ and any $t_k \in T_k$ such that $Z_{k,t_k} \neq \varnothing$. The rest of the proof therefore assumes $Z = Z_{k,t_k}$.

By the definitions of V^n_{k,t_k}, L_{k,t_k}, and the matrix $[\mathbf{M}_+, \mathbf{M}_-]$, if $z \notin \{k\} \times I_2 \times \{t_k\}$ then $[z](V^n_{k,t_k}) = [z](L_{k,t_k}) = 0$ for all n. It then follows from (30) that

$$\left[S = V^n_{k,t_k} \text{ or } S = L_{k,t_k}\right] \Longrightarrow \sum_{z \in Z_{k,t_k}} \beta_z [z](S) \geq 0. \tag{41}$$

Define

$$\begin{aligned} Z^- &:= \{z \in Z_{k,t_k} \mid \beta_z < 0\} \\ Z^+ &:= Z_{k,t_k} \setminus Z^- \; (= \{z \in Z_{k,t_k} \mid \beta_z > 0\}). \end{aligned}$$

If $Z^- = \varnothing$ then (40) holds trivially, with $Z_* := Z^+$, $H := \varnothing$ and $\tilde{\beta}_{z'} := \beta_{z'}$ for all $z' \in Z_*$. Thus, let $Z^- \neq \varnothing$. For any $n \in \{2, 3, \ldots, N\}$, define

$$\begin{aligned} \psi_+(n) &:= \sum_{z \in Z^+} \beta_z [z](V^n_{k,t_k}) \\ \psi_-(n) &:= \sum_{z \in Z^-} |\beta_z| [z](V^n_{k,t_k}). \end{aligned}$$

By (41), $\psi_+ \geqq \psi_-$ on $\{2, 3, \ldots, N\}$.

To prove (32), construct the index set H recursively. Start with the $\succ_{k,t_k}$-minimum within Z^- and denote it by z_1. Let n_1 be the rank of z_1 in the list (28), so that z_1 refers to object j^{n_1} if $n_1 > 2$, and either objects $j^1(k, t_k)$ or $j^2(k, t_k)$ if $n_1 = 2$.[16] Observe that there exists $z' \in Z^+$ for which $z' \succeq_{k,t_k} z_1$. Otherwise, as $(V^n_{k,t_k})_{n=2}^N$ is nested,

$$\psi_+(n_1) = 0 < |\beta_{z_1}| = |\beta_{z_1}| [z_1](V^{n_1}_{k,t_k}) \leq \psi_-(n_1),$$

contradicting $\psi_+ \geqq \psi_-$. Thus, let z'_1 be the $\succeq_{k,t_k}$-minimum in $\{z' \in Z^+ \mid z' \succeq_{k,t_k} z_1\}$. Let

$$\tilde{\beta}_1 := \min\left\{|\beta_{z_1}|, |\beta_{z'_1}|\right\} \; \left(= \min\left\{-\beta_{z_1}, \beta_{z'_1}\right\}\right). \tag{42}$$

Claim: For any $n \in \{2, 3, \ldots, N\}$,

$$\psi_+(n) - \tilde{\beta}_1 [z'_1](V^n_{k,t_k}) \geq \psi_-(n) - \tilde{\beta}_1 [z_1](V^n_{k,t_k}). \tag{43}$$

This is illustrated by Figure 4. Formally, let n'_1 be the rank of z'_1 in the list (28), so that z'_1 refers to object $j^{n'_1}$ if $n'_1 > 2$, and either objects $j^1(k, t_k)$ or $j^2(k, t_k)$ if $n'_1 = 2$.

[16] The elements of Z are indexed in reverse order of $\succ_{k,t_k}$ here (for notation convenience), opposite to the indexing used in the heuristics for this lemma in the main text.

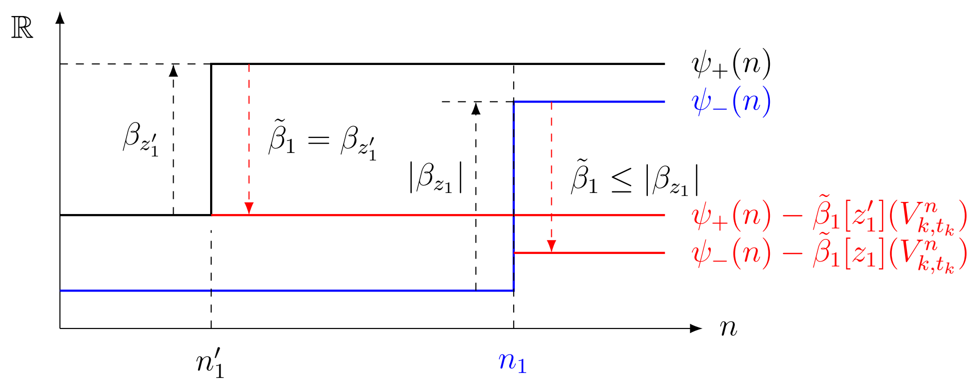

Figure 4: Shaving ψ_+ and ψ_- by the same drop $\tilde{\beta}_1$

Since $z_1' \succeq_{k,t_k} z_1$, $n_1' \leq n_1$. Then $\psi_+ \geqq \psi_-$ directly implies (43) for all $n < n_1'$ (because $[z_1'](V^n_{k,t_k}) = 0 = [z_1](V^n_{k,t_k})$) and for all $n \geq n_1$ (because $[z_1'](V^n_{k,t_k}) = 1 = [z_1](V^n_{k,t_k})$). For any $n_1' \leq n < n_1$,

$$\psi_-(n) - \tilde{\beta}_1[z_1](V^n_{k,t_k}) = \psi_-(n) \quad \leq \quad \psi_-(n_1) - |\beta_{z_1}| \leq \psi_-(n_1) - \tilde{\beta}_1 \leq \psi_+(n_1) - \tilde{\beta}_1 = \psi_+(n) - \tilde{\beta}_1,$$

where the "=" to the right is because $n_1' \leq n < n_1$ and z_1' is the $\succeq_{k,t_k}$-minimum in $\{z' \in Z^+ \mid z' \succeq_{k,t_k} z_1\}$ (so ψ_+ has no jump point between n_1' and n_1 except n_1'). Thus (43) follows.

Note that $\sum_{r \in Z_{k,t_k}} \beta_r[r]$ is equal to

$$\tilde{\beta}_1 \left([z_1'] - [z_1]\right) + \sum_{r \in Z^+ \setminus \{z_1'\}} \beta_r[r] + \sum_{r \in Z^- \setminus \{z_1\}} \beta_r[r] + \left(\beta_{z_1'} - \tilde{\beta}_1\right)[z_1'] + \left(\beta_{z_1} + \tilde{\beta}_1\right)[z_1]. \tag{44}$$

Initiate the value of H by $H := \{1\}$ and update:

$$\begin{aligned} \beta_{z_1} &:= -(|\beta_{z_1}| - \tilde{\beta}_1) \\ \beta_{z_1'} &:= \beta_{z_1'} - \tilde{\beta}_1. \end{aligned}$$

According to the updated values of β_{z_1} and $\beta_{z_1'}$, update:

$$Z^+ := \begin{cases} Z^+ & \text{if } \beta_{z_1'} \neq 0 \\ Z^+ \setminus \{z_1'\} & \text{else} \end{cases}$$

$$Z^- := \begin{cases} Z^- & \text{if } \beta_{z_1} \neq 0 \\ Z^- \setminus \{z_1\} & \text{else} \end{cases}$$

Then (44) is equal to

$$\sum_{h \in H} \tilde{\beta}_h \left([z_h'] - [z_h]\right) + \sum_{r \in Z^+ \sqcup Z^-} \beta_r[r]. \tag{45}$$

Let $\psi'_+(n) := \psi_+(n) - \tilde{\beta}_1[z'_1](V^n_{k,t_k})$ and $\psi'_-(n) := \psi_-(n) - \tilde{\beta}_1[z_1](V^n_{k,t_k})$ for all $n \in \{2, \ldots, N\}$. Then $\psi'_+ \geqq \psi'_-$ by (43). Consequently, the previous reasoning on $\sum_{r\in Z_{k,t_k}} \beta_r[r]$ applies to the sum $\sum_{r\in Z^+\sqcup Z^-} \beta_r[r]$ to extract z_2 and z'_2 respectively from (the updated) Z^- and Z^+, so that z_2 is the $\succeq_{k,t_k}$-minimum in Z^-, and z'_2 the $\succeq_{k,t_k}$-minimum among those $z' \in Z^+$ for which $z' \succeq_{k,t_k} z_2$. Then define $\tilde{\beta}_2 := \min\{|\beta_{z_2}|, |\beta_{z'_2}|\}$ (with the updated values of β_r) and update $H := H \sqcup \{2\}$. Consequently, (45) is equal to

$$\sum_{h\in H} \tilde{\beta}_h \left([z'_h] - [z_h]\right) + \sum_{r\in(Z^+\sqcup Z^-)\setminus\{z_2,z'_2\}} \beta_r[r] + \left(\beta_{z'_2} - \tilde{\beta}_2\right)[z'_2] + \left(\beta_{z_2} + \tilde{\beta}_2\right)[z_2].$$

Update the values of β_{z_2}, $\beta_{z'_2}$, Z^+ and Z^- as in the previous step. Then the formula displayed just above becomes the same as (45) except for the updated values of H, β_r, Z^+ and Z^-.

Continue the procedure until the updated Z^- is empty. Then let Z_* be the updated Z^+, and $\tilde{\beta}_{z'}$ be the updated $\beta_{z'}$ for any $z' \in Z_*$. Consequently, (45) becomes the right-hand side of (40). Since the procedure from (44) to the recursively updated (45) is just rewriting the left-hand side of (40), (40) is proved, as desired. ■

Lemma 6 Let $j \in I_2$. By the definitions of U^n_j and the matrix $[\mathbf{M}_+, \mathbf{M}_-]$, if $h \notin H_j$ then neither z_h nor z'_h belong to $\mathscr{Z}_j$ and hence $[z'_h](U^n_j) - [z_h](U^n_j) = 0$ for all n. Thus it follows from (30) and (32) that

$$\forall n \in \{1, \ldots, |\mathscr{Z}_j|\} : \sum_{h\in H_j} \tilde{\beta}_h \left([z'_h](U^n_j) - [z_h](U^n_j)\right) \geq 0. \tag{46}$$

As explained at the preamble, z'_h and z_h cannot be both in $\mathscr{Z}_j$. Thus $H_j = H^+_j \sqcup H^-_j$ where

$$\begin{aligned} H^+_j &:= \{h \in H_j \mid z'_h \in \mathscr{Z}_j\} \\ H^-_j &:= \{h \in H_j \mid z_h \in \mathscr{Z}_j\}. \end{aligned} \tag{47}$$

If $H^-_j = \varnothing$, then set $H_* := H$ and we have $z_h \notin \mathscr{Z}_j$ for all $h \in H_*$, and hence (33) is true with $W := \varnothing$. Thus assume, without loss, that $H^-_j \neq \varnothing$. Since $H^+_j \cap H^-_j = \varnothing$, (46) becomes

$$\sum_{h\in H^+_j} \tilde{\beta}_h \left([z'_h](U^n_j) - [z_h](U^n_j)\right) \geq \sum_{h\in H^-_j} \tilde{\beta}_h \left([z_h](U^n_j) - [z'_h](U^n_j)\right)$$

for any n. Since z'_h and z_h cannot be both in $\mathscr{Z}_j$, the $[z_h](U^n_j)$ on the left-hand side, and the $[z'_h](U^n_j)$ on the right-hand side, are both zero. Thus, for any $n = 1, \ldots, |\mathscr{Z}_j|$,

$$\sum_{h\in H^+_j} \tilde{\beta}_h[z'_h](U^n_j) \geq \sum_{h\in H^-_j} \tilde{\beta}_h[z_h](U^n_j). \tag{48}$$

Then mimic the proof of (40) in that of Lemma 5. The (Z^+, Z^-) there is replaced by (H_j^+, H_j^-) here, Z_* replaced by H_* here, the function ψ_+ replaced by $n \mapsto \sum_{h \in H_j^+} \tilde{\beta}_h[z'_h](U_j^n)$, and the function ψ_- replaced by $n \mapsto \sum_{h \in H_j^-} \tilde{\beta}_h[z_h](U_j^n)$. Due to (48), $\psi_+ \geqq \psi_-$ holds. Then successively extract the element h_m from H_j^- such that z_{h_m} is the $\succ_j$-minimum among those z_h for which h remains in H_j^-. Correspondingly, extract the element h_m^* from H_j^+ such that $z'_{h_m^*}$ is the $\succ_j$-minimum among those z'_h for which h remains in H_j^+ and $z'_h \succ_j z_{h_m}$. Add (h_m, h_m^*) to the set W recursively (which is legitimate because (h_m, h_m^*) is distinct from the incumbents in W, as each round of extraction exhausts the weight of at least one of z_{h_m} and $z'_{h_m^*}$; cf. Eq. (42)). Define $\widehat{\beta}_{h,h^*} := \min\left\{\tilde{\beta}_h, \tilde{\beta}_{h^*}\right\}$ for any $(h, h^*) \in W$. Repeat until the weights $\tilde{\beta}_h$ of all elements of H_j^- are exhausted. Then let H_* be the set of the elements in H_j^+ whose weights $\tilde{\beta}_h$ are not exhausted. Thus (33) obtains. The final clause of the lemma holds: Every step of the extraction puts only those (w, w_*) into W such that $\{z'_{w_*}, z_w\} \subseteq \mathscr{Z}_j$ and $z'_{w_*} \succeq_j z_w$; $z_h \notin \mathscr{Z}_j$ for any $h \in H_*$ because $h \notin H_j^-$ as H_j^- has become empty, all its elements having been extracted. Thus the lemma follows. ■