The Economics of Characterizing Reduced-Form Auctions*

Charles Z. Zheng†

November 23, 2023

Abstract

A necessary step in mechanism design with nonlinear design objectives, the mathematical problem of characterizing the reduced forms of ex post feasible allocations is analogous to the economic problem of decentralizing a socially optimal production plan. This paper proposes a method to verify the characterization à la Border (1991) despite the combinatorial complexity among multiple objects and bidders with arbitrary numbers of types such as assignment problems. First, find a social planner's solution that allocates each "inputs" (ex post state) to its compatible "outputs" (interim states). Second, derive from this solution a set of partially revealed preferences among the outputs. Third, prove existence of a price function (shadow prices of Border's conditions) supported by the upper or lower contour sets with respect to these revealed preferences. The method applies easily to generalize the received result in the mainstream model (paramodularity) and to establish a counterpart to a contemporary result (total unimodularity). The method applies nontrivially to the assignment problems between N objects and two bidders, $N \geq 2$ and arbitrary number of types per bidder, with the third step using the hyper-rectangle cover theory.

JEL Classification: C61, D44, D82

Keywords: Auctions, reduced forms, multiple objects, assignment, paramodularity, total unimodularity, interim allocation, implementability, cover theory

*The latest version is posted at this link. I thank the Social Science and Humanities Research Council of Canada for Insight Grant R4809A04.

†Department of Economics, University of Western Ontario, London, ON, Canada, charles.zheng@uwo.ca.

1 Introduction

Like the prisoners in Plato's Cave who, chained to an inside wall, see only the shadows projected on the wall from objects of higher dimensions, a player in a Bayesian game often sees only a projection from the actual game. For instance, in an assignment situation where a player cannot get more than one object, if player k gets object j then any different object $j' \neq j$ can only be allocated to a different player $k' \neq k$. This constraint has to be respected by any equilibrium outcome say $((q_{kj}(t_1, \ldots, t_m))_{j=1}^n)_{k=1}^m$ conditional on the realized type profile $(t_1, \ldots, t_m)$ across all players, with q_{kj} associating to each type profile a probability with which objet j goes to player k. However, at the interim stage of decision making, any player k is uncertain about the types t_{-k} of the other players and hence the variable that affects his decision on behalf of $((q_{kj})_{j=1}^n)_{k=1}^m$ is only the reduced form $(Q_{kj}(t_k))_{j=1}^n$, each $Q_{kj}(t_k)$ equal to the expected value of $q_{kj}(t_k, t_{-k})$ with t_{-k} being the random variable. The design of a Bayesian game therefore often amounts to designing a profile $((Q_{kj})_{j=1}^n)_{k=1}^m$ of such marginals, called *interim allocation*, as opposed to the underlying $((q_{kj})_{j=1}^n)_{k=1}^m$, called *ex post allocation*. The question is how to tell whether a profile $((Q_{kj})_{j=1}^n)_{k=1}^m$, each Q_{kj} being a function of only player k's type, is indeed the reduced form of some feasible ex post allocation $((q_{kj})_{j=1}^n)_{k=1}^m$, each q_{kj} being a function of the type profile across all players.

This question in the single-object case has long been recognized to be a necessary step in mechanism design problems where the designer's objective is not a linear function of the reduced form $((Q_{kj})_{j=1}^n)_{k=1}^m$.[1] Such nonlinearity is hard to avoid in multiple-object situations, with feasibility requirements often tied to various combinations of players and objects. We need a *reduced form characterization*, namely, a necessary and sufficient condition for a $((Q_{kj})_{j=1}^n)_{k=1}^m$ to be generated by a feasible ex post allocation.[2]

A pattern common among a majority of the literature on this issue is that the characterization being offered is a test of the candidate against arbitrary sets of interim states, each

[1] Examples include auctions with risk aversion, dating back to Maskin and Riley [23] and Matthews [24], endogenous valuation such as Gershkov et al. [12], information acquisition such as Li [21], and budget constraints such as Boulatov and Severinov [4].

[2] While the reduced form issue could be partially avoided through restricting attention to dominance- instead of Bayesian incentive compatible mechanisms, the restriction has loss of generality because the equivalence between dominance- and Bayesian incentive compatibility requires other conditions such as that each bidder's type be one-dimensional (cf. Manelli and Vincent [22] and Gershkov et al. [11]).

interim state being a pairing between a bidder and an object together with a realized type of the bidder. Such a characterization is first obtained by Border [2] for single-unit symmetric auctions, extended to asymmetric single-unit models by Border [3], Manelli and Vincent [22], Mierendorff [25], Goeree and Kushnir [13][3] and Yang [29], and generalized to multiple units of a single object by Che et al. [7]. It is recently advanced by the contemporary works of Lang and Yang [20] and of Valenzuela-Stookey [28] to handle some multiple-object models. These "Border-like" characterizations, requiring only a test against arbitrary sets of interim states from individual players' parochial viewpoints, constitute a significant advance from the feasibility definition of reduced forms, which would require a test against arbitrary *ex post states*, each being a profile $(t_1, \ldots, t_m)$ of realized types across all bidders, representing a perfect hindsight of the realized types across all players.[4]

The question is how general such Border-like characterizations can be. The generalization by Che et al. [7] relies on a paramodularity assumption, which does not incorporate assignment models. While the total unimodularity model of Lang and Yang [20] includes assignment models, their application to assignments is restricted to cases where a bidder can have at most two types.[5] Valenzuela-Stookey's [28] assignment model allows for arbitrary numbers of types per bidder, though his characterization is "approximate," namely, sufficient but not necessary for reduced forms. The frontier of the literature calls for a general method to tackle the combinatorial complications exemplified by assignment problems.

This paper proposes such a method. It starts with a necessary and sufficient condition for a Border-like characterization to be valid. To articulate the condition without more notations, let us think of interim states as *outputs*, and ex post states as *inputs*, so that a production plan $((Q_{kj})_{j=1}^{n})_{k=1}^{m})$ is feasible iff it results from an input supply plan (ex post allocation $((q_{kj})_{j=1}^{n})_{k=1}^{m})$ that distributes each input to its compatible outputs subject to

[3]Goeree and Kushnir also consider social choice models.

[4] A characterization different from the Border format is offered through majorization techniques by Hart and Reny [14] and generalized by Kleiner et al. [17] and Kolesnikov et al. [18], assuming symmetric bidders and symmetric mechanisms. In the single-object multiunit case of [17], Gershkov et al. [12] find a connection between the majorization and the Border-like characterization. Cai et al. [6] offer a characterization in a multiple-object additive-utility model that requires a test on arbitrary weights across the sets of interim states. Toikka et al. [27] obtain a characterization in a particular Bayesian persuasion problem by direct calculations of the projection cones associated with the reduced forms.

[5] Budish et al. [5] has introduced a special case of total unimodularity, bihierarchy, to a complete-information model of assignments.

all transmission capacity constraints. Given any linear valuation of the outputs, a *social planner's solution* is an input supply plan that maximizes the total value of outputs subject to the said feasibility constraints. In this language, Theorem 1 says that a Border-like characterization is valid iff, given any linear valuation of the outputs, the social planner's solution can be decentralized into a set of prices associated with the output bundles that are "universally binding" in a particular sense: Associated with every output bundle (a set of interim states) are two possible, nonnegative prices. One is the price of the permit that allows any input supplier to exceed a ceiling capacity in supplying his input to its compatible outputs in the bundle, and the other is the price of the permit that allows him to fall short of a minimum level of supply within the said bundle. For an output bundle to be associated with a positive price for exceeding the ceiling, the bundle needs to be *upward universally binding* in the sense that, for every input supplier, the ceiling is reached if he were to follow the social planner's solution in supplying to the compatible outputs in the bundle. Symmetrically, for an output bundle to be associated with a positive price for falling below the floor, the bundle needs to be *downward universally binding* in that every input supplier barely reaches the floor if he were to follow the social planner's solution in supplying to the bundle.

The universal binding condition provides an insight on how to find a desired price system thereby to verify the Border-like characterization. The social planner's solution decides for every unit of each input whether to distribute it to one output or another among the compatible outputs. If these choices lead to a revealed preference relation among all outputs (interim states) then we obtain the corresponding *upper or lower contour sets* in the output space. An upper contour set is upward universally binding because its elements are revealed to be preferred to those outside the set. For every input (ex post state), the social planner would max out the total quantity that the input can contribute to the compatible members of the upper contour set before she allocates the input elsewhere. Consequently, the input faces a binding ceiling constraint in contributing to the upper contour set according to the planner's solution. This true for all inputs, the universality of the binding condition is met. The reasoning for lower contour sets is symmetric.

Thus, a main step to obtain a Border-like characterization is to derive from the social planner's solution a set of partial orders on the outputs, each partial order rationalizing the planner's solution within a subset of the output space. Due to combinatorial complications, the derivation does not come directly from the arbitrary linear valuation parametric to the planner's solution: An output of high value may need to be produced simultaneously with another output of low value, and hence the social planner may forgo this high-value output for a middle-value one whose complement has a sufficiently larger value. To fully capture the revealed preferences of the social planner's solution, multiple partial orders are often needed so that every output is covered by some upper or lower contour sets.

With a sufficiently large family of upper or lower contour sets comes the final step to obtain a Border-like characterization. This is to verify that a price function supported by these contour sets exists to decentralize the social planner's solution. The step boils down to proving existence of a nonnegative solution in a system of linear equations. In simple cases, the system can be directly solved. In general, the existence proof is to verify that no Gaussian elimination on the linear system can produce a contradictory equation (Lemma 3). This suffices the existence thanks to Chu et al.'s [8] hyper-rectangle cover theory.

The method sketched above applies easily to Che et al.'s [7] paramodularity model, slightly extended here for multiple objects (Theorem 2). It is easy because paramodularity is the special case where any linear valuation of the outputs (interim states) implies directly a total order on the output space that rationalizes the social planner's solution. The implication is direct because paramodularity guarantees that the social planner's problem is solved by the greedy-generous algorithm in descending order of the linear valuation. The order being total, the equation system in the final step of the method is trivial to solve.

The method applies relatively easily to establish a counterpart (Theorem 3) to Lang and Yang's [20] total unimodularity result. An implication of their total unimodularity assumption is that any linear valuation of the outputs is a conic combination of $\{1, 0, -1\}$-valued extreme rays. I replace total unimodularity with a slightly weaker assumption that also has the said implication. The social planner's solution can then be rationalized by multiple binary relations, each—corresponding to an extreme ray—partitioning the output space into only three indifference sets, the good (valued at 1), the bad (valued -1), and the neutral (valued zero). These are the upper or lower contour sets to support the desired price function, the existence of which comes directly from the conic combination.

The method applies nontrivially to two assignment models, each concerning the assignment of N objects between two bidders, with $N \geq 2$ and an arbitrary number of types for each bidder (Theorems 4 and 5). The crucial constraint in both models is that no bidder can have more than one object. The *partial assignment* model allows a bidder to end with no object. The *full assignment* model, which Valenzuela-Stookey's [28] model does not include, requires each bidder to end with some object.

In the social planner's choice associated with an assignment problem, an interim state (a player–object-type triple) has two kinds of rivals. Because an object cannot go to different bidders, an interim state say "bidder L, good 2 and type A of bidder L" competes with all the interim states associated with the other bidder for the same object, such as "bidder R, good 2 and type B of bidder R" and "bidder R, good 2 and type C of bidder R." Think of these rivals in a *row*. Meanwhile, since no bidder can have more than one object, the interim state also competes with those referring to the same bidder-type for different objects, such as "bidder L, good 3 and type A of bidder L" and "bidder L, good 4 and type A of bidder L." Think of these rivals in a *column*. Two interim states that have neither row nor column in common are complements, as the two bidder-object pairs constitute a feasible allocation outcome. Given any linear valuation of the interim states, the social planner's solution chooses for every ex post state a pair of complements that has the largest total value in the full assignment model, or the largest total positive value in the partial assignment model (with negative values eliminated by non-assignment).

Within every row or column, I construct a partial order to rationalize the social planner's solution. In the full assignment model, the partial order within a column is based on the ordinal ranking among its members according to the said linear valuation: The higher is one's value among its column rivals, the more it can contribute to the total value when paired with a complement. The *comparative advantage* of an interim state is defined to be its value minus the highest value among its column rivals. The partial order within a row is based on the ordinal ranking among its members according to their comparative advantages: Within a row, the competition is among the top contenders (highest values) from their own columns. Between any two of such top contenders, the one with the larger comparative advantage gets chosen by the social planner to be paired with the second-place contender in the column of the other, because having a larger comparative advantage means a larger total value when paired with a complement. Thus I obtain a family of partial revealed preferences of the social planner's solution, and hence the corresponding upper or lower contour sets. The partial assignment model is handled in a similar manner, with the definition of comparative advantages modified to consider only the positive part of the valuation.

Supported by these upper or lower contour sets, a price function exists to decentralize the social planner's solution. That is verified through proving that none of the Gaussian eliminations on the equation system for the price function can result in a contradictory equation. Each equation in this system corresponds to an interim state, with the unknowns being the prices of the upper or lower contour sets that contain the interim state. Any Gaussian elimination amounts to a linear combination of these equations. I reduce it to the desired outcome by inductively removing a kind of quadruples from any such linear combination: Within each column, the prices of the contour sets that contain one interim state and exclude another are captured through subtracting the equation corresponding to the former by the equation corresponding to the latter. For each column there is a "difference equation" obtained from subtracting the equation for the top contender by the equation for the second highest contender in the column. Then, within each row, the prices of the contour sets that contains one top contender and exclude another are captured through subtracting the former's difference equation by the latter's. This "difference of differences" operation corresponds to a quadruple that I remove from the said linear combination. That is also roughly the intuition how the final step for the Border-like characterization is achieved.

The next section introduces the basic definitions including the Border-like characterization. Then Section 3 introduces the method, which is applied to the paramodularity model in Section 4 and to a counterpart to the total unimodularity model in Section 5. Section 6 then explains at length the application to both assignment models. Extensions to arbitrary infinite type spaces are presented in Section 7. Appendix A contains all delayed details.

2 Basic Definitions

Let I_1 be the set of bidders, I_2 the set of objects, both assumed to be finite. Let

$$I := I_1 \times I_2$$

denote the set of all possible bidder-object pairs. For any $i_1 \in I_1$, let T_{i_1} be the set of the possible types of bidder i_1. Any (i_1, i_2, t_{i_1}), with $(i_1, i_2) \in I_1 \times I_2$ and $t_{i_1} \in T_{i_1}$, is called *interim state*, often denoted by (i, t_{i_1}) with $i := (i_1, i_2) \in I$. Let $T := \prod_{i_1 \in I_1} T_{i_1}$ be the type space, and $(T, \mathscr{T}, \mu)$ the measure space such that the *ex post state*, or the profile

$$t := (t_{i_1})_{i_1 \in I_1}$$

of types across bidders, is drawn from T according to probability measure μ. If the cardinality $|T|$ of T is finite, assume $\mu\{t\} > 0$ for all $t \in T$, $\mu\{t\}$ the measure of the singleton $\{t\}$.

Denote $\mathscr{R}$ for either the set $\mathbb{R}$ of real numbers or the set $\mathbb{Z}$ of integers, and let $\mathscr{R}_+$ be the set of nonnegative elements of $\mathscr{R}$. For any $t \in T$, let $X_t \subseteq \mathscr{R}^I$ be the set of allocation outcomes that are feasible when t is the ex post state. An element of X_t is in the form of $x := (x_i)_{i \in I}$, or $(x_{i_1,i_2})_{(i_1,i_2) \in I}$, with x_{i_1,i_2} being the quantity of object i_2 allocated to bidder i_1. (Thus, quantities are divisible if $\mathscr{R} = \mathbb{R}$, and indivisible if $\mathscr{R} = \mathbb{Z}$. Also note that the set X_t of feasible allocation outcomes may vary with the ex post state t.) The mapping $t \mapsto X_t$ is called *constraint structure*. For each $t \in T$, denote $\mathrm{cv}X_t$ for the convex hull of X_t. Assume that X_t is nonempty and compact for all $t \in T$.

An *ex post allocation* is a profile $(q_i)_{i \in I}$ of measurable functions $q_i : T \to \mathbb{R}$ ($\forall i \in I$) such that $(q_i(t))_{i \in I} \in \mathrm{cv}X_t$ for μ-a.e. $t \in T$.[6] Thus, an ex post allocation can randomize on the feasible allocation outcomes, and its range is a subset of $\mathbb{R}^I$ even when $X_t \subseteq \mathbb{Z}^I$. An *interim allocation* is a profile $Q := (Q_i)_{i \in I}$ of measurable functions $Q_i : T_{i_1} \to \mathbb{R}$ ($\forall i := (i_1, i_2) \in I$).

For each $i_1 \in I_1$, let μ_{i_1} be the marginal measure of μ onto T_{i_1}; let $T_{-i_1} := \prod_{j \in I_1 \setminus \{i_1\}} T_j$, and denote $\mu_{-i_1}(\cdot|t_{i_1})$ for the conditional measure on T_{-i_1} according to μ conditional on t_{i_1}. An interim allocation $(Q_i)_{i \in I}$ is said to be the *reduced form* of some ex post allocation $(q_i)_{i \in I}$ if and only if Q_i is the marginal of q_i onto T_{i_1} for all $i := (i_1, i_2) \in I$, namely,

$$Q_i(t_{i_1}) = \int_{T_{-i_1}} q_i(t_{i_1}, t_{-i_1}) d\mu_{-i_1}(t_{-i_1}|t_{i_1}) \tag{1}$$

for any $i := (i_1, i_2) \in I$ and any $t_{i_1} \in T_{i_1}$. Denote $\mathscr{Q}$ for the set of interim allocations that are reduced forms of some ex post allocations.

Economic Interpretation Think of an ex post state $(t_{i'_1})_{i'_1 \in I_1}$ as an "input," and an interim state (i, t'_{i_1}) as an "output." Then (1) requires that the quantity of output (i, t'_{i_1}) be equal to the sum of the quantities of those inputs that are supplied to this output and are compatible with it, with input $(t_{i'_1})_{i'_1 \in I_1}$ *compatible* with output (i, t'_{i_1}) iff $t_{i_1} = t'_{i_1}$. For it to be supplied to a compatible output (i, t'_{i_1}), an input t needs to go through the "pipeline" i. And the feasibility set X_t is the constraint on the input's transmission quantities through various sets of pipelines. An interim allocation Q therefore corresponds to a profile of output quantities. For Q to belong to $\mathscr{Q}$, the quantity of every output needs to be supplied by

[6] It is immaterial to strengthen the condition $(q_i(t))_{i \in I} \in X_t$ to all $t \in T$ (cf. Border [2, Proposition 3.1]).

its compatible inputs, and the allocation from every input needs to satisfy its feasibility constraints. Thus, checking the feasibility of an interim allocation directly would require checking the feasibility of every input allocation plan, input by input. An alternative is to check the feasibility by considering only the constraints on the outputs implied by the input-wise constraints. Proving that this alternative suffices is our task.

Denote the set of interim states by

$$\mathscr{X} := \bigcup_{(i_1,i_2)\in I} \left(\{(i_1,i_2)\} \times T_{i_1}\right).$$

For any $S \subseteq \mathscr{X}$ and any $t := (t_{i_1})_{i_1\in I_1} \in T$, define

$$I(S,t) := \{(i_1,i_2) \in I \mid (i_1,i_2,t_{i_1}) \in S\}. \tag{2}$$

Continuing with the economic interpretation, think of $I(S,t)$ as the set of pipelines through which the input t needs to go through in order to be supplied to the compatible outputs in S. The constraint X_t on the input t implies restrictions on the total quantity that the input can be transmitted through these pipelines. To capture such restrictions, define

$$f(S,t) := \max_{x\in X_t} \sum_{i\in I(S,t)} x_i, \tag{3}$$

$$g(S,t) := \min_{x\in X_t} \sum_{i\in I(S,t)} x_i. \tag{4}$$

Think of $f(S,t)$ and $g(S,t)$ as the upper and lower bunds of the total quantity of input t allowed by the feasibility set X_t to be supplied to the compatible outputs in S. It is therefore conceivable that satisfaction of such upper and lower bounds, as in the Border condition defined next, is necessary for an interim allocation to be feasible. What is nontrivial, however, is whether satisfaction thereof is also sufficient for an interim allocation to be feasible.

An interim allocation $(Q_i)_{i\in I}$ is said to satisfy the *Border condition* if and only if

$$\int_T g(S,t)d\mu(t) \le \sum_{i\in I}\int_T Q_i(t_{i_1})\chi_S(i,t_{i_1}))d\mu(t) \le \int_T f(S,t)d\mu(t) \tag{5}$$

for all measurable $S \subseteq \mathscr{X}$ (χ_S being the characteristic function of S; $S \subseteq \mathscr{X}$ said *measurable* iff $\{t_{i_1} \in T_{i_1} \mid (i,t_{i_1}) \in S\}$ is measurable for all $i \in I$).[7] Denote $\mathscr{Q}_{\mathrm{B}}$ for the set of interim

[7] In Border's [2] single-unit symmetric model, bidders are ex ante identical, so an interim state needs no label for the bidder, and hence an $S \subseteq \mathscr{X}$ is simply a set of some possible types of any bidder.

allocations that satisfy the Border condition. The characterization of reduced forms is the claim $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$. Proving $\mathscr{Q}_{\mathrm{B}} \supseteq \mathscr{Q}$ is easy (Appendix A.1). It is the converse that we shall investigate: When is $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ true?

3 The Universal Binding Condition

The rest of the paper, except Section 7, assumes the cardinality $|T|$ of the type space T to be finite. Then the set of interim allocations is a Euclidean space $\mathbb{R}^{\mathscr{Z}}$, and hence any continuous linear operator on $\mathscr{Q}$ corresponds to a vector $\alpha := (\alpha(z))_{z\in\mathscr{Z}} \in \mathbb{R}^{\mathscr{Z}}$. The next lemma (proved in Appendix A.2), due to the separating hyperplane theorem, provides a starting point to establish $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.

Lemma 1 *Suppose that $|T|$ is finite. Then $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if, for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$,*

$$\max_{Q\in\mathscr{Q}_{\mathrm{B}}} \sum_{i\in I} \sum_{t_{i_1}\in T_{i_1}} \alpha(i,t_{i_1}) Q_i(t_{i_1}) \mu_{i_1}\{t_{i_1}\} \leq \sum_{t\in T} \mu\{t\} \max_{(q_i(t))_{i\in I}\in \mathrm{cv} X_t} \sum_{i\in I} q_i(t)\alpha(i,t_{i_1}). \tag{6}$$

Continuing with the economic interpretation, think of the maximization problem on the right-hand side of (6) as the *social planner's problem*: Given any linear valuation α of the outputs (interim states (i, t_{i_1})), choose for each input t (ex post state) a plan $(q_i(t))_{i\in I}$ to allocate the input to the various compatible outputs subjects to the feasibility constraint X_t, input by input, in order to maximize the total value of the outputs thereby produced. By contrast, the left-hand-side problem is subject to the constraint $Q \in \mathscr{Q}_{\mathrm{B}}$ (the Border condition (5)), which is uniform to all inputs. To see that more clearly, consider the dual of the left-hand-side problem. Its choice variable is the shadow prices for the Border condition, one price per set S of outputs, uniform to all inputs t. Thus the constraint $Q \in \mathscr{Q}_{\mathrm{B}}$ may be too broad for Q to satisfy the input-by-input constraint on the right-hand side, upsetting (6).

Theorem 1 observes the exact condition for (6) to hold, thereby suggesting a general method to validate Border-like characterizations of reduced forms. The proof (Appendix A.3) is based on comparing the dual of the left-hand side in (6) with the right-hand side.

Theorem 1 *Assume that $|T|$ is finite. Then $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$ there exist $(p_+, p_-) : 2^{\mathscr{Z}} \to \mathbb{R}^2_+$ and $q^* := (q^*_i)_{i\in I} : t \mapsto (q^*_i(t))_{i\in I} \in \mathrm{cv} X_t$ such that*

$$\alpha(z) = \sum_{S\subseteq\mathscr{Z}} p_+(S)\chi_S(z) - \sum_{S\subseteq\mathscr{Z}} p_-(S)\chi_S(z) \tag{7}$$

for all $z \in \mathscr{Z}$,

$$q^*(t) \in \arg\max_{(x_i)_{i\in I}\in \mathrm{cv} X_t} \sum_{i\in I} x_i \alpha(i, t_{i_1}) \tag{8}$$

for all $t := (t_{i_1})_{i_1 \in I_1} \in T$*, and, for all* $S \subseteq \mathscr{Z}$,

$$\begin{aligned} p_+(S) > 0 \quad &\Rightarrow \quad \forall t \in T \left[f(S,t) = \textstyle\sum_{i\in I} q_i^*(t) \chi_S(i, t_{i_1}) \right] \\ p_-(S) > 0 \quad &\Rightarrow \quad \forall t \in T \left[g(S,t) = \textstyle\sum_{i\in I} q_i^*(t) \chi_S(i, t_{i_1}) \right]. \end{aligned} \tag{9}$$

The condition identified by Theorem 1 is that any linear valuation α parametric to the social planner's problem can be decomposed through Eq. (7) into a linear combination of nonnegative prices $p_+(S)$ and $p_-(S)$ that are associated with a particular kind of bundles S of outputs subject to (9): If all inputs are allocated according to the social planner's solution q^* (specified in (8)), then the total quantity of every input t supplied to the compatible outputs in the bundle S needs to be either maxed out to the ceiling $f(S,t)$ for the price $p_+(S)$ to be positive, or reduced to the floor $g(S,t)$ for the price $p_-(S)$ to be positive. This binding condition of S is universal in the sense of being required for all inputs t.

To understand the universality of the binding condition, think of $p_+(S)$ as the price of a tradable permit for any input supplier who holds it to exceed the ceiling capacity by one unit in supplying the input to his compatible outputs in S. If $p_+(S) > 0$ for some S such that one input supplier reaches his ceiling capacity while another one is below hers, an arbitrage opportunity exists and the price is not in equilibrium. The case for those S for which $p_-(S) > 0$ is analogous, with $p_-(S)$ interpreted as the market price of a tradable permit to fall short of the minimum requirement of supply by one unit.

The Method The universal binding condition gives a clue on how to construct a price vector (p_+, p_-) thereby obtaining a Border-like characterization of reduced forms.[8] Note that the condition is defined with respect to q^*. Thus, for any set S of outputs that is *upward* universally binding in the sense of $f(S,t) = \sum_{i\in I} q_i^*(t)\chi_S(i, t_{i_1})$ for all $t \in T$ (so that $p_+(S)$ can be positive), the planner acts as if she strictly prefers any element in S to any element outside S, always raising any input t's supply to S to the ceiling $f(S,t)$. Analogously, for any set S of outputs that is *downward* universally binding in the sense of $g(S,t) = \sum_{i\in I} q_i^*(t)\chi_S(i, t_{i_1})$ for all $t \in T$ (so that $p_-(S)$ can be positive), the planner acts

[8] That is to exploit the sufficiency observation of Theorem 1. Its necessity observation can be useful to establish impossibility results (e.g., Supplement B.1).

as if she strictly prefers any element outside S to any element in S, always reducing any input t's supply to S down to the floor $g(S,t)$. Therefore, the support of p_+ consists only of the upper contour sets, and the support of p_- only the lower contour sets, with respect to such revealed preferences of the planner's solution q^*. This idea the next lemma formalizes.

Lemma 2 *Let $q^* := (q_i^*)_{i\in I}$ map every $t \in T$ to a $(q_i^*(t))_{i\in I} \in \mathrm{cv}X_t$, and let $\{U^n \mid n = 1,\dots,n_*\}$ and $\{L^n \mid n = 1,\dots,n^*\}$ be collections of some subsets of $\mathscr{X}$ such that*

$$\varnothing = U^0 \subsetneq U^1 \subsetneq \cdots \subsetneq U^{n_*-1} \subsetneq U^{n_*}, \tag{10}$$

$$L^{n^*} \supsetneq L^{n^*-1} \supsetneq \cdots \supsetneq L^1 \supsetneq L^0 = \varnothing. \tag{11}$$

i. If for any $t := (t_{i_1})_{i_1\in I_1} \in T$ and any $n = 1,\dots,n_*$,

$$\sum_{(i,t_{i_1})\in U^n\setminus U^{n-1}} q_i^*(t) = f(U^n,t) - f(U^{n-1},t), \tag{12}$$

then U^n is upward universally binding for any $n = 1,\dots,n_*$.

ii. If for any $t := (t_{i_1})_{i_1\in I_1} \in T$ and any $n = 1,\dots,n^*$,

$$\sum_{(i,t_{i_1})\in L^n\setminus L^{n-1}} q_i^*(t) = g(L^n,t) - g(L^{n-1},t), \tag{13}$$

then L^n is downward universally binding for any $n = 1,\dots,n^*$.

Proof To prove claim (i), note from (10) that, for any $k \in \{1,\dots,n_*\}$, $U^k = \bigsqcup_{n=1}^k (U^n \setminus U^{n-1})$. Thus, for any $t := (t_{i_1})_{i_1\in I_1} \in T$,

$$\begin{aligned}
\sum_{i\in I} q_i^*(t)\chi_{U^k}(i,t_{i_1}) &= \sum_{n=1}^k \sum_{i\in I} q_i^*(t)\chi_{U^n\setminus U^{n-1}}(i,t_{i_1}) \\
&= \sum_{n=1}^k \sum_{(i,t_{i_1})\in U^n\setminus U^{n-1}} q_i^*(t) \\
&= \sum_{n=1}^k \big(f(U^n,t) - f(U^{n-1},t)\big) \\
&= f(U^k,t),
\end{aligned}$$

with the second last line due to (12), and the last line due to cancelation and $f(\varnothing,t) = 0$. Thus, U^k is upward universally binding. The proof for claim (ii) is analogous. ■

To unpack the implications of Lemma 2, note from (10) and (12) that the choice q^*, when restricted within U^{n_*}, behaves as if following a greedy algorithm according to a preference relation $\succeq_U$ that strictly prefers any element of U^k to any element of $U^n \setminus U^k$ for all $n > k$, and is indifferent among the elements in $U^n \setminus U^{n-1}$ for each n.[9] That is, q^* maxes out the feasible allocation to any preferred output before allocating any quantity at all to any less preferred output. Analogously, from (11) and (13) one can see that q^*, when restricted within L^{n^*}, acts as if following a generous algorithm—the mirror image of the greedy algorithm—according to a preference relation $\succeq_L$ that strictly prefers any element of L^k to any element of $L^n \setminus L^k$ for all $n > k$, and is indifferent among the elements in $L^n \setminus L^{n-1}$ for each n. That is, $q^*|_{L^{n^*}}$ follows the greedy algorithm on L^{n^*} such that the role of f is played by $-g$.[10] Thus, the sets U^n are the upper contour sets with respect to $\succeq_U$, and the sets L^n the lower contour sets with respect to $\succeq_L$. By Lemma 2, these sets satisfy the universal binding condition. They suit our need to support the price vector (p_+, p_-). In other words, to construct the support for the (p_+, p_-) required in Theorem 1, it suffices to look for such partially revealed preferences and hence the corresponding upper or lower contour sets.

Such revealed preferences, in general, cannot be directly inferred from the linear valuation α, despite α being a parameter in the social planner's problem. For example, consider an assignment problem between two bidders and two objects such that each bidder is to be allocated exactly one object, and vice versa. There are four bidder-object pairs, $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$. Suppose when the ex post state is (t_1, t_2), the α values are $\alpha(1,1,t_1) = 10$, $\alpha(1,2,t_1) = 4$, $\alpha(2,1,t_2) = 9$ and $\alpha(2,2,t_2) = 2$. The ranking according to the α values is then $(1,1,t_1) \succ (2,1,t_2) \succ (1,2,t_1) \succ (2,2,t_2)$. Had the greedy algorithm been valid according to this ranking, the social planner would have given a full quantity 1 to $(1,1,t_1)$, namely, $q^*_{1,1}(t_1,t_2) = 1$, and hence zero quantity to $(2,1,t_2)$ (since good 1 cannot be assigned to both bidders). But the planner is to maximize the total α-value of the outputs among all feasible allocation outcomes, which can only be either $\{(1,1),(2,2)\}$ (giving good 1 to bidder 1, and good 2 to bidder 2) or $\{(2,1),(1,2)\}$ (giving good 1 to bidder 2, and good 2 to bidder 1). The maximum is $\{(2,1),(1,2)\}$, yielding a total value equal to $9 + 4 = 13$

[9] To see why (12) corresponds to a greedy algorithm, plug $n = 1$ into (12) to see $\sum_{U^1} q_i^*(t) = f(U^1, t)$, and then plug $n = 2$ into (12) to see $\sum_{U^2 \setminus U^1} q_i^*(t) = f(U^2, t) - f(U^1, t)$, and so on. See Schrijver [26, Ch. 40] for an extensive coverage of greedy algorithms. The hierarchical allocation familiar in the optimal auction literature is a special case thereof, with the said preference relation being the ranking of virtual utilities.

[10] See Hassin [15] for a definition of generous algorithms.

instead of a total value $10+2=12$ from the alternative. Thus, the correct solution has $q^*_{1,1}(t_1,t_2)=0$ and $q^*_{2,1}(t_1,t_2)=1$. So the revealed preference between $(1,1,t_1)$ and $(2,1,t_2)$ is $(2,1,t_2)\succ(1,1,t_1)$, opposite to the ranking according to the α values.

Due to the combinatorial complexity illustrated by this example, in general the arbitrarily given α does not yield directly a total order on the entire set $\mathscr{Z}$ of outputs that rationalizes the social planner's solution q^*. Rather, close inspection of q^* is needed to obtain multiple partial orders on $\mathscr{Z}$ each of which rationalizes q^* restricted within a subset of $\mathscr{Z}$. Within any such a subset, the corresponding partial order yields the upper or lower contour sets. The collection of such upper or lower contour sets, across these partial orders, can then support the price vector (p_+,p_-).

With such upper or lower contour sets, to satisfy the condition required in Theorem 1 it suffices to prove that the (p_+,p_-) as a function of these sets satisfies (7). That is, we need to prove that the linear system (7) has a nonnegative solution for (p_+,p_-). In simple cases, this can be done through directly solving the equation system. To handle general cases, the following lemma provides a method based on the hyper-rectangle cover theory.

Let $\mathscr{S}_+$ and $\mathscr{S}_-$ be two collections of subsets in $\mathscr{Z}$. (In applications, the elements of $\mathscr{S}_+$ are the upward universally binding sets to support p_+, and the elements of $\mathscr{S}_-$ the downward universally binding sets to support p_-.) Let $\mathbf{M}_+$ be a matrix with $|\mathscr{Z}|$ rows and $|\mathscr{S}_+|$ columns, so that rows are indexed by $\mathscr{Z}$, and columns by $\mathscr{S}_+$. For each $z\in\mathscr{Z}$ and each $S\in\mathscr{S}_+$, let the entry at the intersection between row z and column S be equal to $\chi_S(z)$. Analogously, let $\mathbf{M}_-$ be a $|\mathscr{Z}|$-by-$|\mathscr{S}_-|$ matrix whose rows are indexed by $\mathscr{Z}$ and columns by $\mathscr{S}_-$, and whose entry at the intersection between row z and column S be equal to $-\chi_S(z)$ $(\forall z\in\mathscr{Z}\forall S\in\mathscr{S}_-)$. Thus $[\mathbf{M}_+,\mathbf{M}_-]$ is a matrix with $|\mathscr{Z}|$ rows and $|\mathscr{S}_+|+|\mathscr{S}_-|$ columns. Denote $\boldsymbol{p}$ for the column vector

$$\boldsymbol{p}:=\left[(p_+(S)_{S\in\mathscr{S}_+},(p_-(S)_{S\in\mathscr{S}_-}\right]^{\mathsf{T}},$$

and $\boldsymbol{\alpha}$ for the column vector

$$\boldsymbol{\alpha}:=[(\alpha(z)_{z\in\mathscr{Z}}]^{\mathsf{T}}.$$

Then (7) is equivalent to $[\mathbf{M}_+,\mathbf{M}_-]\,\boldsymbol{p}=\boldsymbol{\alpha}$, namely, with $\boldsymbol{\alpha}$ moved to the left-hand side,

$$[\mathbf{M}_+,\mathbf{M}_-,-\boldsymbol{\alpha}]\,\boldsymbol{p}=\mathbf{0}.$$

Lemma 3 *For any $\alpha \in \mathbb{R}^{\mathscr{Z}}$, and any $\mathscr{S}_+, \mathscr{S}_- \subseteq 2^{\mathscr{Z}}$, with the associated matrices $\mathbf{M}_+$ and $\mathbf{M}_-$, if no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ can produce any nonnegative row whose entry at the $-\boldsymbol{\alpha}$ position is (strictly) positive, then there exist $p_+ : \mathscr{S}_+ \to \mathbb{R}_+$ and $p_- : \mathscr{S}_- \to \mathbb{R}_+$ that satisfy (7).*[11]

Proof As explained previously, the existence of the (p_+, p_-) specified thereof is equivalent to the existence of a nonnegative solution of $\boldsymbol{p}$ for $[\mathbf{M}_+, \mathbf{M}_-]\,\boldsymbol{p} = \boldsymbol{\alpha}$. By Theorem 2 of Chu et al. [8], such a nonnegative solution exists if the cover order of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ is less than or equal to that of $[\mathbf{M}_+, \mathbf{M}_-]$. According to the procedure in their Section 4, the cover order of any matrix say $\mathbf{A}$ is equal to the maximum number of (strictly) positive entries among all the nonnegative rows that any Gaussian elimination on $\mathbf{A}$ can produce. Since the only difference between $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ and $[\mathbf{M}_+, \mathbf{M}_-]$ is the $-\boldsymbol{\alpha}$ column, any nonnegative row produced by a Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ that has zero at the entry for $-\boldsymbol{\alpha}$ can be produced by the same Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-]$. Thus, the desired inequality of the cover orders between them follows from the hypothesis in the lemma. ■

To understand the intuition behind Lemma 3, consider two instances of (7) for interim states $z', z'' \in \mathscr{Z}$ such that $\alpha(z') < \alpha(z'')$. Subtract the instance of (7) when $z = z'$ by the instance of (7) when $z = z''$, so the left-hand side is a negative number. Then a contradiction occurs if the right-hand side is nonnegative, which occurs if $\chi_S(z') \geq \chi_S(z'')$ whenever $p_+(S) > 0$ ($S \in \mathscr{S}_+$) and $-\chi_S(z') \geq -\chi_S(z'')$ whenever $p_-(S) > 0$ ($S \in \mathscr{S}_-$). That is, the subtraction between the two instances of (7) produces a nonnegative row vector $\left[(\chi_S(z') - \chi_S(z''))_{S \in \mathscr{S}_+}, (-\chi_S(z') + \chi_S(z''))_{S \in \mathscr{S}_-}\right]$, while the corresponding $-\alpha(z') - (-\alpha(z''))$ is positive. This contradictory case is ruled out by the hypothesis in the lemma. The hypothesis merely strengthens to cover all Gaussian eliminations so that it rules out any linear combination of such subtractions that produces a contradiction to (7).

Thus comes a road map to obtain reduced-form characterizations:

1. For any linear valuation $\alpha \in \mathbb{R}^{\mathscr{Z}}$, find a solution q^* to the social planner's problem (8).

2. Derive from q^* some partial orders $\succeq_Z$ on $\mathscr{Z}$ that partially rationalizes the planner's choice q^*: For each $\succeq_Z$, q^* restricted to the corresponding $Z \subseteq \mathscr{Z}$ behaves as if following a greedy-generous algorithm according to $\succeq_Z$. The upper or lower contour sets with respect to $\succeq_Z$ satisfy the universal binding condition.

[11] Extend (p_+, p_-) to $2^{\mathscr{Z}}$ trivially by setting $p_+(S) := 0$ and $p_-(S') := 0$ for all $S \notin \mathscr{S}_+$ and $S' \notin \mathscr{S}_-$.

3. Prove that (7) has a nonnegative solution for (p_+, p_-) for which p_+ is supported by the upper contour sets, and p_- supported by the lower contour sets.

4 Paramodularity

The method sketched above applies easily to the paramodularity model, where the social planner's solution reflects the ranking of the linear valuation α directly. For any $t \in T$ and any $E \subseteq I$, define

$$f_t(E) := \max_{x \in X_t} \sum_{i \in E} x_i, \tag{14}$$

$$g_t(E) := \min_{x \in X_t} \sum_{i \in E} x_i. \tag{15}$$

A constraint structure $(X_t)_{t \in T}$ is said to be *paramodular* iff two conditions hold for any $t \in T$: (i) the pair (f_t, g_t), defined above completely characterizes X_t in the sense that

$$X_t = \left\{ (x_i)_{i \in I} \in \mathscr{R}^I \;\middle|\; \forall E \subseteq I \left[g_t(E) \leq \sum_{i \in E} x_i \leq f_t(E) \right] \right\}, \tag{16}$$

and (ii) (f_t, g_t) is *paramodular* on 2^I: first, f_t and $-g_t$ are each *submodular* in the sense that

$$\begin{aligned} f_t(E) - f_t(E \cap E') &\geq f_t(E \cup E') - f_t(E') \\ g_t(E) - g_t(E \cap E') &\leq g_t(E \cup E') - g_t(E') \end{aligned}$$

for all $E, E' \subseteq I$; second, (f_t, g_t) is *compliant* in the sense that, for all $E, E' \subseteq I$,

$$f_t(E') - f_t(E' \setminus E) \geq g_t(E) - g_t(E \setminus E').$$

Thus, the feasible allocation outcomes are defined by the upper and lower bounds of the total quantities for various sets of bidder-object pairs. The paramodularity assumption regulates these bounds so that their marginal changes are the binding constraints when we add an element to, or remove it from, a set of bidder-object pairs. Submodularity of f_t is a notion of diminishing marginal upper bounds, and submodularity of $-g_t$ a notion of increasing marginal lower bounds, with respect to set inclusion (Figure 1). Compliance of (f_t, g_t) is a notion of marginal upper bounds never falling below marginal lower bounds.[12]

[12] One can prove that compliance is equivalent to $e \in E \subseteq I \Rightarrow f_t((I \setminus E) \cup \{e\}) - f_t(I \setminus E) \geq g_t(E) - g_t(E \setminus \{e\})$.

$$
\begin{tikzcd}
E & & & E \cup E' \\
& {} \arrow[r, "f : \geq", "g : \leq"'] & {} & \\
E \cap E' \arrow[uu] & & & E \arrow[uu]
\end{tikzcd}
$$

Figure 1: Submodularity of f and $-g$

The paramodular model includes as special cases single-unit auctions (I_2 is singleton, $f_t(E) = 1$ for all $\varnothing \neq E \subseteq I$ and $g_t \equiv 0$), multiunit auctions (I_2 is singleton, as in Che et al. [7]), and two-person bargainings (I_2 is singleton and, with I simplified to I_1, $g_t\{1,2\} = 1$, $g_t\{1\} = g\{2\} = 0$, and $f_t\{1,2\} = f_t\{1\} = f_t\{2\} = 1$; detailed in Corollary 3, Section 7). The following theorem includes these cases as well as the case of multiple heterogeneous objects and state-dependent allocation constraints.

Theorem 2 *If a constraint structure is paramodular and if $|T|$ is finite, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.*

Proof For any $t \in T$ and any $S \subseteq \mathscr{Z}$ we have $f(S,t) = f_t(I(S,t))$ by (3) and (14), and $g(S,t) = g_t(I(S,t))$ by (4) and (15). From the definition of $I(S,t)$, it is easy to prove that $I(S \cup S', t) = I(S,t) \cup I(S',t)$ and $I(S \cap S', t) = I(S,t) \cap I(S',t)$ for all $S, S' \subseteq \mathscr{Z}$ and all $t \in T$. It then follows from the paramodularity of (f_t, g_t) on 2^I that the pair $(f(\cdot,t), g(\cdot,t))$ is paramodular on $2^{\mathscr{Z}}$. It also follows that the X_t in (16) is isomorphic to

$$
\tilde{X}_t := \left\{ (x_z)_{z\in\mathscr{Z}} \in \mathscr{R}^{\mathscr{Z}} \;\middle|\; \forall S \subseteq \mathscr{Z} \left[g(S,t) \leq \sum_{z \in S} x_z \leq f(S,t) \right] \right\},
$$

because for any $z = (i, t'_{i_1}) \in \mathscr{Z}$ such that $t'_{i_1} \neq t_{i_1}$, $I(\{z\},t) = \varnothing$ and hence $f(\{z\},t) = g(\{z\},t) = 0$, forcing $x_z = 0$.

Thus, for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$ and any $t := (t_{i_1})_{i_1 \in I_1} \in T$, the social planner's problem is equivalent to

$$
\max_{(x_z)_{z\in\mathscr{Z}} \in \mathrm{cv}\tilde{X}_t} \sum_{z \in \mathscr{Z}} x_z \alpha(z). \tag{17}
$$

List all interim states as $\left(z^1, z^2, \cdots, z^{n_*}, \cdots, z^{|\mathscr{Z}|}\right)$ in descending order of $\alpha(z)$, so that

$$
\alpha(z^1) \geq \alpha(z^2) \geq \cdots \geq \alpha(z^{n_*}) \geq 0 > \alpha(z^{n_*+1}) \geq \cdots \geq z(\alpha^{|\mathscr{Z}|}).
$$

For any $n = 1, \ldots, |\mathscr{Z}|$, define

$$S^n := \begin{cases} \{z^k \mid 1 \le k \le n\} & \text{if } n \le n_* \\ \{z^k \mid n \le k \le |\mathscr{Z}|\} & \text{if } n \ge n_* + 1. \end{cases}$$

Since $\tilde{X}_t$ is defined by the paramodular pair $(f(\cdot,t), g(\cdot,t))$, problem (17) is solved by the greedy-generous algorithm (Hassin [15, Theorems 4 & 5]):[13]

$$q^*_{i,t'_{i_1}}(t) = \begin{cases} f(S^n,t) - f(S^{n-1},t) & \text{if } n \le n_* \\ g(S^n,t) - g(S^{n+1},t) & \text{if } n > n_* \end{cases}$$

for all $(i, t'_{i_1}) \in \mathscr{Z}$. The argument t in the notation $q^*_{i,t'_{i_1}}(t)$ reflects the fact that t is parametric to (17). Apply the equation displayed above to (i, t_{i_1}) for all $i \in I$ and suppress the redundant t_{i_1} in the subscript of $q^*_{i,t_{i_1}}(t)$ to obtain, for any $i \in I$ such that $(i, t_{i_1}) = z^n$,

$$q^*_i(t) = \begin{cases} f(S^n,t) - f(S^{n-1},t) & \text{if } n \le n_* \\ g(S^n,t) - g(S^{n+1},t) & \text{if } n > n_*. \end{cases}$$

This, applied to all $t \in T$, is the social planner's solution that Step 1 in our method needs.

By the definition of S^n, $\varnothing = S^0 \subsetneq S^1 \subsetneq S^2 \subsetneq \cdots \subsetneq S^{n_*}$, as the hierarchy $(U^n)_{n=1}^{n^*}$ in (10); and $S^{n_*+1} \supsetneq \cdots \supsetneq S^{|\mathscr{Z}|-1} \supsetneq S^{|\mathscr{Z}|} \supsetneq S^{|\mathscr{Z}|+1} = \varnothing$, as the hierarchy $(L^n)_n$ in (11). Then the q^* obtained above satisfies (12) and (13), because $S^n \setminus S^{n-1} = \{z^n\}$ for all $n \le n_*$, and $S^n \setminus S^{n+1} = \{z^n\}$ for all $n > n_*$. It then follows from Lemma 2 that S^n is upward universally binding for all $n \le n_*$, and downward universally binding for all $n > n_*$.

Consequently, by Theorem 1, it suffices to show that (7) has a nonnegative solution for $((p_+(S^n))_{n=1}^{n_*}, (p_-(S^n)_{n=n_*+1}^{|\mathscr{Z}|}))$. Since all the sets for which p_+ is defined form a nested sequence, and so do all the sets for which p_- is defined, directly solve the system (7) to obtain $p_+(S^n) = \alpha(z^n) - \alpha(z^{n+1})$ for all $n < n_*$, $p_+(S^{n_*}) = \alpha(z^{n_*})$, $p_-(S^{n_*+1}) = -\alpha(z^{n_*+1})$, and $p_-(S^n) = \alpha(z^{n-1}) - \alpha(z^n)$ for all $n > n_* + 1$. ■

We see from the proof that the greedy-generous algorithm is to raise the quantity for a high-α output up to its marginal upper bound, and to reduce the quantity for a low-α one down to its marginal lower bound. The paramodularity assumption is just to ensure that such a hierarchical solution is feasible and optimal. Thus the revealed preference of the planner's choice q^* presents itself as the ordinal ranking of the α values. Consequently, it is trivial to solve (7) for a price system that Theorem 1 requires.

[13] The compliance condition in Hassin's Theorem 4 is meant to be assumed for all subsets rather than only those related by set inclusion.

5 Decomposability

Lang and Yang [20] adopt a total unimodularity assumption in the combinatorial optimization literature (e.g., Edmonds [9], Frank et al. [10], Hoffman [16]) to handle some cases outside the paramodular model. From the perspective of our method, the main drive in total unimodularity is to guarantee that any linear valuation α parametric to the social planner's problem can be decomposed into a conic combination of $\{-1, 0, 1\}$-valued extreme rays. This coupled with a linearity assumption on the constraint structure turns out to imply that the social planner's solution can be rationalized by multiple partial orders, one for each of the $\{-1, 0, 1\}$-valued extreme rays. That achieves Step 2 in our method, the other steps trivial in this case. The outcome is a counterpart to Lang and Yang's characterization.

Consider constraint structures in the form of $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ where $\mathscr{F}, \mathscr{G} \subseteq 2^I$, $\hat{f} : \mathscr{F} \to \mathbb{Z}_+$, $\hat{g} : \mathscr{G} \to \mathbb{Z}_+$, $\hat{f} \geqq \hat{g}$ on $\mathscr{F} \cap \mathscr{G}$, $\varnothing \in \mathscr{F} \Rightarrow \hat{f}(\varnothing) = 0$, $\varnothing \in \mathscr{G} \Rightarrow \hat{g}(\varnothing) = 0$ and X_t is a nonempty compact set X constant to all $t \in T$:

$$X = \left\{ (x_i)_{i \in I} \in \mathbb{Z}^I \;\middle|\; \forall E \in \mathscr{F} \left[\sum_{i \in E} x_i \leq \hat{f}(E) \right]; \forall E \in \mathscr{G} \left[\sum_{i \in E} x_i \geq \hat{g}(E) \right] \right\}. \tag{18}$$

Given $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$, let $\mathcal{P}$ be the set of vectors $(\alpha, \varphi, \gamma) \in \mathbb{R}^{\mathscr{X}} \times \mathbb{R}_+^{\mathscr{F} \times T} \times \mathbb{R}_+^{\mathscr{G} \times T}$ such that

$$\forall i \in I \, \forall t \in T : \alpha(i, t_{i_1}) - \sum_{F \in \mathscr{F}} \varphi(F, t) \chi_F(i) + \sum_{G \in \mathscr{G}} \gamma(G, t) \chi_G(i) = 0. \tag{19}$$

We can think of $\varphi(F, t)$ as the Lagrange multiplier for the ceiling constraint $\sum_{i \in F} x_i \leq \hat{f}(F)$, and $\gamma(G, t)$ the Lagrange multiplier for the floor constraint $\sum_{i \in G} x_i \geq \hat{g}(G)$, in the social planner's problem specifically for the single input t. Note that these Lagrange multipliers are tailored individually for each specific input t. By contrast, the price vector (p_+, p_-) sought after in our method needs to be uniform across all inputs t.

Lang and Yang's *total unimodular* assumption stipulates that the matrix $[\mathbf{M}_1, \mathbf{M}_2]^{\mathrm{T}}$ for which $\mathcal{P} = \{\boldsymbol{v} \mid \mathbf{M}_1 \boldsymbol{v} = \mathbf{0}; \mathrm{M}_2 \boldsymbol{v} \geqq \mathbf{0}\}$ is totally unimodular, namely, the determinant of every square submatrix of every order is 0, 1, or -1. The main implication of this assumption is that every $(\alpha, \varphi, \gamma) \in \mathcal{P}$ is a conic combination of some $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ for which each α_k is $\{-1, 0, 1\}$-valued, and all φ_k and γ_k are $\{0, 1\}$-valued. Instead, the decomposability assumption I propose next requires only the elements of a subset of $\mathcal{P}$ to be conic combinations of some $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ for which only every α_k is required to be $\{-1, 0, 1\}$-valued.

A constraint structure $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ is said to be *decomposable* iff for any $\alpha \in \mathbb{R}^{\mathscr{X}}$ there exists a solution (φ^*, γ^*) to the problem

$$\min_{(\varphi,\gamma)\in\mathbb{R}_+^{\mathscr{F}\times T}\times\mathbb{R}_+^{\mathscr{G}\times T}} \quad \textstyle\sum_{t\in T}\mu\{t\}\left(\sum_{F\in\mathscr{F}}\hat{f}(F)\varphi(F,t)-\sum_{G\in\mathscr{G}}\hat{g}(G)\gamma(G,t)\right) \qquad \text{subject to (19)} \tag{20}$$

such that $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ is a conic combination among some $(\alpha_1, \varphi_1, \gamma_1), \dots, (\alpha_K, \varphi_K, \gamma_K) \in \mathcal{P}$ (for some integer K) and

$$\forall k \in \{1, \dots, K\}\,\forall i \in I\,\forall t_{i_1} \in T_{i_1} : \alpha_k(i, t_{i_1}) \in \{0, 1, -1\}. \tag{21}$$

One can verify that total unimodularity implies decomposability (Appendix A.5).

I also assume another implication of total unimodularity noted by Lang and Yang, that

$$\mathrm{cv}X = \left\{(x_i)_{i\in I} \in \mathbb{R}^I \;\middle|\; \forall E \in \mathscr{F}\left[\sum_{i\in E} x_i \leq \hat{f}(E)\right]; \forall E \in \mathscr{G}\left[\sum_{i\in E} x_i \geq \hat{g}(E)\right]\right\}. \tag{22}$$

That is, the set of randomized ex post allocation outcomes is characterized by $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$.

The other assumption I make is linearity of a function h defined by

$$h(F, G) := \max_{x\in\mathrm{cv}X}\left(\sum_{i\in F} x_i - \sum_{i\in G} x_i\right) \tag{23}$$

for any $F, G \subseteq I$ such that $F \cap G = \varnothing$. The function h is said to be *linear* iff

$$h(F, G) = h(F, \varnothing) + h(\varnothing, G)$$

for all $F, G \subseteq I$ such that $F \cap G = \varnothing$.[14]

Theorem 3 *Assume that $|T|$ is finite, that the construct structure $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ is decomposable and satisfies (22), and that h is linear. Then $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.*

[14] Without the linearity assumption, Lang and Yang's characterization is a condition that needs to be checked for any pair $(S_+, S_-) \in \left(2^{\mathscr{X}}\right)^2$ of sets of interim states such that $S_+ \cap S_- = \varnothing$. By contrast, my characterization, the Border condition $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$, needs only to be checked for any set $S \in 2^{\mathscr{X}}$ of interim states. As noted in their Corollary 1, their condition reduces to mine if the linearity is assumed. Thus the next Theorem 3 is slightly stronger than their main result in conclusions, and corresponds to a slight generalization of their Corollary 1, which also assumes linearity. Supplement B.2 has more on the difference between decomposability and total unimodularity.

Proof To apply Theorem 1, pick any $\alpha \in \mathbb{R}^{\mathscr{Z}}$. By the decomposability assumption, there exist $(\varphi^*, \gamma^*) \in \mathbb{R}_+^{\mathscr{F} \times T} \times \mathbb{R}_+^{\mathscr{G} \times T}$, $(\alpha_k, \varphi_k, \gamma_k)_{k=1}^K$ and $(\beta_k)_{k=1}^K \in \mathbb{R}_{++}^K$ for some integer K such that (φ^*, γ^*) is a solution to problem (20), $(\alpha_k, \varphi_k, \gamma_k) \in \mathcal{P}$ and satisfies (21) for any $k = 1, \ldots, K$, and $\alpha = \sum_{k=1}^K \beta_k \alpha_k$, $\varphi^* = \sum_{k=1}^K \beta_k \varphi_k$, and $\gamma^* = \sum_{k=1}^K \beta_k \gamma_k$. Now that (20) has a solution, its dual

$$\begin{aligned}
\max_{(q_i)_{i \in I} \in (\mathbb{R}^T)^I} \quad & \sum_{t \in T} \sum_{i \in I} \alpha(i, t_{i_1}) q_i(t) \mu\{t\} \\
\text{s.t.} \quad & \left(\hat{f}(F) - \sum_{i \in F} q_i(t) \right) \mu\{t\} \geq 0 \quad (\forall F \in \mathscr{F}\, \forall t \in T) \\
& \left(\sum_{i \in G} q_i(t) - \hat{g}(G) \right) \mu\{t\} \geq 0 \quad (\forall G \in \mathscr{G}\, \forall t \in T),
\end{aligned}$$

has a solution $q^* := (q_i^*)_{i \in I}$. By (22) and $\mu\{t\} > 0$ for all $t \in T$ ($|T| < \infty$), the dual of problem (20) is equivalent to the social planner's problem,

$$\max_{(q(t))_{t \in T} \in \prod_{t \in T} \mathrm{cv} X} \sum_{i \in I} q_i(t) \alpha(i, t_{i_1}).$$

Thus, q^* satisfies (8), which is what Step 1 in our method needs.

For any $k = 1, \ldots, K$, define

$$\begin{aligned}
S^{k,+} &:= \{z \in \mathscr{Z} \mid \alpha_k(z) = 1\}, \\
S^{k,-} &:= \{z \in \mathscr{Z} \mid \alpha_k(z) = -1\}.
\end{aligned}$$

Due to the decomposability and linearity assumptions, one can prove (Appendix A.4):

Claim 1 *For every $k \in \{1, \ldots, K\}$ and every $t := (t_{i_1})_{i_1 \in I_1} \in T$,*

$$\begin{aligned}
\sum_{i \in I} q_i^*(t) \chi_{S^{k,+}}(i, t_{i_1}) &= f\left(S^{k,+}, t\right), \\
\sum_{i \in I} q_i^*(t) \chi_{S^{k,-}}(i, t_{i_1}) &= g\left(S^{k,-}, t\right).
\end{aligned}$$

For every $k \in \{1, \ldots, K\}$, let $\varnothing =: U^0 \subsetneq U^1 := S^{k,+}$, as the hierarchy (10), and $S^{k,-} =: L^1 \supsetneq L^0 := \varnothing$, as the hierarchy (11). Since $U^1 \setminus U^0 = U^1 = S^{k,+}$ and $L^1 \setminus L^0 = L^1 = S^{k,-}$, the two equations in Claim 1 are the same as the (12) and (13) in Lemma 2. It then follows from the lemma that $S^{k,+}$ is upward universally binding, and $S^{k,-}$ downward universally binding, for all $k = 1, \ldots, K$. Thus we obtain the supports for p_+ and p_- that Step 2 in our method needs. To achieve Step 3 in the method, simply use the conic combination condition

$$\forall i \in I \forall t_{i_1} \in T_{i_1} : \alpha(i, t_{i_1}) \;=\; \sum_{k=1}^{K} \beta_k \left(\chi_{S^{k,+}}(i, t_{i_1}) - \chi_{S^{k,-}}(i, t_{i_1})\right)$$

to obtain a solution for (7): $p_+(S^{k,+}) = p_-(S^{k,-}) = \beta_k$ for all k and $p_+(S) = p_-(S) = 0$ for all other subsets S of $\mathscr{X}$. Thus, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ follows from Theorem 1. ■

As said earlier, the drive in the decomposability assumption is to decompose any linear valuation for the social planner's problem into a conic combination of $\{-1, 0, 1\}$-valued valuations. That means the social planner's choice can be rationalized by multiple preference relations among the interim states (outputs), each partitioning the outputs into only three indifference sets, the good (those z with $\alpha_k(z) = 1$), the bad (those with $\alpha_k(z) = -1$), and the neutral ($\alpha_k(z) = 0$). They give the upper or lower contour sets that our method needs.

6 Assignments

It is known that the paramodular model cannot cover assignment problems, due to the latter's typical constraint that a bidder cannot get more than one unit of an object (Supplement B.3). While the total unimodular model allows for such constraint structures, its application is restricted to at most two possible types per bidder.[15] This section thus considers assignment models between N objects and two bidders such that each bidder has any finite number of types, and $N \geq 2$. Our method applies nontrivially. The support for the price system is derived from multiple partial orders, each rationalizing the social planner's solution locally. The existence proof of the price system relies on the hyper-rectangle cover theory in Lemma 3.

Let the set of bidders be $I_1 := \{1, 2\}$, and the set of objects $I_2 := \{1, \ldots, N\}$, $N \geq 2$. There is exactly one unit for each object, indivisible. An allocation outcome is in the form of $((x_{kj})_{k=1}^{2})_{j=1}^{N} \in \{0, 1\}^{2N}$, signifying bidder k getting object j.[16] It is convenient to represent an allocation outcome $((x_{kj})_{k=1}^{2})_{j=1}^{N}$ equivalently as a set $M \subseteq I$ $(= I_1 \times I_2)$ such that

[15] Although one can show that the decomposability assumption, an implication of and substitute for total unimodularity, includes paramodularity as a special case and hence allows for more than two types per bidder (Supplement B.2), it is unknown whether its application beyond paramodularity can do so.

[16] In what follows the index for bidders and that for objects will play distinct roles. Thus bidders will often be indexed by k, and objects by j.

$(k,j) \in M \iff x_{k,j} = 1$ (note $x_{k,j} \neq 1 \Rightarrow x_{k,j} = 0$). The assignment constraint,

$$\forall k \in I_1 : \quad \textstyle\sum_{j \in I_2} x_{kj} \leq 1 \tag{24}$$

$$\forall j \in I_2 : \quad \textstyle\sum_{k \in I_1} x_{kj} \leq 1, \tag{25}$$

is then equivalent to the condition

$$M = \{(k,j),(k',j')\} \Longrightarrow [k \neq k', j \neq j'] \, . \tag{26}$$

6.1 Full Assignment

In the *full assignment* model, the set X of feasible allocation outcomes is constant to all $t \in T$ and is defined as the set of all $((x_{kj})_{k=1}^{2})_{j=1}^{N} \in \{0,1\}^{2N}$ that satisfy (24), (25), and the full assignment condition that

$$\forall k \in I_1 : \sum_{j \in I_2} x_{kj} \geq 1. \tag{27}$$

This coupled with (24) requires that each bidder get exactly one object. If each $x \in X$ is represented as a subset of I as specified in the preamble, X is equivalent to the set $\mathscr{M}$ of two-element subsets M of I that satisfy (26).

Theorem 4 *In the full assignment model such that $|I_1| = 2$ and $|T|$ is finite, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.*

The proof follows the road map in Section 3. For Step 1, pick any $\alpha \in \mathbb{R}^{\mathscr{X}}$ and consider the social planner's problem for any ex post state (input) $t := (t_1, t_2) \in T$:

$$\max_{x \in \mathrm{cv} X} \sum_{(k,j) \in I} x_{k,j} \alpha(k,j,t_k) = \max_{x \in X} \sum_{(k,j) \in I} x_{k,j} \alpha(k,j,t_k) = \max_{M \in \mathscr{M}} \sum_{(k,j) \in M} \alpha(k,j,t_k),$$

where the first equality is due to the linear programming with X finite, and the second equality due to the definition of $\mathscr{M}$. Obviously the problem is solved by coupling the first- or second-highest $\alpha(1,j,t_1)$ among $j \in I_2$ with the first- or second-highest $\alpha(2,j,t_2)$ among $j \in I_2$ such that the couple have different j-coordinates. This is illustrated in the next table (with three objects), where the entry at row j and column (k, t_k) signifies $\alpha(k,j,t_k)$. The social planner's solution is $\{(1,2),(2,1)\}$ (giving good 2 to bidder 1, and good 1 to bidder 2) when the ex post state is (t_1,t_2), and $\{(1,3),(2,2)\}$ (giving good 3 to bidder 1, and good 2 to bidder 2) when the state is (t_1, t'_2).

	$(1,t_1)$	$(2,t_2)$	$(1,t_1)$	$(2,t_2')$
1	-1	3	-1	$1/2$
2	4	0	4	3
3	2	$1/2$	2	0

To define a solution to the social planner's problem in general, let

$$\begin{aligned} j^1(k,t_k) &:= \min\left(\arg\max_{j\in I_2}\alpha(k,j,t_k)\right) \\ j^2(k,t_k) &:= \min\left(\arg\max_{j=I_2\setminus\{j^1(k,t_k)\}}\alpha(k,j,t_k)\right) \end{aligned}$$

for any $t := (t_1,t_2) \in T$ and any $k \in \{1,2\}$. For any $j \in I_2$ $(= \{1,\ldots,N\})$, let

$$\delta(k,j,t_k) := \alpha(k,j,t_k) - \max_{j'\in I_2\setminus\{j\}}\alpha(k,j',t_k). \tag{28}$$

Define $M_*(t_1,t_2) \in \mathscr{M}$ by:

a. if $j^1(1,t_1) \neq j^1(2,t_2)$, let $M_*(t_1,t_2) := \{(1,j^1(1,t_1)),(2,j^1(2,t_2))\}$;

b. else $(j^1(1,t_1) = j^1(2,t_2))$ then:

   i. if $\delta(1,j^1(1,t_1),t_1) \geq \delta(2,j^1(1,t_1),t_2)$ $(= \delta(2,j^1(2,t_2),t_2))$, let

$$M_*(t_1,t_2) := \left\{(1,j^1(1,t_1)),(2,j^2(2,t_2))\right\};$$

   ii. else $(\delta(1,j^1(1,t_1),t_1) < \delta(2,j^1(1,t_1),t_2))$, let

$$M_*(t_1,t_2) := \left\{(1,j^2(1,t_1)),(2,j^1(2,t_2))\right\}.$$

In other words, if the highest $\alpha(1,j,t_1)$ among all j and the highest $\alpha(2,j,t_2)$ among all j are of different objects, couple them as the solution (Case (a)). Otherwise (Case (b)), they refer to the same object and so coupling them is infeasible with respect to (26); thus the solution is either coupling the highest $\alpha(1,j,t_1)$ with the second highest $\alpha(2,j,t_2)$, or coupling the second highest $\alpha(1,j,t_1)$ with the highest $\alpha(2,j,t_2)$, whichever is larger in total α-values, and break the tie by favoring bidder 1. To see that case (b) is exactly what is said, note from (28) and the definition of j^2 that $\delta(1,j^1(1,t_1),t_1) \geq \delta(2,j^1(2,t_2),t_2)$ is equivalent to

$$\alpha(1,j^1(1,t_1),t_1) + \alpha(2,j^2(2,t_2),t_2) \geq \alpha(2,j^1(2,t_2),t_2) + \alpha(1,j^2(1,t_1),t_1).$$

It is clear that $M_*(t)$ is a solution to the social planner's problem for every $t \in T$ and hence M_* corresponds to the q^* that Step 1 in our method needs.

Step 2 is to construct a family of partial revealed preferences of the social planner's solution M_* that is rich enough to cover every interim state. Every interim state (k, j, t_k) competes with the other interim states that refer to the same object j (due to the feasibility condition (25)) and with those that refer to the same bidder-type (k, t_k) (condition (24). Thus, for each "row" consisting of the interim states referring to a same object j, and each "column" referring to a same bidder-type (k, t_k), we construct a partial order.

For any $k \in I_1$ and any $t_k \in T_k$, both the top contender $j^1(k, t_k)$ and the second highest contender $j^2(k, t_k)$ in the column $\{k\} \times I_2 \times \{t_k\}$ may get coupled with some $(-k, j')$ to be the solution in $M_*(t_k, t_{-k})$ for some t_{-k}, while the other elements of the column have no chance by the definition of M_*. Thus define a binary relation $\succeq_{k,t_k}$ by:

i. $j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k)$;

ii. for any $\{j, j'\} \neq \{j^2(k, t_k), j^1(k, t_k)\}$, let $j \succ_{k,t_k} j'$ iff

$$\alpha(k, j, t_k) > \alpha(k, j', t_k) \text{ or } [\alpha(k, j, t_k) = \alpha(k, j', t_k) \text{ and } j < j'] .$$

List the elements of I_2 (cardinality N) as

$$j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k) \succ_{k,t_k} j^3 \succ_{k,t_k} j^4 \succ_{k,t_k} \cdots \succ_{k,t_k} j^N. \tag{29}$$

Consider the upper contour sets $(V^n_{k,t_k})^N_{n=2}$, and a lower contour set L_{k,t_k}, with respect to $\succeq_{k,t_k}$:

$$\begin{aligned} V^n_{k,t_k} &:= \{k\} \times \left(\{j^1(k, t_k), j^2(k, t_k)\} \cup \{j^m \mid 3 \leq m \leq n\}\right) \times \{t_k\} \quad (\forall n = 2, \ldots, N) \\ L_{k,t_k} &:= \{k\} \times I_2 \times \{t_k\}. \end{aligned}$$

By Lemma 2, V^n_{k,t_k} is upward universally binding, and L_{k,t_k} downward universally binding, for any $k \in I_1$, any $t_k \in T_k$ and any $n \in \{2, \ldots, N\}$ (Lemma 9, Appendix A.6). The proof is similar to (albeit lengthier) its counterparts in the previous sections.

For any $j \in I_2$, among the interim states in the row j, the rivalry is only between top contenders: Given any ex post state (t_k, t_{-k}), if (k, j, t_k) is the top contender from its column while the other $(-k, j, t_{-k})$ is not, the top contender gets chosen by $M_*(t_k, t_{-k})$ to be paired with a top contender that belongs to a different row than j. Thus, within the subset

$$\mathscr{Z}_j := \{(k, j, t_k) \mid k \in I_1; t_k \in T_k; j = j^1(k, t_k)\}$$

of the row, define $\succ_j$ by: for any $(k, j, t_k), (k', j, t'_{k'}) \in \mathscr{Z}_j$, let $(k, j, t_k) \succ_j (k', j, t'_{k'})$ iff:

i. either $\delta(k, j, t_k) > \delta(k', j, t'_{k'})$

ii. or $\delta(k, j, t_k) = \delta(k', j, t'_{k'})$ and $k < k'$ (i.e., $k = 1$ and $k' = 2$)

iii. or $\delta(k, j, t_k) = \delta(k', j, t'_{k'})$ and $k = k'$ and $t_k \rhd t'_k$ ($\rhd$ being a strict total order on T_k).

Conditions (i) and (ii), together with the definition of M_*, guarantee that $(k, j, t_k) \succ_j (-k, j, t_{-k})$ is equivalent to $(k, j) \in M_*(t_k, t_{-k})$ for any $(k, j, t_k), (-k, j, t_{-k}) \in \mathscr{Z}_j$. Condition (iii) is to break the tie between two interim states that refer to the same bidder k. Although (k, j, t_k) and (k, j, t'_k) are not in rivalry, as any ex post state t'' is compatible with at most one of them, we need to rank them for $\succ_j$ to be a total order on $\mathscr{Z}_j$.

It is clear from the definition that $\succ_j$ is a total order on $\mathscr{Z}_j$. Thus, enumerate the elements of $\mathscr{Z}_j$ in descending order of $\succ_j$ so that

$$z^1 \succ_j z^2 \succ_j \cdots \succ_j z^{|\mathscr{Z}_j|}. \tag{30}$$

Correspondingly, for every $n = 1, \ldots, |\mathscr{Z}_j|$ define the upper contour set of z^n as

$$U_j^n := \{z^1, \ldots, z^n\}.$$

By Lemma 2, U_j^n is upward universally binding for any $j \in I_2$ and any $n = 1, \ldots, |\mathscr{Z}_j|$ (Lemma 10, Appendix A.6).

In Figure 2, the interim state (k, j^1, t_k) is the top contender within its column. Thus it is contained in all the upper contour sets V_{k,t_k}^m in the column. As a top contender, (k, j^1, t_k) belongs to the row $\mathscr{Z}_j$ for which $j = j^1$. Within $\mathscr{Z}_j$, it ranks third, labeled as z^3. Thus it is not contained in the upper contour sets U_j^1 or U_j^2, but it is in the other U_j^n's in that row. In Figure 3, by contrast, (k, j^3, t_k) is not a top contender, but rather ranks third in its column. Thus it is contained only by the V_{k,t_k}^n for which $n \geq 3$. Regardless of their ranks in the column, however, all the members (k, j^n, t_k) in the column are contained by L_{k,t_k}, as it is defined to be the entire column.

Following Step 3 in our road map, let

$$\begin{aligned}
\mathscr{S}_+ &:= \left\{U_j^n \mid j \in I_2; n = 1, \ldots, |\mathscr{Z}_j|\right\} \bigcup \left\{V_{k,t_k}^n \mid k \in I_1; t_k \in T_k; n = 2, \ldots, N\right\} \\
\mathscr{S}_- &:= \left\{L_{k,t_k} \mid k \in I_1; t_k \in T_k\right\}.
\end{aligned}$$

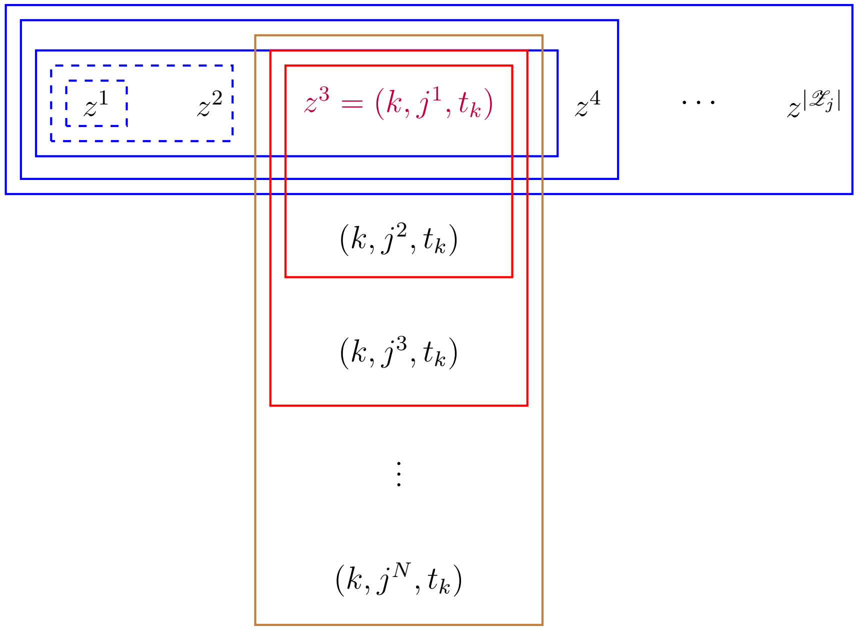

Figure 2: Solid-line boxes: the U_j^n (blue) or V_{k,t_k}^m (red or brown) that contain z^3; dotted-line boxes: upper contour sets that do not contain z^3; the brown box is also L_{k,t_k}

We prove existence of price vectors p_+ and p_- supported by $\mathscr{S}_+$ and $\mathscr{S}_-$ respectively. According to Lemma 3, let $[\mathbf{M}_+, \mathbf{M}_-]$ be the matrix defined there with respect to the $\mathscr{S}_+$ and $\mathscr{S}_-$ here. For any $z \in \mathscr{Z}$, denote $[z]$ for the row of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ corresponding to z, so $[z](S)$ is the entry in the row at the intersection with column S, and $[z](-\boldsymbol{\alpha}) := -\alpha(z)$.

By Lemma 3, to verify the condition required in Theorem 1 it suffices to prove that no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ can produce any nonnegative row whose entry at the $-\boldsymbol{\alpha}$ position is positive. Since any Gaussian elimination on a matrix corresponds to a linear combination of its rows, it suffices to prove that there exist no $Z \subseteq \mathscr{Z}$ and $(\beta_z)_{z \in Z} \in (\mathbb{R} \setminus \{0\})^Z$ for which

$$\sum_{z \in Z} \beta_z [z](S) \geq 0 \quad \forall S \in \mathscr{S}_+ \sqcup \mathscr{S}_- \quad \text{and} \tag{31}$$

$$\sum_{z \in Z} \beta_z \alpha(z) < 0. \tag{32}$$

Thus, to prove Theorem 4, suppose (31) for some $\varnothing \neq Z \subseteq \mathscr{Z}$ and some $(\beta_z)_{z \in Z} \in (\mathbb{R} \setminus \{0\})^Z$, and we shall prove that (32) does not hold.

Intuitively speaking, suppose that the negative outcome (32) could be achieved in one operation. That would involve subtracting a row $[z]$ in $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ from another

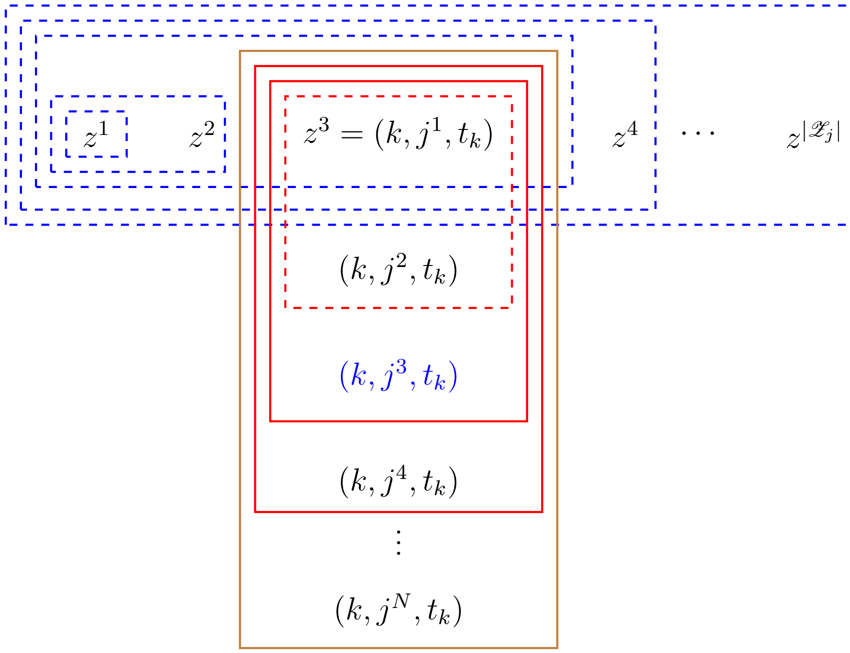

Figure 3: Solid-line boxes: the upper or lower contour sets that contain (k, j^3, t_k)

row $[z']$ for which $\alpha(z) > \alpha(z')$. Meanwhile, for the operation to satisfy the nonnegativity condition (31), for every $S \in \mathscr{S}_+$ the entry $[z](S)$ in row $[z]$ has to be no greater than the corresponding entry $[z'](S)$ in row $[z']$. That is, $z \in S \Rightarrow z' \in S$. Thus the likely case is that z and z' belong to the same "column" $\{k\} \times I_2 \times \{t_k\}$, where the upper contour sets are nested, as the (k, j^m, t_k)s in Figures 2 and 3. Then either $z' \succ_{k,t_k} z$ (so that any V^m_{k,t_k} that contains z also contains z') or $z \sim_{k,t_k} z'$ (so z and z' are contained by the same family of V^m_{k,t_k}s). Since $\alpha(z) > \alpha(z')$, $z' \succ_{k,t_k} z$ is impossible. Thus the only possibility is $z \sim_{k,t_k} z'$. That is, z is the top contender in $\{k\} \times I_2 \times \{t_k\}$, and z' the second-highest one therein. Now that z is on the top (like the z^3 in Figure 2), it belongs to some upper contour set U^n_j of the top rivals in $\mathscr{Z}_j$. But since the operation is $[z'] - [z]$, it produces a row whose entry at the U^n_j position is negative, $-[z](U^n_j) = -1$. Therefore, to maintain the nonnegativity condition (31), we need to add to $[z'] - [z]$ another row $[z'_*]$ in $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ so that $[z'_*](U^n_j) \geq [z](U^n_j)$. That means $z'_* \succ_j z$, since the upper contour sets in $\mathscr{Z}_j$ are nested. Then the definition of $\succ_j$ implies that $\delta(z'_*) \geq \delta(z)$. In other words, the α-value differential of z'_*, compared to its highest rival say z_*, is no less than the α-value differential of z compared to its highest rival, which is z'. Thus, even if we minimize the positive contribution of z'_* by subtracting from $[z'_*]$ its highest rival $[z_*]$, the "difference of differences" $[z'_*] - [z_*] - ([z] - [z'])$ would still have a nonnegative net value of α, to the opposite of (32).

The formal argument is just to decompose the sum in (31) into a conic combination of such "difference of differences" quadruples so that (32) cannot hold. The first step is to rearrange the sum in (31) into a conic combination of the differences $[z'] - [z]$ such that z and z' belong to the same "column" ($\{k\} \times I_2 \times \{t_k\}$ for some k and some t_k).

Lemma 4 *For any $\varnothing \neq Z \subseteq \mathscr{Z}$ and any $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$ that satisfy (31), there exist a finite set H and a positive vector $(\tilde{\beta}_h)_{h\in H} \in \mathbb{R}^H_{++}$ for which*

$$\sum_{z\in Z} \beta_z[z] = \sum_{h\in H} \tilde{\beta}_h \left([z'_h] - [z_h]\right) \tag{33}$$

such that for every $h \in H$ there exist $k \in I_1$ and $t_k \in T_k$ for which $z_h = (k, j, t_k)$ and $z'_h = (k, j', t_k)$ for some $j, j' \in I_2$ such that $j' \succeq_{k,t_k} j$ ($\succ_{k,t_k}$ or $\sim_{k,t_k}$) and $j' \neq j$.

Proof For any $k \in I_1$ and any $t_k \in T_k$, define

$$Z_{k,t_k} := \{z \in Z \mid z = (k, j, t_k) \text{ for some } j \in I_2\} \quad (= Z \cap (\{k\} \times I_2 \times \{t_k\})).$$

Note that Z can be partitioned into the family of disjoint subsets Z_{k,t_k} across k and across t_k. Thus, it suffices to prove (33) in the case where Z is replaced by Z_{k,t_k} for any $k \in I_1$ and any $t_k \in T_k$ such that $Z_{k,t_k} \neq \varnothing$. The rest of the proof therefore focuses on such a Z_{k,t_k}.

By the definitions of V^n_{k,t_k}, L_{k,t_k}, and the matrix $[\mathbf{M}_+, \mathbf{M}_-]$, if $z \notin \{k\} \times I_2 \times \{t_k\}$ then $[z](V^n_{k,t_k}) = [z](L_{k,t_k}) = 0$ for all n. It then follows from (31) that

$$\left[S = V^n_{k,t_k} \text{ or } S = L_{k,t_k}\right] \Longrightarrow \sum_{z\in Z_{k,t_k}} \beta_z[z](S) \geq 0. \tag{34}$$

Define

$$\begin{aligned} Z^- &:= \{z \in Z_{k,t_k} \mid \beta_z < 0\} \\ Z^+ &:= Z_{k,t_k} \setminus Z^-. \end{aligned}$$

If $Z^- \neq \varnothing$, let $Z^+_* := Z^+$ and $H := \varnothing$ to obtain (39) at the end of this proof. Thus, assume $Z^- \neq \varnothing$ without loss. For any $n \in \{2, 3, \ldots, N\}$, define

$$\begin{aligned} \psi_+(n) &:= \sum_{z\in Z^+} \beta_z[z](V^n_{k,t_k}) \\ \psi_-(n) &:= \sum_{z\in Z^-} |\beta_z|[z](V^n_{k,t_k}). \end{aligned}$$

By (34), $\psi_+ \geqq \psi_-$ on $\{2,3,\dots,N\}$.

To prove (33), construct the index set H recursively. Start with the $\succ_{k,t_k}$-minimum within Z^- and denote it by z_1. Let n_1 be the rank of z_1 in the list (29), so that $z_1 = (k, j^{n_1}, t_k)$ if $n_1 > 2$, and z_1 is either $(k, j^1(k,t_k), t_k)$ or $(k, j^2(k,t_k), t_k)$ if $n_1 = 2$. Observe that there exists $z' \in Z^+$ for which $z' \succeq_{k,t_k} z_1$. Otherwise, as $(V^n_{k,t_k})_{n=2}^N$ is nested,

$$\psi_+(n') = 0 < |\beta_{z_1}| = |\beta_{z_1}|[z_1](V^n_{k,t_k}) \le \psi_-(n')$$

for all $2 \le n' \le n_1$, contradicting the fact $\psi_+ \geqq \psi_-$. Thus, let z_1' be the $\succeq_{k,t_k}$-minimum in $\{z' \in Z^+ \mid z' \succeq_{k,t_k} z_1\}$. Let

$$\tilde{\beta}_1 := \min\left\{|\beta_{z_1}|, |\beta_{z_1'}|\right\}\ \left(= \min\left\{-\beta_{z_1}, \beta_{z_1'}\right\}\right).$$

Claim: For any $n \in \{2,3,\dots,N\}$,

$$\psi_+(n) - \tilde{\beta}_1[z_1'](V^n_{k,t_k}) \ge \psi_-(n) - \tilde{\beta}_1[z_1](V^n_{k,t_k}). \tag{35}$$

To show that, let n_1' be the rank of z_1' in the list (29), so that $z_1' = (k, j^{n_1'}, t_k)$ if $n_1' > 2$, and z_1' is either $(k, j^1(k,t_k), t_k)$ or $(k, j^2(k,t_k), t_k)$ if $n_1' = 2$. Since $z_1' \succeq_{k,t_k} z_1$, $n_1' \le n_1$. Ineq. (35) follows from $\psi_+ \geqq \psi_-$ for all $n < n_1'$, where $[z_1'](V^n_{k,t_k}) = 0 = [z_1](V^n_{k,t_k}) = 0$, and it follows for all $n \ge n_1$, where $[z_1'](V^n_{k,t_k}) = 1 = [z_1](V^n_{k,t_k})$. For any $n_1' \le n < n_1$,

$$\psi_-(n) - \tilde{\beta}_1[z_1](V^n_{k,t_k}) = \psi_-(n) \le \psi_-(n_1) - |\beta_{z_1}| \le \psi_-(n_1) - \tilde{\beta}_1 \le \psi_+(n_1) - \tilde{\beta}_1 = \psi_+(n) - \tilde{\beta}_1,$$

where the "=" to the right is because $n_1' \le n < n_1$ and z_1' is the $\succeq_{k,t_k}$-minimum in $\{z' \in Z^+ \mid z' \succeq_{k,t_k} z_1\}$ (so the step function ψ_+ has no jump point between n_1' and n_1 except n_1'). Thus (35) follows.

Note that the sum $\sum_{r \in Z_{k,t_k}} \beta_r[r]$ is equal to

$$\tilde{\beta}_1\left([z_1'] - [z_1]\right) + \underbrace{\sum_{r \in Z^+ \setminus \{z_1'\}} \beta_r[r] + \sum_{r \in Z^- \setminus \{z_1\}} \beta_r[r] + \left(\beta_{z_1'} - \tilde{\beta}_1\right)[z_1'] + \left(\beta_{z_1} + \tilde{\beta}_1\right)[z_1]}_{R}. \tag{36}$$

Initiate the value of H by $H := \{1\}$. Let $\tilde{Z}^+ := Z^+$ if $\beta_{z_1'} - \tilde{\beta}_1 \neq 0$, and $\tilde{Z}^+ := Z^+ \setminus \{z_1'\}$ if $\beta_{z_1'} - \tilde{\beta}_1 = 0$. Analogously, let $\tilde{Z}^- := Z^-$ if $\beta_{z_1} + \tilde{\beta}_1 \neq 0$, and $\tilde{Z}^- := Z^- \setminus \{z_1\}$ if $\beta_{z_1} + \tilde{\beta}_1 = 0$. Then the previous reasoning on $\sum_{r \in Z_{k,t_k}} \beta_r[r]$, extracting z_1 and z_1' from Z^- and Z^+ due to $\psi_+ \geqq \psi_-$, applies to the sum R in (36) thereby extracting z_2 and z_2' respectively from $\tilde{Z}^-$

and $\tilde{Z}^+$ due to (35). Just like $z_1' \succeq_{k,t_k} z_1$, we have $z_2' \succeq_{k,t_k} z_2$. Update $H := H \sqcup \{2\}$, and update the values of β by

$$\beta_{z_1} := -(|\beta_{z_1}| - \tilde{\beta}_1) \tag{37}$$

$$\beta_{z_1'} := \beta_{z_1'} - \tilde{\beta}_1. \tag{38}$$

Then define $\tilde{\beta}_2 := \min\{|\beta_{z_2}|, |\beta_{z_2'}|\}$. Rewrite the R in (36) analogously to (36). Thus, $\sum_{r\in Z_{k,t_k}} \beta_r[r]$ is equal to

$$\sum_{h\in H} \tilde{\beta}_h\left([z_h'] - [z_h]\right) + \underbrace{\sum_{r\in\tilde{Z}^+\setminus\{z_2'\}} \beta_r[r] + \sum_{r\in\tilde{Z}^-\setminus\{z_2\}} \beta_r[r] + (\beta_{z_2'} - \tilde{\beta}_2)[z_2'] + (\beta_{z_2} + \tilde{\beta}_2)[z_2]}_{R'}.$$

Then apply the previous reasoning to the sum R' and extract z_3' and z_3 from the remainders in $\tilde{Z}^+$ and $\tilde{Z}^-$, so that $z_3' \succeq_{k,t_k} z_3$. Continue the procedure until all the elements of Z^- are removed (to become a z_h for some h in the updated H). Let Z_*^+ be the set of the elements of Z^+ that have not been removed. Then

$$\sum_{r\in Z_{k,t_k}} \beta_r[r] = \sum_{h\in H} \tilde{\beta}_h\left([z_h'] - [z_h]\right) + \sum_{z'\in Z_*^+} \beta_{z'}[z'], \tag{39}$$

where $\beta_{z'}$ has been updated in every step in the same manner as (37) and (38) if a portion of $\beta_{z'}[z']$ has been used to cover some element of Z^-. Thus, in (39), $\beta_{z'} \geq 0$ for all $z' \in Z_*^+$.

Finally, $\sum_{z'\in Z_*^+} \beta_{z'}[z'] = \mathbf{0}$. That is because every $z' \in Z_*^+$ belongs to Z_{k,t_k} and hence is contained by L_{k,t_k}. Thus $[z'](L_{k,t_k}) = -1$ by the definition of the matrix $\mathbf{M}_-$. Consequently, if $Z_*^+ \neq \varnothing$ then (34) implies $\beta_{z'} = 0$ for all $z' \in Z_*^+$. Then (39) becomes (33), as desired. ■

The second step is to rearrange the right-hand side of (33), a conic combination of differences, into a conic combination of "differences of differences." For any $j \in I_2$ and given the set H obtained by Lemma 4, define

$$H_j := \{h \in H \mid z_h \in \mathscr{Z}_j \text{ or } z_h' \in \mathscr{Z}_j\}.$$

Note: $H = \bigsqcup_{j\in I_2} H_j$. That is because for any (k, t_k) there is at most one $j \in I_2$ for which $j = j^1(k, t_k)$ (by the definition of j^1). Since z_h and z_h' refer to the same (k, t_k) (Lemma 4), it is impossible for $z_h \in \mathscr{Z}_j$ and $z_h' \in \mathscr{Z}_{j'}$ for some $j, j' \in I_2$. Thus, the sum in (33) is equal to $\sum_{j\in I_2}\sum_{h\in H_j} \tilde{\beta}_h\left([z_h'] - [z_h]\right)$. The next lemma is to reorganize each $\sum_{h\in H_j} \tilde{\beta}_h\left([z_h'] - [z_h]\right)$. The proof is quite analogous to that of Lemma 4.

Lemma 5 *Assume the conclusion of Lemma 4. Then for any $j \in I_2$ there exist $W \subset H_j^2$ and $H_* \subseteq H_j$ and a nonnegative vector $(\widehat{\beta}_w)_{w \in W \sqcup H_*} \in \mathbb{R}_+^{W \sqcup H_*}$ for which*

$$\sum_{h \in H_j} \tilde{\beta}_h \left([z'_h] - [z_h]\right) = \sum_{(w,w_*) \in W} \widehat{\beta}_w \left([z'_{w_*}] - [z_{w_*}] - ([z_w] - [z'_w])\right) + \sum_{h \in H_*} \widehat{\beta}_h \left([z'_h] - [z_h]\right) \quad (40)$$

such that, for any $(w, w_*) \in W$, $z'_{w_*}, z_w \in \mathscr{Z}_j$ and $z'_{w_*} \succeq_j z_w$, and for any $h \in H_*$, $z_h \notin \mathscr{Z}_j$.

Proof Let $j \in I_2$. By the definitions of U_j^n and the matrix $[\mathbf{M}_+, \mathbf{M}_-]$, if $h \notin H_j$ then neither z_h nor z'_h belong to $\mathscr{Z}_j$ and hence $[z'_h](U_j^n) - [z_h](U_j^n) = 0$ for all n. Thus it follows from (31) and (33) that

$$\forall n \in \{1, \ldots, |\mathscr{Z}_j|\} : \sum_{h \in H_j} \tilde{\beta}_h \left([z'_h](U_j^n) - [z_h](U_j^n)\right) \geq 0. \quad (41)$$

As explained at the preamble, z'_h and z_h cannot be both in $\mathscr{Z}_j$. Thus $H_j = H_j^+ \sqcup H_j^-$ where

$$\begin{aligned} H_j^+ &:= \{h \in H_j \mid z'_h \in \mathscr{Z}_j\} \\ H_j^- &:= \{h \in H_j \mid z_h \in \mathscr{Z}_j\}. \end{aligned} \quad (42)$$

If $H_j^- = \varnothing$, then set $H_* := H$ and we have $z_h \notin \mathscr{Z}_j$ for all $h \in H_*$, and hence (40) is true with $W := \varnothing$. Thus assume, without loss, that $H_j^- \neq \varnothing$. Since $H_j^+ \cap H_j^- = \varnothing$, (41) becomes

$$\sum_{h \in H_j^+} \tilde{\beta}_h \left([z'_h](U_j^n) - [z_h](U_j^n)\right) \geq \sum_{h \in H_j^-} \tilde{\beta}_h \left([z_h](U_j^n) - [z'_h](U_j^n)\right)$$

for any n. Since z'_h and z_h cannot be both in $\mathscr{Z}_j$, the $[z_h](U_j^n)$ on the left-hand side, and the $[z'_h](U_j^n)$ on the right-hand side, are both zero. Thus, for any $n = 1, \ldots, |\mathscr{Z}_j|$,

$$\sum_{h \in H_j^+} \tilde{\beta}_h [z'_h](U_j^n) \geq \sum_{h \in H_j^-} \tilde{\beta}_h [z_h](U_j^n). \quad (43)$$

Then mimic the proof of (39) in that of Lemma 4. The (Z^+, Z^-) there is replaced by (H_j^+, H_j^-) here, Z_*^+ replaced by H_* here, the function ψ_+ replaced by $n \mapsto \sum_{h \in H_j^+} \tilde{\beta}_h [z'_h](U_j^n)$, and the function ψ_- replaced by $n \mapsto \sum_{h \in H_j^-} \tilde{\beta}_h [z_h](U_j^n)$. Thus, the property $\psi_+ \geqq \psi_-$ is preserved as (43). With this property, successively extract the element h_m from H_j^- such that z_{h_m} is the $\succ_j$-minimum among those z_h for which h remains in H_j^-. Correspondingly, extract the element h_m^* from H_j^+ such that $z'_{h_m^*}$ is the $\succ_j$-minimum among those z'_h for which h remains in H_j^+ and $z'_h \succ_j z_{h_m}$. Add (h_m, h_m^*) to the set W recursively (which is legitimate because (h_m, h_m^*) is distinct from the incumbents in W, as each round of extraction exhausts the weight of at least one of z_{h_m} and $z'_{h_m^*}$). Define $\widehat{\beta}_{h,h^*} := \min\left\{\tilde{\beta}_h, \tilde{\beta}_{h^*}\right\}$ for any $(h, h^*) \in W$. Repeat until the weights $\tilde{\beta}_h$ of all elements of H_j^- are exhausted. Then let H_* be the set of the elements in H_j^+ whose weights $\tilde{\beta}_h$ are not exhausted. Thus (40) obtains. For any $h \in H_*$, h does not belong to the updated H_j^- as it has become empty, all elements having been extracted. Thus $z_h \notin \mathscr{Z}_j$ by (42). ■

To complete the proof of Theorem 4, combine (33) and (40), together with the fact that the sum in (33) equals $\sum_{j\in I_2}\sum_{h\in H_j}\tilde{\beta}_h\left([z'_h] - [z_h]\right)$ explained prior to Lemma 5, to have

$$\sum_{z\in Z}\beta_z[z] = \sum_{(w,w_*)\in W}\widehat{\beta}_w\left([z'_{w_*}] - [z_{w_*}] + [z'_w] - [z_w]\right) + \sum_{h\in H_*}\widehat{\beta}_h\left([z'_h] - [z_h]\right) \tag{44}$$

such that, for any $h \in H_*$, $z_h \notin \mathscr{Z}_j$, and for any $(w, w_*) \in W$, $z'_{w_*} \succeq_j z_w$ for some $j \in I_2$, and $z'_{w_*} \succeq_{k^*,t^*_{k^*}} z_{w_*}$ and $z'_w \succeq_{k,t_k} z_w$ for some $(k^*, t^*_{k^*})$ and some (k, t_k).

For any $h \in H_*$, $z'_h \succeq_{k',t'_{k'}} z_h$ for some $(k', t'_{k'})$ by Lemma 4. By Lemma 5, $z_h \notin \mathscr{Z}_j$. Thus z_h is not the top contender in the "column" $\{k'\} \times I_2 \times \{t'_{k'}\}$, namely, $\alpha(z'_h) \geq \alpha(z_h)$. This being true for all $h \in H_*$, we have

$$\sum_{h\in H_*}\widehat{\beta}_h\left([z'_h](-\boldsymbol{\alpha}) - [z_h](-\boldsymbol{\alpha})\right) = \sum_{h\in H_*}\widehat{\beta}_h\left(-\alpha(z'_h) + \alpha(z_h)\right) \leq 0. \tag{45}$$

For any $(w, w_*) \in W$, note from $z'_{w_*} \succeq_j z_w$ and the definition of $\succeq_j$ that $\delta(z_w) \leq \delta(z'_{w_*})$. That means, by the definition of δ (as well as the fact $z'_{w_*}, z_w \in \mathscr{Z}_j$, the bidder-type (k, t_k) referred to by z'_w and z_w, and the $(k^*, t^*_{k^*})$ referred to by z'_{w_*} and z_{w_*}),

$$\alpha(z_w) - \max_{j'\in I_2\setminus\{j\}}\alpha(k, j', t_k) \leq \alpha(z'_{w_*}) - \max_{j'\in I_2\setminus\{j\}}\alpha(k^*, j', t^*_{k^*}).$$

By Lemma 4, $z'_w \succeq_{k,t_k} z_w$. It then follows from the definition of $\succeq_{k,t_k}$ and the fact $z_w \in \mathscr{Z}_j$ that z'_w is the second highest contender in its column, namely,

$$\alpha(z'_w) = \max_{j'\in I_2\setminus\{j\}}\alpha(k, j', t_k).$$

Meanwhile, the fact that $z_{w_*} \notin \mathscr{Z}_j$ implies that

$$\alpha(z_{w_*}) \leq \max_{j'\in I_2\setminus\{j\}}\alpha(k^*, j', t^*_{k^*}).$$

Combine the three formulas displayed above to obtain

$$\alpha(z_w) - \alpha(z'_w) \leq \alpha(z'_{w_*}) - \alpha(z_{w_*}).$$

This being true for all $(w, w_*) \in W$, we have

$$\sum_{(w,w_*)\in W} \widehat{\beta}_w \left([z'_{w_*}] - [z_{w_*}] + [z'_w] - [z_w]\right)(-\boldsymbol{\alpha}) = \sum_{(w,w_*)\in W} \widehat{\beta}_w \left(-\alpha(z'_{w_*}) + \alpha(z_{w_*}) - \alpha(z'_w) + \alpha(z_w)\right) \leq 0.$$

Plug this and (45) into (44) to see that (32) cannot hold. Thus the theorem is proved.

6.2 Partial Assignment

In the *partial assignment* model, the set X of feasible allocation outcomes is constant to all $t \in T$ and is defined to be the set of all $((x_{kj})_{k=1}^{2})_{j=1}^{N} \in \{0,1\}^{2N}$ that satisfy (24) and (25), without requiring the full assignment condition (27). That is, each bidder gets at most one object and may get none. In its set-representation, a feasible allocation outcome is a subset of I that satisfies (26). In other words, X is equivalent to the set $\mathscr{M}_{\mathrm{P}}$ of the subsets M of I $(= \{1,2\} \times \{1,\ldots,N\})$ that satisfy (26). Different from the $\mathscr{M}$ in the full assignment model, the cardinality of an element of $\mathscr{M}_{\mathrm{P}}$ may be less than two.

Theorem 5 *In the partial assignment model such that $|I_1| = 2$ and $|T|$ is finite, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.*

As in the full assignment model, the proof follows the road map in Section 3. Pick any $\alpha \in \mathbb{R}^{\mathscr{Z}}$ and consider the social planner's problem for any $t := (t_1, t_2) \in T$:

$$\begin{aligned}
\max_{x\in \mathrm{cv}X} \sum_{i\in I} x_i \alpha(i, t_{i_1}) &= \max_{x\in X} \sum_{i\in I} x_i \alpha(i, t_{i_1}) = \max_{M\in \mathscr{M}_{\mathrm{P}}} \sum_{i\in M} \alpha(i, t_{i_1}) \\
&= \max_{j\in I_2} \max_{j'\in I_2\setminus\{j\}} \left(\max\{0, \alpha(1,j,t_1)\} + \max\{0, \alpha(2,j',t_2)\}\right),
\end{aligned}$$

where the first two "=" are the same as in the previous model, and the last "=" highlights the difference, that there is no loss for a solution to assign zero quantity to any interim state whose α-value is nonpositive. Thus, the social planner's problem is solved by coupling the first- or second-highest *positive* $\alpha(1,j,t_1)$ among $j \in I_2$ with the first- or second-highest *positive* $\alpha(2,j,t_2)$ among $j \in I_2$ such that the couple have different j-coordinates. To illustrate, consider the following table that displays the α-values given ex post state (t_1, t_2), with rows corresponding to objects, and columns bidder-types:

	$(1, t_1)$	$(2, t_2)$
1	2	3
2	-4	0
3	-1	$1/2$

The solution in the previous model would have been $\{(1,1),(2,3)\}$ (giving good 1 to bidder 1, and good 3 to bidder 2). That would produce a total α-value $2+1/2$, while the second best would be $\{(1,3),(2,1)\}$, producing a total α-value $-1+3=2$. In the partial assignment model, by contrast, the solution is $\{(2,1)\}$ (giving good 1 to bidder 2 and none to bidder 1). That produces a total positive α-value $\max\{0,-1\}+3=3$, while the second best, $2+1/2$.

To define a solution to the social planner's problem in general, let

$$\begin{aligned}
j^1(k,t_k) &:= \begin{cases} \min\left(\arg\max_{j\in I_2}\alpha(k,j,t_k)\right) & \text{if } \max_{j\in I_2}\alpha(k,j,t_k)>0 \\ 0 & \text{else} \end{cases} \\
j^2(k,t_k) &:= \begin{cases} \min\left(\arg\max_{j\in I_2\setminus\{j^1(k,t_k)\}}\alpha(k,j,t_k)\right) & \text{if } \max_{j\in I_2\setminus\{j^1(k,t_k)\}}\alpha(k,j,t_k)>0 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

for any $k\in I_2$ and any $t_k\in T_k$. For any $j\in I_2$ $(=\{1,\ldots,N\})$, let

$$\delta_{\mathrm{P}}(k,j,t_k) := \max\{0,\alpha(k,j,t_k)\} - \max_{j'\in I_2\setminus\{j\}}\max\{0,\alpha(k,j',t_k)\}.$$

For any $t:=(t_1,t_2)\in T$, define $M_*(t_1,t_2)\in\mathscr{M}_{\mathrm{P}}$ by:

a. if $j^1(1,t_1)\neq j^1(2,t_2)$, let $M_*(t_1,t_2):=\{(1,j^1(1,t_1)),(2,j^1(2,t_2))\}\setminus\{(1,0),(2,0)\}$;

b. else $(j^1(1,t_1)=j^1(2,t_2))$ then:

   i. if $\delta_{\mathrm{P}}(1,j^1(1,t_1),t_1)\geq\delta_{\mathrm{P}}(2,j^1(1,t_1),t_2)$ $(=\delta_{\mathrm{P}}(2,j^1(2,t_2),t_2))$, let

$$M_*(t_1,t_2):=\left\{(1,j^1(1,t_1)),(2,j^2(2,t_2))\right\}\setminus\{(1,0),(2,0)\};$$

   ii. else $(\delta_{\mathrm{P}}(1,j^1(1,t_1),t_1)<\delta_{\mathrm{P}}(2,j^1(1,t_1),t_2))$, let

$$M_*(t_1,t_2):=\left\{(1,j^2(1,t_1)),(2,j^1(2,t_2))\right\}\setminus\{(1,0),(2,0)\}.$$

This definition of M_* parallels that of M_* in the previous model except that $M_*(t)$ here excludes any bidder-object pair with nonpositive α-value at state t (through "$\setminus\{(1,0),(2,0)\}$"). Clearly, $M_*(t)$ is a solution to the social planner's problem for every $t\in T$, hence M_* corresponds to the q^* that Step 1 of our method needs.

Step 2, as in the previous model, is to construct partial orders that rationalize M_* partially, one for every "column" of interim states referring to a common bidder-type (k,t_k) and every "row" of interim states referring to a common object j.

For any $j \in I_2$, define the "row"

$$\mathscr{L}_j := \left\{(k, j, t_k) \mid k \in I_1; t_k \in T_k; j = j^1(k, t_k) \neq 0\right\}.$$

The definition is the same as its counterpart in the previous model except for the nonzero condition on the right-hand side here, which excludes those "top" contenders whose α-values are nonpositive. The binary relation $\succ_j$ on $\mathscr{L}_j$ is defined in the same way as that in the full assignment model, where δ is replaced by δ_{P} here. Correspondingly, list the elements of $\mathscr{L}_j$ in descending order of $\succ_j$ as in (30), and define its upper contour sets U_j^n as there. By the same proof of Lemma 10 (Appendix A.6),[17] one verifies that U_j^n is upward universally binding for any $n = 1, \ldots, |\mathscr{L}_j|$.

The partial order $\succeq_{k,t_k}$ within a "column" $\{k\} \times I_2 \times \{t_k\}$ is needed only when both $j^1(k, t_k)$ and $j^2(k, t_k)$ are nonzero; otherwise the tradeoff within the column is trivial. Thus, consider any $k \in I_1$ and any $t_k \in T_k$ for which $j^2(k, t_k) \neq 0$ (which implies $\alpha(k, j^2(k, t_k), t_k) > 0$ and hence $j^1(k, t_k) \neq 0$). Define $\succeq_{k,t_k}$ by:

a. let $j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k)$;

b. for any $\{j, j'\} \neq \{j^2(k, t_k), j^1(k, t_k)\}$:

   i. if $\alpha(k, j, t_k) > 0$ and $\alpha(k, j', t_k) > 0$, let $j \succ_{k,t_k} j'$ iff

   $$\alpha(k, j, t_k) > \alpha(k, j', t_k) \text{ or } [\alpha(k, j, t_k) = \alpha(k, j', t_k) \text{ and } j < j'];$$

   ii. if $\alpha(k, j, t_k) > 0 \geq \alpha(k, j', t_k)$, let $j \succ_{k,t_k} j'$.

Let there be N_{k,t_k} distinct elements $j \in I_2$ for which $\alpha(k, j, t_k) > 0$. Note $N_{k,t_k} \geq 2$ by the supposition $j^2(k, t_k) > 0$. List all these elements (whose α-values are positive) in descending order of $\succeq_{k,t_k}$, just like (29):

$$j^1(k, t_k) \sim_{k,t_k} j^2(k, t_k) \succ_{k,t_k} j^3 \succ_{k,t_k} j^4 \succ_{k,t_k} \cdots \succ_{k,t_k} j^{N_{k,t_k}}. \tag{46}$$

Define the upper contour sets $(V_{k,t_k}^n)_{n=2}^{N_{k,t_k}}$ with respect to $\succeq_{k,t_k}$ just as in the full assignment model. For any $j \in I_2$ such that $\alpha(k, j, t_k) \leq 0$, define (a lower contour set)

$$L(k, j, t_k) := \{(k, j, t_k)\}.$$

[17] With a trivial modification of replacing the clause "and $j = j^1(k, t_k)$ by the definition of $\mathscr{L}_j$" in the third last sentence of the proof (Appendix A.6) by "and $j = j^1(k, t_k) \neq 0$ by the definition of $\mathscr{L}_j$."

The verification of V^n_{k,t_k} being upward universally binding for all n, and $L(k,j,t_k)$ downward universally binding for all $\alpha(k,j,t_k) \le 0$ (Lemma 11, Appendix A.6), is similar to its counterpart in the full assignment model.

Then comes Step 3 in the road map. Let

$$
\begin{aligned}
\mathscr{S}_+ &:= \left\{U^n_j \mid j \in I_2; \mathscr{Z}_j \neq \varnothing; n = 1, \ldots, |\mathscr{Z}_j|\right\} \\
&\quad \bigcup \left\{V^n_{k,t_k} \mid k \in I_1; t_k \in T_k; j^2(k,t_k) \neq 0; n = 2, \ldots, N_{k,t_k}\right\}, \\
\mathscr{S}_- &:= \{L(k,j,t_k) \mid k \in I_1; t_k \in T_k; \alpha(k,j,t_k) \le 0\}.
\end{aligned}
$$

Different from its counterpart in the previous model, $\mathscr{S}_-$ consists of the singletons of interim states whose α-values are nonpositive (and hence excluded by M_*).

Following Lemma 3, define the matrix $[\mathbf{M}_+, \mathbf{M}_-]$ with respect to the $\mathscr{S}_+$ and $\mathscr{S}_-$ here. For any $z \in \mathscr{Z}$, denote $[z]$ for the row of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ corresponding to z, so $[z](S) = \chi_S(z)$ when S is some U^n_j or V^n_{k,t_k}, and $[z](S) = -\chi_S(z)$ when S is some $L(k,j,t_k)$; and $[z](-\boldsymbol{\alpha}) := -\alpha(z)$. For notation convenience, denote $[0]$ for the zero vector in the space spanned by the row vectors of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$. That is, $[0](S) = [0](-\boldsymbol{\alpha}) = 0$ for all $S \in \mathscr{S}_+ \sqcup \mathscr{S}_-$.

To prove Theorem 5 by Lemma 3, it suffices to prove that, for any nonempty subset Z of $\mathscr{Z}$ and any $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$, if (31) holds then (32) does not hold. To that end, we can exclude from Z all the interim states z for which $\alpha(z) \le 0$:

Lemma 6 *If (31) implies $\sum_{z\in Z} \beta_z \alpha(z) \ge 0$ for any $Z \subseteq \mathscr{Z}$ such that $\alpha(z) > 0$ for all $z \in Z$ and $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$, then (31) implies $\sum_{z\in Z} \beta_z \alpha(z) \ge 0$ for any $Z \subseteq \mathscr{Z}$ and any $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$.*

Proof For $Z \subseteq \mathscr{Z}$ and $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$ that satisfy (31), define

$$
\begin{aligned}
Z^+ &:= \{z \in Z \mid \beta_z > 0\}, \\
Z^- &:= Z \setminus Z^+ \ (= \{z \in Z \mid \beta_z < 0\}).
\end{aligned}
$$

Observe that $\alpha(z) \le 0 \Rightarrow z \notin Z^+$. That is because $\alpha(z) \le 0$ implies $[z](L(z)) = -1$. Since $L(z) = \{z\}$, $[z'](L(z)) = 0$ for all $z' \neq z$. Thus, $z \in Z^+$, combined with the fact $Z^+ \cap Z^- = \varnothing$ and the hypothesis $\beta_z > 0$, contradicts (31).

Thus, if $z \in Z$ and $\alpha(z) \le 0$, then $z \in Z^-$. Let

$$
Z' := Z \setminus \{z \in Z^- \mid \alpha(z) \le 0\}.
$$

Note: Z' satisfies (31) where the role of Z is replaced by Z'. That is because $\alpha(z) \leq 0$ implies that $[z](S) = 0$ for all $S \in \mathscr{S}_+ \sqcup S_- \setminus \{L(z)\}$, $[z](L(z)) = -1$, and $[z](-\boldsymbol{\alpha}) = -\alpha(z) \geq 0$. Thus, removing $\beta_z[z]$ from $\sum_{z' \in Z} \beta_{z'}[z']$ has no effect on $\sum_{z' \in Z} \beta_{z'}[z'](S)$ if $S \neq L(z)$ and, when $S = L(z)$, merely turns

$$\sum_{z' \in Z} \beta_{z'}[z'](L(z)) = \beta_z(-1) = -\beta_z = |\beta_z|$$

into $\sum_{z' \in Z \setminus \{z\}} \beta_{z'}[z'](L(z)) = \beta_z(-1) = 0$ because $[z'](L(z) = 0$ for all $z' \neq z$. Consequently, Z' with all such z removed preserves (31). Then, by the hypothesis of the lemma, $\sum_{z' \in Z'} \beta_{z'}\alpha(z') \geq 0$. Thus the desired conclusion follows:

$$\sum_{z \in Z} \beta_z \alpha(z) = \sum_{z' \in Z'} \beta_{z'}\alpha(z') + \sum_{z \in Z^- :\, \alpha(z) \leq 0} \beta_z \alpha(z) \geq 0. \quad \blacksquare$$

By Lemma 6, let us assume, without loss, that $\alpha(z) > 0$ for any $z \in Z$ in (31). Also assume $Z^- \neq \varnothing$, which is without loss because $Z^- = \varnothing \Rightarrow \sum_{z \in Z} \beta_z \alpha(z) \geq 0$, as $\alpha(z) > 0$ for all $z \in Z$. Then follows the analogous observation to the previous Lemma 4:

Lemma 7 *For any subset $Z \subseteq \mathscr{X}$ such that $\alpha(z) > 0$ for all $z \in Z$ and $Z^- \neq \varnothing$, and for any $(\beta_z)_{z \in Z} \in (\mathbb{R} \setminus \{0\})^Z$, if (31) is true then there exist a set H and a positive $(\tilde{\beta}_h)_{h \in H} \in \mathbb{R}^H_{++}$ for which*

$$\sum_{z \in Z} \beta_z[z] = \sum_{h \in H} \tilde{\beta}_h \left([z'_h] - [z_h]\right) \tag{47}$$

such that for every $h \in H$ there exist $k \in I_1$ and $t_k \in T_k$ that satisfy one of the following:

i. $z_h = (k, j, t_k)$ and $z'_h = (k, j', t_k)$ for some $j, j' \in I_2$ such that $j \neq j'$, $\alpha(z_h) > 0$, $\alpha(z'_h) > 0$, and $j' \succeq_{k,t_k} j$ ($\succ_{k,t_k}$ or $\sim_{k,t_k}$);

ii. or $[z_h] = [0]$ and $z'_h = (k, j', t_k)$ such that $j' = j^1(k, t_k) > 0 = j^2(k, t_k)$;

iii. or $[z'_h] = [0]$ and $z_h = (k, j, t_k)$ such that $j = j^1(k, t_k) > 0 = j^2(k, t_k)$.

Proof As in the case of Lemma 4, let us assume, without loss of generality, that $Z \subseteq \{k\} \times I_2 \times \{t_k\}$ for some $k \in I_1$ and some $t_k \in T_k$.

First suppose $j^2(k, t_k) = 0$. Then $\alpha(k, j, t_k) \leq 0$ for all $j \in I_2 \setminus \{j^1(k, t_k)\}$. Consequently, if $z \in Z$ then $\alpha(z) > 0$ (by hypothesis of the lemma) and hence $z = (k, j^1(k, t_k), t_k)$ and $j^1(k, t_k) > 0$. Thus, $Z = \{z\}$ ($Z \subseteq \{k\} \times I_2 \times \{t_k\}$ by the previous paragraph). Since

$Z = Z^+ \sqcup Z^-$, it follows that z cannot be in both Z^+ and Z^-. If $z \in Z^+$ then (47) holds trivially, with $H := \{1\}$ and $z'_1 := z$, which is case (ii) in the conclusion. Else, $z \in Z^-$, then (47) holds trivially, with $H := \{1\}$ and $z_1 := z$, which is case (iii) of the conclusion.

Next suppose $j^2(k, t_k) \neq 0$. Then the binary relation $\succeq_{k,t_k}$ is defined on the column $\{k\} \times I_2 \times \{t_k\}$, and $V^n_{k,t_k} \in \mathscr{S}_+$ for all $n \in \{2, \ldots, N_{k,t_k}\}$. Thus, if n is the rank of z in the list (46) (with rank of both $j^1(k, t_k)$ and $j^2(k, t_k)$ be 2), then $[z](V^m_{k,t_k})$ is equal to one for all $m \geq n$, and equal to zero for all $m < n$. Consequently, we can apply the proof of Lemma 4 that results in Eq. (39) therein. Now that (39) obtains, if $Z^+_* \neq \varnothing$, simply rewrite the $\sum_{z' \in Z^+_*} \beta_{z'}[z']$ on its right-hand side as $\sum_{z' \in Z^+_*} \beta_{z'}([z'] - [0])$ and modify the definition of H there by $H := H \sqcup Z^+_*$. Then (47) obtains, with the elements of Z^+_* belonging to case (ii), and all the other elements of H in case (i). ■

Therefore, Lemma 5, a consequence of Lemma 4 in the previous model, can be replicated with the same proof in the current model as a consequence of Lemma 7.[18] Then Theorem 5 is proved by the same argument at the end of Section 6.1.

7 Infinite Type Spaces

The next theorem, proved in Appendix A.7, relaxes the previous assumption $|T| < \infty$.

Theorem 6 *For any $(f, g) : 2^I \to \mathscr{R}^2_+$, if there exists $\epsilon > 0$ such that for any integer $m > 1/\epsilon$, $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$ holds given any $|T| < \infty$ and any t-independent constraint structure defined by (16) where $(f_t, g_t) = (f, g^m)$ for all t and, for any $m \in \{1, 2, \ldots\}$ and any $E \subseteq I$,*

$$g^m(E) = \max\{0, g(E) - 1/m\}, \tag{48}$$

then $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$ given the t-independent constraint structure defined by (16) where $(f_t, g_t) = (f, g)$ for all t, whether $|T|$ is finite or not.

[18] Case (i) in the conclusion of Lemma 7 is in line with the last clause in the statement of Lemma 4. Neither cases (ii) nor (iii) cause any complication. That is because the proof of Lemma 5 is to balance the (negative) weight $\tilde{\beta}_h$ of every $h \in H$ for which $z_h \in \mathscr{Z}_j$ by the (positive) weights $\tilde{\beta}_{h_*}$ of some elements h_* of H for which $z'_{h_*} \in \mathscr{Z}_j$. Thus case (ii) is a nonissue because the z_h in (ii) is replaced by the zero vector. In case (iii), $z_h = (k, j, t_k) \in \mathscr{Z}_j$ as $j = j^1(k, t_k)$. Then its weight is balanced in the said manner due to the nonnegativity condition (31). The fact that $[z'_h] = [0]$ has no impact on the reasoning because in the proof of Lemma 5 the z'_h has no impact either, since it belongs to the column in which z_h is the top contender and hence z'_h is not a top contender. In the current model, when the proof results in (40), the set H_* therein does not include any of the z_h in case (iii), as no element of H_* belongs to $\mathscr{Z}_j$, while here $z_h \in \mathscr{Z}_j$.

The proof of Theorem 6 is an extension of Che et al.'s [7, Online Appendix B.2] passing-to-limit argument. The main assumption around (48) is to ensure that, when any given interim allocation is being approximated from below by the nearest nonnegative grid points, the floor constraints in the Border condition within the discretized model is satisfied. This assumption is true given paramodularity or partial assignment. Thus follow the next two corollaries (proved in Appendix A.8). By contrast, the assumption is not satisfied in the full assignment model, because its floor constraint involves positive integers and hence cannot be perturbed downward. By the same token, neither is the assumption satisfied by Lang and Yang's total unimodular model, which allows for positive integer floor constraints.

Corollary 1 *If (f, g) is constant across ex post states $t \in T$ and is paramodular on 2^I, and if $\mathscr{R} = \mathbb{R}$, then $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$.*

Corollary 2 *In the partial assignment model, $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$.*

Corollary 1 also gives the Border-like characterization in the two-player bargaining model, a special case of paramodularity (proved in Appendix A.8).[19]

Corollary 3 *In the two-player bargaining model defined in Section 4, $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$.*

8 Conclusion

Characterizing reduced-form allocations is a necessary step in the design of optimal Bayesian allocation mechanisms when agents' interim expected payoffs are nonlinear in their interim allocations. The frontier of the literature calls for a method to tackle the combinatorial complications among multiple objects and asymmetrically informed players, exemplified by the assignment problems between bidders with nontrivial type spaces. This paper contributes to this frontier with a unifying, economically meaningful method: Establishing a tractable characterization of reduced-form allocations is analogous to decentralizing a social planner's solution in a linear production economy. First, for any linear valuation of the outputs (interim states) find a solution of the social planner's problem to maximize the total value of the outputs among the feasible production plans (ex post allocation) that allocate every input

[19] Lang [19] obtains a special case of the characterization based on an assumption that $|T|$ is finite and that at least one of the two players has only a binary type space.

(ex post state) to its compatible outputs. Second, derive from the social planner's solution a family of partial orders on the outputs that partially rationalize the solution without having to take into account of the input-by-input feasibility constraints. Third, prove existence of a price vector that is supported by the upper or lower contour sets of these partial orders thereby obtaining the decentralizing prices (shadow prices for the Border condition of reduced forms). The major received result, based on the paramodularity model, and the counterpart to a contemporary result that goes beyond that model, turn out to be easy applications of the method. Following the method I obtain a novel finding, the tractable characterization of the reduced forms in the assignment problems between N objects and two bidders, $N \geq 2$ and the number of types arbitrary per bidder, whether or not full assignment is required as part of the feasibility constraint. With the exact characterization of reduced forms available, the approach of optimal mechanism design, which has been unavailable to the assignment literature until now, is at hand for a nontrivial set of these problems.

References

[1] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag: Berlin, 3rd edition, 2006. A.2

[2] Kim C. Border. Implementation of reduced form auctions: A geometric approach. *Econometrica*, 59(4):1175–1187, 1991. 1, 6, 7, A.1, A.7

[3] Kim C. Border. Reduced form auctions revisited. *Economic Theory*, 31, 2007. 1

[4] Alexei Boulatov and Sergei Severinov. Optimal and efficient mechanisms with asymmetrically budget constrained buyers. *Games and Economic Behavior*, 127:155–178, 2021. 1

[5] Eric Budish, Yeon-Koo Che, Fuhito Kojima, and Paul Milgrom. Designing random allocation mechanism theory and applications. *American Economic Review*, 103(2):585–623, 2013. 5

[6] Yang Cai, Constantinos Daskalakis, and Matthew Weinberg. An algorithmic characterization of multi-dimensional mechanisms. Working Paper, MIT, December 2011. 4

[7] Yeon-Koo Che, Jinwoo Kim, and Konrad Mierendorff. Generalized reduced-form auctions: A network-flow approach. *Econometrica*, 81(6):2487–2520, 2013. 1, 4, 7, B.1, 20

[8] Xiaoxuan Chu, Kon Max Wong, Jun Chen, and Jiankang Zhang. Systems of linear equations with non-negative constraints: Hyper-rectangle cover theory and its applications. *Mathematics*, 11:2328, 2023. 1, 3

[9] Jack Edmonds. Submodular functions, matroids, and certain polyhedra. In M. Jünger et al., editor, *Combinatorial Optimization (Edmonds Festschrift), LNCS 2570*, pages 11–26. Springer-Verlag Berlin Heidelberg, 2003. 5

[10] András Frank, Tamás Király, Júlia Pap, and David Pritchard. Characterizing and recognizing generalized polymatroids. *Mathematical Programming*, 146:146–273, 2014. 5, B.2

[11] Alex Gershkov, Jacob K. Goeree, Alexey Kushnir, Benny Moldovanu, and Xianwen Shi. On the equivalence of Bayesian and dominant strategy implementation. *Econometrica*, 81(1):197–220, January 2013. 2

[12] Alex Gershkov, Benny Moldovanu, Phillip Strack, and Mengxi Zhang. A theory of auctions with endogenous valuations. *Journal of Political Economy*, 129:1011–1051, 2021. 1, 4

[13] Jacob Goeree and Alexey Kushnir. A geometric approach to mechanism design. *JPE Micro*, 2022. Forthcoming. 1

[14] Sergiu Hart and Philip J. Reny. Implementation of reduced form mechanisms: A simple approach and a new characterization. *Economic Theory Bulletin*, 3:1–8, 2015. 4

[15] Refael Hassin. Minimum cost flow with set-constraints. *Networks*, 12(12):1–21, Spring 1982. 10, 4

[16] Alan J. Hoffman. Total unimodularity and combinatorial theorem. *Linear Algebra and Its Applications*, 13:103–108, 1976. 5, A.5

[17] Andreas Kleiner, Benny Moldovanu, and Philipp Strack. Extreme points and majorization: Economic applications. *Econometrica*, 89:1557–1593, 2021. 4

[18] Alexander V. Kolesnikov, Fedor Sandomirskiy, Aleh Tsyvinski, and Alexander P. Zimin. Beckmann's approach to multi-item multi-bidder auctions. arXiv:2203.06837, Sep 6, 2022. 4

[19] Xu Lang. Reduced-form allocations with complementarity: A 2-person case. Working Paper, Southwestern University of Finance and Economics, February 22, 2022. 19

[20] Xu Lang and Zaifu Yang. Reduced-form allocations for multiple indivisible objects under constraints. Working Paper, University of York, August 26, 2022. 1, 5, A.5, B.2, B.2, 20

[21] Yunan Li. Efficient mechanisms with information acquisition. *Journal of Economic Theory*, 182:279–328, 2019. 1

[22] Alejandro M. Manelli and Daniel R. Vincent. Bayesian and dominant-strategy implementation in the independent private-values model. *Econometrica*, 78(6):1905–1938, 2010. 2, 1

[23] Eric Maskin and John Riley. Optimal auctions with risk averse buyers. *Econometrica*, 52(6):1473–1518, November 1984. 1

[24] Steven A. Matthews. On the implementability of reduced form auctions. *Econometrica*, 52(6):1519–1522, November 1984. 1

[25] Konrad Mierendorff. Asymmetric reduced form auctions. *Economics Letters*, 110:41–44, 2011. 1

[26] Alexander Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*, volume B. Springer-Verlag: New York, 2003. 9

[27] Juuso Toikka, Akhil Vohra, and Rakesh Vohra. Bayesian persuasion: Reduced form approach. PIER Working Paper 22-018, University of Pennsylvania, May 2002. 4

[28] Quitzé Valenzuela-Stookey. Approximate interim allocations. Working Paper, Department of Economics, UC Berkeley, Jan. 30, 2023. 1

[29] Erya Yang. Reduced-form mechanism design and ex post fairness. *Economic Theory Bulletin*, 9:269–293, 2021. 1

A Delayed Details

A.1 Necessity of the Border Condition

Lemma 8 $\mathscr{Q}_{\mathrm{B}} \supseteq \mathscr{Q}$.

Proof Since X_t is assumed compact for all $t \in T$, every element of $\mathscr{Q}$ is μ-essentially bounded due to (1). Thus, following Border [2], treat both $\mathscr{Q}$ and $\mathscr{Q}_{\mathrm{B}}$ as subsets of the $L_\infty(\mu)$-space of functions $\mathscr{X} \to \mathbb{R}$. For any L_∞-function $Q := (Q_i)_{i\in I} \in \mathbb{R}^{\mathscr{X}}$ and any $L_1(\mu)$-function $\varphi \in \mathbb{R}^{\mathscr{X}}$, define

$$\langle Q, \varphi\rangle := \int_T \sum_{i\in I} Q_i(t_{i_1})\varphi(i, t_{i_1})d\mu(t).$$

Then $\langle \cdot, \varphi\rangle$ is a continuous linear functional on the L_∞-space of interim allocations. Since $\mathscr{Q}$ is a subset of this space, $\langle Q, \varphi\rangle \le \sup_{Q'\in\mathscr{Q}}\langle Q', \varphi\rangle$ for all $Q \in \mathscr{Q}$. This inequality implies $Q \in \mathscr{Q}_{\mathrm{B}}$, namely, Q satisfies the Border condition (5): Pick any measurable $S \subseteq \mathscr{X}$. To obtain the second inequality in (5), apply $\langle Q, \varphi\rangle \le \sup_{Q'\in\mathscr{Q}}\langle Q', \varphi\rangle$ to the case where $\varphi = \chi_S$. Since $Q' \in \mathscr{Q}$, Q' satisfies (1) with respect to some ex post allocation q'. Thus

$$\begin{aligned}
\sup_{Q'\in\mathscr{Q}} \langle Q', \chi_S\rangle &= \sup_{(q'(t))_{t\in T}\in\prod_{t\in T}\mathrm{cv}X_t} \sum_{i\in I}\int_{T_{i_1}}\int_{T_{-i_1}} q'_i(t_{i_1}, t_{-i_1})\chi_S(i, t_{i_1})d\mu_{-i_1}(t_{-i_1}|t_{i_1})d\mu_{i_1}(t_{i_1}) \\
&= \int_T \sup_{q'(t)\in\mathrm{cv}X_t} \sum_{i\in I} q'_i(t))\chi_S(i, t_{i_1})d\mu(t) \\
&= \int_T \sup_{q'(t)\in X_t} \sum_{i\in I} q'_i(t))\chi_S(i, t_{i_1})d\mu(t) \\
&\le \int_T f(S, t)d\mu(t),
\end{aligned}$$

with the third line due to the fact that X_t contains all extremal points of its convex hull, and the last line due to (3). Thus the second inequality in (5) follows. The first inequality in (5) is analogous via $\varphi := -\chi_S$. Thus $Q \in \mathscr{Q}_{\mathrm{B}}$. ■

A.2 Proof of Lemma 1

By the definition of $\mathscr{Q}$ and (1), the right-hand side of (6) is

$$\begin{aligned}
\sum_{t\in T}\mu\{t\} \max_{(q_i(t))_{i\in I}\in\mathrm{cv}X_t} \sum_{i\in I} q_i(t)\alpha(i, t_{i_1}) &= \max_{((q_i(t))_{i\in I})_{t\in T}\in\prod_{t\in T}(\mathrm{cv}X_t)} \sum_{t\in T}\sum_{i\in I} q_i(t)\alpha(i, t_{i_1})\mu\{t\} \\
&= \max_{Q\in\mathscr{Q}} \sum_{i\in I}\sum_{t_{i_1}\in T_{i_1}} \alpha(i, t_{i_1})Q_i(t_{i_1})\mu_{i_1}\{t_{i_1}\}.
\end{aligned}$$

For any $Q := (Q_i(t_{i_1}))_{(i,t_{i_1})\in\mathscr{Z}} \in \mathbb{R}^{\mathscr{Z}}$ and $\alpha := (\alpha(i,t_{i_1}))_{(i,t_{i_1})\in\mathscr{Z}} \in \mathbb{R}^{\mathscr{Z}}$, denote

$$\langle Q, \alpha \rangle := \sum_{i\in I} \sum_{t_{i_1}\in T_{i_1}} Q_i(t_{i_1})\alpha(i,t_{i_1})\mu_{i_1}\{t_{i_1}\}.$$

Then (6) is equivalent to

$$\max_{Q\in\mathscr{Q}_{\mathrm{B}}} \langle Q,\alpha\rangle \le \max_{Q\in\mathscr{Q}} \langle Q,\alpha\rangle.$$

Note that the set $\mathscr{Q}$ of reduced forms is convex and compact, since the set of ex post allocation is convex and compact (each assigning to each state t an element of the convex hull of the compact X_t) and the mapping from ex post allocations to their reduced forms, Eq. (1), is linear and continuous. It then follows from a finite-dimensional application of the separating hyperplane theorem (Theorem 7.51 of Aliprantis and Border [1, p288]) that

$$\mathscr{Q} = \left\{ \bar{Q} \in \mathbb{R}^{\mathscr{Z}} \,\middle|\, \forall \alpha \in \mathbb{R}^{\mathscr{Z}} \left[\left\langle \bar{Q},\alpha \right\rangle \le \max_{Q\in\mathscr{Q}} \langle Q,\alpha\rangle \right] \right\}.$$

Claim: $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if (6) holds for all $\alpha \in \mathbb{R}^{\mathscr{Z}}$. Clearly $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ implies (6) for all $\alpha \in \mathbb{R}^{\mathscr{Z}}$. To prove the converse, suppose that (6) is true for all $\alpha \in \mathbb{R}^{\mathscr{Z}}$. If $\bar{Q} \notin \mathscr{Q}$ then, by the equation displayed above, there exists an α given which $\left\langle \bar{Q},\alpha \right\rangle > \max_{Q\in\mathscr{Q}} \langle Q,\alpha\rangle$. Then $\left\langle \bar{Q},\alpha \right\rangle > \max_{Q\in\mathscr{Q}_{\mathrm{B}}} \langle Q,\alpha\rangle$ by (6). Thus $\bar{Q} \notin \mathscr{Q}_{\mathrm{B}}$. ∎

A.3 Proof of Theorem 1

By the definition of $\mathscr{Q}_{\mathrm{B}}$ (condition (5)), the left-hand side of (6) is equivalent to

$$\begin{aligned}
\max_{(Q_i)_{i\in I}\in\mathbb{R}^I} \quad & \sum_{i\in I} \sum_{t_{i_1}\in T_{i_1}} \mu_{i_1}\{t_{i_1}\} Q_i(t_{i_1})\alpha(i,t_{i_1}) \\
\forall S \subseteq \mathscr{Z}: \quad & \sum_{i\in I} \sum_{i_1\in T_{i_1}} \mu_{i_1}\{t_{i_1}\} Q_i(t_{i_1})\chi_S(i,t_{i_1}) \le \sum_{t\in T} \mu\{t\} f(S,t) \\
& \sum_{i\in I} \sum_{i_1\in T_{i_1}} \mu_{i_1}\{t_{i_1}\} Q_i(t_{i_1})\chi_S(i,t_{i_1}) \ge \sum_{t\in T} \mu\{t\} g(S,t).
\end{aligned}$$

Treat the $Q_i(t_{i_1})\mu_{i_1}\{t_{i_1}\}$ in this problem as a choice variable to obtain its dual:

$$\begin{aligned}
\min_{(p_+,p_-):2^{\mathscr{Z}}\to\mathbb{R}^2_+} \quad & \textstyle\sum_{t\in T} \mu\{t\} \sum_{S\subseteq\mathscr{Z}} \left(p_+(S)f(S,t) - p_-(S)g(S,t)\right) \\
\forall z \in \mathscr{Z} \quad & \alpha(z) = \textstyle\sum_{S\subseteq\mathscr{Z}} \left(p_+(S) - p_-(S)\right)\chi_S(z).
\end{aligned}$$

Thus, it follows from Lemma 1 that $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if for any $\alpha \in \mathbb{R}^{\mathscr{Z}}$ there exists $(p_+,p_-): 2^{\mathscr{Z}} \to \mathbb{R}^2_+$ for which (7) holds for all $z \in \mathscr{Z}$, and

$$\sum_{t\in T} \mu\{t\} \sum_{S\subseteq\mathscr{Z}} \left(p_+(S)f(S,t) - p_-(S)g(S,t)\right) \le \sum_{t\in T} \mu\{t\} \max_{(q_i(t))_{i\in I}\in \mathrm{cv} X_t} \sum_{i\in I} q_i(t)\alpha(i,t_{i_1}). \quad (49)$$

The right-hand side of this inequality is equal to

$$\sum_{t\in T}\mu\{t\}\sum_{i\in I}q_i^*(t)\alpha(i,t_{i_1})$$

for any q^* that satisfies (8). Thus, (49) is equivalent to

$$\sum_{t\in T}\mu\{t\}\sum_{S\subseteq\mathscr{Z}}(p_+(S)f(S,t)-p_-(S)g(S,t))\le\sum_{t\in T}\mu\{t\}\sum_{i\in I}q_i^*(t)\alpha(i,t_{i_1}).$$

Plug (7) into the right-hand side of this inequality to rewrite the inequality as

$$\begin{aligned}&\sum_{t\in T}\mu\{t\}\sum_{S\subseteq\mathscr{Z}}(p_+(S)f(S,t)-p_-(S)g(S,t))\\ \le\ &\sum_{t\in T}\mu\{t\}\sum_{i\in I}q_i^*(t)\left(\sum_{S\subseteq\mathscr{Z}}p_+(S)\chi_S(i,t_{i_1})-\sum_{S\subseteq\mathscr{Z}}p_-(S)\chi_S(i,t_{i_1})\right).\end{aligned}$$

Rearrange terms to rewrite this inequality as

$$\begin{array}{r}\sum_{S\subseteq\mathscr{Z}}p_+(S)\sum_{t\in T}\mu\{t\}\left(f(S,t)-\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1})\right)\\ +\sum_{S\subseteq\mathscr{Z}}p_-(S)\sum_{t\in T}\mu\{t\}\left(-g(S,t)+\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1})\right)\end{array}\le 0. \tag{50}$$

Thus, the existence of (p_+,p_-) that satisfies (7) and (49) is equivalent to the existence of (p_+,p_-) that satisfies (7) and (50) for some q^* that satisfies (8). For any $S\subseteq\mathscr{Z}$ and $t\in T$,

$$f(S,t)=\max_{x\in X_t}\sum_{i\in I(S,t)}x_i=\max_{x\in\mathrm{cv}X_t}\sum_{i\in I(S,t)}x_i\ge\sum_{i\in I(S,t)}q_i^*(t)=\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1}),$$

with the first "=" due to (3), the second "=" due to X_t containing all extremal points of $\mathrm{cv}X_t$, the inequality due to $q_i^*(t)\in\mathrm{cv}X_t$, and the last "=" due to the definition of $I(S,t)$. By the same token,

$$-g(S,t)+\sum_{i\in I}q_i^*(t)\chi_S(i,t_{i_1})\ge 0$$

for all $S\subseteq\mathscr{Z}$ and all $t\in T$. This, coupled with the assumption that $\mu\{t\}>0$ for all $t\in T$, implies that (50) holds if and only if (9) holds for all $S\subseteq\mathscr{Z}$. ■

A.4 Proof of Claim 1 in Theorem 3

Since (φ^*,γ^*) is a solution to (20) and q^* a solution to the dual thereof, (q^*,φ^*,γ^*) is a saddle point of the Lagrangian

$$\begin{aligned}L(q,\varphi,\gamma)\ :=\ &\sum_{i\in I}\sum_{t\in T}\alpha(i,t_{i_1})q_i(t)\mu\{t\}+\sum_{F\in\mathscr{F}}\sum_{t\in T}\varphi(F,t)\left(\hat f(F)-\sum_{i\in F}q_i(t)\right)\mu\{t\}\\ &+\sum_{G\in\mathscr{G}}\sum_{t\in T}\gamma(G,t)\left(\sum_{i\in F}q_i(t)-\hat g(G)\right)\mu\{t\}.\end{aligned}$$

For any $t \in T$, $k \in \{1, \dots, K\}$, $q(t) \in \mathrm{cv}X$, $\varphi_k(\cdot, t) \in \mathbb{R}_+^{\mathscr{F}}$ and any $\gamma(\cdot, t) \in \mathbb{R}_+^{\mathscr{G}}$, define

$$\begin{aligned} L_k^t(q(t), \varphi(\cdot, t), \gamma(\cdot, t)) \;\; &:= \;\; \sum_{i \in I} \alpha_k(i, t_{i_1}) q_i(t) \mu\{t\} + \sum_{F \in \mathscr{F}} \varphi_k(F, t) \left(\hat{f}(F) - \sum_{i \in F} q_i(t) \right) \mu\{t\} \\ &\quad + \sum_{G \in \mathscr{G}} \gamma_k(G, t) \left(\sum_{i \in G} q_i(t) - \hat{g}(G) \right) \mu\{t\}. \end{aligned}$$

Claim: For any $k = 1, \dots, K$ and any $t \in T$, $(q^*(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ is a saddle point of $L_k^t(q(t), \varphi(\cdot, t), \gamma(\cdot, t))$. First, since $(\alpha_k, \varphi_k^*, \gamma_k^*)$ belongs to $\mathcal{P}$ and hence satisfies (19), $q^*(t)$ maximizes $L_k(\cdot, \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$. Second, note that

$$L\left(q, \sum_k \beta_k \varphi_k, \sum_k \beta_k \gamma_k \right) = \sum_{k=1}^{K} \beta_k \sum_{t \in T} L_k^t(q(t), \varphi_k(\cdot, t), \gamma_k(\cdot, t)).$$

Since $(q^*, \varphi^*, \gamma^*)$ is a saddle point of L, L is minimized by $(\sum_k \beta_k \varphi_k^*, \sum_k \beta_k \gamma_k^*)$ given $q = q^*$, as $\varphi^* = \sum_k \beta_k \varphi_k^*$ and $\gamma^* = \sum_k \beta_k \gamma_k^*$. Consequently, since $\beta_k > 0$ for all k and $(\varphi_k(\cdot, t), \gamma_k(\cdot, t))$ does not enter $L_{k'}^{t'}$ for any $(k', t') \neq (k, t)$, $(\varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ minimizes L_k^t given $q(t) = q^*(t)$, for any k and any t. Thus, $(q^*(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ is a saddle point of L_k^t, as claimed.

By the claim proved above, for any $k = 1, \dots, K$ and any $t \in T$, $q^*(t)$ solves

$$\begin{aligned} \max_{(q_i(t))_{i \in I} \in \mathbb{R}^I} \quad & \sum_{i \in I} \alpha_k(i, t_{i_1}) q_i(t) \mu\{t\} \\ \text{s.t.} \quad & \left(\hat{f}(F) - \sum_{i \in F} q_i(t) \right) \mu\{t\} \geq 0 \quad (\forall F \in \mathscr{F}) \\ & \left(\sum_{i \in G} q_i(t) - \hat{g}(G) \right) \mu\{t\} \geq 0 \quad (\forall G \in \mathscr{G}), \end{aligned}$$

because L_k^t is the Lagrangian associated to this problem. This, combined with $\mu\{t\} > 0$ and (22), implies that for any $k = 1, \dots, K$ and any $t \in T$, $q^*(t)$ solves $\max_{x \in \mathrm{cv}X} \sum_{i \in I} \alpha_k(i, t_{i_1}) x_i$.

For any $k = 1, \dots, K$ and any $t := (t_{i_1})_{i_1 \in I_1} \in T$, since α_k is $\{0, 1, -1\}$-valued by (21),

$$\max_{x \in \mathrm{cv}X_t} \sum_{i \in I} \alpha_k(i, t_{i_1}) x_i = \max_{x \in \mathrm{cv}X_t} \left(\sum_{i:\, \alpha_k(i, t_{i_1}) = 1} x_i - \sum_{i:\, \alpha_k(i, t_{i_1}) = -1} x_i \right) = h\left(I(S^{k,+}, t), I(S^{k,-}, t) \right),$$

where the last "$=$" uses (23), which applies because $I(S^{k,+}, t) \cap I(S^{k,-}, t) = \varnothing$ by the definitions of $S^{k,+}$ and $S^{k,-}$. Since h is assumed linear,

$$\begin{aligned} h\left(I(S^{k,+}, t), I(S^{k,-}, t) \right) \;\; &= \;\; \max_{x \in \mathrm{cv}X} \sum_{i \in I(S^{k,+}, t)} x_i - \min_{x \in \mathrm{cv}X} \sum_{i \in I(S^{k,-}, t)} x_i \\ &= \;\; \max_{x \in X} \sum_{i \in I(S^{k,+}, t)} x_i - \min_{x \in X} \sum_{i \in I(S^{k,-}, t)} x_i, \end{aligned}$$

where the first line is due to linearity of h and (23), and the second line due to the fact that X contains all extremal points of its convex hull. Let $(x_i^+)_{i\in I}$ be a solution to $\max_{x\in X}\sum_{i\in I(S^{k,+},t)} x_i$, and $(x_i^-)_{i\in I}$ a solution to $\min_{x\in X}\sum_{i\in I(S^{k,-},t)} x_i$. Since $(q_i^*(t))_{i\in I}$ is a solution to $\max_{x\in \mathrm{cv}X_t}\sum_{i\in I}\alpha_k(i,t_{i_1})x_i$, the two formulas displayed above together yield

$$\sum_{i\in I(S^{k,+},t)} q_i^*(t) - \sum_{i\in I(S^{k,-},t)} q_i^*(t) = \sum_{i\in I(S^{k,+},t)} x_i^+ - \sum_{i\in I(S^{k,-},t)} x_i^-.$$

Since $I(S^{k,+},t)\cap I(S^{k,-},t) = \varnothing$, the above equation implies that $q_i^*(t) = x_i^+$ for all $i \in I(S^{k,+},t)$, and $q_i^*(t) = x_i^-$ for all $i\in I(S^{k,-},t)$. Thus,

$$\begin{aligned}
\sum_{i\in I} q_i^*(t)\chi_{S^{k,+}}(i,t_{i_1}) &= \max\nolimits_{x\in X_t}\sum\nolimits_{i\in I\left(S^{k,+},t\right)} x_i &= f\left(S^{k,+},t\right),\\
\sum_{i\in I} q_i^*(t)\chi_{S^{k,-}}(i,t_{i_1}) &= \min\nolimits_{x\in X_t}\sum\nolimits_{i\in I\left(S^{k,-},t\right)} x_i &= g\left(S^{k,-},t\right),
\end{aligned}$$

where the last "=" on each line uses the notation defined in (3) and (4). ■

A.5 Total Unimodularity Implies Decomposability

Remark 1 *For any constraint structure $(\mathscr{F},\mathscr{G},\hat{f},\hat{g})$ such that $|T|<\infty$, if it satisfies total unimodularity then it is decomposable.*

Proof Let $\alpha\in\mathbb{R}^{\mathscr{X}}$. Since X is assumed nonempty and compact, so is $\mathrm{cv}X$. Thus the social planner's problem has a finite optimum, and hence its dual, problem (20), has a solution (φ^*,γ^*). It follows that $(\alpha,\varphi^*,\gamma^*)\in\mathcal{P}$. By total unimodularity, every element of $\mathcal{P}$ is a conic combination of some $\{(\alpha_k,\varphi_k,\gamma_k)\mid k=1,\dots,K\}\subset\mathcal{P}$ for some integer K such that (21) is true and the ranges of both φ_k and γ_k are $\{0,1\}$ (Lang and Yang's [20] Lemma 13, or Hoffman's [16] Lemma 3.1). Thus, $(\alpha,\varphi^*,\gamma^*)$ is the conic combination of $\{(\alpha_k,\varphi_k,\gamma_k)\mid k=1,\dots,K\}\subset\mathcal{P}$ satisfying (21), and hence decomposability is satisfied. ■

A.6 The Universal Binding Property of U_j^n, V_{k,t_k}^n, L_{k,t_k} and $L(k,j,t_k)$

Lemma 9 *For any $k\in I_1$, $t_k\in T_k$ and $n\in\{2,\dots,N\}$, V_{k,t_k}^n is upward universally binding, and L_{k,t_k} downward universally binding, in the full assignment model.*

Proof By the definition of V_{k,t_k}^n and L_{k,t_k}, we have $\varnothing =: V_{k,t_k}^1 \subsetneq V_{k,t_k}^2 \subsetneq V_{k,t_k}^3 \subsetneq \cdots \subsetneq V_{k,t_k}^N$ and $L_{k,t_k}\supsetneq L_{k,t_k}^0 := \varnothing$, as in the (10) and (11) in Lemma 2. To verify (12) and (13) there, combine (3) with (24), and combine (4) with (27), to obtain for any $t' \in T$ and any $t_k \in T_k$:

$$[\varnothing \neq S \subseteq \{k\} \times I_2 \times \{t_k\}, t'_k = t_k] \Longrightarrow f(S, t') = 1 \tag{51}$$

$$S \subsetneq \{k\} \times I_2 \times \{t_k\} \Longrightarrow g(S, t') = 0 \tag{52}$$

$$t'_k = t_k \Longrightarrow g(\{k\} \times I_2 \times \{t_k\}, t') = 1. \tag{53}$$

It suffices (12) and (13) to consider all $t' \in T$ such that $t'_k = t_k$, because if $t'_k \neq t_k$ then $I(\cdot, t') = \varnothing$ and (12) and (13) are trivial. Thus, pick any $t' \in T$ for which $t'_k = t_k$. For any $n > 2$, $V^n_{k,t_k} \setminus V^{n-1}_{k,t_k} = \{(k, j^n, t_k)\}$. The social planner's solution M_* always assigns zero quantity to (k, j^n) because $n > 2$. That is, $q^*_{k,j^n}(t') = 0$. Thus

$$\sum_{(k,j,t'_k) \in V^n_{k,t_k} \setminus V^{n-1}_{k,t_k}} q^*_{k,j}(t') = q^*_{k,j^n}(t') = 0 = f(V^n_{k,t_k}, t') - f(V^{n-1}_{k,t_k}, t'),$$

with the third "=" due to (51). For $n = 2$, $V^2_{k,t_k} \setminus V^1_{k,t_k} = \{(k, j^2(k, t_k), t_k), (k, j^1(k, t_k), t_k)\}$, and hence by the definition of M_*

$$\sum_{(k,j,t'_k) \in V^2_{k,t_k} \setminus V^1_{k,t_k}} q^*_{k,j}(t') = \sum_{j \in \{j^2(k,t_k), j^1(k,t_k)\}} q^*_{k,j}(t') = 1,$$

which is equal to $f(V^2_{k,t_k}, t') - f(V^1_{k,t_k}, t')$ by (51) and $V^1_{k,t_k} = \varnothing$. Thus (12) is verified. To verify (13), note that $L_{k,t_k} \setminus L^0_{k,t_k} = \{k\} \times I_2 \times \{t_k\}$. Thus

$$\sum_{(k,j,t'_k) \in L_{k,t_k} \setminus L^0_{k,t_k}} q^*_{k,j}(t') = \sum_{j \in I_2} q^*_{k,j}(t') = 1$$

by the definition of M_*. Meanwhile,

$$g(L_{k,t_k}, t') - g(L^0_{k,t_k}, t') = 1 - 0 = 1,$$

with the first "=" due to (52) and (53). Thus (13) is satisfied. ∎

Lemma 10 *For any $j \in I_2$ and any $n = 1, \ldots, |\mathscr{Z}_j|$, U^n_j is upward universally binding in the full assignment model.*

Proof The definition of U^n_j implies $\varnothing =: U^0 \subsetneq U^1_j \subsetneq \cdots \subsetneq U^{|\mathscr{Z}_j|}_j$, as the (10) in Lemma 2. To verify (12) there, pick any $n \in \{1, \ldots, |\mathscr{Z}_j|\}$, so $\{z^n\} = U^n \setminus U^{n-1}$. Pick any $t := (t_1, t_2) \in T$. To avoid triviality, suppose that $z^n = (k, j, t_k)$ (so $(k, j) \in I(\{z^n\}, t)$). First consider the case where there exists $k < n$ for which $z^k = (k', j, t_{k'})$ (so $(k', j) \in I(U^k, t)$). Since $z^n \neq z^k$, we have $k' \neq k$ and hence $k' = -k$. The premise $k < n$ (namely $z^k \succ_j z^n$), together with $k' = -k$, implies that $(k', j) \in M_*(t_1, t_2)$ (and hence $(k, j) \notin M_*(t_1, t_2)$). Thus, $q^*_{k,j}(t_1, t_2) = 0$. Meanwhile, since both $I(U^{n-1}, t)$ and $I(U^n, t)$ are nonempty (each containing $(-k, j)$),

$$f(U^n, t) - f(U^{n-1}, t) = 1 - 1 = 0,$$

with the first "=" due to (25) coupled with (3). Thus (12) is satisfied in this case. In the other case, there is no $k < n$ for which $z^k = (k', j, t_{k'})$, then $I(U^{n-1}, t) = \varnothing$ and hence $f(U^{n-1}, t) = 0$. Whereas, $I(U^n, t) = \{(k, j)\}$ since $z^n \in U^n$, hence $f(U^n, t) = 1$ by (3) and (25). Thus

$$f(U^n, t) - f(U^{n-1}, t) = 1 - 0 = 1.$$

Meanwhile, since there is no $(-k, j, t_{-k}) \succ_j (k, j, t_k)$ in this case, and $j = j^1(k, t_k)$ by the definition of $\mathscr{Z}_j$, we have $(k, j) \in M_*(t_1, t_2)$ by the definition of M_*. That is, $q^*_{k,j}(t_1, t_2) = 1$. Thus (12) is satisfied in both cases. ■

Lemma 11 *In the partial assignment model, for any $k \in I_1$ and any $t_k \in T_k$ such that $j^2(k, t_k) > 0$, V^n_{k,t_k} is upward universally binding for any $n \in \{2, \ldots, N_{k,t_k}\}$, and $L(k, j, t_k)$ downward universally binding whenever $\alpha(k, j, t_k) \leq 0$.*

Proof The definition of V^n_{k,t_k} implies $\varnothing =: V^1_{k,t_k} \subsetneq V^2_{k,t_k} \subsetneq V^3_{k,t_k} \subsetneq \cdots \subsetneq V^{N_{k,t_k}}_{k,t_k}$, as in the (10) in Lemma 2. To verify (12) in that lemma, pick any $t' := (t'_k, t'_{-k}) \in T$. If $t'_k \neq t_k$ then $I(V^n_{k,t_k}, t') = \varnothing$ for all n and the proof is trivial. Suppose $t'_k = t_k$. Then for any $n \in \{2, \ldots, N_{k,t_k}\}$, $I(V^n_{k,t_k}, t') = \{k\} \times \{j^1(k, t_k), j^2(k, t_k)\} \cup \{j^m \mid 2 \leq m \leq n\}$. This, combined with (3) and (24), implies $f(V^n_{k,t_k}, t') = 1$. Thus,

$$f(V^2_{k,t_k}, t') - f(V^1_{k,t_k}, t') = 1 - 0 = 1$$
$$\forall n \geq 3 : f(V^n_{k,t_k}, t') - f(V^{n-1}_{k,t_k}, t') = 1 - 1 = 0.$$

Meanwhile, since both $j^1(k, t_k)$ and $j^2(k, t_k)$ are nonzero (hypothesis of the lemma),

$$\sum_{(k,j,t_k) \in V^n_{k,t_k} \setminus V^{n-1}_{k,t_k}} = \sum_{j \in \{j^1(k,t_k), j^2(k,t_k)\}} = q^*_{k,j^1(k,t_k),t_k}(t') + q^*_{k,j^2(k,t_k),t_k)}(t') = 1$$

for all $n \in \{2, \ldots, N_{k,t_k}\}$, by the definition of M_*. Thus (12) is satisfied, and hence V^n_{k,t_k} is upward universally binding.

For any $j \in I_2$ such that $\alpha(k, j, t_k) \leq 0$, we have $L(k, j, t_k) =: L^1 \supsetneq L^0 := \varnothing$, a simple case of the (11) in Lemma 2. To verify (13), note from (4), (24) and (25) that $g(S, t') = 0$ for all $S \subseteq \mathscr{X}$. Thus $g(L^1, t') - g(L^0, t') = 0$. Meanwhile, $\alpha(k, j, t_k) \leq 0$ implies $q^*_{k,j',t_k}(t') = 0$ by the definition of M_*. Thus (13) is true, and $L(k, j, t_k)$ downward universally binding. ∎

A.7 Proof of Theorem 6

Lemma 8 already has $\mathscr{Q}_{\mathrm{B}} \supseteq \mathscr{Q}$. The proof of $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ is a passing-to-limit argument: Given any type space T and any constraint structure X defined by (f, g), pick any $Q \in \mathscr{Q}_{\mathrm{B}}$. Construct a sequence $(Q^m)_{m=1}^{\infty}$ of finite-type interim allocations converging to Q so that, for all sufficiently large m, $Q^m \in \mathscr{Q}_{\mathrm{B}}$ given any constraint structure defined by (f, g^m). Then the hypothesis of the theorem implies $Q^m \in \mathscr{Q}$ with respect to constraint structure (f, g^m). Consequently, the convergence of $Q^m \to Q$ and $g^m \to g$ implies $Q \in \mathscr{Q}$ with respect to constraint structure (f, g). Since Q can be any element of $\mathscr{Q}_{\mathrm{B}}$ given constraint structure (f, g), we have $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ given T and (f, g). Next are the details of this argument.

Recall that I_1 is the set of bidders, and I_2 the set of objects. A *cell* in $\mathbb{R}^{I_2}$ is a set $\left\{(x_{i_2})_{i_2 \in I_2} \in \mathbb{R}^{I_2} \mid \forall i_2 \in I_2 \left[y_{i_2} \leq x_{i_2} < y'_{i_2}\right]\right\}$ for some real numbers $y_{i_2} < y'_{i_2}$ $(\forall i_2 \in I_2)$.

Let $Q := (Q_{i_1,i_2})_{(i_1,i_2) \in I_1 \times I_2} \in \mathscr{Q}_{\mathrm{B}}$ given any type space T and constraint structure (f, g). Write Q as $(Q_{i_1})_{i_1 \in I_1}$ such that $Q_{i_1} := (Q_{i_1,i_2})_{i_2 \in I_2}$ for each $i_1 \in I_1$. For any $m = 1, 2, 3, \ldots$, partition $\mathbb{R}^{I_2}$ into a collection $\mathscr{C}_m$ of cells each of which has diameter at most $1/m$. For each bidder $i_1 \in I_1$ and each cell $C \in \mathscr{C}_m$ that has nonempty intersection with the range of Q_{i_1}, denote $([\min C]_{i_2})_{i_2 \in I_2}$ for the coordinate-wise minimum among all elements of C, with $[\min C]_{i_2}$ being its coordinate in the i_2th dimension, and define

$$Q^m_{i_1,i_2}(t_{i_1}) := \max\left\{0, [\min C]_{i_2}\right\}$$

for all $i_2 \in I_2$ and all t_{i_1} in the inverse image $Q^{-1}_{i_1}(C)$ of C. Thus,

$$\max\left\{0, Q_{i_1,i_2}(t_{i_1}) - 1/m\right\} \leq Q^m_{i_1,i_2}(t_{i_1}) \leq Q_{i_1,i_2}(t_{i_1})$$

for each m, each $(i_1, i_2) \in I$ $(= I_1 \times I_2)$ and each $t_{i_1} \in T_{i_1}$. (The second inequality follows from the definition of $Q^m_{i_1,i_2}(t_{i_1})$ and the fact that there is no loss to restrict the range of Q_{i_1} to $\mathbb{R}^{I_2}_{+}$, as f and g are both nonnegative-valued.) Since $Q \in \mathscr{Q}_{\mathrm{B}}$ with respect to (f, g), Q satisfies (5) with respect to (f, g). Thus, by the above-displayed inequalities, $Q^m := (Q^m_{i_1,i_2})_{(i_1,i_2) \in I}$ satisfies the (5) with respect to (f, g^m), where g^m is defined by (48).

Since there is no loss to restrict the range of Q_{i_1} to a bounded set (as f and g are each finite-valued), for each m there are only finitely many cells in $\mathscr{C}_m$ that intersect with the range of Q_{i_1}. Thus, $Q^m_{i_1}$ is equivalent to a function defined on the finite type space

$$T^m_{i_1} := \left\{ Q^{-1}_{i_1}(C) \mid Q^{-1}_{i_1}(C) \neq \varnothing; C \in \mathscr{C}_m \right\}.$$

It follows that for any m, $Q^m \in \mathscr{Q}_{\mathrm{B}}$ given type space $T^m := \prod_{i_1 \in I_1} T^m_{i_1}$ and constraint structure (f, g^m). Thus, by the hypothesis in the theorem, for all sufficiently large m, Q^m belongs to the $\mathscr{Q}$ given T^m and (f, g^m). For any such m, by the definition of $\mathscr{Q}$, there exists an ex post (feasible) allocation q^m given T^m and (f, g^m). Consequently, one can extract a subsequence $(q^{m_k})^\infty_{k=1}$ converging to some ex post (feasible) allocation q given the original type space T and original constraint structure (f, g). Furthermore, following the reasoning (and topologies) in Border [2], $\lim_{k\to\infty} Q^{m_k}$ is the reduced form of q, and $Q = \lim_{k\to\infty} Q^{m_k}$. That is, $Q \in \mathscr{Q}$ given T and (f, g), as desired. ∎

A.8 Proofs of Corollaries 1, 2 and 3

Corollariy 1 We shall prove that, for any sufficiently large integer m, $-g^m$ is submodular and (f, g^m) is compliant. (Since $\mathscr{R} = \mathbb{R}$ by the assumption of the corollary, the g^m defined in (48) is a legitimate constraint function.) Submodularity of $-g^m$ means

$$g^m(E) + g^m(E') \leq g^m(E \cup E') + g^m(E \cap E') \tag{54}$$

for all $E, E' \subseteq I$. Since 2^I is finite, it suffices to show, given any $E, E' \subseteq I$, that (54) holds for all sufficiently large m. If $g(E) > 0$ and $g(E') > 0$, then (48), the definition of g^m, implies that, for any large enough m, $g^m(E) = g(E) - 1/m$ and $g^m(E') = g(E') - 1/m$; meanwhile, the right-hand side of (54) is never less than $g(E \cup E') + g(E \cap E') - 2/m$ (by (48)). Thus (54) follows from $g(E) + g(E') \leq g(E \cup E') + g(E \cap E')$ (submodularity of $-g$) for all large m. If $g(E) = 0$ and $g(E') = 0$, then $g^m(E) = g^m(E') = 0$ by the definition of g^m, and (54) follows trivially because its right-hand side is always nonnegative (by the definition of g^m). Else, one of $g(E)$ and $g(E')$ is zero, and the other positive. Then $g(E \cup E') > 0$ and $g(E \cap E') = 0$ (monotonicity of g, due to submodularity of $-g$). Without loss of generality, say $g(E) > 0 = g(E')$. Then for any m sufficiently large, (54) becomes $g(E) - 1/m \leq g(E \cup E') - 1/m$, which is true by $g(E') \leq g(E \cup E')$ (submodularity of $-g$). Thus, (54) is true for any sufficiently large m.

Compliance of (f, g^m) means

$$f(E') - f(E' \setminus E) \geq g^m(E) - g^m(E \setminus E') \tag{55}$$

for all $E, E' \subseteq I$. Suppose that (55) does not hold no matter how large m is. Then, it follows from the fact $f(E') - f(E' \setminus E) \geq g(E) - g(E \setminus E')$ (compliance of (f, g)) that $g^m(E \setminus E') = g(E \setminus E') - 1/m$ and $g^m(E) = 0$ for any m. Then by the definition of g^m we have $g(E) = 0 < g(E \setminus E')$, contradicting the monotonicity of g noted previously. Thus, (55) holds for all sufficiently large m. Since there are only finitely many subsets of I, (55) holds for all subsets of I when m is sufficiently large. Thus, for any sufficiently large m, (f, g^m) is paramodular. Then the conclusion follows from Theorems 2 and 6. ■

Corollary 2 In the partial assignment model, there is no nonzero floor constraint. Thus (15) implies that $g(E) = 0$ for all $E \subseteq I$. Then $g^m = 0 = g$ for all m. The conclusion therefore follows from Theorems 5 and 6. ■

Corollary 3 In the two-player bargaining model, $I_1 = \{1, 2\}$ and I_2 is singleton (single object). Thus we can identify I with I_1 and denote $I := \{1, 2\}$. The constraint structure is defined by (f, g) via (16) such that $f\{1, 2\} = f\{1\} = f\{2\} = 1$, $g\{1\} = g\{2\} = 0$ and $g\{1, 2\} = 1$, as well as the standard $f(\varnothing) = g(\varnothing) = 0$. By Corollary 1, it suffices $\mathscr{Q}_{\mathrm{B}} = \mathscr{Q}$ to prove that (f, g) is paramodular on 2^I. Submodularity of f follows directly from $f\{1, 2\} = f\{1\} = f\{2\} = 1$, and submodularity of $-g$ directly from $g\{1\} = g\{2\} = 0$ and $g\{1, 2\} = 1$. We claim that (f, g) is compliant, namely,

$$f(E') - f(E' \setminus E) \geq g(E) - g(E \setminus E') \tag{56}$$

for all $E, E' \subseteq \{1, 2\}$. Note, from the g defined above, that $g(E) - g(E \setminus E') \in \{0, 1\}$ and $f(E') - f(E' \setminus E) \geq 0$. Thus, if $g(E) - g(E \setminus E') = 0$ then (56) follows trivially. Suppose that $g(E) - g(E \setminus E') = 1$. Then by the definition of this g, $E \setminus E'$ is either singleton or empty, and $E = \{1, 2\}$. Thus, $E' \neq \varnothing$ and $E' \setminus E = \varnothing$. Consequently, $f(E') - f(E' \setminus E) = f(E') - f(\varnothing) = 1$, so (56) holds. Thus, (f, g) is compliant, as desired. ■

B Online Supplement

B.1 An Example to Apply the Necessity Part of Theorem 1

Let us apply the necessity observation in Theorem 1 to an example in Che et al. [7]. Multiple units of a homogeneous object are to be allocated to three bidders, named 1, 2 and 3. Now that the set I_2 of objects is singleton, without loss denote $I := I_1 := \{1, 2, 3\}$. Each bidder's type is drawn from the same set $\{\underline{\theta}, \overline{\theta}\}$, so $T = \{\underline{\theta}, \overline{\theta}\}^3$. The set X of feasible allocation outcomes is defined by (16) such that $f\{1\} = f\{2\} = f\{3\} = 3$, $f\{1, 2\} = f\{2, 3\} = f\{3, 1\} = 4$, $f\{1, 2, 3\} = 6$, and $f(\varnothing) = g(E) = 0$ for all $E \subseteq I$. One readily sees that the f in this example is not submodular on 2^I. Thus Theorem 2 does not apply. Che et al. note $\mathscr{Q}_\mathrm{B} \nsubseteq \mathscr{Q}$ in this example based on unpublished computations. Here I prove $\mathscr{Q}_\mathrm{B} \nsubseteq \mathscr{Q}$ through showing that the universal binding condition, necessary for $\mathscr{Q}_\mathrm{B} \subseteq \mathscr{Q}$ by Theorem 1, cannot be satisfied in this example.

In this example, the set $\mathscr{X}$ of interim states is $\{(i, t_i) \mid \in \{1, 2, 3\}, t_i \in \{\underline{\theta}, \overline{\theta}\}\}$. For all $i \in \{1, 2, 3\}$, let

$$\alpha(i, \underline{\theta}) := 1, \quad \alpha(i, \overline{\theta}) := 3.$$

For any $t \in T$ $(= \{\underline{\theta}, \overline{\theta}\}^3)$, let

$$q^*(t) \in \arg\max_{x \in \mathrm{cv} X} \sum_{i \in I} x_i \alpha(i, t_i).$$

One readily sees that $q^*(t) = (2, 2, 2)$ if $t = (\underline{\theta}, \underline{\theta}, \underline{\theta})$ or $t = (\overline{\theta}, \overline{\theta}, \overline{\theta})$, and for any other t, $q^*(t) \in \{(3, 1, 1), (1, 3, 1), (1, 1, 3)\}$ such that one of the bidders whose types are $\overline{\theta}$ is assigned 3 units, and one of the bidders whose types are $\underline{\theta}$ is assigned 1 unit.

By the necessity assertion in Theorem 1, it suffices $\mathscr{Q}_\mathrm{B} \nsubseteq \mathscr{Q}$ to prove that that there exists no $(p_+, p_-) : 2^{\mathscr{X}} \to \mathbb{R}^2_+$ that satisfies both (7) and (9) given any q^* with the above-stated property. Suppose, to the contrary, that there is such a (p_+, p_-).

Claim: For any $S \subseteq \mathscr{X}$, $(i, \underline{\theta}) \in S$ implies $p_+(S) = 0$. To prove the claim, without loss, let $(1, \underline{\theta}) \in S$. First, suppose that $(i', \underline{\theta}) \in S$ for some $i' \neq 1$. Without loss, let $i' := 2$. Let $t := (\underline{\theta}, \underline{\theta}, \overline{\theta})$. Then $q^*(t) = (1, 1, 3)$. Meanwhile, $I(S, t) \supseteq \{1, 2\}$ and hence $f(S, t)$ is equal to either 4 (when $I(S, t) = \{1, 2\}$) or 6 (when $I(S, t) = \{1, 2, 3\}$). In either case, $f(S, t) > \sum_{i \in I(S,t)} q^*_i(t)$, because $\sum_{i \in I(S,t)} q^*_i(t) = q^*_1(t) + q^*_2(t) = 2$ when $I(S, t) = \{1, 2\}$, and $\sum_{i \in I(S,t)} q^*_i(t) = \sum_{i=1}^3 q^*_i(t) = 5$ when $I(S, t) = \{1, 2, 3\}$. Thus (9) is violated unless $p_+(S) = 0$.

Thus, if $p_+(S) > 0$ then $(i', \underline{\theta}) \notin S$ for any $i' \neq 1$. Let $t' := (\underline{\theta}, \underline{\theta}, \underline{\theta})$. Then $q^*(t') = (2, 2, 2)$ and $I(S, t') = \{1\}$. Thus $f(S, t') = 3$ whereas $\sum_{i \in I(S,t')} q_i^*(t') = q_1^*(t') = 2$. But then $p_+(S) > 0$ violates (9): contradiction. Thus $p_+(S) = 0$, and the claim is proved.

By the claim just proved, Eq. (7) for any $i \in \{1, 2, 3\}$ implies

$$\alpha(i, \underline{\theta}) = -\sum_{S \subseteq \mathscr{X}} p_-(S) \chi_S(i, \underline{\theta}) \leq 0,$$

which contradicts the fact that $\alpha(i, \underline{\theta}) = 1$. Thus, there exists no $(p_+, p_-) : 2^{\mathscr{X}} \to \mathbb{R}_+^2$ that satisfies both (7) and (9) given any social planner's solution q^*, as asserted. ∎

B.2 Paramodularity Implies Decomposability

In contrast to Lang and Yang's [20] total unimodularity assumption, for which they do not offer any example where a player may have more than two types, the assumptions of Theorem 3 allow for arbitrary numbers of types per player. That is because paramodularity implies decomposability as well as the other assumptions in the theorem, as observed next.

Remark 2 *If $|T| < \infty$ and if the set X of feasible allocation outcomes is paramodular, then h (defined in (23)) is linear, and X satisfies (18) for some $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ that is decomposable and satisfies (22).*

Proof Let X be paramodular and defined by $(f, g) : 2^I \to \mathbb{R}_+^2$ via (16). To prove linearity of h, pick any $F, G \subseteq I$ with $F \cap G = \varnothing$. By (23) the definition of h,

$$h(F, G) = \max_{x \in \mathrm{cv} X} \sum_{i \in I} (\chi_F(i) - \chi_G(i))\, x_i.$$

This problem is solved by the greedy-generous algorithm (due to paramodularity). Thus,

$$h(F, G) = f(F) - g(G) = \max_{x \in X} \sum_{i \in F} x_i - \min_{x \in X} \sum_{i \in G} x_i = h(F, \varnothing) + h(\varnothing, G),$$

with the second equality due to (14) and (15). Thus h is linear.

To prove the other parts of the remark, define $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ by $\mathscr{F} := \mathscr{G} := 2^I$, $\hat{f} := f$, and $\hat{g} := g$. Then (18) holds, and $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ is a constraint structure considered in Section 5. To prove that (22) is satisfied, note that $\mathrm{cv} X$ is equal to the $\mathscr{Q}$ in the special case where everyone's type is common knowledge, i.e., T_{i_1} is singleton for all $i_1 \in I_1$. Thus, by paramodularity, Theorem 2 implies

$$\mathrm{cv}X = \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \;\middle|\; \forall E \subseteq I \left[g(E) \leq \sum_{i \in E} x_i \leq f(E) \right] \right\},$$

which becomes (22) because $\mathscr{F} = \mathscr{G} = 2^I$, $\hat{f} = f$ and $\hat{g} = g$.

To prove that $(\mathscr{F}, \mathscr{G}, \hat{f}, \hat{g})$ is decomposable, pick any $\alpha \in \mathbb{R}^{\mathscr{X}}$. Since X is assumed nonempty and compact, so is $\mathrm{cv}X$. Thus the problem on the right-hand side of (6) has a finite optimum, and hence so does its dual. Since $\mathscr{F} = \mathscr{G} = 2^I$, $\hat{f} = f$ and $\hat{g} = g$, this dual is problem (20). Then, by Frank et al. [10, Prop. 2] due to the paramodular assumption, there exists a solution (φ^*, γ^*) to problem (20) such that, for any $t \in T$, the supports

$$\begin{aligned} \mathrm{supp}\,\varphi^*(\cdot, t) &:= \{F \subseteq I \mid \varphi^*(F, t) > 0\} \quad \text{and} \\ \mathrm{supp}\,\gamma^*(\cdot, t) &:= \{G \subseteq I \mid \gamma^*(G, t) > 0\} \end{aligned}$$

are both laminar families on disjoint ground sets. Denote

$$\begin{aligned} \alpha^+ &:= (\max\{0, \alpha(i, t_{i_1})\})_{(i, t_{i_1}) \in \mathscr{X}}, \\ \alpha^- &:= (\min\{0, \alpha(i, t_{i_1})\})_{(i, t_{i_1}) \in \mathscr{X}}. \end{aligned}$$

Apply Lang and Yang [20, Lemmas 6 & 7] to see that $(\alpha^+, \varphi^*, \mathbf{0})$ belongs to

$$\mathcal{P}_1 := \{(\alpha, \varphi, \gamma) \in \mathcal{P} \mid \alpha \geqq \mathbf{0}, \gamma = \mathbf{0}, [F \notin \mathrm{supp}\,\varphi^*(\cdot, t) \Rightarrow \varphi(F, t) = 0]\},$$

$(\alpha^-, \mathbf{0}, \gamma^*)$ belongs to

$$\mathcal{P}_2 := \{(\alpha, \varphi, \gamma) \in \mathcal{P} \mid \alpha \leqq \mathbf{0}, \varphi = \mathbf{0}, [G \notin \mathrm{supp}\,\gamma^*(\cdot, t) \Rightarrow \gamma(G, t) = 0]\},$$

and $\mathcal{P}_1 \cup \mathcal{P}_2$ is contained in the cone generated by some finite subset $\{(\alpha_k, \varphi_k, \gamma_k) \mid k \in \mathcal{K}\} \subset \mathcal{P}$ that satisfies (21). Thus, there exist finite sets K_1 and K_2, $(\beta_k^1)_{k \in K_1} \in \mathbb{R}^{K_1}_{++}$, $(\beta_k^2)_{k \in K_2} \in \mathbb{R}^{K_2}_{++}$, $(\alpha_k^1, \varphi_k^1, \gamma_k^1)_{k \in K_1} \in \mathcal{P}^{K_1}$, and $(\alpha_k^2, \varphi_k^2, \gamma_k^2)_{k \in K_2} \in \mathcal{P}^{K_2}$, such that every α_k^j satisfies (21), and

$$\begin{aligned} (\alpha^+, \varphi^*, \mathbf{0}) &= \sum_{k \in K_1} \beta_k^1 \left(\alpha_k^1, \varphi_k^1, \gamma_k^1\right), \\ (\alpha^-, \mathbf{0}, \gamma^*) &= \sum_{k \in K_2} \beta_k^2 \left(\alpha_k^2, \varphi_k^2, \gamma_k^2\right). \end{aligned}$$

Sum the two equations and note $\alpha = \alpha^+ + \alpha^-$ to obtain

$$(\alpha, \varphi^*, \gamma^*) = \sum_{k \in K_1} \beta_k^1 \left(\alpha_k^1, \varphi_k^1, \gamma_k^1\right) + \sum_{k \in K_2} \beta_k^2 \left(\alpha_k^2, \varphi_k^2, \gamma_k^2\right).$$

That is, $(\alpha, \varphi^*, \gamma^*)$ is a conic combination of $\bigcup_{j=1}^2 \{(\alpha_k^j, \varphi_k^j, \gamma_k^j) \mid k \in K_j\}$, and each α_k^j is $\{0, 1, -1\}$-valued. Thus the constraint structure is decomposable. ■

Different from decomposability, the total unimodularity assumption has not been observed to include paramdoularity as a special case. The reason is that total unimodularity requires that the entire set $\mathcal{P}$ be generated by extreme rays whose α-components are $\{0, 1, -1\}$-valued, and φ- and γ-components are $\{0, 1\}$-valued, whereas decomposability requires only a subset of $\mathcal{P}$ be generated by extreme rays whose α-components are $\{0, 1, -1\}$-valued, and hence decomposability is broad enough to include paramodularity. Total unimodularity may be an unnecessary condition for reduced-form characterization.[20]

B.3 Non-Paramodularity of the Assignment Models

In both assignment models, the set X of feasible allocation outcomes satisfies (16) such that

$$\forall i_2 \in I_2 : \quad f(I_1 \times \{i_2\}) = 1, \tag{57}$$

$$\forall i_1 \in I_1 \; \forall E \subsetneq I_2 : \quad g(\{i_1\} \times E) = 0, \tag{58}$$

$$\forall i \in I : \quad f\{i\} = 1, \tag{59}$$

$$\forall M \subseteq I : \quad [(i_1, i_2), (i_1', i_2') \in M, i_1 \neq i_1', i_2 \neq i_2'] \Rightarrow f(M) = 2, \tag{60}$$

and either

$$\forall i_1 \in I_1 : g(\{i_1\} \times I_2) = 1 \tag{61}$$

in the full assignment model, or

$$\forall i_1 \in I_1 : f(\{i_1\} \times I_2) = 1 \quad \text{and} \quad \forall E \subseteq I : g(E) = 0 \tag{62}$$

in the partial assignment model.

The full assignment model violates the compliance condition:

$$f\{(1,1),(2,1)\} - f(\{(1,1),(2,1)\} \setminus (\{1\} \times I_2)) = 1 - f\{(2,1)\} = 1 - 1 = 0,$$

[20] Lang and Yang [20] cite an example from Che et al. [7] for which neither the reduced-form characterization nor the total unimodularity assumption are valid. Needless to say, existence of such an example does not imply that total unimodularity is a necessary condition for the reduced-form characterization to be valid.

with the first equality due to (57) and the second due to (59). Whereas, by (61),

$$g(\{1\} \times I_2) - g((\{1\} \times I_2) \setminus \{(1,1),(2,1)\}) = 1 - g(\{1\} \times (I_2 \setminus \{1\})) = 1 - 0 = 1,$$

with the second last "=" due to (58). Thus (f, g) violates compliance.

The partial assignment model violates the submodular condition for f: $f\{(1,1),(2,1)\}+f\{(2,1),(2,2)\} = 1+1 = 2$ by (57) and (62), whereas

$$\begin{aligned} & f(\{(1,1),(2,1)\} \cup \{(2,1),(2,2)\}) + f(\{(1,1),(2,1)\} \cap \{(2,1),(2,2)\}) \\ = \ & f\{(1,1),(2,1),(2,2)\} + f\{(2,1)\} = 2 + 1 = 3, \end{aligned}$$

where $f\{(1,1),(2,1),(2,2)\} = 2$ by (60). Thus f is not submodular. ■