Locally Greedy Characterizations of Interim Allocations*

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Abstract

The received characterizations of feasible interim allocations are mostly in the spirit of Border (1991): Fix a family of sets, each containing some bidder-types, and test the interim allocation under consideration against all these sets. In the published literature, such Border-like characterizations are known to be valid only in a framework that rules out combinatorial complexities such as matchings. This paper proves a Border-like characterization in a matching model between two players and N objects that allows each player to have arbitrary numbers of types. In proving the characterization the paper also develops a method based on a necessary and sufficient condition for Border-like characterizations to be valid in general models. It implies that the validity of the characterization requires that any interim allocation about to become infeasible be locally greedy: that its domain be covered by a family of subsets within each of which the underlying ex post allocation follows some greedy algorithm.

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1 Introduction

This paper reports a progress regarding the classic problem of characterizing the set of feasible interim allocations. To have a concrete picture of the problem, consider a special case where two bidders, named 1 and 2, are to be matched with two of three objects (tasks), named a, b and c. Bidder 1's type t_1 is commonly known, and bidder 2's type is either t_2 or t'_2. The *type profile* across bidders is thus (t_1, t_2) or (t_1, t'_2), jointly distributed according to some probability measure μ. As in any standard mechanism design model, by the revelation principle the designer can restrict attention to direct revelation mechanisms, the chief components of which are *ex post allocations*, which are functions that map any type profile (τ_1, τ_2) to a lottery $(q_1^j(\tau_1, \tau_2), q_2^j(\tau_1, \tau_2))_{j=a,b,c} \in \Delta(\in \{0, 1\}^{2\times 3})$ that outputs a matching between the bidders and the objects so that each bidder gets exactly one object, and each object goes to at most one bidder. Individual bidders, however, know only their own types during the interim stage of decisions, so their perspectives correspond to *object-bidder-type triples* such as $a2t_2$ that concerns good a going to bidder 2 of type t_2, $c2t'_2$ concerning good c going to bidder 2 of type t'_2, and so forth. Thus, the chief instrument to incentivize individuals is a function, called *interim allocation*, that maps any object-bidder-type triple say $ji\tau_i$ to a probability $Q_i^j(\tau_i) \in [0, 1]$ with which bidder-type (i, τ_i) gets object j. For an interim allocation Q to be *feasible*, it needs to be the reduced form of some underlying ex post allocation q, namely, $Q_i^j(\tau_i) = \sum_{\tau_{-i}} q_i^j(\tau_i, \tau_{-i})\mu(\tau_{-i}|\tau_i)$ for all object-bidder-type triples $ji\tau_i$. The problem is: How can we tell whether or not an interim allocation Q is feasible purely by inspecting Q?

This problem is often unavoidable in many models because the interim allocation is more tractable than its ex post allocation counterpart.[1] The literature has located specific conditions about the primitives that guarantee for this problem a relatively tractable solution, in the form of Border [2] or its generalization such as Che et al. [6]. Such Border-like solutions are relatively tractable because they require only linear inequalities, and the smallest set thereof, to test against any interim allocation under consideration. However, these conditions are mostly restricted by a matroid assumption or, more generally, paramodularity, which

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[1] The importance of this problem was first recognized by Maskin and Riley [19] and Matthews [20] in their studies of risk-averse bidders and continues to current research such as the recent studies of Gershkov et al. [9] on auctions with endogenous valuations and Haghpanah et al. [12] on procurements from a group of heterogeneous sellers.

restricts the environment to either a single item for sale or multiple items that involve no nontrivial combinatorial complication. Matching models, including even the special case sketched above, are outside the framework. The lack of Border-like characterizations in combinatorially nontrivial multiple-object models causes a bottleneck in research so that only single-agent cases are being considered in such models (cf. Daskalakis et al. [8]). Gopalan et al. [11] cast a doubt on the possibility of Border-like solutions outside the paramodularity framework, claiming that such solutions exist only if a particular computational problem that is widely believed to be hard turns out to be easy.

This paper makes an observation that a Border-like solution obtains in a matching model between N objects and two bidders such that each bidder can have any finite number of (possibly multidimensional) types (Theorem 1). This is the first Border-like characterization outside the said mainstream framework without restricting each player to at most two types. The characterization is a step toward the direction of bringing the mechanism design approach to the matching literature, which is mostly confined to the approach of finding a mechanism satisfying some properties that are assumed good rather than designing an optimal mechanism subject to constraints.[2]

The observation is based on a necessary and sufficient condition for Border-like solutions to hold with or without the paramodularity assumption (Theorem 2). The idea behind the condition is that the validity of Border-like solutions comes hand-in-hand with the local validity of the greedy algorithm well-known in combinatorial optimization. A Border-like characterization requires that every boundary point of the set of feasible interim allocations be "locally greedy": that the domain of all interim allocations be covered by a family of restricted domains within each of which the underlying ex post allocation for the boundary point acts like a greedy solution with respect to some ranking among the elements within (Corollary 1). The recognition of the role that greedy algorithms play in exact Border-like characterizations without the confine of paramadularity is new to the literature.[3]

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[2] According to Budish [4], part of the reason for this restriction is the lack of tools to characterize the constraints in matching problems.

[3] A well-known economic example of greedy algorithms is Myerson's [22] optimal auction: Score every bidder-type by its virtual utility; for every profile of realized types across bidders, rank the bidders by their scores that the bidders with their associated realized types have; then one-by-one in descending order of the rank, allocate to each bidder the maximum expected quantity down to either the last bit of resource or the last bidder who has a positive score. Border [2] uses a similar greedy algorithm (with different scoring

The literature of characterizing feasible interim allocations is initiated by Border [2] for a single-unit symmetric auction model, significantly extended by Border [3], Manelli and Vincent [18], Mierendorff [21], Goeree and Kushnir [10],[4] and further generalized to multiunit models by Che et al. [6] and Cai et al. [5]. These works except Cai et al. [5] are all within the paramodularity framework mentioned before. As pointed out by Che et al. [6, Supplemental Appendix E.1], the (non-Border-like) characterization provided by Cai et al. [5], though without the paramodularity assumption, requires a continuum of constraints (despite finite type space) to test an interim allocation. A recent discovery in the literature is the equivalence between Border-like solutions and a majorization condition within a framework that assumes paramodularity and restricts attention to symmetric monotone mechanisms, such as Hart and Reny [13], Kleiner et al. [15] and Kolesnikov et al. [16]. Different from the literature, Vohra et al. [25] provide a non-Border-like characterization for a Bayesian persuasion problem through a direct meticulous calculation.

Contemporarily, Lang and Yang [17] provide a Border-like characterization in a total unimodularity model, outside the paramodularity framework and applicable to matching models, though it is unknown whether the model applies to cases where a player has more than two types. Recently, Valenzuela-Stookey [24] proposes to approximate the set of feasible interim allocations with a superset and a subset generated by some greedy solutions.

2 Border-Like Characterizations

Consider a matching model between two bidders and a finite set J of objects (with cardinality $|J| > 2$)[5] such that the matching from bidders to objects has to be one-to-one, and each bidder can have any finite number of types. An allocation outcome is in the form of $x := ((x_i^j)_{i=1}^2)_{j\in J} \in \{0,1\}^{2|J|}$, with $x_i^j = 1$ signifying bidder i getting object j. Denote the set of

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rules) to obtain his solution. While the role of greedy algorithms is not explicit in Che et al. [6], they rely on Hassin's [14] theorem of network flows based on the paramodularity assumption, which Hassin relates to greedy algorithms explicitly. Thus, there is a sense in the literature that greedy algorithms are somehow connected to Border-like solutions within the paramadularity framework, but the connection has not been spelled out before, and the connection appeared as if restricted to paramodularity.

[4] Goeree and Kushnir also consider social choice models.

[5] If $|J| = 2$ then the model is trivialized to a single unit model because the allocation of one item to a bidder implies the allocation of the other item to the other bidder.

feasible allocation outcomes by X, so X consists of all the $((x_i^j)_{i=1}^2)_{j\in J} \in \{0,1\}^{2|J|}$ for which

$$\forall i = 1,2: \ \textstyle\sum_{j\in J} x_i^j \ = \ 1 \tag{1}$$

$$\forall j \in J: \ \textstyle\sum_{i=1}^2 x_i^j \ \leq \ 1. \tag{2}$$

For each $i = 1, 2$, let T_i be the set of the types of bidder i. Write $T := T_1 \times T_2$ and let μ be the probability measure on T that has T as the support, and let $\mu(t) := \mu(\{t\})$ for any singleton $\{t\} \subset T$. For each $i = 1, 2$, let μ_i be the marginal measure of μ onto T_i, and $\mu_{-i}(\cdot|t_i)$ the conditional measure on T_{-i} according to μ conditional on t_i. Assume that T is a finite set.

Any element of T is a type profile, and any object-bidder-type triple jit_i consists of some $j \in J$, $i \in \{1,2\}$ and $t_i \in T_i$. An ex post allocation is a function (q_1^j, q_2^j) defined on T such that $(q_1^j(t), q_2^j(t))_{j\in J}$ belongs to the convex hull $\mathrm{cv}(X)$ of X for every type profile $t := (t_1, t_2) \in T$. An interim allocation is a function Q defined on the set of all object-bidder-type triples such that $Q(jit_i) = Q_i^j(t_i) \in \mathbb{R}$ for every $j \in J$, $i \in \{1,2\}$ and $t_i \in T_i$. An interim allocation Q is feasible iff there exists an ex post allocation (q_1^j, q_2^j) such that

$$Q_i^j(t_i) = \sum_{t_{-i}\in T_{-i}} q_i^j(t_i, t_{-i})\mu_{-i}(t_{-i}|t_i) \tag{3}$$

for every object-bidder-type triple jit_i.

How should the feasibility condition for an interim allocation Q look like? Since Q is a function of object-bidder-type triples, any such a condition should be about the behavior of Q on various sets of such triples. Let us start with a single triple, say $a1\tau_1$, namely, the interim perspective of bidder 1 of type τ_1 concerning the prospect of getting object a. That amounts to figuring out the projection of the underlying ex post allocation (q_1^j, q_2^j) onto $a1\tau_1$. The allocation outcome $(q_1^j(t), q_2^j(t))_{j\in J}$ is a random vector determined by the stochastic type profile $t := (t_1, t_2)$. From the perspective of $a1\tau_1$, the random vector registers a projection $q_1^a(\tau_1, t_2)$ if the realized type profile (t_1, t_2) happens to have $t_1 = \tau_1$, and it registers nothing if $t_1 \neq \tau_1$. When the type profile t ranges randomly in T, the projection (or lack thereof) that $(q_1^j(t), q_2^j(t))_{j\in J}$ registers in the dimension of $a1\tau_1$ is equal to $Q_1^a(\tau_1)$ in expectation due to (3). Meanwhile, since the random vector is bounded within the convex hull $\mathrm{cv}(X)$ of the set X of feasible allocation outcomes, all these projections in the dimension of $a1\tau_1$ are bounded between $\min_{x\in X} x_1^a$ and $\max_{x\in X} x_1^a$. Since the random vector registers a projection on $a1\tau_1$ iff $t_1 = \tau_1$, it follows that

$$\sum_{t\in T} \underbrace{\left(\min_{x\in X} x_1^a\right)}_{=0} \chi_{\{\tau_1\}}(t_1)\mu(t) \le Q_1^a(\tau_1)\mu_1(\tau_1) \le \sum_{t\in T} \underbrace{\left(\max_{x\in X} x_1^a\right)}_{=1} \chi_{\{\tau_1\}}(t_1)\mu(t), \tag{4}$$

where χ_A denotes the characteristic function of set A, and the bounds $\min_X x_1^a$ and $\max_X x_1^a$ are equal to zero and one respectively due to (1) and (2). We can think of (4) as a "test" of Q against $a1\tau_1$ that any feasible Q should pass.

For Q to be feasible, it should manifest the combinatorial constraints (1) and (2) across multiple object-bidder pairs that the underlying ex post allocation q satisfies. Thus, it is not enough to test Q against all single object-bidder-type triples such as the above $a1\tau_1$. We also need to test it against combinations of such triples. Let us test Q against a combination S of three triples, say $S := \{a1\tau_1, a2\tau_2, c2\tau_2\}$: Collect the projections that the random allocation outcome $(q_1^j(t), q_2^j(t))_{j\in J}$ registers on any of the three elements in S; add all such projections up and divide the sum by the total number of all instances of the type profile, then in average we should get $Q_1^a(\tau_1) + Q_2^a(\tau_2) + Q_2^c(\tau_2)$ according to (3). Meanwhile, at every instance of the type profile $t := (t_1, t_2) \in T$, the random vector $(q_1^j(t), q_2^j(t))_{j\in J}$ registers—

1. either a single projection $q_1^a(\tau_1, t_2)$ when $t_1 = \tau_1$ and $t_2 \neq \tau_2$;
2. or two projections simultaneously, $q_2^a(t_1, \tau_2) + q_2^c(t_1, \tau_2)$, when $t_1 \neq \tau_1$ and $t_2 = \tau_2$;
3. or three projections simultaneously, $q_1^a(\tau_1, \tau_2) + q_2^a(\tau_1, \tau_2) + q_2^c(\tau_1, \tau_2)$, when $t_1 = \tau_1$ and $t_2 = \tau_2$;
4. else, no projection at all ($t_1 \neq \tau_1$ and $t_2 \neq \tau_2$).

Since the random vector is bounded within $\mathrm{cv}(X)$, the projection in Case (i), as in the previous singleton case for $\{a1\tau_1\}$, is bounded within $[\min_{x\in X} x_1^a, \max_{x\in X} x_1^a]$, namely, $[0, 1]$. In Case (ii). the ceiling of $q_2^a(t_1, \tau_2) + q_2^c(t_1, \tau_2)$ is equal to one because bidder 2 can get up to one object between a and c; and the floor is equal to zero because bidder 2 can be matched with object b (with bidder 1 matched with another object) so that $q_2^a(t_1, \tau_2) = q_2^c(t_1, \tau_2) = 0$. Thus the sum of projections in Case (ii) is again bounded within $[0, 1]$. In Case (iii), the sum of projections can go up to two because we can match bidder 1 with object a, and bidder 2 with c, or $(1 \to a, 2 \to c)$ for short, so that $q_1^a(\tau_1, \tau_2) + q_2^a(\tau_1, \tau_2) + q_2^c(\tau_1, \tau_2) = 1 + 0 + 1 = 2$. And the sum can be as low as zero because $(1 \to c, 2 \to b)$ is a feasible match, so that

S	$a1t_1$	$a1t_1'$	$b1t_1$	$b1t_1''$	$b1t_1'''$	$a2t_2$	$a2t_2'$
t_1	$a1t_1$		$b1t_1$				
t_2						$a2t_2$	
	$(a,1)$		$(b,1)$			$(a,2)$	
$I(S,(t_1,t_2))$	$(1,a)$		$(1,b)$			$(2,a)$	

Table 1: type profile (t_1, t_2) selects those $ji\tau_i \in S$ for which $\tau_i = t_i$

$q_1^a(\tau_1, \tau_2) = q_2^a(\tau_1, \tau_2) = q_2^c(\tau_1, \tau_2) = 0$. Thus, the total in Case (iii) is bounded within $[0, 2]$. The total in Case (iv) is trivially zero. Consequently, $Q_1^a(\tau_1) + Q_2^a(\tau_2) + Q_2^c(\tau_2)$ is bounded between the expected values of these lower and upper bounds. More precisely, the condition that Q should satisfy when tested against the set S is obtained by replacing the sandwiched term $Q_1^a(\tau_1)$ in (4) by $Q_1^a(\tau_1) + Q_2^a(\tau_2) + Q_2^c(\tau_2)$, and replacing the $\max_{x\in X} x_1^a$ on the right-hand side by the upper bounds noted above, which can be zero (Case (iv)), one (Case (i) or (ii)), or two (Case (iii)), depending on which case the random type profile t belongs to.

The lower bound in (4) may differ from zero when Q is tested against other sets of object-bidder-type triples. In general, it is cumbersome to write down the expected values of the upper and lower bounds explicitly, which would require collecting all the instances of the cases listed above, each corresponding to a particular expression of the upper or lower bound. For succinct expressions of the expected values, I adopt a notation from Che et al [6]: For any set S of object-bidder-type triples and any type profile $t := (t_1, t_2) \in T$, define

$$I(S, t) := \{(i, j) \in \{1, 2\} \times J \mid jit_i \in S\}. \tag{5}$$

The elements of $I(S, t)$ correspond to those in S that simultaneously receive projections from the random allocation outcome $(q_1^j(t), q_2^j(t))_{j\in J}$ at one single instance where the type profile is t. For example, in the three-element S described previously, $I(S, t)$ is $\{(1, a)\}$ in Case (i), $\{(2, a), (2, c)\}$ in Case (ii), $\{(1, a), (2, a), (2, c)\}$ in Case (iii), and empty in Case (iv). Put differently, a type profile t filters out the elements jit_i' of S whose types t_i' do not match the corresponding components of t, thereby producing the set $I(S, t)$ of relevant bidder-object pairs under t. Table 1 illustrates that with a two-bidder two-object example.

From the standpoint of $I(S, t)$, the upper and lower bounds motivated previously are

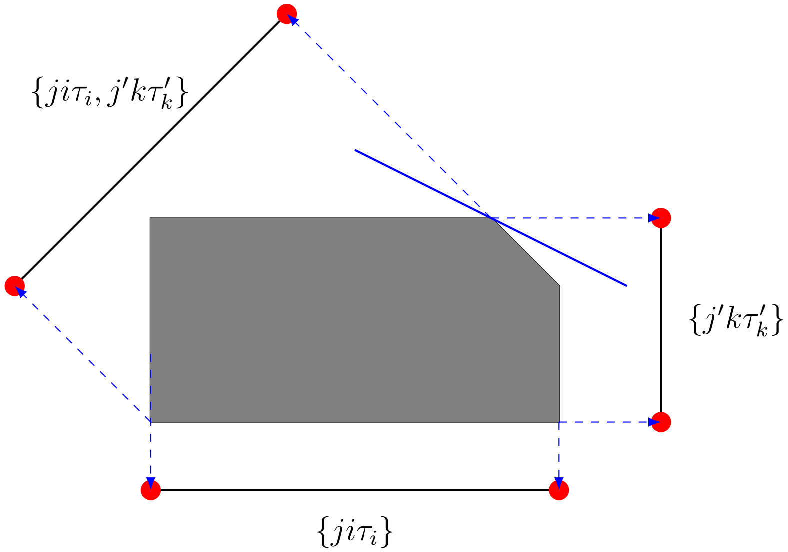

Figure 1: The projections with respect to $\{ji\tau_i\}$, $\{j'k\tau'_k\}$ and $\{ji\tau_i, j'k\tau'_k\}$

defined for any set S of object-bidder-type triples and any type profile $t \in T$:

$$f(S,t) := \max_{x \in X} \sum_{(i,j) \in I(S,t)} x_i^j \tag{6}$$

$$g(S,t) := \min_{x \in X} \sum_{(i,j) \in I(S,t)} x_i^j. \tag{7}$$

Then, when the type profile t ranges in T, the lower bound in all cases can be expressed as $g(S,t)$, and the upper bound in all cases, $f(S,t)$, and hence the general format of (4) is

$$\sum_{t \in T} g(S,t)\mu(t) \leq \sum_{jit_i \in S} Q_i^j(t_i)\mu_i(t_i) \leq \sum_{t \in T} f(S,t)\mu(t) \tag{8}$$

for every set S of object-bidder-type triples.

We have thus come across a necessary condition for any interim allocation Q to be feasible: that Ineq. (8) holds for all sets S of object-bidder-types. The intuition is roughly in Figure 1, where the beings confined to the flatland see the shaded region as merely various segments from various perspectives. Their perspectives correspond to our sets S of object-bidder-type triples. And their effort to characterize the shaded region is to try defining the region by the red dots that define the various segments. Each segment corresponds to an instance of (8), based on a particular set S, with the lower and upper bounds of the inequality being the two endpoints of the segment.

The above intuition is substantially distinct from a perhaps better-known idea of support functions, which is to characterize the shaded region by its supporting hyperplanes such as the blue solid line in Figure 1. The problem of the latter idea, albeit natural, is that the family of supporting hyperplanes is often too large, and almost all its members may turn out to be redundant. For example, the said blue line is a convex combination between, and hence can be replaced by, the two edges of the region that the blue line intersects. According to the support function idea, every instance of (8) would be based on a linear combination of sets S, S', S'', etc., rather than a single set S. Even if we restrict attention to a fixed collection of such sets, there is still a continuum of linear combinations among them. In fact, in almost every endeavor to characterize feasible interim allocations, the support function idea is the starting point, while the characterization by (8) is the final outcome, which corresponds to only a basis of the supporting hyperplanes.

Based on the above, it is easy to prove that satisfaction of (8) for all S is necessary for an interim allocation Q to be feasible. The rest of this paper proves the nontrivial converse:

Theorem 1 *For any interim allocation Q in the matching model, Q is feasible if Q satisfies (8) for every set S of object-bidder-type triples.*

Together with the necessity assertion, Theorem 1 constitutes the Border-like characterization in this model. The Border reference is because (8) becomes Border's [2] condition if we replace the matching model by the single-unit n-symmetric-biidder model ($J = \{a\}$, $T_i = T_1$, $\mu_i = \mu_1$ and $Q_i^a = Q_1^a$ for all bidders i, and $X = \{(x_i)_{i=1}^n \in [0,1]^n \mid \sum_{i=1}^n x_i \leq 1\}$ for all $t \in T$). That condition in turn is known to be equivalent to Kleiner et al.'s [15] majorization condition in the symmetric unidimensional-type model with monotone Q_i^j. Condition (8) also specializes to Che et al.'s [6] condition when the matching model is replaced by the multiunit homogeneous object model ($J = \{a\}$).

These known results, however, are inapplicable to the matching model because they rely on a matroid assumption or, more generally, a paramodularity assumption about the set X of feasible allocation outcomes, which the matching model does not satisfy. The next section will explain that together with the proof idea for Theorem 1.

3 The Idea for the Proof

Theorem 1 claims that the inequality system (8) suffices to characterize the set of feasible interim allocations. That is, for any boundary point of the set of feasible interim allocations there exists at least one instance of (8) that is binding. Since every instance of (8) corresponds to a particular set S of object-bidder-type triples, proving the existence of a binding instance of (8) amounts to locating a set S such that the associated inequality in (8) is binding for the boundary point. Thus, to prove the theorem, it suffices to locate for any boundary point a "binding set" S.

How to locate a binding set for an arbitrarily chosen boundary point of the set of interim allocations? The idea is to think of any such a boundary point as a stochastic choice among object-bidder-type triples: If $q := (q_1^j, q_2^j)_{j \in J}$ is the underlying ex post allocation of the boundary point, then for any type profile $t := (t_1, t_2) \in T$, q induces a function $q(\cdot|t)$ of object-bidder-type triples:

$$q(ji\tau_i|t_1, t_2) := \begin{cases} q_i^j(t_1, t_2) & \text{if } \tau_i = t_i \\ 0 & \text{else} \end{cases} \tag{9}$$

for any $j \in J$, any $i \in \{1, 2\}$ and any $\tau_i \in T_i$. Then (8) is the same as

$$\sum_{t \in T} g(S, t)\mu(t) \leq \sum_{t \in T} \sum_{ji\tau_i \in S} q(ji\tau_i|t)\mu(t) \leq \sum_{t \in T} f(S, t)\mu(t).$$

Thus, for (8) to be binding with respect to S, say reaching the right-hand side $\sum_{t \in T} f(S, t)\mu(t)$, we need $\sum_{t \in T} \sum_{ji\tau_i \in S} q(ji\tau_i|t)\mu(t) = \sum_{t \in T} f(S, t)\mu(t)$ and hence, since $\sum_{ji\tau_i \in S} q(ji\tau_i|t) \leq f(S, t)$ for all t by the definitions of f and $q(\cdot|t)$, it needs to be that

$$\forall t \in T : \sum_{ji\tau_i \in S} q(ji\tau_i|t) = f(S, t).$$

In other words, the total projections that the set S of object-bidder-type triples can collect from the underlying ex post allocation should always reach the maximum $f(S, t)$ when the random type profile t ranges in T. That means, no matter which type profile t is realized, $q(\cdot|t)$ always chooses any element of S over any element outside S. Symmetrically, for (8) to be binding on the left-hand side with respect to S, $q(\cdot|t)$ should always choose any element outside S over any element inside S for any type profile t.

Therefore, to locate a binding set for an arbitrarily chosen boundary point, we need only to search among the upper or lower contour sets with respect to the "chosen over" binary relation that is revealed by the choice $q(\cdot|t)$ under various type profiles t.

Let me illustrate the idea with the simple case where the matching model is replaced by the mainstream model in which the set X of feasible allocation outcomes is assumed to be a matroid or its multiunit generalization.[6] This case gives us the convenience that the "chosen over" relation revealed by $q(\cdot|t)$ is a linear order among all object-bidder-type triples and is independent of the stochastic type profile t. Then the upper contour sets constitute a chain, or a nested sequence of sets; and the lower contour sets constitute another chain in reverse order. It is then easy to locate a binding set in one of the two chains. In going through the easy case, we will see the sticking point for its extension to general cases, and how the sticking point could be resolved.

Pick any boundary point Q of the set of feasible interim allocations and let w be the gradient of a supporting hyperplane for Q. The gradient w is a function that maps every object-bidder-type triple jit_i to a $w(jit_i) \in \mathbb{R}$ and w is not constantly zero. And Q maximizes

$$\sum_{j \in J} \sum_{i \in \{1,2\}} \sum_{t_i \in T_i} \tilde{Q}_i^j(t_i) w(jit_i) \mu_i(t_i)$$

among all feasible interim allocations $\tilde{Q}$. Let q be the underlying ex post allocation for Q. One readily sees from (3) that for every type profile $t := (t_1, t_2) \in T$, $(q_1^j(t), q_2^j(t))_{j \in J}$ solves

$$\max_{(x_1^j, x_2^j)_{j \in J} \in \mathrm{cv}(X)} \sum_{i=1}^{2} \sum_{j \in J} w(jit_i) x_i^j. \tag{10}$$

When X is a matroid (or the paramodular generalization thereof), this problem is solved by the greedy algorithm (cf. Hassin [14]): List all object-bidder-type triples as

$$z^1, z^2, \ldots, z^{n_*}, z^{n_*+1} \ldots, z^m,$$

with m being the total number of object-bidder-type triples, such that

$$w(z^1) \geq w(z^2) \geq \ldots \geq w(z^{n_*}) \geq 0 > w(z^{n_*+1}) \geq \ldots \geq w(z^m);$$

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[6] See Schrijver [23, Vol. B] or Vohra [26] on the definition of matroid. The single unit model of Border [2] is a simple case of matroid, so is Kleiner et al.'s [15] symmetric bidder model. Che et al.'s [6] model, a generalized polymatroid that satisfies paramodularity, is the multiunit generalization of matroid that allows for ceilings and floors of the total quantities for various sets of bidders; the paramdularity assumption regulates the mapping from any set of bidders to the corresponding ceiling and floor.

for any $n = 0, 1, \ldots, m$, let

$$\begin{aligned} U^n &:= \{z^k \mid 1 \leq k \leq n\} \\ L^n &:= \{z^k \mid 1 \leq m-(k-1) \leq n\}; \end{aligned}$$

for every type profile $t \in T$, define $q(\cdot|t)$ by

$$\forall n = 1, \ldots, n_* : \ \textstyle\sum_{z \in U^n \setminus U^{n-1}} q(z|t) = f(U^n, t) - f(U^{n-1}, t) \tag{11}$$

$$\forall n = 1, \ldots, m - n_* : \ \textstyle\sum_{z \in L^n \setminus L^{n-1}} q(z|t) = g(L^n, t) - g(L^{n-1}, t); \tag{12}$$

then $(q_1^j(t), q_2^j(t))_{j \in J}$ defined by $q_i^j(t) := q(jit_i|t)$ for all object-bidder pairs (j, i) is a solution to problem (10).

Obviously, for all type profile $t \in T$ the revealed preference of this $q(\cdot|t)$ is a linear order $\succ$ among all object-bidder-type triples:

$$z^1 \succ z^2 \succ \ldots \succ z^{n_*} \succ z^{n_*+1} \succ \ldots \succ z^m,$$

and the U^n's (resp. L^n's) are all the upper (resp. lower) contour sets with respect to $\succ$. Due to the chain structure $U^0 \subsetneq U^1 \subsetneq \cdots \subsetneq U^m$ and $L^0 \subsetneq L^1 \subsetneq \cdots \subsetneq L^m$, it is easy to find a solution of the shadow prices $p_+(U^n) \in \mathbb{R}_+$ and $p_-(L^n) \in \mathbb{R}_+$ for U^n and L^n ($\forall n$) to satisfy the complementarity condition

$$w(jit_i) = \sum_{n=1}^{m} \left(p_+(U^n)\chi_{U^n}(jit_i) - p_-(L^n)\chi_{L^n}(jit_i)\right)$$

for every object-bidder-type triple jit_i. Consequently, since $w \neq \mathbf{0}$, there exists some U^n or L^n for which $p_+(U^n) > 0$ or $p_-(L^n) > 0$. Say it is $p_+(U^n) > 0$. Then at the instance of (8) corresponding to $S = U^n$, the right-hand side constraint is binding, namely, U^n is a binding set for the arbitrarily chosen boundary point Q.

The easy method described above relies on that the greedy algorithm, (11)–(12), solves the problem (10), so that the revealed preference $\succ$ of q is total, determined by the ordinal ranking of the w-values, and independent of the random type profile t. However, without a matroid structure for the set X of feasible allocation outcomes, the greedy algorithm need not solve problem (10), and hence the revealed preference of q in general is neither total nor deterministic. That is the only sticking point we have to resolve to extend the simple method described above to our matching model.

	$(1, t_1)$	$(2, t_2)$
a	1.05	1
b	0.8	0.7
c	0.5	0

Table 2: Suboptimality of the greedy solution

For example, Table 2 lists the w-values of several object-bidder-type triples in a matching model between three objects (a, b and c) and two bidders (1 and 2). The greedy algorithm starts with the triple $a1t_1$ since its w-value 1.05 is the largest. Then it goes to the triple whose w-value is the largest among the rest, which is $b1t_1$ with w-value 0.8. But it is infeasible to allocate any quantity to $b1t_1$ because it demands object b for bidder 1 while bidder 1 has been matched with object a entirely at the first step. Thus the algorithm moves on to $b2t_2$, whose w-value 0.7 is the next largest, and raises its quantity up to the full, feasible amount. The matching so produced is $(a \mapsto 1, b \mapsto 2)$, with total w-value 1.75.

By contrast, the optimal matching is to assign object a to bidder 2 and object b to bidder 1, producing a total w-value 1.8. The reason why the greedy algorithm fails to be optimal is that the set of feasible matchings is not a matroid: Among the three triples $a1t_1$, $a2t_2$ and $b1t_1$, once the matching $(a \mapsto 1)$ is made then the other two triples $a2t_2$ and $b1t_1$ become infeasible and so we can extract only one match from the three. Whereas, if the matching $(b \mapsto 1)$ is made first, one of the other triples, $a2t_2$, is still feasible, and so we can extract out of the three two matches, which combine to produce a larger total value. In other words, the greedy algorithm misses some combinations.

This bottleneck, however, is only that the greedy algorithm does not apply globally to all object-bidder-type triples. If we restrict the choice to a sufficiently small set of object-bidder-type triples, the optimal solution for problem (10) is still consistent with a greedy solution that follows some ranking among these triples. For example, in Table 2, we can tell that within the row a, $a2t_2$ ranks above $a1t_1$, and within the column $(2, t_2)$, $a2t_2$ ranks above $b2t_2$. Since $c2t_2$ has the lowest w-value in the column, it never has a chance to be matched as only two objects are needed for the two bidders; hence we can rank $b2t_2$ above $c2t_2$. Then, within each row or column, the optimal solution is still *locally greedy*, consistent with a greedy algorithm that follows some partial order (which need not be the ranking of the w-values). Therefore, within each row or column, from the partial order we have a chain of upper or lower contour sets. Since every object-bidder-type triple belongs to a row (for the object) and a column (for the bidder-type), it is contained in some upper contour sets and some lower ones. Then the complementarity condition in the easy case can be more generally written as

$$w(jit_i) = \sum_{S \in \mathscr{S}_+} p_+(S)\chi_S(jit_i) - \sum_{S \in \mathscr{S}_-} p_-(S)\chi_S(jit_i) \tag{13}$$

for every object-bidder-type triple ji_i, where $\mathscr{S}_+$ denotes the family of upper contour sets obtained in the above manner, and $\mathscr{S}_-$, the family of lower contour sets. To have a binding set for the arbitrarily chosen boundary point, it suffices to prove that (13) admits a nonnegative solution for the shadow prices $p_+(S) \in \mathbb{R}_+$ for all $S \in \mathscr{S}_+$, and $p_-(S) \in \mathbb{R}_+$ for all $S \in \mathscr{S}_-$.

In sum, Theorem 1 is proved via the following steps (formalized and generalized to Theorem 2 and Corollary 1 in the appendix):

1. For any function $w : jit_i \mapsto w(jit_i) \in \mathbb{R}$, find an ex post allocation q such that for each type profile $t \in T$, $(q_1^j(t), q_2^j(t))_{j \in J}$ solves problem (10), the optimal matching problem.

2. For each object $j \in J$ find the revealed preference $\succ_j$ of q among the object-bidder-type triples referring to j; for each bidder-type (i, t_i) find the revealed preference $\succ_{it_i}$ of q among the triples referring to (i, t_i); let $\mathscr{S}_+$ (resp. $\mathscr{S}_-$) be a collection of upper (resp. lower) contour sets, each being an upper (resp. lower) contour set with respect to some object $j \in J$ or some bidder-type (i, t_i).

3. Prove existence of $p_+ : \mathscr{S}_+ \to \mathbb{R}_+$ and $p_- : \mathscr{S}_- \to \mathbb{R}_+$ that satisfies (13) for every object-bidder-type triple jit_i.

The following sections proceed according to the three steps with the example at the start of this paper. The general case is handled in Appendix A.4.

4 Optimal Matching

Let me illustrate the proof of Theorem 1 with the example mentioned at the start of the paper, with two bidders (1 and 2), three objects (a, b and c), one type t_1 for bidder 1, and two types (t_2 and t_2') for bidder 2. There are totally nine object-bidder-type triples. According

	$(1, t_1)$	$(2, t_2)$	$(2, t'_2)$
a	-1	3	$1/2$
b	4	0	3
c	2	$1/2$	0

Table 3: A gradient w for three objects and three bidder-types

	$(1, t_1)$	$(2, t_2)$
a	-1	3
b	4	0
c	2	$1/2$

Table 4: The optimal match, colored red, under type profile (t_1, t_2)

to Step 1 of the method, we start with any gradient w of the supporting hyperplane for some boundary point of the set of feasible interim allocation. Table 3 displays one such w, with the entry at row j and column (i, τ_i) being the w-value $w(ji\tau_i)$ of object j matched with bidder i of type τ_i (e.g., $w(a1t_1) = -1$).

Given this w, consider the optimal matching problem, (10). When the type profile is (t_1, t_2), the solution $q(t_1, t_2)$ to problem (10) is to allocate object a to bidder 2, and object b to bidder 1. That is because, under the type profile (t_1, t_2), bidder 1 (of type t_1) gets larger w-value from object b than from any other object, and bidder 2 (of type t_2) gets larger w-value from object a than from any other object (Table 4). That is, in Table 4, the *top contender* $b1t_1$ in column $(1, t_1)$ and the top contender $a2t_2$ in column $(2, t_2)$ are compatible as they refer to different objects, and hence they constitute the maximum of the total w-value among all feasible allocation outcomes under type profile (t_1, t_2). By contrast, consider the other type profile, (t_1, t'_2). With bidder 2's realized type changed to t'_2, the column $(2, t_2)$ in the previous table is replaced by the column $(2, t'_2)$ in Table 5. The top contender $b2t'_2$ in the new column is incompatible with the top contender $b1t_1$ in column $(1, t_1)$, as they are after the same object b. Thus, one of the top contenders needs to be replaced by the second-highest contender in the corresponding column, either the $c1t_1$ in column $(1, t_1)$, or the $a2t'_2$ in column $(2, t'_2)$. The former in pairing with the top contender $b2t'_2$ yields $2+3 = 5$, whereas the latter in pairing with the top contender $b1t_1$ yields $1/2 + 4 = 9/2$. Thus the

	$(1, t_1)$	$(2, t_2')$
a	-1	$1/2$
b	4	3
c	2	0

Table 5: The optimal match, colored red, under type profile (t_1, t_2')

solution $q(t_1, t_2')$ to problem (10) is to allocate object b to bidder 2, and object c to bidder 1.

5 Revealed Preferences and Contour Sets

According to Step 2, let us figure out the revealed preference of the optimal matching q within each row or column of Table 3. Start with the columns. Since only two objects are needed for the two bidders, only the top or second highest contenders within the column—according to the w-values—is chosen at all by the optimal matching q, while any contender below them is never chosen. Thus, from Tables 4 and 5 we obtain the following partial orders separately for the three columns:

$$\begin{aligned}\text{column } (1, t_1): &\quad b1t_1 \sim c1t_1 \succ a1t_1\\ \text{column } (2, t_2): &\quad a2t_2 \succ c2t_2 \succ b2t_2\\ \text{column } (2, t_2'): &\quad b2t_2' \succ a2t_2' \succ c2t_2'\end{aligned}$$

where $b1t_1 \sim c1t_1$ because $q(\cdot|\tau_1, \tau_2)$ chooses $b1t_1$ over $c1t_1$ when $\tau_2 = t_2$, and chooses the opposite when $\tau_2 = t_2'$

Next, consider the revealed preference within each row. Given any type profile $(\tau_1, \tau_2) \in T$, the optimal matching $q(\cdot|\tau_1, \tau_2)$, as demonstrated in Tables 4 and 5, makes the following choices: If the top contenders in columns $(1, \tau_1)$ and $(2, \tau_2)$ refer to different objects, choose both contenders; else pair the top contender in one column with the second-highest contender in the other column, and pick the pair that yields the larger total w-value. Thus, the revealed preference of q in a row $j \in J$ is the ordinal ranking of

$$\delta(ji\tau_i) := w(ji\tau_i) - \max_{j' \neq j} w(j'i\tau_i) \quad \forall j \in J \; \forall i \in \{1, 2\} \; \forall \tau_i \in T_i. \tag{14}$$

For example, in row b, $\delta(b1t_1) = 4 - 2 = 2$, $\delta(b2t_2) = 0 - 3 = -3$, $\delta(b2t_2') = 3 - \frac{1}{2} = 5/2$,

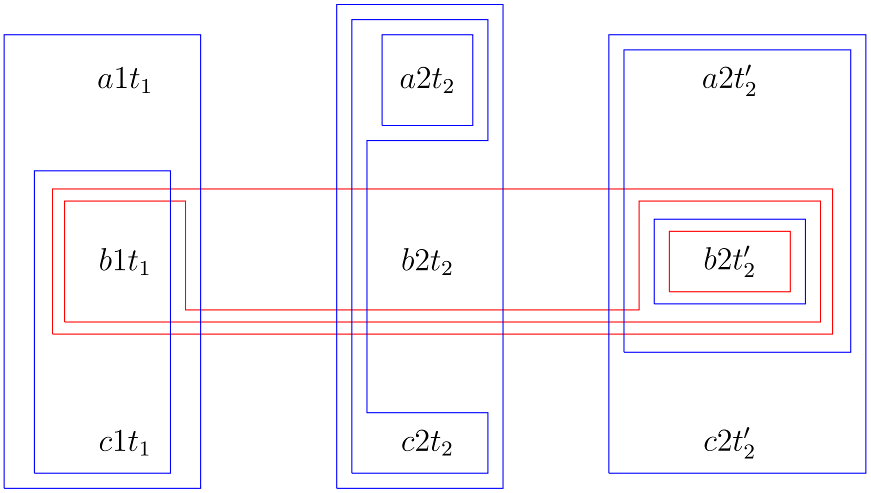

Figure 2: The upper contour sets in the three columns and row b

and hence the revealed preference

$$\text{row } b: \quad b2t'_2 \succ b1t_1 \succ b2t_2.$$

From the four partial orders displayed above, we obtain the upper or lower contour sets in Figure 2, where each blue box defines an upper contour set within the column that the box belongs to, and each red box defines an upper contour set within row b. In each column, the blue box that contains the entire column is also a lower contour set. Note the chain structure of the contour sets within each column and within row b. The inner is a box contained within other boxes of the same color, the higher are the ranks of its elements.

The choice within the other two rows in Table 3 is trivial. Neither row contains more than one top contenders, and hence optimal matching never has to choose one over the other to resolve the conflict between two bidder-types going after the same object. Thus, we do not bother to find the revealed preference of q within either row.

Within each row or column with the partial order $\succ$ (or $\sim$) described above, let $\succeq$ denote the weak counterpart such that $z \succeq z'$ if and only if $z \succ z'$ or $z \sim z'$ or $z = z'$, for any object-bidder-type triples z and z' within the same row or the same column. For each bidder-type (i, τ_i), define

$$U_b(i\tau_i) := \{bk\tau_k \mid bk\tau_k \succeq bi\tau_i; k \in \{1,2\}; \tau_k \in T_k\},$$

the upper contour set in row b that has $bi\tau_i$ as a $\succeq$-minimum; for every $j \in J$, define

$$V_{i\tau_i}(j) := \{j'i\tau_i \mid j'i\tau_i \succeq ji\tau_i; j' \in J\},$$

the upper contour set in column (i, τ_i) that has $ji\tau_i$ as a $\succeq$-minimum; also define

$$L_{i\tau_i} := \{ji\tau_i \mid j \in J\},$$

a lower contour set in column (i, τ_i) that contains the entire column. We have thus obtained a family of upper contour sets:

$$\mathscr{S}_+ := \left\{U_b(2t_2'), U_b(1t_1), U_b(2t_2), V_{1t_1}(b), V_{1t_1}(a), V_{2t_2}(a), V_{2t_2}(c), V_{2t_2}(b), V_{2t_2'}(b), V_{2t_2'}(a), V_{2t_2'}(c)\right\},$$

where $V_{1t_1}(c)$ is omitted because $V_{1t_1}(c) = V_{1t_1}(b)$; and a family of lower contour sets:

$$\mathscr{S}_- := \{\{b1t_1, c1t_1, a1t_1\}, \{a2t_2, c2t_2, b2t_2\}, \{b2t_2', a2t_2', c2t_2'\}\}.$$

Note that $\mathscr{S}_+ \sqcup \mathscr{S}_-$ ($\sqcup$ denoting disjoint union) covers all object-bidder-type triples.

6 Shadow Prices for the Binding Sets

Step 3 is to prove existence of a pair (p_+, p_-) of shadow price functions, $p_+ : \mathscr{S}_+ \to \mathbb{R}_+$ and $p_- : \mathscr{S}_- \to \mathbb{R}_+$, that satisfy (13) for all object-bidder-type triples jit_i. To do that for the general model, I use the following lemma based on the hyper-rectangle cover theory.

Let $\mathscr{Z}$ denote the set of all object-bidder-type triples. Let $\mathscr{S}_+$ be a collection of upper contour sets, and $\mathscr{S}_-$ a collection of lower contour sets, each being a contour set with respect to some partial order that is revealed by the optimal matching within some subset of $\mathscr{Z}$. Let $\mathbf{M}_+$ be a $|\mathscr{Z}|$-by-$|\mathscr{S}_+|$ matrix whose rows are indexed by $\mathscr{Z}$ and columns by $\mathscr{S}_+$, and whose entry at the intersection between row z and column S is equal to $\chi_S(z)$ ($\forall z \in \mathscr{Z} \forall S \in \mathscr{S}_+$). Analogously, let $\mathbf{M}_-$ be a $|\mathscr{Z}|$-by-$|\mathscr{S}_-|$ matrix whose rows are indexed by $\mathscr{Z}$ and columns by $\mathscr{S}_-$, and whose entry at the intersection between row z and column S is equal to $-\chi_S(z)$ ($\forall z \in \mathscr{Z} \forall S \in \mathscr{S}_-$). Thus $[\mathbf{M}_+, \mathbf{M}_-]$ is a matrix with $|\mathscr{Z}|$ rows and $|\mathscr{S}_+| + |\mathscr{S}_-|$ columns. Let $\boldsymbol{p}$ denote the column vector

$$\boldsymbol{p} := \left[(p_+(S))_{S \in \mathscr{S}_+}, (p_-(S))_{S \in \mathscr{S}_-}\right]^\intercal,$$

and $\boldsymbol{w}$ the column vector

$$\boldsymbol{w} := [(w(z))_{z \in \mathscr{Z}}]^\intercal.$$

Then the equation system (13) is the same as

$$[\mathbf{M}_+, \mathbf{M}_-]\, \boldsymbol{p} = \boldsymbol{w}. \tag{15}$$

For each $ji\tau_i \in \mathscr{Z}$, let $[ji\tau_i]$ denote the *row* of the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ corresponding to $ji\tau_i$, namely,

$$[ji\tau_i] := \left[(\chi_S(ji\tau_i))_{S \in \mathscr{S}_+}, (-\chi_S(ji\tau_i))_{S \in \mathscr{S}_-}, -w(ji\tau_i) \right].$$

The *row space* of matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ consists of all linear combinations of the rows of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$.

Lemma 1 *Eq. (15) has a nonnegative solution for $\boldsymbol{p}$ if in the row space of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$, any nonnegative row has zero as the entry at the position for $-\boldsymbol{w}$.*

To understand the intuition behind Lemma 1, consider two instances of (13) for some $z', z'' \in \mathscr{Z}$ such that $w(z') < w(z'')$. Subtract the instance of (13) when $z = z'$ by the instance of (13) when $z = z''$, so the left-hand side is a negative number. Then a contradiction occurs if the right-hand side is nonnegative, which occurs if $\chi_S(z') \geq \chi_S(z'')$ whenever $p_+(S) > 0$ ($S \in \mathscr{S}_+$), and $-\chi_S(z') \geq -\chi_S(z'')$ whenever $p_-(S) > 0$ ($S \in \mathscr{S}_-$). That is, the subtraction between the two instances of (13) produces a nonnegative row $\left[(\chi_S(z') - \chi_S(z''))_{S \in \mathscr{S}_+}, (-\chi_S(z') + \chi_S(z''))_{S \in \mathscr{S}_-}, -w(z') - (-w(z'')) \right]$, with the last positive component signifying the contradictorily negative left-hand side $w(z') - w(z'')$. This contradictory case is ruled out by the hypothesis in the lemma, which also rules out any linear combination of similar subtractions that produces a contradiction to (13).

With Lemma 1, Step 3 is achieved by proving that it is impossible for any sequence of row operations on $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ to produce a positive entry at $-\boldsymbol{w}$ if the entire row that results from the operations is nonnegative. Let us now go back to the example considered in previous sections to see the impossibility assertion heuristically.

First, note that a contradictory case cannot be produced by an addition or subtraction between two rows of the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ that refer to different bidder-types. That is, if $(i, \tau_i) \neq (k, \tau'_k)$ then neither $[ji\tau_i] + [j'k\tau'_k]$ nor $[ji\tau_i] - [j'k\tau'_k]$ can produce a nonnegative row, because at the position for $L_{i\tau_i}$, $[ji\tau_i]$ has a negative entry -1 while $[j'k\tau'_k]$ has a zero entry, and at the position for $L_{k\tau'_k}$ the situation is symmetric. Thus, the row that results from the operation is not a nonnegative vector.

It follows that a contradictory case has to result from an operation between two rows of the matrix that refer to the same bidder-type say (i, τ_i), and the operation has to be a subtraction in order to eliminate the negative entry at the position for $L_{i\tau_i}$. Furthermore,

	$U_b(1t_1)$	$U_b(2t_2')$	$U_b(2t_2)$	$V_{1t_1}(b)$	$V_{1t_1}(a)$	$V_{2t_2'}(b)$	L_{1t_1}	$L_{2t_2'}$	$\cdots$	$-w$
$[c1t_1]$	0	0	0	1	1	0	-1	0	$\cdots$	-2
$[b1t_1]$	1	0	1	1	1	0	-1	0	$\cdots$	-4
$[c1t_1] - [b1t_1]$	-1	0	-1	0	0	0	0	0	$\cdots$	2

Table 6: A row subtraction

in order to produce a positive entry at the position for $-\boldsymbol{w}$, the subtraction has to be from a low-w-value object-bidder-type to a high-w-value one. That is, the operation has to be $[ji\tau_i] - [j'i\tau_i]$ for some bidder-type (i, τ_i) and some objects $j \neq j'$ such that $w(j, i, \tau_i) < w(j', i, \tau_i)$. Meanwhile, for $[ji\tau_i] - [j'i\tau_i]$ to be nonnegative in every entry, the -1 entry in $-[j'i\tau_i]$ at the position of $V_{i\tau_i}(j')$ needs to be canceled out by a positive 1 entry in $[ji\tau_i]$ at the same position. That requires, by the definition of the matrix $\mathbf{M}_+$, $ji\tau_i \in V_{i\tau_i}(j')$, namely, $ji\tau_i \succeq j'i\tau_i$. In sum, a contradictory case has to result from a subtraction $[ji\tau_i] - [j'i\tau_i]$ for which $w(ji\tau_i) < w(j'i\tau_i)$ and $ji\tau_i \succeq j'i\tau_i$. In our example, there is only one such possibility: $[c1t_1] - [b1t_1]$, namely, the second-highest contender in the column $(1, t_1)$ of Table 3 subtracted by the top contender in the same column of the table, as the two are indifferent according to the revealed preference of the optimal matching q.

Table 6 displays $[c1t_1] - [b1t_1]$. The ellipses there stand for the omitted, less relevant entries, which are zero for both vectors. Right away we see that there are negative entries in $[c1t_1] - [b1t_1]$: At the position for $U_b(1t_1)$, for example, the entry is -1, because $b1t_1 \in U_b(1t_1)$ while $c1t_1 \notin U_b(1t_1)$ (namely, $\chi_{U_b(1t_1)}(c1t_1) - \chi_{U_b(1t_1)}(b1t_1) = 0 - 1$). Thus, the operation has yet to produce a nonnegative vector.

Consequently, to produce a contradictory case the operation has to involve another object-bidder-type $j'k\tau_k'$ such that the vector $[j'k\tau_k']$ has a positive entry at the position for $U_b(1t_1)$ to cancel out the negative entry in $[c1t_1] - [b1t_1]$ at that position. This $j'k\tau_k'$ is $b2t_2'$, because $b2t_2'$ is the only element of $U_b(1t_1)$ other than $b1t_1$, as $b2t_2' \succ b1t_1$ by the revealed preference of q within row b of Table 3. Hence the operation requires an addition of $[b2t_2']$. Since $[b2t_2']$ has a -1 entry at the position for $L_{2t_2'}$, the addition of $[b2t_2']$ in turn requires the subtraction of either $[a2t_2']$ or $[c2t_2']$ to cancel out the -1. Say it is $[a2t_2']$, so the operation aimed at producing a contradictory case entails a quadruple

$$[c1t_1] - [b1t_1] + [b2t_2'] - [a2t_2'].$$

	$U_b(1t_1)$	$U_b(2t'_2)$	$U_b(2t_2)$	$V_{1t_1}(b)$	$V_{1t_1}(a)$	$V_{2t'_2}(b)$	$V_{2t'_2}(a)$	L_{1t_1}	$L_{2t'_2}$	$\cdots$	$-w$
$[c1t_1]$	0	0	0	1	1	0	0	-1	0	$\cdots$	-2
$[b1t_1]$	1	0	1	1	1	0	0	-1	0	$\cdots$	-4
$[c1t_1] - [b1t_1]$	-1	0	-1	0	0	0	0	0	0	$\cdots$	2
$[b2t'_2]$	1	1	1	0	0	1	1	0	-1	$\cdots$	-3
$[a2t'_2]$	0	0	0	0	0	0	1	0	-1	$\cdots$	$-1/2$
$[b2t'_2] - [a2t'_2]$	1	1	1	0	0	1	0	0	0	$\cdots$	$-5/2$
net total	0	1	0	0	0	1	0	0	0	$\cdot$	$-1/2$

Table 7: A quadruple operation $[c1t_1] - [b1t_1] + [b2t'_2] - [a2t'_2]$

Table 7 displays this quadruple operation. It shows, on the final row that results from operation, the entry at the $-\boldsymbol{w}$ position is nonpositive, $-1/2$. In other words,

$$w(c1t_1) - w(b1t_1) + w(b2t'_2) - w(a2t'_2) \geq 0.$$

That is no coincidence, because $b2t'_2 \succ b1t_1$ implies, by the revealed preference of the optimal matching q within the row b in Table 3, that $\delta(b2t'_2) \geq \delta(b1t_1)$. By (14) the definition of δ, this inequality in turn implies the inequality displayed above. The displayed inequality remains intact if we replace the $[a2t'_2]$ in the quadruple by the other possible element, $[c2t'_2]$, because $w(b2t'_2) - w(c2t'_2) > \delta(b2t'_2)$, with $w(c2t'_2) < w(a2t'_2)$.

Now that we have exhausted the last resort to produce a nonnegative row whose entry at the $-\boldsymbol{w}$ position is positive, we see heuristically that the equation system (15) has no contradiction and hence admits a solution for the shadow price functions.

The proof, deferred to Appendix A.4.4, is more involving than the above to handle two complications. One is that in an arbitrary sequence of row operations on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$, a row operation such as $[c1t_1] - [b1t_1]$ may be more generally $\alpha[c1t_1] - \beta[b1t_1]$ for some positive real numbers α and β. The other complication is that there may be multiple quadruples like the one shown above. The first complication is handled by dissecting the row with the larger coefficient, say $-\beta[b1t_1]$ with $\beta > \alpha$, into two portions, one being $-\alpha[b1t_1]$ to be paired with $\alpha[c1t_1]$ so that the pair becomes $[c1t_1] - [b1t_1]$ multiplied by α, and the other being $-(\beta - \alpha)[b1t_1]$, treated as a separate row in the sequence of operations (which in turn can be paired with another row due to the nonnegative-row condition). The second complication is handled by recursively removing such quadruples from the sequence of row operations until the sequence is reduced to null. At each step of the induction, a quadruple is selected so that both the nonnegative-row condition and the positivity of the $-\boldsymbol{w}$ entry are unchanged by its removal, and hence the inductive step on the remaining sequence follows. The sequence is eventually exhausted without occurrence of any contradictory cases. Thus by Lemma 1 a solution for the equation system (15) exists.

7 Conclusion

Characterizing feasible interim allocations is often an unavoidable step in the design of optimal mechanisms, and Border-like solutions have been the prototype of such characterizations. The literature has located only sufficient conditions to obtain Border-like solutions, and those sufficient conditions are mostly confined to the paramodularity framework that rules out nontrivial combinatorial complications such as matchings. This paper adds to the literature a Border-like characterization in a matching model outside the paramodularity framework. In proving the characterization, the paper develops a method potentially applicable to other models, with or without paramodularity. The method is intuitive. A Border-like characterization is to provide the exact boundary of feasible allocations from the interim perspectives of some privately informed individuals considering some specific objects. The method tells us how to locate such interim perspectives that correspond to the boundary. If the Border-like characterization is valid, any allocation at the boundary is locally greedy in its choice among various interim perspectives: the interim perspectives that constitute the boundary are either a top portion or a bottom portion of the priority list regarding the choice among objects for the same bidder-type, or the choice among bidders for the same object. Only the final step of the method involves some complication, and the complication is entailed only in the logical steps to determine whether a finite linear equation system has a nonnegative solution. Thus, the complication may be reduced significantly, and the method may apply to more general models, if the linear equation system is handled by computational tools.

A Appendix

A.1 The Condition to Validate Border-Like Characterizations

The three steps introduced in Section 3 to obtain Border-like characterizations are based on the following theorem, which applies generally to resource allocation problems, not confined to our matching model or the mainstream, matroid or paramodularity framework. Here we allow for any finite number of bidders, and denote the set of bidders by I. The set X of feasible allocation outcomes is generally defined to be a nonempty compact subset of $\mathscr{R}^{I\times J}$, where $\mathscr{R}$ denotes either the set $\mathbb{R}$ of real numbers (divisible quantities) or a discrete subset of $\mathbb{R}$ (indivisible quantities). As before, each bidder i's set T_i of types is assumed finite. Let $T := \prod_{i\in I} T_i$ denote the set of all type profiles, and μ the joint probability measure on T, with T being the support, and μ_i the marginal onto T_i. An ex post allocation is a function $q := ((q_i^j)_{j\in J})_{i\in I}$ that maps any $t \in T$ to a an element $q(t) \in \mathrm{cv}(X)$. Denote the set of all object-bidder-type triples by $\mathscr{X}$:

$$\mathscr{X} := \{jit_i \mid j \in J; i \in I; t_i \in T_i\}.$$

Then an interim allocation is a function Q that maps any $jit_i \in \mathscr{X}$ to a $Q_i^j(t_i) \in \mathbb{R}$.

Apply (6) and (7) to this general model to define the ceiling $f(S,t)$ and floor $g(S,t)$ are defined for all $S \subseteq \mathscr{X}$ and all $t \in T$.

Let $\mathscr{Q}$ denote the set of feasible interim allocations, namely, the interim allocations that satisfy (3) for all $jit_i \in \mathscr{X}$. Let $\mathscr{Q}_{\mathrm{B}}$ denote the set of interim allocations that satisfy (8) for all $S \subseteq \mathscr{X}$. The Border-like characterization is $\mathscr{Q} = \mathscr{Q}_{\mathrm{B}}$, which is mainly $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$, as the converse is trivial.

Theorem 2 *The following three statements are equivalent:*

a. $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$.

b. For any $w \in \mathbb{R}^{\mathscr{X}}$ *there exist functions* $p_+, p_- : 2^{\mathscr{X}} \to \mathbb{R}_+$ *and an ex post allocation* $((q_i^j)_{i\in I})_{j\in J}$ *such that* $((q_i^j(t))_{i\in I})_{j\in J}$ *solves (is an optimum of)*

$$\max_{(x_i^j)_{(i,j)\in I\times J}\in \mathrm{cv}(X)} \sum_{(i,j)\in I\times J} w(jit_i)x_i^j \tag{16}$$

for all $t := (t_i)_{i \in I} \in T$,

$$w(z) = \sum_{S \subseteq \mathscr{Z}} p_+(S)\chi_S(z) - \sum_{S \subseteq \mathscr{Z}} p_-(S)\chi_S(z) \tag{17}$$

for all $z \in \mathscr{Z}$*, and*

$$\begin{array}{lll} p_+(S) > 0 & \Rightarrow & \forall t \in T \left[f(S,t) = \sum_{(i,j)\in I\times J} q_i^j(t)\chi_S(jit_i) \right] \\ p_-(S) > 0 & \Rightarrow & \forall t \in T \left[g(S,t) = \sum_{(i,j)\in I\times J} q_i^j(t)\chi_S(jit_i) \right] \end{array} \tag{18}$$

for all $S \subseteq \mathscr{Z}$*.*

c. *For any* $w \in \mathbb{R}^{\mathscr{Z}}$ *and any* q *such that* $q(t)$ *solves* (16) *for all* $t \in T$ *there exists* $p_+, p_- : 2^{\mathscr{Z}} \to \mathbb{R}_+$ *that satisfies* (17) *for all* $z \in \mathscr{Z}$ *and* (18) *for all* $S \subseteq \mathscr{Z}$*.*

Proof For any $Q, w \in \mathbb{R}^{\mathscr{Z}}$, denote the bilinear operation by

$$\langle Q, w \rangle := \sum_{jit_i \in \mathscr{Z}} Q_i^j(t_i) w(jit_i)\mu_i(t_i).$$

The set $\mathscr{Q}$ is convex and compact because the set of ex post allocation is convex and compact by its definition and the mapping from ex post allocations to their reduced forms, Eq. (3), is linear and continuous. Then an easy application of the separating hyperplane theorem (e.g., Theorem 7.51 of Aliprantis and Border [1, p288]) says that $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if for every $w \in \mathbb{R}^{\mathscr{Z}}$ we have

$$\max_{Q \in \mathscr{Q}_{\mathrm{B}}} \langle Q, w \rangle \leq \max_{Q \in \mathscr{Q}} \langle Q, w \rangle.$$

Replace the left-hand side of this inequality by its dual to obtain the fact that $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if for any $w \in \mathbb{R}^{\mathscr{Z}}$ there exist $p_+, p_-\colon 2^{\mathscr{Z}} \to \mathbb{R}_+$ that satisfies (17)—the constraint in the dual—for all $z \in \mathscr{Z}$ and

$$\sum_{t \in T} \mu(t) \sum_{S \subseteq \mathscr{Z}} \left(p_+(S) f(S,t) - p_-(S) g(S,t)\right) \leq \max_{Q \in \mathscr{Q}} \langle Q, w \rangle.$$

Rewrite the right-hand side of this inequality by the definition of $\langle Q, w \rangle$ and Eq. (3) to see that for any $\bar{q}$ such that $\bar{q}(t)$ solves (16) for all $t \in T$, the inequality is equivalent to

$$\sum_{t \in T} \mu(t) \sum_{S \subseteq \mathscr{Z}} \left(p_+(S) f(S,t) - p_-(S) g(S,t)\right) \leq \sum_{t \in T} \mu(t) \sum_{(i,j)\in I\times J} \bar{q}_i^j(t) w(j,i,t_i),$$

which combined with (17) is the same as

$$\begin{array}{r} \sum_{S \subseteq \mathscr{Z}} p_+(S) \sum_{t \in T} \mu(t) \left(f(S,t) - \sum_{(i,j)\in I\times J} \bar{q}_i^j(t)\chi_S(jit_i) \right) \\ + \sum_{S \subseteq \mathscr{Z}} p_-(S) \sum_{t \in T} \mu(t) \left(-g(S,t) + \sum_{(i,j)\in I\times J} \bar{q}_i^j(t)\chi_S(jit_i) \right) \end{array} \leq 0. \tag{19}$$

For any $S \subseteq \mathscr{Z}$ and $t \in T$,

$$f(S,t) = \max_{x \in X} \sum_{(i,j) \in I(S,t)} x_i^j = \max_{x \in \mathrm{cv}(X)} \sum_{(i,j) \in I(S,t)} x_i^j \geq \sum_{(i,j) \in I(S,t)} \bar{q}_i^j(t) = \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(jit_i),$$

with the first "=" due to (6), the second "=" due to X containing all extremal points of $\mathrm{cv}(X)$, the inequality due to $\bar{q}_i^j(t) \in \mathrm{cv}(X)$, and the last "=" due to (5) the definition of $I(S,t)$. By the same token,

$$-g(S,t) + \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(jit_i) \geq 0$$

for all $S \subseteq \mathscr{Z}$ and all $t \in T$. This, coupled with $\mu(t) > 0$ for all $t \in T$ (T being finite and the support of μ), implies that (19) holds if and only if (18) holds for all $S \subseteq \mathscr{Z}$.

In sum, for any $\bar{q}$ such that $\bar{q}(t)$ solves (16) for all $t \in T$, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ if and only if for any $w \in \mathbb{R}^{\mathscr{Z}}$ there exists (p_+, p_-): $2^{\mathscr{Z}} \to \mathbb{R}_+^2$ that satisfies both (17) for all $z \in \mathscr{Z}$ and (18) for all $S \subseteq \mathscr{Z}$. That immediately implies "(b) $\Rightarrow$ (a) $\Rightarrow$ (c)." Meanwhile, "(a) $\Rightarrow$ (b)" holds because there always exists an ex post allocation $\bar{q}$ such that $\bar{q}(t)$ solves (16) for all $t \in T$. ■

A.2 Local Greediness of Border-Like Characterizations

The connection between Border-like characterizations in general models and the local validity of greedy solutions is formalized here as an implication of Theorem 2. Recall from (9) that any ex post allocation q induces a stochastic choice function $q(\cdot|t) : \mathscr{Z} \to \mathbb{R}$ for every $t \in T$.

Corollary 1 *Assume the general model in Section A.1. If $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$, $w \in \mathbb{R}^{\mathscr{Z}}$ and $q(t)$ solves (16) for all $t \in T$, then there exist index sets $\mathscr{K}_+$ and $\mathscr{K}_-$ and $(S_k)_{k \in \mathscr{K}_+ \sqcup \mathscr{K}_-}$ such that, for each $k \in \mathscr{K}_+ \sqcup \mathscr{K}_-$, $\varnothing \neq S_k \subseteq \mathscr{Z}$,*

$$\{z \in \mathscr{Z} \mid w(z) \neq 0\} \subseteq \bigcup_{k \in \mathscr{K}_+ \sqcup \mathscr{K}_-} S_k \tag{20}$$

and, for some m_k subsets $S_k^1, \ldots S_k^{m_k}$ of $\mathscr{Z}$,

$$\varnothing = S_k^0 \subsetneq S_k^1 \subsetneq S_k^2 \subsetneq \cdots \subsetneq S_k^{m_k} = S_k \tag{21}$$

and, for all $t \in T$ and all $n = 1, \ldots, m_k$,

$$\sum_{z \in S_k^n \setminus S_k^{n-1}} q(z|t) = \begin{cases} f(S_k^n, t) - f(S_k^{n-1}, t) & \text{if } k \in \mathscr{K}_+ \\ g(S_k^n, t) - g(S_k^{n-1}, t) & \text{if } k \in \mathscr{K}_-. \end{cases} \tag{22}$$

Proof Let $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$, $w \in \mathbb{R}^{\mathscr{Z}}$, and $q(t)$ solves (16) for all $t \in T$. By Statement (c) of Theorem 2, let $\mathscr{S}_+ := \{S \subseteq \mathscr{Z} \mid p_+(S) > 0\}$, and $\mathscr{S}_- := \{S \subseteq \mathscr{Z} \mid p_-(S) > 0\}$. For any $z \in \mathscr{Z}$ such that $w(z) \neq 0$, Eq. (17) requires that $z \in S$ for some $S \in \mathscr{S}_+ \cup \mathscr{S}_-$. Thus $\{z \in \mathscr{Z} \mid w(z) \neq 0\} \subseteq \bigcup_{S \in \mathscr{S}_+ \cup \mathscr{S}_-} S$. Start with any $S \in \mathscr{S}_+$. It belongs to a collection of elements of $\mathscr{S}_+$ such that the set inclusion relation $\subseteq$ is total on the collection. Extend the collection to maximal to obtain a maximal chain (nested sequence)

$$\varnothing =: S_1^0 \subsetneq S_1^1 \subsetneq S_1^2 \subsetneq \cdots \subsetneq S_1^{m_1} =: S_1$$

for some positive integer m_1 and some $S_1^n \in \mathscr{S}_+$ for all $n = 1, \dots, m$ such that $S_1^n = S$ for one of these n. Initiate $\mathscr{K}_+ := \{1\}$. Pick any element S' of $\mathscr{S}_+$ outside $\{S_1^n\}_{n=1}^{m_1}$ and construct another maximal chain $\{S_2^n\}_{n=1}^{m_2}$ in $\mathscr{S}_+$ that contains S'. Update $\mathscr{K}_+ := \mathscr{K}_+ \cup \{2\}$. Repeat until every element of $\mathscr{S}_+$ is contained in some of the chains already constructed. Thus the index set $\mathscr{K}_+$ is constructed. Do the same for $\mathscr{S}_-$ to construct the index set $\mathscr{K}_-$. By construction, both (20) and (21) are satisfied. To verify (22), pick any $k \in \mathscr{K}_+$, so $S_k^n \in \mathscr{S}_+$ for all $n = 1, \dots, m_k$. Then (18), coupled with the notation (9), implies

$$\forall t \in T \, \forall n = 1, \dots, m_k : \sum_{z \in S_k^n} q(z|t) = f(S_k^n, t).$$

Apply this to the cases of $n = k$ and $n = k - 1$ (combined with $f(\varnothing, t) = 0$) to obtain

$$\sum_{z \in S_k^n \setminus S_k^{n-1}} q(z|t) = \sum_{z \in S_k^n} q(z|t) - \sum_{z \in S_k^{n-1}} q(z|t) = f(S_k^n, t) - f(S_k^{n-1}, t)$$

for all $t \in T$. The case of $k \in \mathscr{K}_-$ is similar. Thus, (22) is satisfied as well. ■

A.3 Proof of Lemma 1

By Theorem 2 of Chu et al. [7], a nonnegative solution of $\boldsymbol{p}$ for $[\mathbf{M}_+, \mathbf{M}_-]\,\boldsymbol{p} = \boldsymbol{w}$ exists if the cover order of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ is less than or equal to the cover order of $[\mathbf{M}_+, \mathbf{M}_-]$. According to the procedure in their Section 4, the cover order of any matrix say $\mathbf{A}$ is equal to the maximum number of (strictly) positive entries among all the nonnegative rows that any Gaussian elimination on $\mathbf{A}$ can produce. Since the only difference between $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ and $[\mathbf{M}_+, \mathbf{M}_-]$ is the $-\boldsymbol{w}$ column, any nonnegative row produced by a Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ that has zero at the entry for $-\boldsymbol{w}$ can be produced by the same Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-]$. Thus, the desired inequality between their cover orders follows from the hypothesis in the lemma. ■

A.4 Proof of Theorem 1

It suffices to prove statement (b) in Theorem 2. To that end, pick any $w \in \mathbb{R}^{\mathscr{X}}$ and consider the linear programming problem (16), which in our matching model is the optimal matching problem (10). Since X in the matching model is a finite set, the problem has an integral solution. Thus, we can pick any integral solution q of (10) and prove the theorem by proving existence of nonnegative functions p_+ and p_- on $2^{\mathscr{X}}$ that satisfy both (17) with respect to w and (18) with respect to this q. This is done through the following subsections.

A.4.1 The Optimal Matching

Let $>$ be any linear order on J (transitive, total and antisymmetric). For any $i \in \{1, 2\}$ and any $t_i \in T_i$, define the top and second-highest contenders with respect to bidder-type (i, t_i):

$$\begin{aligned} j^1(i,t_i) &:= \min_{>} \left(\arg\max_{j \in J} w(jit_i) \right) \\ j^2(i,t_i) &:= \min_{>} \left(\arg\max\nolimits_{j \in J \setminus \{j^1(i,t_i)\}} w(jit_i) \right), \end{aligned}$$

where $\min_>$ denotes the minimum with respect to $>$. For every $jit_i \in \mathscr{X}$, define

$$\delta(jit_i) := w(jit_i) - \max_{k \in J \setminus \{j\}} w(kit_i). \tag{23}$$

For any $t \in T$, define a matching $q(t) \in X$ as follows:

i. if $j^1(1,t_1) \neq j^1(2,t_2)$ then match bidder 1 with object $j^1(1,t_1)$, and bidder 2 with object $j^1(2,t_2)$, or $(1 \mapsto j^1(1,t_1), 2 \mapsto j^1(2,t_2))$ for short;

ii. else, if $\delta(j^1(1,t_1)1t_1) \geq \delta(j^1(2,t_2)2t_2)$, then the matching is $(1 \mapsto j^1(1,t_1), 2 \mapsto j^2(2,t_2))$ (matching bidder 1 with the top contender for bidder-type $(1,t_1)$, and bidder 2 with the second-highest contender for $(2,t_2)$);

iii. else, the matching is $(1 \mapsto j^2(1,t_1), 2 \mapsto j^1(2,t_2))$.

Obviously, $q(t)$ is an integral solution of the optimal matching problem (10).

A.4.2 The Revealed Preferences

Different Objects for the same Bidder-Type For any $i \in I$ and any $t_i \in T_i$, the choice of q within the "column" (cf. Table 3) $\{jit_i \mid j \in J\}$ is consistent with the ordinal ranking of the w-values of its elements, as it chooses either the top or second-highest contender and never chooses any other element in the set. Thus, for each bidder-type (i, t_i), we can arrange the elements of $\{jit_i \mid j \in J\}$ into a list $(j^n it_i)_{n=1}^N$ ($N := |J|$) according to the revealed preference $\succeq_{it_i}$ of q within $\{jit_i \mid j \in J\}$ (with symmetric part $\sim_{it_i}$ and strict part $\succ_{it_i}$):

$$j^1 it_i \sim_{it_i} j^2 it_i \succ_{it_i} j^3 it_i \succ_{it_i} \cdots \succ_{it_i} j^{N-1} it_i \succ_{it_i} j^N it_i,$$

so that $j^1 = j^1(i, t_i)$, $j^2 = j^2(i, t_i)$, and $n < k \Rightarrow w(j^n it_i) \geq w(j^k it_i)$. Correspondingly, define all upper contour sets and one lower contour set with respect to $\succeq_{it_i}$:

$$\begin{aligned} V_{it_i}^n &:= \left\{j^k it_i \mid 1 \leq k \leq n+1\right\} \quad \forall n = 1, \ldots, N-1, \\ L_{it_i} &:= \{jit_i \mid j \in J\}. \end{aligned}$$

Different Bidder-Types for the Same Object For any $j \in J$, the choice of the optimal matching q within the "row" (cf. Table 3) $\{(jit_i \mid i \in I; t_i \in T_i\}$ has the following patterns. First, q never picks a member jit_i in the row that is neither the top nor the second-highest contender in its column $\{jit_i \mid j \in J\}$. Second, a member jit_i of the row that is not the top contender in its column is never chosen by q over a member jkt'_k that is the top contender in its column $\{jkt'_k \mid j \in J\}$: The two elements are both relevant to q only when the type profile is (t_i, t'_k). Now that jit_i is not the top contender in its column, the top contender $j^1(i, t_i)it_i$ in that column gets to be chosen together with jkt'_k, and so jkt'_k is not crowded out. Thus, the competition within a row is only among the members that are the top contenders in their own columns, namely, among the members of the set

$$\mathscr{Z}_j := \left\{jit_i \mid i \in I; t_i \in T_i; j = j^1(i, t_i)\right\}.$$

Within $\mathscr{Z}_j$, the choice of q is consistent with a (strict) linear order $\succ_j$ such that $jit_i \succ_j jkt'_k$ if and only if either $\delta(jit_i) > \delta(jkt'_k)$ or "$\delta(jit_i) = \delta(jkt'_k)$ and either $i < k$ or $i = k$ and $t_i \lhd t'_k$" ($\lhd$ can be any linear order on T_i). List the elements of $\mathscr{Z}_j$ into $(z^n)_{n=1}^{|\mathscr{Z}_j|}$ so that

$$z^1 \succ_j z^2 \succ_j \cdots \succ_j z^{|\mathscr{Z}_j|}. \tag{24}$$

For any $n = 1, \ldots, |\mathscr{Z}_j|$, let

$$U_j^n := \{z^k \mid 1 \leq k \leq n\}.$$

A.4.3 The Binding Sets

To prove statement (b) in Theorem 2 (thereby proving Theorem 1), I will use the upper contour sets U_j^n and $V_{it_i}^n$ and the lower contour sets L_{it_i} constructed in the previous subsection as the elements of the supports for the shadow price functions p_+ and p_-. To that end. let us verify that these contour sets do satisfy the condition (18) in statement (b):

$$\sum_{jit_i \in S} q_i^j(\tau_1, \tau_2) = \begin{cases} f(S; \tau_1, \tau_2) & \text{if } S = U_j^n \text{ or } V_{it_i}^n \\ g(S; \tau_1, \tau_2) & \text{if } S = L_{it_i} \end{cases}$$

for all $(\tau_1, \tau_2) \in T$. For any upper contour set say U_j^n, all elements therein refer to the same object j, and hence for any $\tau \in T$, $f(U_j^n, \tau) = 1$ if $I(U_j^n, \tau) \neq \varnothing$, and $f(U_j^n, \tau) = 0$ if $I(U_j^n, \tau) = \varnothing$. Meanwhile, for any $\tau := (\tau_1, \tau_2) \in T$, $q_i^j(\tau) = 1$ for the bidder i such that $ji\tau_i$ is the $\succ_j$-maximum among all the $jk\tau_k \in \mathscr{Z}_j$ $(k = 1, 2)$, and $q_h^j(\tau) = 0$ for all $h \in I \setminus \{i\}$. Thus, if $I(U_j^n, \tau) \neq \varnothing$, i.e., U_j^n contains at least one $z \in \mathscr{Z}_j$ such that $z = jk\tau_k$, then U_j^n, being an upper contour set, contains the $\succ_j$-maximum among them, which is $ji\tau_i$. Thus,

$$\sum_{jht_h \in S} q_h^j(\tau) = q_i^j(\tau) = 1 = f(U_j^n, \tau).$$

Else, $I(U_j^n, \tau) = \varnothing$, U_j^n contains no member z of $\mathscr{Z}_j$ such that $z = jk\tau_k$, and hence $\sum_{jht_h \in S} q_h^j(\tau) = 0 = f(U_j^n, \tau)$. Thus, U_j^n satisfy (18). The reasoning for $V_{it_i}^n$ is similar. For the lower contour set L_{it_i}, by its definition, either $I(L_{it_i}, \tau) = \{(i,j) \mid j \in J\}$ (and hence $g(L_{it_i}, \tau) = 1$) if $t_i = \tau_i$, or $I(L_{it_i}, \tau) = \varnothing$ (and hence $g(L_{it_i}, \tau) = 0$) if $t_i \neq \tau_i$. Since $\sum_{j \in J} q_i^j(\tau) = 1$ by (1), L_{it_i} satisfies (18) as well.

A.4.4 Existence of the Shadow Price Function

Let

$$\begin{aligned} \mathscr{S}_+ &:= \left\{U_j^n \mid j \in J; n = 1, \ldots, |\mathscr{Z}_j|\right\} \bigcup \left\{V_{it_i}^n \mid i \in I; t_i \in T_i; n = 1, \ldots, N-1\right\} \\ \mathscr{S}_- &:= \left\{L_{it_i} \mid i \in I; t_i \in T_i\right\}. \end{aligned}$$

The final step is to prove existence of $p_+ : \mathscr{S}_+ \to \mathbb{R}_+$ and $p_- : \mathscr{S}_- \to \mathbb{R}_+$ that satisfy (17). Let $[\mathbf{M}_+, \mathbf{M}_-]$ be the matrix according to the definition preceding Lemma 1, with respect to the $\mathscr{S}_+$ and $\mathscr{S}_-$ here. That is, for any $jit_i \in \mathscr{Z}$, let $[jit_i]$ denote the row of $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$:

$$\begin{aligned} [jit_i] &:= \Big(([jit_i](S))_{S \in \mathscr{S}_+ \sqcup \mathscr{S}_-}, [jit_i](-w) \Big) \\ &:= \left((\chi_S(jit_i))_{S \in \mathscr{S}_+}, (-\chi_S(jit_i))_{S \in \mathscr{S}_-}, -w(jit_i) \right). \end{aligned}$$

Then (17) is the same as $[\mathbf{M}_+, \mathbf{M}_-]\,\boldsymbol{p} = \boldsymbol{w}$. By Lemma 1, it suffices to prove that no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$ can produce a nonnegative row whose entry at the $-\boldsymbol{w}$ position is positive. Since any Gaussian elimination consists of a sequence of row-wise operations, through addition or subtraction with each row multiplied by a coefficient, the row that results from any elimination procedure is equal to $\sum_{z\in Z} \beta_z[z]$ for some $Z \subseteq \mathscr{Z}$ such that $\beta_z \in \mathbb{R} \setminus \{0\}$ is the coefficient for the row $[z]$ in the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{w}]$. The nonnegativity condition of the row that results from the procedure means $\sum_{z\in Z} \beta_z[z] \geqq \mathbf{0}$.

Thus, pick any $Z \subseteq \mathscr{Z}$ and any $(\beta_z)_{z\in Z} \in (\mathbb{R} \setminus \{0\})^Z$ for which $\sum_{z\in Z} \beta_z[z] \geqq \mathbf{0}$. For any $i \in I$ and any $t_i \in T_i$, define

$$Z_{it_i} := \{jit_i \mid j \in J\} \cap Z.$$

Note that $Z = \sqcup_{i\in I} \sqcup_{t_i\in T_i} Z_{it_i}$. Thus,

$$\sum_{z\in Z} \beta_z[z] = \sum_{i\in I} \sum_{t_i\in T_i} \sum_{z\in Z_{it_i}} \beta_z[z]. \tag{25}$$

By the definition of $L_{kt'_k}$,

$$[jit_i]\left(L_{kt'_k}\right) = \begin{cases} -1 & \text{if } (i, t_i) = (k, t'_k) \\ 0 & \text{else.} \end{cases} \tag{26}$$

Similarly, $[jit_i](V^n_{kt'_k}) = 0$ for all n whenever $(k, t'_k) \neq (i, t_i)$. Thus,

$$\left[S = V^n_{it_i} \text{ or } S = L_{it_i}\right] \Longrightarrow \sum_{z\in Z} \beta_z[z](S) = \sum_{z\in Z_{it_i}} \beta_z[z](S).$$

By (25) and $\sum_{z\in Z} \beta_z[z] \geq \mathbf{0}$,

$$\left[S = V^n_{it_i} \text{ or } S = L_{it_i}\right] \Longrightarrow \sum_{z\in Z_{it_i}} \beta_z[z](S) \geq 0. \tag{27}$$

Rearrangement for the Same Bidder-Type Let $i \in I$ and $t_i \in T_i$. Define

$$\begin{aligned} Z^-_{it_i} &:= \{z \in Z_{it_i} \mid \beta_z < 0\} \\ Z^+_{it_i} &:= Z_{it_i} \setminus Z^- \; (= \{z \in Z_{k,t_k} \mid \beta_z > 0\}). \end{aligned}$$

Thus,

$$\sum_{z\in Z_{it_i}} \beta_z[z] = \sum_{z\in Z^+_{it_i}} |\beta_z|\,[z] - \sum_{z\in Z^-_{it_i}} |\beta_z|\,[z].$$

Lemma 2 *For each* (i, t_i) *there exist a finite set* K*, functions* $\alpha : K \to Z^-_{it_i}$ *and* $\zeta : K \to Z^+_{it_i}$*, and a positive vector* $(\tilde{\beta}_k)_{k \in K} \in \mathbb{R}^K_{++}$ *such that* $\zeta(k) \succeq_{it_i} \alpha(k)$ *for each* $k \in K$ *and*

$$\sum_{z \in Z_{it_i}} \beta_z[z] = \sum_{k \in K} \tilde{\beta}_k \left([\zeta(k)] - [\alpha(k)]\right).$$

Proof Let $u : \{jit_i \mid j \in J\} \to \{1, \ldots, N-1\}$ be a utility function representing the preference relation $\succeq_{it_i}$ on $\{jit_i \mid j \in J\}$ ($N - 1 = |J| - 1$ is the number of $\sim_{it_i}$-indifference sets in $\{jit_i \mid j \in J\}$). By the definition of $(V^n_{it_i})^{N-1}_{n=1}$, for any $m \in \{1, \ldots, N-1\}$,

$$\{jit_i \mid u(jit_i) \geq m; j \in J\} = V^{N-m}_{it_i}. \tag{28}$$

Let us construct the set K recursively. Start with any element of $Z^-_{it_i}$ that minimizes u on $Z^-_{it_i}$, and denote this element by $\alpha(1)$. Initiate K by $K := \{1\}$. There exists $z' \in Z^+_{it_i}$ for which $z' \succeq_{it_i} \alpha(1)$. That is because $[\alpha(1)](V^{N-u(\alpha(1))}_{it_i}) = 1$ and hence $\alpha(1)$ contributes a negative entry to $\sum_{z \in Z_{it_i}} \beta_z[z](V^{N-u(\alpha(1))}_{it_i})$ since $\alpha(1) \in Z^-_{it_i}$; the nonnegativity of this sum then requires that there be $z' \in Z^+_{it_i}$ for which $[z'](V^{N-u(\alpha(1))}_{it_i}) = 1$ to cancel out the negative entry. That means $z' \in V^{N-u(\alpha(1))}_{it_i}$ and hence $u(z') \geq u(\alpha(1))$. Let $\zeta(1)$ denote the minimizer of $u(z')$ among all the $z' \in Z^+_{it_i}$ such that $z' \succeq_{it_i} \alpha(1)$. Let

$$\tilde{\beta}_1 := \min\left\{|\beta_{\alpha(1)}|, |\beta_{\zeta(1)}|\right\} \;\; (= \min\left\{-\beta_{\alpha(1)}, \beta_{\zeta(1)}\right\}). \tag{29}$$

For any $z \in Z_{it_i}$, let

$$\beta'_z := \begin{cases} \beta_{\alpha(1)} + \tilde{\beta}_1 & \text{if } z = \alpha(1) \\ \beta_{\zeta(1)} - \tilde{\beta}_1 & \text{if } z = \zeta(1) \\ \beta_z & \text{else} \end{cases} \tag{30}$$

$$\widetilde{Z}^-_{it_i} := \begin{cases} Z^-_{it_i} \setminus \{\alpha(1)\} & \text{if } \beta'_{\alpha(1)} = 0 \\ Z^-_{it_i} & \text{else} \end{cases}$$

$$\widetilde{Z}^+_{it_i} := \begin{cases} Z^+_{it_i} \setminus \{\zeta(1)\} & \text{if } \beta'_{\zeta(1)} = 0 \\ Z^+_{it_i} & \text{else.} \end{cases}$$

Then

$$\sum_{z \in Z_{it_i}} \beta_z[z] = \sum_{k \in K} \tilde{\beta}_k \left([\zeta(k)] - [\alpha(k)]\right) + \sum_{z \in \widetilde{Z}^-_{it_i} \sqcup \widetilde{Z}^+_{it_i}} \beta'_z[z]. \tag{31}$$

Claim: (27) is preserved when Z_{it_i} is replaced by $\widetilde{Z}^+_{it_i} \sqcup \widetilde{Z}^-_{it_i}$, and β_z replaced by β'_z. The case for $S = L_{it_i}$ follows from (26), as $[\zeta(k)]$ and $[\alpha(k)]$ refer to the same (i, t_i) and hence their entries at the position for L_{it_i} are canceled out. For the other case, we need to show

$$\sum_{z \in \widetilde{Z}^-_{it_i} \sqcup \widetilde{Z}^+_{it_i}} \beta'_z[z](V^{N-m}_{it_i}) \geq 0 \tag{32}$$

for all $m = 1, \ldots, N-1$. To that end, plug (30) into the left-hand side of (32) to have

$$\begin{aligned} \sum_{z \in \widetilde{Z}^-_{it_i} \sqcup \widetilde{Z}^+_{it_i}} \beta'_z[z](V^{N-m}_{it_i}) &= \sum_{z \in \widetilde{Z}^+_{it_i}} \beta'_z[z](V^{N-m}_{it_i}) - \sum_{z \in \widetilde{Z}^-_{it_i}} \left|\beta'_z[z]\right|(V^{N-m}_{it_i}) \\ &= \sum_{z \in Z^+_{it_i}} \beta_z[z](V^{N-m}_{it_i}) - \tilde{\beta}_1[\zeta(1)](V^{N-m}_{it_i}) \\ &\quad - \sum_{z \in Z^-_{it_i}} \left|\beta_z[z]\right|(V^{N-m}_{it_i}) + \tilde{\beta}_1[\alpha(1)](V^{N-m}_{it_i}). \end{aligned}$$

The right-hand side of the equation displayed above reduces to

$$\sum_{z \in Z^+_{it_i}} \beta_z[z](V^{N-m}_{it_i}) - \sum_{z \in Z^-_{it_i}} \left|\beta_z[z]\right|(V^{N-m}_{it_i})$$

when $m \leq u(\alpha(1))$ or $m > u(\zeta(1))$. That is because $m \leq u(\alpha(1))$ implies that both $\zeta(1)$ and $\alpha(1)$ belong to $V^{N-m}_{it_i}$ by (28) and so $[\zeta(k)](V^{N-m}_{it_i}) = [\alpha(k)](V^{N-m}_{it_i})$, and $m > u(\zeta(1))$ implies that neither $\zeta(1)$ nor $\alpha(1)$ belong to such $V^{N-m}_{it_i}$ and hence again $[\zeta(k)](V^{N-m}_{it_i}) = [\alpha(k)](V^{N-m}_{it_i})$. Thus, by the (27) for the original Z_{it_i}, (32) holds for all such m. For any other m, namely, $u(\zeta(1)) \geq m > u(\alpha(1))$, recall that $\zeta(1)$ is a minimum of $u(z')$ among all $z' \in Z^+_{it_i}$ such that $z' \succeq_{it_i} \alpha(1)$. Thus, decreasing m from $u(\zeta(1))$ to $u(\alpha(1))$ cannot enlarge $\sum_{z \in Z^+_{it_i}} \beta_z[z](V^{N-m}_{it_i})$. Hence

$$\begin{aligned} &\sum_{z \in Z^+_{it_i}} \beta_z[z](V^{N-m}_{it_i}) - \tilde{\beta}_1[\zeta(1)](V^{N-m}_{it_i}) - \sum_{z \in Z^-_{it_i}} \left|\beta_z[z]\right|(V^{N-m}_{it_i}) + \tilde{\beta}_1[\alpha(1)](V^{N-m}_{it_i}) \\ \geq\ &\sum_{z \in Z^+_{it_i}} \beta_z[z](V^{N-u(\alpha(1))}_{it_i}) - \tilde{\beta}_1[\zeta(1)](V^{N-m}_{it_i}) - \sum_{z \in Z^-_{it_i}} \left|\beta_z[z]\right|(V^{N-m}_{it_i}) + \tilde{\beta}_1[\alpha(1)](V^{N-m}_{it_i}) \\ =\ &\sum_{z \in Z^+_{it_i}} \beta_z[z](V^{N-u(\alpha(1))}_{it_i}) - \tilde{\beta}_1 - \sum_{z \in Z^-_{it_i} \setminus \{\alpha(1)\}} \left|\beta_z[z]\right|(V^{N-m}_{it_i}), \end{aligned}$$

with the last line due to $u(\zeta(1)) \geq m > u(\alpha(1))$ and hence $\alpha(1) \notin V^{N-m}_{it_i} \ni \zeta(1)$, namely, $[\alpha(1)](V^{N-m}_{it_i}) = 0$ and $\zeta(1)](V^{N-m}_{it_i}) = 1$. Since $(V^n_{it_i})^{N-1}_{n=1}$ is a nested increasing sequence and

$m > u(\alpha(1))$, $z \in V_{it_i}^{N-m} \Rightarrow z \in V_{it_i}^{N-u(\alpha(1))}$. Thus,

$$\sum_{z\in Z^-_{it_i}\setminus\{\alpha(1)\}} |\beta_z[z]|\,(V_{it_i}^{N-m}) \le \sum_{z\in Z^-_{it_i}\setminus\{\alpha(1)\}} |\beta_z[z]|\,(V_{it_i}^{N-u(\alpha(1))}).$$

Plug this into the two multiline formulas displayed above to obtain

$$\sum_{z\in \widetilde{Z}^-_{it_i}\sqcup \widetilde{Z}^+_{it_i}} \beta'_z[z](V_{it_i}^{N-m}) \ \ge \ \sum_{z\in Z^+_{it_i}} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}) - \tilde{\beta}_1 - \sum_{z\in Z^-_{it_i}\setminus\{\alpha(1)\}} |\beta_z[z]|\,(V_{it_i}^{N-u(\alpha(1))}).$$

By the definition (Eq. (29)) of $\tilde{\beta}_1$, $\tilde{\beta}_1 \le \left|\beta_{\alpha(1)}\right| = \left|\beta_{\alpha(1)}\right|[\alpha(1)](V_{it_i}^{N-u(\alpha(1))})$ and hence

$$-\tilde{\beta}_1 - \sum_{z\in Z^-_{it_i}\setminus\{\alpha(1)\}} |\beta_z[z]|\,(V_{it_i}^{N-u(\alpha(1))}) \ge -\sum_{z\in Z^-_{it_i}} |\beta_z[z]|\,(V_{it_i}^{N-u(\alpha(1))}).$$

Plug this back into the second last inequality displayed above to obtain

$$\sum_{z\in \widetilde{Z}^-_{it_i}\sqcup \widetilde{Z}^+_{it_i}} \beta'_z[z](V_{it_i}^{N-m}) \ge \sum_{z\in Z^+_{it_i}} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}) - \sum_{z\in Z^-_{it_i}} |\beta_z[z]|\,(V_{it_i}^{N-u(\alpha(1))}) = \sum_{z\in Z_{it_i}} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}),$$

which is nonnegative by the (27) for the original Z_{it_i}. Thus (32) follows.

Now let $\tilde{Z}^+_{it_i}$, $\tilde{Z}^-_{it_i}$ and β'_z take the roles of $Z^+_{it_i}$, $Z^-_{it_i}$ and β_z. This preserves (27) by the above reasoning and reduces the cardinality of $Z^+_{it_i}$ or $Z^-_{it_i}$ by one.

Repeat the above procedure on the updated $Z^+_{it_i}$, $Z^-_{it_i}$ and β_z. Let $\alpha(2)$ be an element of $Z^-_{it_i}$ that minimizes $u(\cdot)$ on $Z^-_{it_i}$, and $\zeta(2)$ an element of $Z^+_{it_i}$ that minimizes $u(z')$ among all $z' \in Z^+_{it_i}$ for which $z' \succeq_{it_i} \alpha(2)$. Update $K := K \sqcup \{2\}$ and define $\tilde{\beta}_2$ as in (29). Then update $Z^+_{it_i}$, $Z^-_{it_i}$ and β_z again accordingly. As before, this update preserves (27) and further reduces the cardinality of $Z^+_{it_i}$ or $Z^-_{it_i}$ by one.

Continue this procedure until $Z^+_{it_i}$ or $Z^-_{it_i}$ becomes empty. In the end, $Z^-_{it_i} \neq \varnothing = Z^+_{it_i}$ is impossible, otherwise the nonnegativity condition in (27) is violated. Neither is $Z^-_{it_i} = \varnothing \neq Z^+_{it_i}$ possible, otherwise the elements of $Z^+_{it_i}$, by (26), have negative entries at the position for L_{it_i} that are not canceled out, again violating the nonnegativity condition. Both sets empty, (31) turns into the equation that the lemma claims. ■

Apply the lemma to all bidder-types to obtain

$$\sum_{z\in Z} \beta_z[z] = \sum_{i\in I}\sum_{t_i\in T_i}\sum_{k\in K_{it_i}} \tilde{\beta}_k\left([\zeta_{it_k}(k)] - [\alpha_{it_k}(k)]\right)$$

such that for every bidder-type (i,t_i), α_{it_k} and ζ_{it_k} are functions $K_{it_i} \to Z^-_{it_i}$ and $K_{it_i} \to Z^+_{it_i}$, and for each $k \in K_{it_i}$, $\zeta_{it_i}(k)$ is a $\succeq_{it_i}$-minimum among $z \in Z^+_{it_i}$ for which $z \succeq_{it_i} \alpha(k)$. Let

$$\mathbb{K} := \sqcup_{i \in I} \sqcup_{t_i \in T_i} K_{it_i}$$

and extend α and ζ to $\mathbb{K}$ by $k \in K_{it_i} \Rightarrow [\alpha(k) := \alpha_{it_i}(k) \text{ and } \zeta(k) := \zeta_{it_i}(k)]$. Then the above equation is more succinctly rewritten as

$$\sum_{z \in Z} \beta_z [z] = \sum_{k \in \mathbb{K}} \tilde{\beta}_k \left([\zeta(k)] - [\alpha(k)] \right). \tag{33}$$

Rearrangement for the Same Object Recall that for every $j \in J$, $\mathscr{Z}_j$ is the set of jit_i across bidder-types (i,t_i) such that jit_i is the top contender in $\{jit_i \mid j \in J\}$. Let

$$\mathbb{K}_j := \{k \in \mathbb{K} \mid \exists i \in I \; \exists t_i \in T_i \, [jit_i \in \mathscr{Z}_j \cap \{\alpha(k), \zeta(k)\}]\}$$

for each $j \in J$. Note that $j \neq j' \Rightarrow \mathbb{K}_j \cap \mathbb{K}_{j'} = \varnothing$. That is because for each bidder-type (i,t_i) there is a unique top contender $j^1(i,t_i)$ in $\{jit_i \mid j \in J\}$ by the definition of $j^1(i,t_i)$ in Section A.4.1. Thus, if one of $\alpha(k)$ and $\zeta(k)$ is a top contender then the other is not.

By the definition of $[\mathbf{M}_+, \mathbf{M}_-]$ and that of the sets $(U^n_j)^{|\mathscr{Z}_j|}_{n=1}$, $[j'it_i](U^n_j) = 0$ for all n unless $j' = j$. Thus, for all $j \in J$ and all $n \in \{1, \ldots, |\mathscr{Z}_j|\}$,

$$\sum_{z \in Z} \beta_z [z](U^n_j) = \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k \left([\zeta(k)](U^n_j) - [\alpha(k)](U^n_j) \right).$$

For all such j and n, by the nonnegative-row condition,

$$\sum_{k \in \mathbb{K}_j} \tilde{\beta}_k \left([\zeta(k)](U^n_j) - [\alpha(k)](U^n_j) \right) \geq 0. \tag{34}$$

For any $j \in J$ such that $\mathbb{K}_j \neq \varnothing$, define

$$\begin{aligned} \mathbb{K}^-_j &:= \{k \in \mathbb{K}_j \mid \alpha(k) \in \mathscr{Z}_j\} \\ \mathbb{K}^+_j &:= \{k \in \mathbb{K}_j \mid \zeta(k) \in \mathscr{Z}_j\}. \end{aligned}$$

Thus, if $k \in \mathbb{K}^-_j$ then $[\alpha(k)]$ enters the Gaussian elimination through subtraction (negative $\beta_{\alpha(k)}$) and, since $\alpha(k) \in \mathscr{Z}_j$, the subtraction $-[\alpha(k)]$ contributes a negative entry at the position for some U^n_j. Analogously, if $k \in \mathbb{K}^+_j$ then $[\zeta(k)]$ enters the Gaussian elimination through addition and contributes a positive entry at the position for some U^n_j.

Since $\alpha(k)$ and $\zeta(k)$ cannot be both top contenders, $\mathbb{K}_j^- \cap \mathbb{K}_j^+ = \varnothing$. Thus, (34) becomes

$$\sum_{k \in \mathbb{K}_j^+} \tilde{\beta}_k \left([\zeta(k)](U_j^n) - [\alpha(k)](U_j^n)\right) \geq \sum_{k \in \mathbb{K}_j^-} \tilde{\beta}_k \left([\alpha(k)](U_j^n) - [\zeta(k)](U_j^n)\right)$$

for all n. Since neither the $\alpha(k)$ on the left-hand side nor the $\zeta(k)$ on the right-hand side are top contenders, their values at U_j^n are both zero. Thus, for any $n = 1, \ldots, |\mathscr{Z}_j|$,

$$\sum_{k \in \mathbb{K}_j^+} \tilde{\beta}_k [\zeta(k)](U_j^n) \geq \sum_{k \in \mathbb{K}_j^-} \tilde{\beta}_k [\alpha(k)](U_j^n). \tag{35}$$

Lemma 3 *For each $j \in J$ such that $\mathbb{K}_j \neq \varnothing$, there exist a finite set H, functions $\theta : H \to \mathbb{K}_j^-$ and $\phi : H \to \mathbb{K}_j^+$, a positive vector $(\beta_h^*)_{h \in H} \in \mathbb{R}_{++}^H$ and a $\mathbb{K}_j^* \subseteq \mathbb{K}_j^+$ such that $\zeta(\phi(h)) \succeq_j \alpha(\theta(h))$ for each $h \in H$ and*

$$\begin{aligned} \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k \left([\zeta(k)] - [\alpha(k)]\right) \quad = \quad & \sum_{h \in H} \beta_h^* \left([\zeta(\phi(h))] - [\alpha(\phi(h))] + [\zeta(\theta(h))] - [\alpha(\theta(h))]\right) \\ & + \sum_{k \in \mathbb{K}_j^*} \tilde{\beta}_k \left([\zeta(k)] - [\alpha(k)]\right). \end{aligned}$$

Proof Mimic the proof of Lemma 2. Let $v : \mathscr{Z} \to \{1, \ldots, |\mathscr{Z}_j|\}$ be a utility function representing the preference relation $\succeq_j$ on $\mathscr{Z}_j$. By the definition of $(U_j^n)_{n=1}^{|\mathscr{Z}_j|}$, for any $m \in \{1, \ldots, |\mathscr{Z}_j|\}$,

$$\{z \in \mathscr{Z}_j \mid v(z) \geq m\} = U_j^{|\mathscr{Z}_j| - m + 1}.$$

To construct the set H, start with any element of $\mathbb{K}_j^-$ that minimizes $v(\alpha(k))$ among all $k \in \mathbb{K}_j^-$ and denote it by $\theta(1)$. Initiate H by $H := \{1\}$. By (35), let $\phi(1)$ be any $k' \in \mathbb{K}_j^+$ that minimizes $v(\zeta(k'))$ among all $k' \in \mathbb{K}_j^+$ for which $\zeta(k') \succeq_j \alpha(\theta(1))$. Let

$$\beta_1^* := \min\left\{\tilde{\beta}_{\theta(1)}, \tilde{\beta}_{\phi(1)}\right\}.$$

Just as in (30) for Lemma 2, dissect the larger one between $\tilde{\beta}_{\theta(1)}$ and $\tilde{\beta}_{\phi(1)}$ into two portions, one equal to β_1^*, and the other equal to the remaining portion. Extract the β_1^* portions of $\theta(1)$ and $\phi(1)$ from $\mathbb{K}_j^-$ and $\mathbb{K}_j^+$ and put them into the set indexed by H. Then, analogous to (31),

$$\begin{aligned} \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k \left([\zeta(k)] - [\alpha(k)]\right) \quad = \quad & \sum_{h \in H} \beta_h^* \left([\zeta(\phi(h))] - [\alpha(\phi(h))] + [\zeta(\theta(h))] - [\alpha(\theta(h))]\right) \\ & + \sum_{k \in \widetilde{\mathbb{K}}_j^- \sqcup \widetilde{\mathbb{K}}_j^+} \tilde{\beta}'_k \left([\zeta(k)] - [\alpha(k)]\right), \end{aligned} \tag{36}$$

with $\widetilde{\mathbb{K}}_j^-$, $\widetilde{\mathbb{K}}_j^+$ and $\tilde{\beta}'_k$ analogously defined. Just like (32), for all $n \in \{1, \ldots, |\mathscr{Z}_j|\}$,

$$\sum_{k \in \widetilde{\mathbb{K}}_j^- \sqcup \widetilde{\mathbb{K}}_j^+} \tilde{\beta}'_k \left([\zeta(k)](U_j^n) - [\alpha(k)](U_j^n)\right) \geq 0$$

due to (35), $(U_j^n)_{n=1}^{|\mathscr{Z}_j|}$ being an increasing nested sequence, and $\phi(1)$ minimizes $v(\zeta(k))$ among the $k \in \mathbb{K}_j^+$ such that $\zeta(k) \succeq_j \alpha(\theta(1))$.

Now let $\widetilde{\mathbb{K}}_j^-$, $\widetilde{\mathbb{K}}_j^+$ and $\tilde{\beta}'_k$ take the roles of $\mathbb{K}_j^-$, $\mathbb{K}_j^+$ and $\tilde{\beta}_k$. This preserves (35) and reduces the cardinality of $\mathbb{K}_j^-$ or $\mathbb{K}_j^+$ by one. Repeat until at least one of the two sets becomes empty. By (35), $\mathbb{K}_j^- \neq \varnothing = \mathbb{K}_j^+$ is impossible. Thus, in the end, $\mathbb{K}_j^- = \varnothing$ and thus the last sum in (36) becomes

$$\sum_{k \in \widetilde{\mathbb{K}}_j^+} \tilde{\beta}'_k \left([\zeta(k)] - [\alpha(k)]\right)$$

with the updated $\widetilde{\mathbb{K}}_j^+$ in the end. Assign this $\widetilde{\mathbb{K}}_j^+$ to $\mathbb{K}_j^*$ and the desired equation obtains. ■

Nonpositivity at $-\boldsymbol{w}$ By Lemma 1, Theorem 1 follows from Theorem 2 if

$$\sum_{z \in Z} \beta_z [z](-\boldsymbol{w}) \leq 0.$$

Suppose, to the contrary, that the inequality does not hold. Then by (33),

$$\sum_{k \in \mathbb{K}} \tilde{\beta}_k \left(w(\zeta(k)) - w(\alpha(k))\right) < 0.$$

This inequality implies that $w(\zeta(k)) - w(\alpha(k) < 0$ for some $k \in \mathbb{K}$, because $\tilde{\beta}_k > 0$ for all k (Lemma 2). Recall that for every $k \in K$, both $\zeta(k)$ and $\alpha(k)$ belong to some $\{jit_i \mid j \in J\}$ and $\zeta(k) \succeq_{it_i} \alpha(k)$. Thus, if $w(\zeta(k)) - w(\alpha(k) < 0$ then $\alpha(k) \succeq_{it_i} \zeta(k)$ and so $\alpha(k) \sim_{it_i} \zeta(k)$, which coupled with $w(\zeta(k)) - w(\alpha(k) < 0$ implies that $\alpha(k)$ is the top contender in $\{jit_i \mid j \in J\}$. Consequently, $\alpha(k) \in \mathscr{Z}_j$ for some $j \in J$. In other words, $w(\zeta(k)) - w(\alpha(k) < 0 \Rightarrow k \in \mathbb{K}_j$ for some $j \in J$, and hence the strict inequality displayed above implies

$$\sum_{j \in J} \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k \left(w(\zeta(k)) - w(\alpha(k))\right) < 0.$$

Since $\tilde{\beta}_k > 0$ for all $k \in \mathbb{K}$, the above inequality in turn implies that for some $j \in J$,

$$\sum_{k \in \mathbb{K}_j} \tilde{\beta}_k \left(w(\zeta(k)) - w(\alpha(k))\right) < 0.$$

Meanwhile, apply the equation asserted by Lemma 3 to the entry for $-\boldsymbol{w}$ and multiply both sides of the equation by -1 to obtain

$$\begin{aligned}\sum_{k\in\mathbb{K}_j}\tilde{\beta}_k\left(w(\zeta(k))-w(\alpha(k))\right) &= \sum_{h\in H}\beta^*_h\left(w\left(\zeta(\phi(h))\right)-w\left(\alpha(\phi(h))\right)+w\left(\zeta(\theta(h))\right)-w\left(\alpha(\theta(h))\right)\right)\\ &\quad+\sum_{k\in\mathbb{K}^*_j}\tilde{\beta}_k\left(w\left(\zeta(k)\right)-w\left(\alpha(k)\right)\right).\end{aligned}$$

Thus, to derive a contradiction it suffices to prove that the right-hand side is nonnegative. First, $\sum_{k\in\mathbb{K}^*_j}\tilde{\beta}_k\left(w\left(\zeta(k)\right)-w\left(\alpha(k)\right)\right)\geq 0$ due to the definition of $\mathbb{K}^*_j$ in Lemma 3: For any $k\in\mathbb{K}^*_j$, with $\mathbb{K}^*_j\subseteq\mathbb{K}^+_j$, $\alpha(k)$ is not a top contender and hence, by the construction of ζ, $w(\zeta(k))\geq w(\alpha(k))$. Thus, the sum is nonnegative ($\tilde{\beta}_k\geq 0$ for all k). Second,

$$w\left(\zeta(\phi(h))\right)-w\left(\alpha(\phi(h))\right)+w\left(\zeta(\theta(h))\right)-w\left(\alpha(\theta(h))\right)\geq 0 \tag{37}$$

for all $h\in H$ due to the property of (θ,ϕ): By Lemma 3, $\zeta(\phi(h))\succeq_j\alpha(\theta(h))$. Thus, $\delta(\zeta(\phi(h)))\geq\delta(\alpha(\theta(h)))$ by the definition of $\succeq_j$. Consequently, by (23) the definition of δ,

$$w\left(\zeta(\phi(h))\right)-w\left(\alpha(\phi(h))\right)\geq\delta(\zeta(\phi(h)))\geq\delta(\alpha(\theta(h))),$$

with the first inequality due to the fact that $\zeta(\phi(h))$ is a top contender (the fact that $\zeta(\phi(h))\succeq_j\alpha(\theta(h))$ requires $\zeta(\phi(h))\in\mathscr{X}_j$). Furthermore, by Lemma 3, $\theta(h)\in\mathbb{K}^-_j$ and hence $\alpha(\theta(h))\in\mathscr{X}_j$, namely, $\alpha(\theta(h))$ is a top contender. This coupled with the fact $\zeta(\theta(h))\succeq_{it_i}\alpha(\theta(h))$ (Lemma 2) implies that $\zeta(\theta(h))$ is the second-highest contender in the $\{jit_i\mid j\in J\}$ where $\alpha(\theta(h))$ is the top contender. Thus, by the definition of δ,

$$\delta(\alpha(\theta(h)))=w\left(\alpha(\theta(h))\right)-w\left(\zeta(\theta(h))\right).$$

This combined with the inequality displayed above gives (37), as desired. ■

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