Reduced-Form Auctions of Multiple Objects*

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Abstract

I prove a condition characterizing the interim allocations generated by ex post allocations in a model of multiple heterogenous objects with multiple units, subject to ceiling and floor capacity constraints specific to various sets of bidder-object configurations, with asymmetric bidders whose types are drawn, independently or not, from general multidimensional spaces. The condition is Border-like (1991) in that it need only be checked for all measurable subsets of each bidder's type space. The characterization is applicable to settings with constraints across bidders and objects such as antitrust restrictions on the number of industries in which a group of firms can operate, minimum equity requirements on the quantity of a category of provisions for a minority group, budget constraints for bidders, and ex post budget balance for the society.

JEL Classification: C61, D44, D82

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1 Introduction

It is well-known in the auction design literature that reduced-form auctions as choice variables are tractable to analyze but difficult to identify.[1] Much work has been devoted to finding conditions for a choice variable to be identified as a reduced-form auction. The first necessary and sufficient condition for reduced-form auctions is obtained by Border [3], assuming symmetric single-unit auctions. Extensions to asymmetric single-unit auctions are made by Border [4], Goeree and Kushnir [16], Manelli and Vincent [23], and Mierendorff [24]. Further extensions to multiunit single-object auctions are made by Alaei, Fu, Haghpanah, Hartline and Malekian [1], who allow for upper-bound capacity constraints for sets of bidders, and by Che, Kim and Mierendorff [8] (henceforth CKM), who allow for not only upper- but also lower-bound capacity constraints thereof. Extension to multiple heterogenous objects is made by Cai, Daskalakis and Weinberg [7] (henceforth CDW), who allow for only upper-bound—though more general—capacity constraints. The conditions by [3], [4], [8], [23] and [24] are *Border-like* in the sense that one needs only to check them for all measurable subsets of each bidder's type space (which in the independent-type case reduce to the upper- and lower-contour sets with respect to the choice variable under consideration). Border-like conditions are computationally simpler than the others.[2]

All the Border-like conditions for reduced forms in the literature are based on the assumption that there is only one kind of objects to allocate, or that the auction is symmetric across bidders and types are unidimensional. Removing this assumption, I shall present a Border-like condition for reduced forms in a general model that allows for multiple heterogenous objects with multiple units, upper- and lower-bounds for capacity constraints, and asymmetric bidders whose types are drawn, independently or not, from general multidi-

[1] See Che, Kim and Mierendorff [8] for reference to the numerous works that treat reduced forms as choice variables, and Gershkov, Moldovanu, Strack and Zhang [14] for a more recent example.

[2] For example, the condition in CDW [7, Theorem 12] needs to be checked not only for every possible type of each bidder but also for every possible weight of every possible triple consisting of a bidder, a type and a good. Another condition that is not Border-like, nonetheless from a majorization perspective, is offered by Hart and Reny [17], assuming symmetric single-unit auctions, and generalized to multiple heterogenous objects by Kleiner, Moldovanu and Strack [21] assuming symmetric bidders, unidimensional types and monotone allocations. In the single-object multiunit case of [21], Gershkov, Moldovanu, Strack and Zhang [14] find a connection between the majorization and Border-like characterizations. Goeree and Kushnir [16] present a support-function perspective on single-unit auctions and some social choice problems.

mensional spaces. The result differs from CKM [8] in allowing for multiple (heterogeneous) objects, and from CDW [7] in offering a Border-like condition and allowing for not only upper- but also lower- bounds for capacity constraints.

Allowing for multiple objects, the result is potentially instrumental to the study of multiple-good mechanism design. The literature thereof, including the recent works by Daskalakis, Deckelbaum and Tzamos [10], and Kleiner and Manelli [20], mostly assumes that there is only one bidder thereby avoiding the issue of identifying reduced forms. Differently, Giannoakopoulos and Koutsoupias [15] consider multiple symmetric bidders, but they restrict attention to dominance- rather than Bayesian incentive compatible mechanisms, and hence again avoid the reduced form issue.[3]

Not only can the upper-bound capacity constraints in my model be across bidders like CKM [8], but they can also be across objects. Thus the characterization result is relevant to antitrust considerations where a firm may be restricted from exceeding a total market share in a set of industries (e.g., a "Glass-Steagall" legislation on internet conglomerates). Analogously, the cross-object, cross-bidder lower-bound capacity constraints capture equity quota policies such as a requirement that a minority group be given at least a certain number of admissions to the colleges in the ivy league category.

An object can be interpreted as money payments. My result therefore provides a condition for a reduced-form payment rule to be feasible with respect to ex post budget balance and some form of budget constraints. While the feasibility of reduced-form payment rules is a nonissue when utilities are assumed quasilinear, it is not so in non-quasilinear models such as when bidders face budget constraints. The conditions for budget-balanced payment rules in the literature (e.g., [5], [9] and [25]) rely on the quasilinearity assumption.

The main technical problem in obtaining a Border-like characterization is how to simplify the family of constraints that a reduced form needs to be checked against to only a family indexed by the measurable subsets of the type space. Through the traditional, separating-hyperplane approach, the family of constraints at the outset would have been as numerous as the set of simple functions on the type space. Border [3] solves this problem through a "hierarchical auction" but the solution relies on the single-unit assumption. Given

[3] This restriction has loss of generality because the equivalence between dominance- and Bayesian incentive compatibility, observed by Manelli and Vincent [23] and Gershkov, Goeree, Kushnir, Moldovanu and Shi [13], requires that each bidder's type be one-dimensional, which is restrictive in models with multiple goods.

the more general model, my solution stems from an intuition that views a reduced form as a matching assignment from suppliers to customers, with *suppliers* corresponding to the profiles of ex post types across bidders, and *customers* the bidder-object-type triples such as (i, j, t_i), whose demand takes the form of "how many of good j should go to bidder i when i's type is t_i." This demand needs to be met by the sum of the quantities of good j allocated to bidder i across all the suppliers (ex post type profiles) that have t_i as the type for i. Each supplier needs to satisfy all capacity constraints in meeting the demands from customers. For example, given an ex post type profile such as (t_i, t_{-i}), if a quantity $q_{i,j}(t_i, t_{-i})$ is to be supplied to (i, j, t_i), and a quantity $q_{-i,j'}(t_i, t_{-i})$ to another customer $(-i, j', t_{-i})$, then $q_{i,j}(t_i, t_{-i}) + q_{-i,j'}(t_i, t_{-i})$ needs to be within the upper and lower bounds of the joint capacity of $\{(i, j), (-i, j')\}$.

This matching viewpoint is similar to the network-flow perspective in CKM [8] except that CKM formulate the problem as a circuit and appeal to Theorem 1 in Hassin [18], while my solution exploits the one-way matching structure and derives the result relatively directly from the primitive assumptions. Hassin's proof of the said Theorem 1 relies on a lengthy computation of the circuit problem and does not offer an intuitive insight to how the result comes from the primitives, namely, the paramodular structure of the constraints. CKM [8] offer a partial explanation for such in their Supplemental Appendix E that does not extend to both upper and lower-bound constraints. My proof is relatively self-contained and clarifies the intuitive linkage from the paramodular constraints to the result (cf. Remark 1).

From the matching perspective described above, validity of a reduced form becomes the condition that, given any weight that varies across the customers, the total supply is greater than or equal to the aggregate of the weighted demands from the customers, when the total supply may be as large as the said capacity constraints allow. That means we can max out the possibility for a reduced form through deriving the total supply from aggregating individual supplies that are each maximized according to the weights across customers and subject to the upper and lower bounds of the joint capacities of various sets of customers. Although such constrained maximization for each individual supplier is a polymatroid intersection problem that has been considered in Theorem 5 of Hassin [18], the condition that Hassin claims to validate the solution thereof is incorrect (Remark 2). I find that the compliance condition in CKM [8], extended to my multiple-object model, is equivalent to a marginal-capacity condition about the upper- and lower-bound capacities (Assumption 2), and that the marginal-capacity condition ensures that the polymatroid intersection problem can be solved by a greedy-generous algorithm. Based on that algorithm, I prove that the arbitrary weights can be factored out of the aforementioned "supply meets demand" condition. That is the key to obtain the Border-like characterization of reduced forms.

The contemporary independent work by Lang and Yang [22] also provides a Border-like characterization of the reduced-form auctions of multiple objects. Their work and mine differ in the assumptions about the primitives that deliver the Border-like characterization. My paper restricts the capacity constraints by the conventional paramodular assumption and allows for arbitrary sets of bidder-object pairs on which such constraints are defined. Lang and Yang restrict how the sets of bidder-object pairs are related to the bidders' types by a total unimodularity condition (TUM) and allow for arbitrary—though required to be integer-valued—capacity constraints. That is, they assume for each bidder a finite number of possible types, and their TUM is a requirement for all nonsingular square submatrices of a matrix that registers how each ex post type profile across bidders is related to each set of bidder-object pairs on which the capacity constraint is defined.[4] Thus, when bidders' type spaces expand in cardinality, my assumption about the primitives remains unchanged, while theirs becomes more and more complicated. Lang and Yang provide a hierarchy condition that implies TUM and is independent of the type space; however, the hierarchy condition is a special case of the paramodular assumption of my model (cf. Lemma 1 of CKM [8]). Although Lang and Yang also provide other sufficient conditions for TUM that need not be special cases of the paramodular assumption, those conditions are each restricted by the assumption that the type spaces are either binary or singleton.

The model and the main theorem are stated in Section 2. Section 3 presents the proof, focusing on the polymatroid intersection problem mentioned above. Section 4 describes several applications. All omitted details are in the Appendix.

2 Notation and Statement of the Theorem

Let there be a finite set I_1 of bidders, and a finite set I_2 of the names of objects to be allocated. Denote $I := I_1 \times I_2$ for the set of all possible bidder-object pairs. For any $i_1 \in I_1$,

[4] Consequently, while a reduced form is meant to free us from considerations of ex post type profiles across bidders, verifying the TUM condition in [22] requires considerations thereof.

let T_{i_1} be the set of possible types of bidder i_1. Let $T := \prod_{i_1 \in I_1} T_{i_1}$, and $(T, \mathscr{T}, \mu)$ the measure space such that the profile $t := (t_{i_1})_{i_1 \in I_1}$ of types across bidders is drawn from T according to probability measure μ. For each $i_1 \in I_1$, let μ_{i_1} be the marginal measure of μ onto T_{i_1}, and $\mathscr{T}_{i_1}$ the corresponding set of measurable subsets of T_{i_1}; let $T_{-i_1} := \prod_{j \in I_1 \setminus \{i_1\}} T_j$, and denote $\mu_{-i_1}(\cdot|t_{i_1})$ for the conditional product measure on T_{-i_1} according to μ conditional on t_{i_1}; let

$$t := (t_{i_1})_{i_1 \in I_1}$$

denote a generic element of T, and $t_{-i_1} := (t_k)_{k \in I_1 \setminus \{i_1\}}$ that of T_{-i_1}.

Let $f, g : 2^I \to \mathbb{R}_+$ be functions such that $f \geqq g$ and $f(\varnothing) = g(\varnothing) = 0$. Let

$$X := \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \;\middle|\; \forall E \subseteq I \left[g(E) \leq \sum_{i \in E} x_i \leq f(E) \right] \right\}. \tag{1}$$

A generic element of X is $x := (x_i)_{i \in I}$ such that $x_i = x_{i_1,i_2} \in \mathbb{R}$ stands for the quantity of object i_2 that bidder i_1 gets $(\forall (i_1, i_2) \in I_1 \times I_2)$. Thus, X represents the set of feasible allocation outcomes, and feasibility is defined by (f, g), with f capturing the ceilings of the total quantity across bidders and across objects, and g the floors thereof.

An *ex post allocation* is a profile $(q_i)_{i \in I}$ of measurable functions $q_i : T \to \mathbb{R}$ $(\forall i \in I)$ such that $(q_i(t))_{i \in I} \in X$ for μ-a.e. $t \in T$.[5] For any $i = (i_1, i_2)$, $q_i(t) := q_{i_1,i_2}(t)$ stands for the expected quantity of object i_2 allocated to bidder i_1 when t is the profile of realized types across bidders.

An *interim allocation* is a profile $(Q_i)_{i \in I}$ of measurable functions $Q_i : T_i \to \mathbb{R}$. We often write Q instead of $(Q_i)_{i \in I}$, with $Q_i(t_{i_1}) := Q_{i_1,i_2}(t_{i_1})$ interpreted as the interim expected value of the quantity of object i_2 to be allocated to bidder i_1 conditional on his realized type t_{i_1}.

An interim allocation $(Q_i)_{i \in I}$ is said to be the *reduced form* of some ex post allocation $(q_i)_{i \in I}$ iff

$$Q_i(t_{i_1}) = \int_{T_{-i_1}} q_i(t_{i_1}, t_{-i_1}) d\mu_{-i_1}(t_{-i_1}|t_{i_1}) \tag{2}$$

for all $i \in I$ and all $t_{i_1} \in T_{i_1}$. Denote $\mathscr{Q}$ for the set of interim allocations that are reduced forms of some ex post allocations.

[5]It is immaterial to strengthen the condition $(q_i(t))_{i \in I} \in X$ to all $t \in T$. In that case, our theorem is extended trivially through modifying q on a set of μ-measure zero as in the proof of Proposition 3.1 in Border [3].

Assumption 1 *f and $-g$ are each submodular*[6] *on 2^I.*

Assumption 2 *If $e \in E \subseteq I$ then*[7]

$$g(E) - g(E \setminus \{e\}) \leq f\left((I \setminus E) \cup \{e\}\right) - f(I \setminus E).$$

Assumption 1 is conventional in the generalized polymatroid literature.[8] Assumption 2 captures the intuition that the marginal effect of any element e on the floor capacity is no larger than its marginal effect on the ceiling capacity. One can prove (Appendix A) that the assumption is equivalent to the other conventional assumption in the said literature, the compliance condition, though the two appear different at first glance. Specialized to the multiunit single-object case (namely, when I_2 is singleton), Assumptions 1 and 2 together are equivalent to the paramodularity assumption in CKM [8].

A special case of this model is the multiple-object setting subject to only object-wise aggregate feasibility constraints: For each $i_2 \in I_2$ there is a constant $m_{i_2} \in \mathbb{R}_{++}$ that represents the total quantity of object i_2 available for allocation. That is, $f(I_1 \times \{i_2\}) = m_{i_2}$ and $g(I_1 \times \{i_2\}) = 0$ for all $i_2 \in I_2$. One can extend (f, g) to all subsets of $I_1 \times I_2$—without adding any restriction—so that both Assumptions 1 and 2 are satisfied (Appendix E)

For any subset S of set Z, denote $\chi_S : Z \to \{0, 1\}$ for the characteristic function of S.

Theorem 1 *If Assumptions 1 and 2 hold, then $(Q_i)_{i \in I} \in \mathscr{Q}$ if and only if, for any $(S_i)_{i \in I}$ such that $S_i \in \mathscr{T}_{i_1}$ for all $i \in I$,*

$$\int_T g\left(\{i \in I \mid t_{i_1} \in S_i\}\right) d\mu(t) \leq \sum_{i \in I} \int_T Q_i(t_{i_1}) \chi_{S_i}(t_{i_1}) d\mu(t) \leq \int_T f\left(\{i \in I \mid t_{i_1} \in S_i\}\right) d\mu(t). \tag{3}$$

When types are independent across bidders, the sets S_i required for the first inequality of (3) can be simplified to the lower contour sets of Q_i, and the sets S_i required for the second inequality of (3), the upper contour sets of Q_i. The condition thereby obtained takes a format similar to those in much of the literature such as [3], [4] and [23].[9]

[6] For any set S, a function $\phi : 2^S \to \mathbb{R}$ is *submodular* on 2^S iff $\phi(E) + \phi(E') \geq \phi(E \cup E') + \phi(E \cap E')$ for all $E, E' \subseteq S$. (If $-\phi$ is submodular, ϕ is said to be supermodular.)

[7] For the rest of the paper, denote $\neg E := I \setminus E$ whenever I is the whole set.

[8] Frank, Király, Pap and Pritchard [11], and Schrijver [27].

[9] CKM [8] has an example showing that the independence assumption is needed for this simplification.

3 Proof of Theorem 1

Along the lines of the proof for Theorem 5 in CKM [8], one can prove (Appendix B) that there is no loss of generality to assume that there are only finitely many possible types:

Lemma 1 *Theorem 1 is true if its statement is true when the cardinality of T is finite.*

Thus, for the rest of the proof, assume that the cardinality of the type space T is finite. Then an interim allocation is a vector in the Euclidean space $\prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$, which is its own dual space via the inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle Q,\alpha\rangle := \sum_{(i_1,i_2)\in I}\sum_{t_{i_1}\in T_{i_1}} Q_{(i_1,i_2)}(t_{i_1})\alpha(i_1,i_2,t_{i_1})\mu_{i_1}(\{t_{i_1}\})$$

for any interim allocation $Q := (Q_i)_{i\in I} \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$ and any $\alpha \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$ denoted by

$$\alpha := ((\alpha(i,t_{i_1}))_{t_{i_1}\in T_{i_1}})_{i\in I} := ((\alpha(i_1,i_2,t_{i_1}))_{t_{i_1}\in T_{i_1}})_{(i_1,i_2)\in I_1\times I_2}.$$

Since $\mathscr{Q}$, by definition and (1), is a convex and compact nonempty subset of $\prod_{i\in I}\left(\mathbb{R}^{T_{i_1}}\right)$, the separating hyperplane theorem[10] implies

$$\mathscr{Q} = \left\{ Q \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right) \;\middle|\; \forall\alpha\in\prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right) : \langle Q,\alpha\rangle \le \max_{Q'\in\mathscr{Q}}\langle Q',\alpha\rangle \right\}. \tag{4}$$

By the definitions of $\mathscr{Q}$ and $\langle\cdot,\cdot\rangle$,

$$\max_{Q'\in\mathscr{Q}}\langle Q',\alpha\rangle = \sum_{t\in T}\mu(\{t\}) \max_{(x_i)_{i\in I}\in X} \sum_{(i_1,i_2)\in I} x_{i_1,i_2}\alpha(i_1,i_2,t_{i_1}) \tag{5}$$

Thus, characterization of $\mathscr{Q}$ boils down to characterization of the maximand

$$M(t) := \max_{x\in X}\sum_{i\in I} x_i\alpha(i,t_{i_1}) \tag{6}$$

for each $t := (t_{i_1})_{i_1\in I} \in T$ and each $\alpha \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$.

To characterize (6), we construct a solution for x, first according to the greedy algorithm in descending order of $\alpha(i,t_{i_1})$ down to zero, maxing out the joint capacity for such i's up to the ceiling allowed by f, and then according to the generous algorithm in ascending order of $\alpha(i,t_{i_1})$ from the most negative up to zero, reducing the joint capacity for such i's down

[10] A finite-dimensional special case of Theorem 7.51 of Aliprantis and Border [2, p288].

to the floor allowed by g. Due to the submodularity of f and $-g$, the solution thereby constructed is an optimal solution to (6) provided that it is feasible, respecting both the ceiling and floor constraints from f and g. Thanks to Assumption 2, the solution is feasible: The greedy algorithm raises the quantity assigned to $i \in I$ up to its marginal contribution to the ceiling capacity with respect to those elements that weigh more than i does in terms of α, and the generous algorithm reduces the quantity assigned to i down to its marginal contribution to the floor capacity relative to those elements that weigh less than i does in terms of α. Ensuring that the former marginal contribution is no less than the latter marginal contribution is exactly what Assumption 2 does. The total quantity assigned to any subset of I can therefore be shown within its upper and lower bounds defined by f and g.

To implement the above plan, we start by labeling the components of $\alpha \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$ in descending order: Denote N for the cardinality of $\bigcup_{(i_1,i_2)\in I}\{(i_1,i_2)\}\times T_{i_1}$, and

$$\mathcal{N} := \{1, 2, \ldots, N\};$$

let $r : \mathcal{N} \to \bigcup_{(i_1,i_2)\in I}\{(i_1,i_2)\}\times T_{i_1}$ be a bijection such that $m < n \Rightarrow \alpha\left(r(m)\right) \geq \alpha\left(r(n)\right)$. For any $n \in \mathcal{N}$, denote

$$\alpha^n := \alpha\left(r(n)\right).$$

Denote $n_* := \max\{n \in \mathcal{N} \mid \alpha^n \geq 0\}$ so that

$$\alpha^1 \geq \alpha^2 \geq \cdots \geq \alpha^{n_*} \geq 0 > \alpha^{n_*+1} \geq \cdots \geq \alpha^N. \tag{7}$$

For any $t := (t_{i_1})_{i_1\in I_1} \in T$, with r^{-1} the inverse of r, let

$$\mathcal{N}(t) := \left\{r^{-1}\left(i_1, i_2, t_{i_1}\right) \mid (i_1,i_2) \in I\right\}$$

and, for every $n \in \mathcal{N}$,

$$\begin{aligned}
U_n^t &:= \{1, \ldots, n-1, n\} \cap \mathcal{N}(t), &(8)\\
V_n^t &:= \{n, n+1, \ldots, N\} \cap \mathcal{N}(t), &(9)\\
U_0^t &:= V_{N+1}^t := \varnothing. &(10)
\end{aligned}$$

Denote $\pi : \bigcup_{(i_1,i_2)\in I}\{(i_1,i_2)\}\times T_{i_1} \to I$ for the projection $\pi(i_1,i_2,t_{i_1}) := (i_1,i_2)$ for all (i_1,i_2,t_{i_1}). Then $\pi\left(r(E)\right) := \{\pi(r(n)) \mid n \in E\}$ is a subset of I for any $E \subseteq \mathcal{N}$.

The next lemma formalizes the said greedy-generous algorithm. It differs from those known in the generalized polymatroid literature by proceeding through a ranking (7) that is determined independently of the realized type profile t. The algorithm in the said literature (e.g., Frank and Tardos [12] and Hassin [18]), if applied to Problem (6), would proceed through a ranking dependent on the t and then the solution would appear to vary intractably with the variable t. Put differently, Eq. (11) provides an important property that the effect of the weights $(\alpha^n)_{n\in\mathcal{N}}$ is separable from the effect of the type profile t. That allows us to factor out the arbitrary α thereby obtaining a Border-like characterization of reduced forms.

Lemma 2 *For any $\alpha \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$, arrange its components as in (7) via bijection r; for any $t \in T$, define $\mathcal{N}(t)$ and (U_n^t, V_n^t) as above. Then*

$$\begin{aligned} M(t) \;=\; & \sum_{n=1}^{n_*-1} f\left(\pi\left(r(U_n^t)\right)\right)\left(\alpha^n - \alpha^{n+1}\right) + f\left(\pi\left(r(U_{n_*}^t)\right)\right)\alpha^{n_*} \\ & + \sum_{n=n_*+2}^{N} g\left(\pi\left(r(V_n^t)\right)\right)\left(\alpha^n - \alpha^{n-1}\right) + g\left(\pi\left(r(V_{n_*+1}^t)\right)\right)\alpha^{n_*+1}. \end{aligned} \tag{11}$$

Proof Let $\alpha \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$ and $t \in T$. The composition $\pi \circ r$ of the projection π and the bijection r induces a bijection $\mathcal{N}(t) \to I$ and hence a one-to-one correspondence

$$(x_i)_{i\in I} \in \mathbb{R}^I \longmapsto (x^n)_{n\in\mathcal{N}(t)} \in \mathbb{R}^{\mathcal{N}(t)} \quad \text{by} \quad x_{\pi(r(n))} \longmapsto x^n.$$

Thus, (6) is equivalent to

$$\begin{aligned} M(t) \;=\; \max_{x\in\mathbb{R}^{\mathcal{N}(t)}} \quad & \sum_{n\in\mathcal{N}(t)} x^n\alpha^n \\ \text{s.t.} \quad & g\left(\pi(r(E))\right) \le \sum_{n\in E} x^n \le f\left(\pi(r(E))\right) \quad (\forall E \subseteq \mathcal{N}(t)). \end{aligned} \tag{12}$$

For any $n \in \mathcal{N}$, define

$$\bar{x}^n := \begin{cases} f\left(\pi\left(r(U_n^t)\right)\right) - f\left(\pi\left(r(U_{n-1}^t)\right)\right) & \text{if } n \le n_* \\ g\left(\pi\left(r(V_n^t)\right)\right) - g\left(\pi\left(r(V_{n+1}^t)\right)\right) & \text{if } n > n_*. \end{cases} \tag{13}$$

By (8)–(10), $(\bar{x}^n)_{n\in\mathcal{N}(t)}$ is the coupling of the greedy algorithm in descending order of α^n—among $n \in \mathcal{N}(t)$—down to zero, and the generous algorithm in ascending order of α^n—among $n \in \mathcal{N}(t)$—up to zero. Thus, as in the proof of Theorem 5 in Hassin [18], $(\bar{x}^n)_{n\in\mathcal{N}(t)}$

solves (12) if it is feasible to (12).[11] To prove the feasibility of $(\bar{x}^n)_{n\in\mathcal{N}(t)}$, pick any nonempty $S \subseteq \mathcal{N}(t)$. By the proof of Theorem 3 in Hassin [18] and the submodularity of $-g$,

$$g\left(\pi(r(S))\right) \leq \sum_{n\in S}\left(g\left(\pi\left(r(V_n^t)\right)\right) - g\left(\pi\left(r(V_{n+1}^t)\right)\right)\right).$$

Analogously, submodularity of f implies

$$\sum_{n\in S}\left(f\left(\pi\left(r(U_n^t)\right)\right) - f\left(\pi\left(r(U_{n-1}^t)\right)\right)\right) \leq f\left(\pi(r(S))\right).$$

Consequently, by the definition of $\bar{x}^n$, it suffices to prove

$$g\left(\pi\left(r(V_n^t)\right)\right) - g\left(\pi\left(r(V_{n+1}^t)\right)\right) \leq f\left(\pi\left(r(U_n^t)\right)\right) - f\left(\pi\left(r(U_{n-1}^t)\right)\right) \tag{14}$$

for every $n \in \mathcal{N}(t)$. To prove (14), first consider the case where the right-hand side is zero. That means, with $\pi \circ r : \mathcal{N}(t) \to I$ bijective, $U_n^t = U_{n-1}^t$, which by (8) implies $n \notin \mathcal{N}(t)$ and hence by (9) $V_n^t = V_{n+1}^t$. Thus the left-hand side is zero as well, so (14) holds. Second, consider the other case, where the right-hand side is nonzero and hence $U_n^t \setminus U_{n-1}^t = \{n\}$. By (8) and (9) and $\pi \circ r : \mathcal{N}(t) \to I$ being bijective, (14) is equivalent to

$$g\left(\pi\left(r(V_n^t)\right)\right) - g\left(\pi\left(r(V_n^t)\right) \setminus \{\pi(r(n))\}\right) \leq f\left(\left(\neg\pi\left(r(V_n^t)\right)\right) \cup \{\pi(r(n))\}\right) - f\left(\neg\pi\left(r(V_n^t)\right)\right),$$

which is true by Assumption 2. Thus, $(\bar{x}^n)_{n\in\mathcal{N}(t)}$ is feasible to (12), and hence solves (12).

Consequently, $M(t) = \sum_{n\in\mathcal{N}(t)} \bar{x}^n\alpha^n$. By (8), (9) and (13), $\bar{x}^n = 0$ if $n \notin \mathcal{N}(t)$. Thus,

$$\begin{aligned}
M(t) &= \sum_{n\in\mathcal{N}(t)} \bar{x}^n\alpha^n + \sum_{n\in\mathcal{N}\setminus\mathcal{N}(t)} \bar{x}^n\alpha^n = \sum_{n\in\mathcal{N}} \bar{x}^n\alpha^n \\
&\overset{(7)}{=} \sum_{n=1}^{n_*} \bar{x}^n\alpha^n + \sum_{n=n_*+1}^{N} \bar{x}^n\alpha^n \\
&\overset{(13)}{=} \sum_{n=1}^{n_*}\left(f\left(\pi(r(U_n^t))\right) - f\left(\pi(r(U_{n-1}^t))\right)\right)\alpha^n + \sum_{n=n_*+1}^{N}\left(g\left(\pi(r(V_n^t))\right) - g\left(\pi(r(V_{n+1}^t))\right)\right)\alpha^n.
\end{aligned}$$

Reordering the terms on the last line of the above-displayed and using $f(\varnothing) = g(\varnothing) = 0$ and (10), we see that the last line thereof is equal to the right-hand side of (11). ■

We are ready to prove that (3) is necessary and sufficient for $Q \in \mathcal{Q}$.

[11] This observation uses the submodularity of f and $-g$ (Assumption 1) and $\pi\circ r$ being a bijection $\mathcal{N}(t) \to I$. See Remark 1 for more explanation.

Necessity Apply (5) to the case where $\alpha(i_1,i_2,t_{i_1}) = \chi_{S_{i_1,i_2}}(t_{i_1})$ for all $(i_1,i_2,t_{i_1}) \in \bigcup_{i\in I}(\{i\}\times T_{i_1})$. Then use (1) to obtain the second inequality of (3). Analogously, apply (5) to $\alpha(i_1,i_2,t_{i_1}) = -\chi_{S_{i_1,i_2}}(t_{i_1})$ for all (i_1,i_2,t_{i_1}) to obtain the first inequality thereof.

Sufficiency Let $Q \notin \mathscr{Q}$. We shall show that (3) is violated. With $Q \notin \mathscr{Q}$, (4) implies $\langle Q,\alpha\rangle > \sum_{t\in T} M(t)\mu(\{t\})$ for some $\alpha \in \prod_{(i_1,i_2)\in I}\left(\mathbb{R}^{T_{i_1}}\right)$. Arrange the components of α as in (7) to rewrite $\langle Q,\alpha\rangle$ as

$$\langle Q,\alpha\rangle = \sum_{n=1}^{n_*}\alpha^n \sum_{t\in T} Q_{\pi(r(n))}\left(t_{\pi_1(r(n))}\right)\mu(\{t\}) + \sum_{n=n_*+1}^{N}\alpha^n \sum_{t\in T} Q_{\pi(r(n))}\left(t_{\pi_1(r(n))}\right)\mu(\{t\}),$$

where $\pi_1(r(n)) := i_1$ whenever $\pi(r(n)) = (i_1,i_2)$ for some $(i_1,i_2)\in I$. Apply the identity

$$\begin{aligned}\sum_{n=1}^{m} a_n b_n &= (a_1-a_2)b_1 + (a_2-a_3)(b_1+b_2) + (a_3-a_4)(b_1+b_2+b_3)\\ &\quad + \cdots + (a_{m-1}-a_m)(b_1+b_2+\cdots+b_{m-1}) + a_m(b_1+b_2+\cdots+b_{m-1}+b_m)\end{aligned}$$

(for any positive integer m and any $a_n, b_n \in \mathbb{R}$ for all $n = 1,\ldots,m$) to the two sums on the right-hand side of the equation displayed previously to obtain

$$\begin{aligned}\langle Q,\alpha\rangle &= \sum_{n=1}^{n_*-1}\left(\alpha^n-\alpha^{n+1}\right)\sum_{s=1}^{n}\sum_{t\in T} Q_{\pi(r(s))}\left(t_{\pi_1(r(s))}\right)\mu(\{t\}) + \alpha^{n_*}\sum_{s=1}^{n_*}\sum_{t\in T} Q_{\pi(r(s))}\left(t_{\pi_1(r(s))}\right)\mu(\{t\})\\ &\quad + \sum_{n_*+2}^{N}\left(\alpha^n-\alpha^{n-1}\right)\sum_{s=n}^{N}\sum_{t\in T} Q_{\pi(r(s))}\left(t_{\pi_1(r(s))}\right)\mu(\{t\}) + \alpha^{n_*+1}\sum_{n_*+1}^{N}\sum_{t\in T} Q_{\pi(r(s))}\left(t_{\pi_1(r(s))}\right)\mu(\{t\}).\end{aligned}$$

Meanwhile, Lemma 2 implies

$$\begin{aligned}\sum_{t\in T} M(t)\mu(\{t\}) &= \sum_{n=1}^{n_*-1}\left(\alpha^n-\alpha^{n+1}\right)\sum_{t\in T} f\left(\pi(r(U_n^t))\right)\mu(\{t\}) + \alpha^{n_*}\sum_{t\in T} f\left(\pi(r(U_{n_*}^t))\right)\mu(\{t\})\\ &\quad + \sum_{n_*+2}^{N}\left(\alpha^n-\alpha^{n-1}\right)\sum_{t\in T} g\left(\pi(r(V_n^t))\right)\mu(\{t\}) + \alpha^{n_*+1}\sum_{t\in T} g\left(\pi(r(V_{n_*+1}^t))\right)\mu(\{t\}).\end{aligned}$$

Thus, $\langle Q,\alpha\rangle - \sum_{t\in T} M(t)\mu(\{t\})$ is equal to

$$\begin{aligned}&\sum_{n=1}^{n_*}\beta^n\left(\sum_{s=1}^{n}\sum_{t\in T} Q_{\pi(r(s))}\left(t_{\pi_1(r(s))}\right)\mu(\{t\}) - \sum_{t\in T} f\left(\pi(r(U_n^t))\right)\mu(\{t\})\right)\\ &+ \sum_{n_*+1}^{N}\beta^n\left(\sum_{s=1}^{n}\sum_{t\in T} Q_{\pi(r(s))}\left(t_{\pi_1(r(s))}\right)\mu(\{t\}) - \sum_{t\in T} g\left(\pi(r(V_n^t))\right)\mu(\{t\})\right),\end{aligned}$$

where $\beta^n := \alpha^n - \alpha^{n+1}$ for all $n \leq n_* - 1$, $\beta^{n_*} := \alpha^{n_*}$, $\beta^n := \alpha^n - \alpha^{n-1}$ for all $n \geq n_* + 2$, and $\beta^{n_*+1} := \alpha^{n_*+1}$. By (7), $\beta^n \geq 0$ for all $n \leq n_*$, and $\beta^n \leq 0$ for all $n > n_*$. Thus, $\langle Q, \alpha\rangle > \sum_{t\in T} M(t)\mu(\{t\})$ implies that

i. either there exists an $n \leq n_*$ for which

$$\sum_{s=1}^{n}\sum_{t\in T} Q_{\pi(r(s))}(t_{\pi_1(r(s))})\mu(\{t\}) > \sum_{t\in T} f\left(\pi(r(U_n^t))\right)\mu(\{t\}), \tag{15}$$

ii. or there exists an $n > n_*$ for which

$$\sum_{s=n}^{N}\sum_{t\in T} Q_{\pi(r(s))}(t_{\pi_1(r(s))})\mu(\{t\}) < \sum_{t\in T} g\left(\pi(r(V_n^t))\right)\mu(\{t\}).$$

In case (i), for each $i := (i_1, i_2) \in I$, let

$$S_i := S_{i_1,i_2} := \left\{t_{i_1} \in T_{i_1} \mid 1 \leq r^{-1}(i_1, i_2, t_{i_1}) \leq n\right\}.$$

Then the left-hand side of (15) is equal to $\sum_{i\in I}\sum_{t\in T} Q_i(t_{i_1})\chi_{S_i}(t_{i_1})\mu(\{t\})$, and the right-hand side equal to $\sum_{t\in T} f\left(\{i \in I \mid t_{i_1} \in S_i\}\right)\mu(\{t\})$. Thus (15) contradicts the second inequality of (3). Case (ii) can be similarly shown to contradict (3). That proves the sufficiency of (3) for $Q \in \mathscr{Q}$, and hence proves Theorem 1. $\square$

Remark 1 The role of Assumption 2 (compliance) is explicit in my proof (also in the paragraph below (6)). The reasoning where Assumption 1 (submodularity) plays a role is in the literature and so my proof is brief about it. To illustrate the role of Assumption 1, consider a $t \in T$ for which $\mathcal{N}(t) = \{n_1, n_2, \ldots, n_{|I|}\}$ and $\alpha^{n_1} \geq \alpha^{n_2} \geq \cdots \geq \alpha^{n_{|I|}} \geq 0$. Thus $U_{n_k}^t = \{n_1, \ldots, n_k\}$ for any $k = 1, \ldots, |I|$. To solve (12), with x^{n_1} weighted the most, it is intuitive to max out x^{n_1} by assigning to it the ceiling capacity $f(\pi(r(n_1)))$. Then the assignment to x^{n_2}, which ranks second in weights, is subject to two constraints, $x^{n_2} \leq f(\pi(r(n_2)))$ individually, and $x^{n_1} + x^{n_2} \leq f(\pi(r(\{n_1, n_2\})))$ jointly. The latter is equivalent to

$$x^{n_2} \leq f(\pi(r(\{n_1, n_2\}))) - f(\pi(r(n_1)))$$

given the assignment of x^{n_1}. Between the two constraints, the latter one is tighter because

$$f(\pi(r(\{n_1, n_2\}))) - f(\pi(r(n_1))) \leq f(\pi(r(n_2)))$$

by the submodularity of f (and hence of $f(\pi(r(\cdot)))$. Thus, it is intuitive to max out x^{n_2} by assigning to it the marginal ceiling capacity $f(\pi(r(\{n_1, n_2\}))) - f(\pi(r(n_1)))$. Continuing in this manner in descending order of α^{n_k}, we heuristically get a solution of (12), which is exactly the greedy algorithm. The general case where

$$\alpha^{n_1} \geq \cdots \geq \alpha^{n_{k^*}} \geq 0 > \alpha^{n_{k^*+1}} \geq \cdots \geq \alpha^{n_{|I|}}$$

is similar. The assignment for $x^{n_{k^*+1}}, \ldots, x^{n_{|I|}}$, whose weights are negative, is the mirror image of the previous case. We just "max out" the negative of their quantities by assigning to them the marginal capacities according to the negative $-g$ of the floor function g, using the submodularity of $-g$. That is exactly the generous algorithm. The greedy and generous algorithms combined, we obtain a solution to (12) except that the floor constraints are relaxed for $x^{n_1}, \ldots, x^{n_{k^*}}$, and the ceiling constraints relaxed for $x^{n_{k^*+1}}, \ldots, x^{n_{|I|}}$. However, as shown in my proof thanks to Assumption 2, the solution also satisfies the relaxed constraints.

Remark 2 Hassin's [18] argument that the greedy-generous algorithm also satisfies the relaxed constraints referred to in Remark 1—namely, the argument for the counterpart of (14)—is incorrect. The proof of Theorem 4 in Hassin [18] claims that the counterpart of (14) follows from a condition

$$[E \subseteq E' \subseteq I \text{ or } E' \subseteq E \subseteq I] \Longrightarrow f(E') - f(E' \setminus E) \geq g(E) - g(E \setminus E'). \tag{16}$$

Whereas, $\pi(r(V_n^t))$ and $\pi(r(U_n^t))$ are not related by $\subseteq$. For example, in the case where $S = \{s\}$ for some element s, Hassin's counterpart of (14) requires that for any other element n with $s < n$,

$$g(\{s, \ldots, n\}) - g(\{s+1, \ldots, n\}) \leq f(\{1, \ldots, s\}) - f(\{1, \ldots, s-1\}). \tag{17}$$

Suppose that this inequality follows from (16). Then $E = \{s, \ldots, n\}$ and $E \setminus E' = \{s + 1, \ldots, n\}$ for some E' such that $E \subsetneq E'$ or $E' \subsetneq E$. Since $E \setminus E' \neq \varnothing$, $E' \subsetneq E$ and hence $E' = \{s\}$. Then (16) implies only that $g(\{s, \ldots, n\}) - g(\{s+1, \ldots, n\}) \leq f(\{s\}$. In order for this inequality to imply (17), one would need $f(\{s\} \leq f(\{1, \ldots, s\}) - f(\{1, \ldots, s-1\})$, which is opposite to the submodularity of f.

4 Applications

4.1 General Sets of Bidder-Object Combinations

Theorem 1 assumes that the set of bidder-object combinations is $I := I_1 \times I_2$. In applications we may be interested in only a subset of I such as one that is not a lattice. For instance, there are two bidders and two goods; bidder 1 (the local bidder) wants only good 1, and bidder 2 (the global bidder) wants both goods 1 and 2. While I includes all four bidder-good pairs, the designer may consider only the three-element set $\{(1,1),(2,1),(2,2)\}$, a proper subset of I.

Let us generalize Theorem 1 to such cases. For any $S \subseteq I$ and functions f and g, say that (S, f, g) is *paramodular* iff $f, g : 2^S \to \mathbb{R}_+$, $f(\varnothing) = g(\varnothing) = 0$, and (f, g) satisfies the counterparts of Assumptions 1 and 2 where I is replaced by S. The next lemma, proved in Appendix C, allows us to extend a paramodular structure from a subset of I to the entire I.

Lemma 3 *For any $S \subseteq I$ and any functions f and g, if (S, f, g) is paramodular and, for any $i \in I \setminus S$, there exist $a_i \geq b_i \geq 0$ such that, for any $E \subseteq I$,*

$$f(E) = f(E \cap S) + \sum_{i \in E \setminus S} a_i, \tag{18}$$

$$g(E) = g(E \cap S) + \sum_{i \in E \setminus S} b_i, \tag{19}$$

then (I, f, g) is paramodular.

If the set of possible bidder-object combinations is only a subset I' of I, and the upper- and lower-bound capacity constraints (f, g) are defined only on subsets of I', the set of feasible allocation outcomes is not the X defined in (1) but rather

$$X' := \left\{ (x_i)_{i \in I'} \in \mathbb{R}^{I'} \;\middle|\; \forall E \subseteq I' \left[g(E) \leq \sum_{i \in E} x_i \leq f(E) \right] \right\}.$$

Instead of $\mathscr{Q}$, let $\mathscr{Q}'$ denote the set of reduced forms in this case, X' replacing X.

Theorem 2 *For any $I' \subseteq I$, if (I', f, g) is paramodular then $(Q_i)_{i \in I'} \in \mathscr{Q}'$ if and only if, for any $(S_i)_{i \in I'}$ such that $S_i \in \mathscr{T}_{i_1}$ for all $i = (i_1, i_2) \in I'$,*

$$\int_T g\left(\{i \in I' \mid t_{i_1} \in S_i\}\right) d\mu(t) \leq \sum_{i \in I'} \int_T Q_i(t_{i_1}) \chi_{S_i}(t_{i_1}) d\mu(t) \leq \int_T f\left(\{i \in I' \mid t_{i_1} \in S_i\}\right) d\mu(t). \tag{20}$$

Proof Extend (f, g) from $2^{I'}$ to 2^I by $f(E) := f(E \cap I')$ and $g(E) := g(E \cap I')$ for all $E \subseteq I$. Lemma 3, applied to the case where $a_i = b_i = 0$ for all $i \in I \setminus I'$, implies that (I, f, g) is paramodular. The extension also implies that $X' \times \{\mathbf{0}\} = X$, where X is the set defined in (1), and $\mathbf{0}$ the zero vector in $\mathbb{R}^{I \setminus I'}$. In other words, $(Q_i)_{i \in I'} \in \mathscr{Q}'$ if and only if $((Q_i)_{i \in I'}, \mathbf{0}) \in \mathscr{Q}$, where $\mathbf{0}$ denotes the array of constant zero functions $t_{i_1} \mapsto \mathbf{0} \in \mathbb{R}^{I_2}$ ((i_1, i_2) ranging in $I \setminus I'$). Then Theorem 1 applies to (I, f, g), and hence $((Q_i)_{i \in I'}, \mathbf{0}) \in \mathscr{Q}$ (namely $(Q_i)_{i \in I'} \in \mathscr{Q}'$) if and only if (3) holds for all $(S_i)_{i \in I}$ such that $S_i \in \mathscr{T}_{i_1}$ for all $i = (i_1, i_2) \in I$. On the left and right sides of (3), by the extension of g and f,

$$
\begin{aligned}
g\left(\{i \in I \mid t_{i_1} \in S_i\}\right) &= g\left(\{i \in I' \mid t_{i_1} \in S_i\}\right), \\
f\left(\{(i \in I \mid t_{i_1} \in S_i\}\right) &= f\left(\{i \in I' \mid t_{i_1} \in S_i\}\right).
\end{aligned}
$$

Thus, any S_i for which $i \notin I'$ can be replaced by $\varnothing$ without loss. Because $Q_i = 0$ for all $i \in I \setminus I'$, the sum in the middle of (3) is equal to $\sum_{i \in I'} \int_T Q_i(t_{i_1}) \chi_{S_i}(t_{i_1}) d\mu(t)$. Thus, (3) for all $(S_i)_{i \in I}$ such that $S_i \in \mathscr{T}_{i_1}$ for all $i = (i_1, i_2) \in I$, is equivalent to (20) for all $(S_i)_{i \in I'}$ such that $S_i \in \mathscr{T}_{i_1}$ for all $i = (i_1, i_2) \in I'$. ■

Theorem 2 applies to multiple-object auctions with cross-bidder, cross-object constraints. For example,

$$\sum_{i \in E_1 \times E_2} x_i \leq f(E_1 \times E_2)$$

for a set E_1 of bidders (e.g., internet conglomerates) and a set E_2 of licenses to operate in various industries (search engine, online shopping, voice recognition, etc.), one license for each industry. Such upper-bound capacity constraints are relevant to antitrust considerations that may restrict a firm from having a significant presence in multiple industries (e.g., a "Glass-Steagall" legislation for the internet). Analogously, there may be a lower-bound constraint

$$\sum_{i \in G_1 \times G_2} x_i \geq g(G_1 \times G_2)$$

for a set G_1 of bidders (say each belonging to a minority group) and a set G_2 of objects (say each being a kind of loans whose interest rates are below a certain level). That is relevant to equity considerations. Furthermore, as noted before, the sets $E_1 \times E_2$ and $G_1 \times G_2$ can be replaced by a subset of I that is not the cartesian product of subsets of I_1 and I_2. Such cases are relevant to situations where a particular bidder may be deemed infeasible a priori to have a particular object that is feasible to other bidders.

4.2 Budget Constraints

Suppose that $I_2 = I_2' \sqcup \{m\}$ such that I_2' is the set of goods for sale, and m represents monetary payment. Then the component $Q_{i_1,m}$, for any $i_1 \in I_1$, represents the expected payment from bidder i_1. In models based on quasilinear utilities, the reduced-form payment rule is implied by the corresponding reduced-form allocation through a tractable, envelope equation, and hence characterization of reduced-form payment rules is a nonissue. Without the quasilinearity assumption, by contrast, the relationship between the reduced-form payment and the reduced-form allocation is often intractable, and the alternative characterization offered by Theorems 1 and 2 may be useful.

Since a bidder can be a payer some time and a payee some other time, to satisfy the assumption in the model that the quantity of any object is nonnegative, suppose that $B \in \mathbb{R}_{++}$ is an exogenous upper bound of the amount of money that a bidder can possibly receive; normalize the quantity of money so that if an allocation outcome has the component $x_{i_1,m} \in \mathbb{R}_+$, then the net monetary payment that bidder i_1 is supposed to deliver is equal to $x_{i_1,m} - B$. (The case $x_{i_1,m} - B < 0$ means that i_1 receives the amount $B - x_{i_1,m}$.)

The next lemma, proved in Appendix D, allows us to extend the paramodular structure from $I_1 \times I_2'$ to $I_1 \times (I_2' \sqcup \{m\})$ so as to apply our characterization results.

Lemma 4 *Suppose: $I_2' \cap I_2'' = \varnothing$, $(I_1 \times I_2', f, g)$ is paramodular, $\mathcal{C}$ is a collection of mutually disjoint subsets of $I_1 \times I_2''$ such that $f \geqq g \geqq 0$ on $\mathcal{C}$, and*

$$P := \left\{ (x_i)_{i \in I_1 \times (I_2' \sqcup I_2'')} \;\middle|\; \forall E \in 2^{I_1 \times I_2'} \cup \mathcal{C} \left[g(E) \le \sum_{i \in E} x_i \le f(E) \right] \right\} \tag{21}$$

is nonempty and bounded. Extend (f, g) to all subsets of $I := I_1 \times (I_2' \sqcup I_2'')$ by

$$f(E) := \max_{(x_i)_{i \in I} \in P} \sum_{i \in E} x_i \quad \text{and} \quad g(E) := \min_{(x_i)_{i \in I} \in P} \sum_{i \in E} x_i \tag{22}$$

for all $E \subseteq I$ $(= I_1 \times (I_2' \sqcup I_2''))$. Then (I, f, g) is paramodular, and P is equal to the X defined by (1) where the (f, g) is the extension in (22).

4.2.1 Individual Budget Constraints

Suppose for each $i_1 \in I_1$ there is a commonly known $w_{i_1} \in \mathbb{R}_+$ equal to the (hard) budget constraint for bidder i_1.[12] That is, for any allocation outcome $(x_i)_{i\in I}$ to be feasible, $-B \leq x_{i_1,m} - B \leq w_{i_1}$ for all $i_1 \in I_1$. In other words,

$$\forall i_1 \in I_1 : 0 \leq x_{i_1,m} \leq w_{i_1} + B.$$

Let I_1' be the set of goods and assume that $(I_1 \times I_1', f, g)$ is paramodular. Let

$$\mathcal{C} := \{\{(i_1, m)\} \mid i_1 \in I_1\}$$

and, for each $i_1 \in I_1$,

$$f(\{(i_1, m)\}) := w_{i_1} + B, \quad g(\{(i_1, m)\}) := 0.$$

Then the set of feasible outcomes (including provisions about monetary payments) that also respect the individual budget constraints is the P defined in (21) with the $\mathcal{C}$ and (f, g) defined here. Thus Lemma 4 applies, which in turn implies that Theorem 1 applies to the case $I = I_1 \times (I_2' \sqcup \{m\})$ and provides (3) as the condition not only for the reduced-form allocation $(Q_{i_1,i_2})_{(i_1,i_2)\in I_1\times I_2'}$ but also for the expected payment rule $(Q_{i_1,m})_{i_1\in I_1}$.

4.2.2 Ex Post Budget Balance

Since we normalize the quantity of monetary payment through raising it by the constant B, ex post budget balance (BB) means that

$$\sum_{i_1\in I_1} x_{i_1,m} = B.$$

To characterize the reduced forms that respect BB, define

$$\mathcal{C} := \{I_1 \times \{m\}\}$$

and

$$f(I_1 \times \{m\}) := g(I_1 \times \{m\}) := B.$$

Then the set of feasible outcomes (including provisions about monetary payments) that also respect BB is the P defined in (21) with the $\mathcal{C}$ and (f, g) defined here. Thus, as in the previous subsection, we can apply Lemma 4 and hence Theorem 1.

[12] Such kind of budget constraints is considered by Boulatov and Severinov [6]. They have also suggested a condition for the reduced forms in their (single-unit independent-type) model.

4.3 Norm-Compactness of Monotone Reduced Forms

The set of reduced-form allocations is often restricted to monotone functions, especially when types are assumed unidimensional. The norm-compactness of the set of such reduced forms guarantees existence of an optimum among them with respect to any design objective that is norm-continuous. That is a stronger result than the existence observations based on the weak* topology of the set of mechanisms (e.g., Kadan, Reny and Swinkels [19]) because there can be design objectives outside the expected utility framework.

Since (3) defines a set of reduced forms closed in pointwise convergence, one can derive from Theorem 1 a norm-compactness observation of monotone reduced forms. To allow for multidimensional types, I adopt Reny's [26] Assumptions G.1 and G.3:

($*$) For every $i_1 \in I_1$: (i) there is a partial ordering $\geq$ for which $(T_{i_1}, \mathscr{T}_{i_1}, \mu_{i_1}, \geq)$ is a partially ordered probability space, and (ii) there is a countable subset $T^0_{i_1}$ of T_{i_1} such that every set in $\mathscr{T}_{i_1}$ assigned positive probability by μ_{i_1} contains two points between which lies a point in $T^0_{i_1}$.

For any $i_1 \in I_1$, a function $Q_{i_1} : T_{i_1} \to \mathbb{R}^{I_2}$ is siad to be *monotone* iff $t'_{i_1} \geq t_{i_1}$ implies $Q_{i_1}(t'_{i_1}) \geq Q_{i_1}(t_{i_1})$ for all $t'_{i_1}, t_{i_1} \in T_{i_1}$, with the latter $\geq$ being the coordinate-wise $\geq$ partial ordering on $\mathbb{R}^{I_2}$. Denote

$$\mathscr{Q}_{\mathrm{mon}} := \left\{ (Q_{i_1,i_2})_{(i_1,i_2)\in I_1\times I_2} \in \mathscr{Q} \mid \forall i_1 \in I_1 \left[(Q_{i_1,i_2})_{i_2\in I_2} \text{ is monotone} \right] \right\}.$$

The next theorem is proved in Appendix F.[13]

Theorem 3 *If ($*$) holds, (I, f, g) is paramodular, and X defined in (1) is nonempty and bounded, then $\mathscr{Q}_{\mathrm{mon}}$ is compact with respect to the $\|\cdot\|_p$-norm for any $1 \leq p < \infty$.*

5 Conclusion

In a general multiunit multiple-object model, this paper provides a complete characterization of reduced-form auctions. The characterization is Border-like in the sense that the condition

[13] From a majorization approach, Kleiner, Moldovanu and Strack [21] have proved norm-compactness of monotone reduced forms in the special case of symmetric bidders and one-dimensional types. They also provide an example where weak*-compactness is insufficient, and norm-compactness needed, to deliver an existence result.

for a reduced form needs only to be checked against all measurable subsets of the type space. Allowing for multiple heterogenous objects, the characterization may be useful to the design of multiple-object auctions. Allowing for both ceilings and floors of the joint capacities across bidders and across objects, the characterization is relevant to cross-industry antitrust policies and cross-category equity considerations.

In obtaining the result I use only two assumptions about the primitives, which remain unchanged when the type space varies arbitrarily, and is not more restrictive than the conditions in the received literature. My proof clarifies their role in the Border-like characterization: The two assumptions, which constitute the paramodularity of the capacity constraints, ensure that the greedy-generous algorithm is a state-by-state optimal assignment from ex post states to interim states. Thanks to the algorithm, we can factor out the effect of the arbitrary weights across interim states thereby obtaining a Border-like characterization. If either assumption is violated, the greedy-generous algorithm might be either suboptimal or infeasible, and the weights across interim states might be entangled with ex post states in the optimal assignment from ex post states to interim states, so that one might not be able to factor out the arbitrary weights. Therefore, an open question is whether a Border-like characterization of reduced forms can still be delivered by conditions weaker than the two paramodularity assumptions.

Appendices

A Assumption 2 and the Compliance Condition

The compliance condition assumed in CKM [8] is:

$$E, E' \subseteq I \Rightarrow g(E) - g(E \setminus E') \leq f(E') - f(E' \setminus E). \tag{23}$$

Theorem 4 *For any (f, g) that satisfies Assumption 1, Assumption 2 $\Longleftrightarrow$ (23).*

Proof To prove the "$\Leftarrow$" part, apply (23) to the case that $E' = (\neg E) \cup \{e\}$. For the converse, suppose Assumption 2. We shall prove (23). First we show—

Claim 1 *For any $E \subseteq I$ and any distinct elements $e_1, \ldots, e_k$ of E,*

$$g(E) - g\left(E \setminus \{e_1, \ldots, e_k\}\right) \leq f\left(\neg\left(E \setminus \{e_1, \ldots, e_k\}\right)\right) - f(\neg E).$$

Proof Suppose, to the contrary, that

$$g(E) - g\left(E \setminus \{e_1, \ldots, e_k\}\right) > f\left(\neg\left(E \setminus \{e_1, \ldots, e_k\}\right)\right) - f(\neg E).$$

Plug into this inequality the statement

$$g(E) \leq g(E \setminus \{e_k\}) + f\left(\neg(E \setminus \{e_k\})\right) - f(\neg E)$$

of Assumption 2 to obtain

$$g(E\setminus\{e_k\}) + f\left(\neg(E \setminus \{e_k\})\right) - f(\neg E) - g\left(E \setminus \{e_1, \ldots, e_k\}\right) > f\left(\neg\left(E \setminus \{e_1, \ldots, e_k\}\right)\right) - f(\neg E).$$

This inequality is equivalent to

$$g(E\setminus\{e_k\}) - g\left((E \setminus \{e_k\}) \setminus \{e_1, \ldots, e_{k-1}\}\right) > f\left(\neg\left[(E \setminus \{e_k\}) \setminus \{e_1, \ldots, e_{k-1}\}\right]\right) - f\left(\neg(E \setminus \{e_k\})\right).$$

Repeat this procedure to eventually obtain

$$\begin{aligned} & g\left(E \setminus \{e_2, \ldots, e_k\}\right) - g\left((E \setminus \{e_2, \ldots, e_k\}) \setminus \{e_1\}\right) \\ > \; & f\left(\neg\left[(E \setminus \{e_2, \ldots, e_k\}) \setminus \{e_1\}\right]\right) - f\left(\neg\left[E \setminus \{e_2, \ldots, e_k\}\right]\right), \end{aligned}$$

contradicting Assumption 2, with $E \setminus \{e_2, \ldots, e_k\}$ playing the role of E. $\square$

It follows from Claim 1 that

$$E'' \subseteq E \subseteq I \Rightarrow g(E) - g(E \setminus E'') \leq f\left(\neg(E \setminus E'')\right) - f(\neg E).$$

To prove (23), pick any $E, E' \subseteq I$. By the above-displayed formula,

$$g(E) - g(E \setminus E') = g(E) - g\left(E \setminus (E \cap E')\right) \leq f\left(\neg\left(E \setminus (E \cap E')\right)\right) - f(\neg E).$$

Thus, it suffices to show

$$f\left(\neg\left(E \setminus (E \cap E')\right)\right) - f(\neg E) \leq f(E') - f(E' \setminus E).$$

This inequality is equivalent to

$$f\left(E' \cup (\neg E)\right) + f\left(E' \cap (\neg E)\right) \leq f(E') + f(\neg E),$$

which is true by submodularity of f (Assumption 1). Thus, (23) holds, as desired. ■

B Proof of Lemma 1

This is similar to the proof in the Supplement Appendix B.2 in CKM [8]: Necessity of (3) for $Q \in \mathscr{Q}$ is trivial. Sufficiency of (3) is proved by a passing-to-limit argument that, for any interim allocation Q satisfying (3), constructs a sequence of finite-type interim allocations converging to Q. The only difference is that, while an interim allocation in the multiunit single-object model of CKM is a profile $(Q_{i_1})_{i_1 \in I_1}$ of real-valued functions $Q_{i_1} : T_{i_1} \to \mathbb{R}$, it is a profile $(Q_{i_1})_{i_1 \in I_1}$ of vector-valued functions $Q_{i_1} : T_{i_1} \to \mathbb{R}^{I_2}$ in our multiple-object model. Given the difference, we just need to modify the construction of the finite-type sequence:

A *cell* in $\mathbb{R}^{I_2}$ is a set $\left\{(x_{i_2})_{i_2 \in I_2} \in \mathbb{R}^{I_2} \mid \forall i_2 \in I_2 \left[a_{i_2} \leq x_{i_2} < b_{i_2}\right]\right\}$ for some $a_{i_2} < b_{i_2}$ ($\forall i_2 \in I_2$). Let $Q := (Q_{i_1,i_2})_{(i_1,i_2) \in I_1 \times I_2}$ be an interim allocation that satisfies (3). Write Q as $(Q_{i_1})_{i_1 \in I_1}$ such that $Q_{i_1} := (Q_{i_1,i_2})_{i_2 \in I_2}$ for each $i_1 \in I_1$. For any $m = 1, 2, 3, \ldots$, partition $\mathbb{R}^{I_2}$ into a collection $\mathscr{C}_m$ of cells each of which has diameter at most $1/m$. For each bidder $i_1 \in I_1$ and each cell $C \in \mathscr{C}_m$ that has nonempty intersection with the range of Q_{i_1}, denote $([\min C]_{i_2})_{i_2 \in I_2}$ for the coordinate-wise minimum among all elements of C, with $[\min C]_{i_2}$ being its coordinate in the i_2th dimension, and define

$$Q^m_{i_1,i_2}(t_{i_1}) := \max\left\{0, [\min C]_{i_2}\right\}$$

for all $i_2 \in I_2$ and all t_{i_1} in the inverse image $Q_{i_1}^{-1}(C)$ of C. Thus,

$$\max\{0, Q_{i_1,i_2}(t_{i_1}) - 1/m\} \leq Q^m_{i_1,i_2}(t_{i_1}) \leq Q_{i_1,i_2}(t_{i_1})$$

for each m, each $(i_1, i_2) \in I$ and each $t_{i_1} \in T_{i_1}$. (The second inequality follows from the definition of $Q^m_{i_1,i_2}(t_{i_1})$ and the fact that there is no loss to restrict the range of Q_{i_1} to $\mathbb{R}_+^{I_2}$, given that the values of f and g are both nonnegative.) Then, since Q satisfies (3), $Q^m := (Q^m_{i_1,i_2})_{(i_1,i_2)\in I}$ satisfies (3) with g replaced by $g_m : 2^I \to \mathbb{R}_+$ defined by

$$g_m(E) := \max\{0, g(E) - 1/m\}$$

for all $E \subseteq I$. It is easy to verify that (f, g_m) satisfies both Assumptions 1 and 2.

Since there is no loss to restrict the range of Q_{i_1} to a bounded set (as f and g are each finite-valued), for each m there are only finitely many cells in $\mathscr{C}_m$ that intersect with the range of Q_{i_1}. Thus, $Q^m_{i_1}$ is equivalent to a function defined on the finite type space

$$T^m_{i_1} := \left\{Q_{i_1}^{-1}(C) \mid Q_{i_1}^{-1}(C) \neq \varnothing; C \in \mathscr{C}_m\right\}.$$

Since Q^m satisfies (3) with (f, g_m) playing the role of (f, g), the finite-type version of Theorem 1 implies that Q^m is the reduced form of an ex post allocation q^m defined on $\prod_{i_1 \in I_1} T^m_{i_1}$. Consequently, as in the proof in CKM, one can extract a subsequence $(q^{m_k})_{k=1}^{\infty}$ converging to some ex post allocation q defined on the original type space T; furthermore, $\lim_{k\to\infty} Q^{m_k}$ is the reduced form of q, and $Q = \lim_{k\to\infty} Q^{m_k}$. That is, (3) implies $Q \in \mathscr{Q}$. ■

C Proof of Lemma 3

By the assumption in the lemma, it suffices to verify Assumptions 1 and 2. To verify Assumptions 1, pick any $E, E' \subseteq I$ to verify

$$f(E) + f(E') \geq f(E \cup E') + f(E \cap E').$$

This inequality, by (18), is equivalent to

$$\begin{aligned} & f\left(E \cap S\right) + f\left(E' \cap S\right) + \sum_{i \in E \setminus S} a_i + \sum_{i \in E' \setminus S} a_i \\ \geq \ & f\left((E \cup E') \cap S\right) + f\left(E \cap E' \cap S\right) + \sum_{i \in (E \cup E') \setminus S} a_i + \sum_{i \in (E \cap E') \setminus S} a_i, \end{aligned}$$

which is true because f is assumed submodular on 2^S. Thus, f is submodular on 2^I. Submodularity of $-g$ on 2^I can be proved symmetrically. To verify Assumption 2, let $e \in E \subseteq I$ to verify

$$g(E) - g(E \setminus \{e\}) \leq f\left((I \setminus E) \cup \{e\}\right) - f(I \setminus E).$$

This inequality, by (18) and (19), is equivalent to

$$\begin{aligned} & g(E \cap S) - g\left((E \setminus \{e\}) \cap S\right) + \sum_{i \in E \setminus S} b_i - \sum_{i \in (E \setminus \{e\}) \setminus S} b_i \\ \leq \quad & f\left([(I \setminus E) \cup \{e\}] \cap S\right) - f\left((I \setminus E) \cap S\right) + \sum_{i \in [(I \setminus E) \cup \{e\}] \setminus S} a_i - \sum_{i \in (I \setminus E) \setminus S} a_i. \end{aligned}$$

If $e \notin S$, the inequality displayed above becomes $b_e \leq a_e$, which is true by assumption of the lemma. If $e \in S$, the inequality displayed above becomes

$$g(E \cap S) - g\left((E \setminus \{e\}) \cap S\right) \leq f\left([(I \setminus E) \cup \{e\}] \cap S\right) - f\left((I \setminus E) \cap S\right),$$

namely,

$$g(E \cap S) - g\left((E \cap S) \setminus \{e\}\right) \leq f\left((S \setminus E) \cup \{e\}\right) - f(S \setminus E),$$

which is true by the counterpart of Assumption 2 where I is replaced by S. Thus, (I, f, g) satisfies Assumption 2. ■

D Proof of Lemma 4

We shall apply Schrijver's [27] Theorem 49.13. To that end, we verify the conditions required by the theorem. First, $2^{I_1 \times I_2'} \cup \mathcal{C}$ is an intersecting family in the sense that for any members E and E' of the family that have nonempty intersection, both $E \cup E'$ and $E \cap E'$ belong to the family. That is because $\mathcal{C}$ by the assumption in the lemma is a collection of mutually disjoint subsets, and each member of $\mathcal{C}$ is disjoint from any subset of $I_1 \times I_2'$. (Clearly $2^{I_1 \times I_2'}$ is an intersecting family itself.) Second, f and $-g$ are each submodular on intersecting pairs in the family $2^{I_1 \times I_2'} \cup \mathcal{C}$. That is because f and $-g$ are each submodular on $2^{I_1 \times I_2'}$, and the condition of submodularity on intersecting pairs is vacuously true for any $E \subseteq I_1 \times I_2'$ and $E' \in \mathcal{C}$, as $E \cap E' = \varnothing$, and likewise vacuously true for any $E, E' \in \mathcal{C}$. Third, (f, g) satisfies the cross inequality

$$g(E) - g(E \setminus E') \leq f(E') - f(E' \setminus E)$$

for all $E, E' \in 2^{I_1 \times I'_2} \cup \mathcal{C}$. If both E and E' are subsets of $I_1 \times I'_2$, the cross inequality is satisfied because $(I_1 \times I'_2, f, g)$ is paramodular by the assumption of the lemma and hence Assumption 2 holds, which in turn implies the cross inequality for all such E and E' (Theorem 4). If E or E' is not a subset of $I_1 \times I'_2$, $E \cap E' = \varnothing$ and the cross inequality holds trivially, both sides equal to zero.

All conditions verified, Schrijver's [27] Theorem 49.13 applies and implies that, on the collection $\mathcal{C}_*$ of all the subsets E of $I_1 \times (I'_2 \sqcup I''_2)$ for which the $f(E)$ and $g(E)$ in the extension (22) of (f, g) are finite, f and $-g$ are each submodular and the cross inequality is satisfied,[14] and that (21) remains true if the collection $\mathcal{C}$ therein is replaced by the $\mathcal{C}_*$ here. Since P is assumed bounded and hence compact, the collection $\mathcal{C}_*$ includes all subsets of $I_1 \times (I'_2 \sqcup I''_2)$. Consequently, P is equal to the X defined in (1) according to the extended (f, g). Thus, $(I_1 \times (I'_2 \sqcup I''_2), f, g)$ satisfies Assumption 1, and (23) holds with $I = I_1 \times (I'_2 \sqcup I''_2)$. Since (23) is equivalent to Assumption 2 (Theorem 4), $(I_1 \times I_2, f, g)$ is paramodular. ∎

E Paramodularity of the Usual Multiple-Object Model

In this model, as described in Section 2, for each $i_2 \in I_2$ there is a constant $m_{i_2} \in \mathbb{R}_{++}$ that represents the total quantity of object i_2 available for allocation, and such object-wise aggregate feasibility condition is the only capacity constraint in the model. This corresponds to the case in Lemma 4 where $I'_2 = \varnothing$, $I''_2 = I_2$,

$$\mathcal{C} = \{I_1 \times \{i_2\} \mid i_2 \in I_2\},$$

and, for each $I_1 \times \{i_2\} \in \mathcal{C}$,

$$f\left(I_1 \times \{i_2\}\right) = m_{i_2} \quad \text{and} \quad g\left(I_1 \times \{i_2\}\right) = 0.$$

Since all members of $\mathcal{C}$ are mutually disjoint, Lemma 4 applies. Thus, the (f, g) here can be extended via (22) to 2^I to become paramodular. Note from (21) and (22) that the extension adds no restriction to the model.

[14] Schrijver's proof verifies the cross inequality only for sets that are not related by $\subseteq$. For sets that are related by $\subseteq$ the cross inequality can be verified easily with the constructs in that proof.

F Proof of Theorem 3

Pick any sequence $(Q^\nu)_{\nu=1}^\infty$ in $\mathscr{Q}_{\rm mon}$. It suffices to extract a subsequence that converges in the $\|\cdot\|_p$-norm topology to some element of $\mathscr{Q}_{\rm mon}$. Since X is assumed bounded, any ex post allocation, as a function $T \to X$, is uniformly bounded by a compact lattice subset of $\mathbb{R}^I$ ($I = I_1 \times I_2$). Thus, with $Q^\nu := (Q^\nu_{i_1})_{i_1 \in I_1} \in \mathscr{Q}$, the range of $Q^\nu_{i_1}$ (as marginals of an ex post allocation) for any $i_1 \in I_1$ can be restricted to a compact lattice subset A_{i_1} of $\mathbb{R}^{I_2}$ for all ν. This, combined with Assumption $(*)$ of the theorem and the fact that each $Q^\nu_{i_1} := (Q^\nu_{i_1,i_2})_{i_2 \in I_2}$ is monotone, ensures Assumptions G.1, G.3 and G.4 of Reny's [26] Lemma A.10 (generalized Helley's selection theorem). Thus, by the said lemma, we can extract from $(Q^\nu)_{\nu=1}^\infty$ a subsequence $(Q^{\nu_k})_{k=1}^\infty$ that converges pointwise (μ-a.e.) to some $Q^* := (Q^*_i)_{i \in I}$ such that $(Q^*_{i_1,i_2})_{i_2 \in I_2}$ is monotone and measurable for all $i_1 \in I_1$.

Since $(Q^{\nu_k})_{k=1}^\infty$ is uniformly bounded by a compact subset, for each $i \in I$ there exist $l_i, h_i \in \mathbb{R}$ such that $l_i \le h_i$ and $l_i \le Q^{\nu_k}_i \le h_i$ for all k. Then the Lebesgue dominated convergence theorem implies that $\lim_{k\to\infty} \|Q^{\nu_k}_i - Q^*_i\|_p = 0$ for all $i \in I$. That is, $(Q^{\nu_k})_{k=1}^\infty$ converges in the norm topology to Q^*.

Thus, the proof is complete if $Q^* \in \mathscr{Q}_{\rm mon}$. Since $(Q^*_{i_1,i_2})_{i_2 \in I_2}$ is monotone for each i_1, it suffices to prove that $Q^* \in \mathscr{Q}$. To prove that, apply Theorem 1 to Q^{ν_k}, which belongs to $\mathscr{Q}$, for each k. Thus, for each k, (3) holds when $Q = Q^{\nu_k}$, for all $(S_i)_{i \in I}$ such that $S_i \in \mathscr{T}_{i_1}$ for all $i \in I$. Given any such $(S_i)_{i \in I}$, since $Q^{\nu_k} \to_k Q^*$ pointwise μ-a.e., the Lebesgue dominated convergence theorem again implies that (3) holds when $Q = Q^*$. Thus, Q^* satisfies (3) for all $(S_i)_{i \in I}$ such that $S_i \in \mathscr{T}_{i_1}$ for all $i \in I$. Then Theorem 1 implies $Q^* \in \mathscr{Q}$, as desired. ∎

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