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Jump bidding and overconcentration in decentralized simultaneous ascending auctions ☆

Charles Z. Zheng

Department of Economics, University of Western Ontario, Social Science Ct., Rm 4071, London, Ontario, N6A 5C2, Canada

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ABSTRACT

A model of English auctions is proposed to incorporate the possibility of jump bidding. When two objects are sold separately via such auctions, bidders signal their willingness to pay via jump bids, thereby forming rational expectations of the prices without relying on any central mediator. Hence a multi-item bidder does not suffer the exposure problem of having to buy an item while he is uncertain about the price of its complement. Single-item bidders, however, free-ride one another in competing against a multi-item bidder. Consequently, the auctions overly concentrate the goods to a multi-item bidder and never overly diffuse them to single-item bidders.

1. Introduction

In decentralized markets, an economic agent hoping to acquire a bundle of goods often runs into the following dilemma. One of the goods is available at a price above its standalone value. Should he buy it or not? The problem is that the prices of the other items in the bundle are uncertain and may be so high that the total price of the bundle exceeds its total value. Without a "Walras auctioneer" to coordinate across markets, the agent cannot postpone his decision on one good to wait for the realization of the prices for its complements.

This dilemma has been crystallized into the *exposure problem* in auction theory. Milgrom (2000) and Bykowsky et al. (2000) have constructed complete-information examples for this problem. A few authors have analyzed the exposure problem in asymmetric-information models. The typical setup is that two objects are being auctioned off via two separate auctions simultaneously. Some bidders are *local* in the sense that they value only one particular object. The others are *global* in the sense that they value both objects as complements. A global bidder faces the exposure problem. Albano et al. (2001, 2006) analyze two variants of a two-object ascending auction. Krishna and Rosenthal (1996) and Rosenthal and Wang (1996) consider simultaneous sealed-bid auctions for possibly more than two objects. The predictions are typically

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E-mail address: charles.zheng@uwo.ca.
URL: http://economics.uwo.ca/faculty/zheng/.

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that various kinds of inefficiency may occur. Sometimes the objects are *overconcentrated* to a single bidder while efficiency requires that they go to different bidders, and sometimes the goods may be *overdiffused* to separate owners while efficiency requires that a single bidder should own them.[1]

This paper analyzes the exposure problem by proposing a new model of English auctions to capture their open ascending nature. The idea is that the transparency of English auctions allows bidders to signal to one another across auctions, thereby forming a rational expectation of the prices. A bidder makes his signal credible via jump bidding, i.e., committing to paying for the good at a pledged price (above its current price) if he wins immediately.[2] Based on this model, a continuation equilibrium is constructed that eliminates the exposure problem conditional on any event that the problem may arise (Proposition 3). Thus, before committing to buying any item, a bidder can find out the price of its complement. A multi-item bidder is therefore no longer deterred by price uncertainty. The prediction of the auctions becomes overconcentration only: In any perfect Bayesian equilibrium that satisfies certain conditions, overdiffusion occurs with zero probability while overconcentration occurs with a strictly positive probability (Proposition 4).[3]

Although sensitive to the two-object assumption, this result conveys the message that, even without central coordination, economic agents may get to form rational expectations of prices through signaling to one another.

2. The primitives

There are two items for sale, A and B, and several bidders. For each bidder i, the values of winning item A alone, item B alone, and both items are, respectively, $u_i(A)$, $u_i(B)$, and $u_i(AB)$. There are three kinds of bidders: one *global bidder*, named bidder γ, who values items A and B as complements, several *A-local* bidders who value only item A, and several *B-local* bidders who value only item B; a *local* bidder means an A- or B-local bidder. That is,

$$\begin{aligned} &u_\gamma(AB) \geqslant u_\gamma(A) + u_\gamma(B), \\ &u_i(AB) - u_i(A) = u_i(B) = 0 \quad \text{if } i \text{ is A-local}, \\ &u_i(AB) - u_i(B) = u_i(A) = 0 \quad \text{if } i \text{ is B-local}. \end{aligned} \tag{1}$$

For each bidder i, it is commonly known whether i is global, A-local, or B-local. For any $x \in \{A, B\}$, the *standalone value* $u_i(x)$ of item x is x-local bidder i's private information and is independently drawn from a commonly known distribution F_{ix}; the standalone values $u_\gamma(A)$ and $u_\gamma(B)$, as well as the synergy $u_\gamma(AB) - u_\gamma(A) - u_\gamma(B)$, are global bidder γ's private information and are independently drawn from commonly known distributions $F_{\gamma A}$, $F_{\gamma B}$ and $F_{\gamma C}$. We assume that these distributions have no atom and no gap and each have zero as the infimum.

A bidder's payoff is equal to his value of the package he wins minus his total monetary payment. He is risk neutral in his payoff.

The two items are auctioned off via separate English auctions that start simultaneously. To be eligible for an item, a bidder needs to participate in its auction from the start. Once he drops out from the auction of an item, a bidder cannot raise his bid for it any more. Once sold, the good is not refundable. Bidders' actions are commonly observed.

For simplicity, we assume that A-local bidders can only bid in the auction for item A, and B-local bidders can only bid for B.

This setup is decentralized in the sense that the auctioneers of the two auctions cannot coordinate with each other on when to close the auctions. Due to this feature, when the global bidder can acquire an item say A, he may be uncertain about the price of the other item B. When the price for A is higher than its standalone value, the bidder faces an *exposure problem*: if he drops out from the auction of item A now, he foregoes the probable opportunity of acquiring both items at a profitable total price; if he buys A now, however, the eventual price for B may turn out to be unprofitably high.

3. The exposure problem under the clock model

Let us illustrate the exposure problem when an English auction is modeled by the traditional "clock model": For each item x, the price p_x starts at zero and rises continuously at an exogenous positive speed until all but one bidder have quit bidding for x, at which point item x is immediately sold to the remaining bidder at the current price.[4]

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[1] The exposure problem may arise in circumstances other than complementarity. For example, a bidder with unit-demand preference may face the exposure problem when he is bidding for multiple homogeneous units simultaneously. The exposure problem may even be driven by a bidder's budget constraint, as in the centralized two-object auction considered by Brusco and Lopomo (2009).

[2] The jump-bidding literature, e.g., Avery (1998) and Gunderson and Wang (1998), typically assumes that jump bidding occurs only at an exogenous stage. In Xiong's (2007) single-unit model, the timing of a jump bid is endogenous but its magnitude is exogenous. In my paper, both the timing and magnitude of a jump bid are endogenous and furthermore a jump bid almost surely fully reveals the bidder's type.

[3] The open transparent nature of English auctions may also facilitate tacit collusion among bidders. For example see Brusco and Lopomo (2002, 2005) and Garratt et al. (2009).

[4] The model in this section is similar to one of the models in Albano et al. (2001, 2006). But my model is much more general because it does not require the two assumptions in their papers: (i) $u_\gamma(A) = u_\gamma(B)$ and $u_\gamma(AB) - u_\gamma(A) - u_\gamma(B)$ is commonly known; (ii) the prices of the two objects rise at the same pace.

Let p_A and p_B denote the current prices for items A and B, respectively. For any $z \in \mathbb{R}$, let $(z)^+ := \max\{z, 0\}$. Let $\mathbb{E}_X[f(X) \mid g(X) \geqslant 0]$ denote the expected value of the function $f(X)$ of the random variable X conditional on $g(X) \geqslant 0$.

An undominated strategy for every local bidder is to continue bidding for his desired item until its price reaches its value. It will be demonstrated that the following strategy for the global bidder γ is a best reply to the undominated strategy.

a. When $p_A = p_B = 0$, participate in both auctions.
b. If neither item has been sold:
   i. If $v_\gamma(A, p_B) > p_A$ and $v_\gamma(B, p_A) > p_B$, where $v_\gamma(A, p_B)$ and $v_\gamma(B, p_A)$ will be defined later, then continue in both auctions.
   ii. If $v_\gamma(A, p_B) \leqslant p_A$ or $v_\gamma(B, p_A) \leqslant p_B$:
      I. If $u_\gamma(x) \leqslant p_x$ for each item $x \in \{A, B\}$, then drop out from both auctions.
      II. If $u_\gamma(x) > p_x$ for some item x, then continue bidding for the item that has the higher expected profit conditional on the current prices and drop out from the other auction.
c. If the bidder has dropped out from the auction of an item x, then continue in the auction for the other item, denoted $-x$, if and only if $p_{-x} < u_\gamma(-x)$.
d. If the bidder has won an item x, then continue in the auction for the item $-x$ if and only if $u_\gamma(AB) - u_\gamma(x) > p_{-x}$.

The strategy described above exhausts all possibilities. It is well-defined if $v_\gamma(A, p_B)$ and $v_\gamma(B, p_A)$ are defined. To define $v_\gamma(A, p_B)$, consider the case where global bidder γ has bought item A at price p_A. Then, given any price $\tilde{p}_B$ for item B, the bidder's payoff will be $u_\gamma(AB) - p_A - \tilde{p}_B$ if he also wins B and $u_\gamma(A) - p_A$ if he loses B. Thus, having bought A, the bidder's optimal action is to continue in the auction for B if and only if $u_\gamma(AB) - u_\gamma(A) > \tilde{p}_B$. (This explains provision (d) in the above strategy.) It follows that, if bidder γ buys A at price p_A and if the eventual price for B is $\tilde{p}_B$, then his ex post payoff is equal to

$$\left(u_\gamma(AB) - u_\gamma(A) - \tilde{p}_B\right)^+ + u_\gamma(A) - p_A.$$

With item A sold to bidder γ, any other bidder j will continue bidding for B up to the standalone value $u_j(B)$, hence

$$\tilde{p}_B = u_{-\gamma}(B) := \max_{j \neq \gamma} u_j(B).$$

Thus, if bidder γ buys A at price p_A, the expected payoff is equal to

$$\mathbb{E}_{u_{-\gamma}(B)}\left[\left(u_\gamma(AB) - u_\gamma(A) - u_{-\gamma}(B)\right)^+ \,\middle|\, u_{-\gamma}(B) \geqslant p_B\right] + u_\gamma(A) - p_A.$$

Hence bidder γ's expected payoff from buying item A at price p_A is positive if and only if

$$\mathbb{E}_{u_{-\gamma}(B)}\left[\left(u_\gamma(AB) - u_\gamma(A) - u_{-\gamma}(B)\right)^+ \,\middle|\, u_{-\gamma}(B) \geqslant p_B\right] + u_\gamma(A) > p_A.$$

Thus, define

$$v_\gamma(A, p_B) := \mathbb{E}_{u_{-\gamma}(B)}\left[\left(u_\gamma(AB) - u_\gamma(A) - u_{-\gamma}(B)\right)^+ \,\middle|\, u_{-\gamma}(B) \geqslant p_B\right] + u_\gamma(A). \tag{2}$$

Analogously, define

$$v_\gamma(B, p_A) := \mathbb{E}_{u_{-\gamma}(A)}\left[\left(u_\gamma(AB) - u_\gamma(B) - u_{-\gamma}(A)\right)^+ \,\middle|\, u_{-\gamma}(A) \geqslant p_A\right] + u_\gamma(B). \tag{3}$$

Proposition 1. *The global bidder's strategy* (a)–(d)*, together with the local bidders' undominated strategy of bidding for the valued item up to its value, constitutes a perfect Bayesian equilibrium when each English auction is a clock auction.*

Proof. By the atomless assumption and Eqs. (2)–(3), $v_\gamma(A, p_B)$ is continuous in p_B, and $v_\gamma(B, p_A)$ is continuous in p_A.

The justification for provisions (a), (c), and (d) in the strategy is obvious. Let us consider the case for provision (b), when neither item has been sold.

First, consider subcase (b.i), where $v_\gamma(A, p_B) > p_A$ and $v_\gamma(B, p_A) > p_B$. By the continuity of $v_\gamma(A, \cdot)$ and $v_\gamma(B, \cdot)$, these strict inequalities will continue to hold for at least a short interval of time. Thus, if the bidder is to continue bidding for A, his expected payoff from staying for item B is positive for at least a while, and the same statement is true when A and B switch roles. This expected payoff is bigger than the expected payoff from staying for only a single item, since $v_\gamma(x, p_{-x}) > u_\gamma(x)$ ($\forall x \in \{A, B\}$) by (2)–(3). Hence it is suboptimal to drop out from one auction now while continuing in the other auction. It is also suboptimal to drop out from both auctions, which yields zero payoff.

Second, consider subcase (b.ii). Without loss, suppose $v_\gamma(A, p_B) \leqslant p_A$. Since $v_\gamma(A, p_B)$ is weakly decreasing in p_B and the prices are strictly increasing in time, $v_\gamma(A, p'_B) < p'_A$ for any price vector (p'_A, p'_B) from now on. Thus, by the construction of $v_\gamma(A, \cdot)$ and $v_\gamma(B, \cdot)$, it is suboptimal to continue bidding for both items. That, however, does not mean the bidder should quit both auctions, because the standalone value of an item may still be above its current price. Hence the justification for provisions (b.ii.I) and (b.ii.II) is obvious. □

The next proposition says that the above equilibrium exhibits at least two kinds of inefficiency. One is *overdiffusion*: the two items go to two separate bidders, while $u_\gamma(AB) > \max_{j\in I} u_j(A) + \max_{k\in I} u_k(B)$, where I denotes the set of all bidders. The second kind is *overconcentration*: for some bidders $i, j \in I$ with $i \neq j$, bidder γ wins both items while $u_\gamma(AB) < u_i(A) + u_j(B)$.

Proposition 2. *In the equilibrium presented in Proposition 1, overdiffusion and overconcentration are events with strictly positive probabilities.*

The proof is in Appendix A. The intuition is: As long as both auctions are still going on after the price of an item has reached its standalone value for a global bidder, the bidder will drop out before the total price of the two items reach their combined value. Hence he "underbids" before winning any item. If he has won an item, however, with the payment for that item sunk, the bidder will bid for the other item up to its marginal value, which may be greater than the total value minus his payment for the won item. Hence the bidder "overbids" after winning an item. Thus, both overdiffusion and overconcentration are probable.

4. The need for an alternative model to capture the dynamics of English auctions

The clock model, albeit widely used in auction theory, has abstracted away most of the dynamic aspects of English auctions. In actual English auctions, bids may be submitted through open outcries, hence a bidder may be able to speed up the rising price through jump bids and adjust his strategy during the intermission between outcries.

These dynamic aspects of English auctions are important for the presence of the exposure problem. A bidder faces the exposure problem when he is about to buy an item at its current price without knowing the eventual price for its complement. If prices ascend through open outcries, however, the bidder may be able to resolve his uncertainty by making a jump-bid. From the rivals' responses, he could at least partially infer the price of the complement. That information might help the bidder to adjust his actions before he has to commit to buying the first item.

5. An alternative model

For each item, the auction is the clock model with the following amendments.

A1. As in the clock model, each *active* bidder, who has not dropped out from the auction, can *continue* bidding by pressing the button for the item.

A2. Besides "continue," an active bidder has the option of *jump bidding*: making a bid higher than the item's current price. This action is done in zero second.

A3. An active bidder can *drop out* from an auction. That is done by either releasing the button (if the price is ascending through the clock) or crying out "out" (if the price clock is pausing due to the following amendment).

A4. If a bidder drops out from an auction, the price clock in the auction pauses at the dropout price for a short interval of time, called a *pause*. The maximum duration of the pause is assumed to be exogenously δ seconds.

a1. During the pause, every bidder still active in the auction has three alternative actions: dropping out, *staying* in the auction, or *resuming* the auction (by crying out "resume").
a2. If the pause has lasted for δ seconds and there is at least one active bidder in the auction, the price clock resumes at the paused level unless there is only one active bidder, in which case the item is immediately sold to this bidder at the paused price.
a3. During the pause, if an active bidder resumes the auction, then the price clock resumes immediately without finishing the δ seconds, and if this bidder is the only active bidder in the auction, the item is immediately sold to this bidder at the paused price.
a4. If every active bidder drops out during the pause, the pause ends without finishing the δ seconds, and the item is sold according to the tie-breaking rule A6 described below.

A5. If a bidder jump-bids in an auction, the price clock in the auction jumps to the jump bid instantly and then pauses at the jump bid. The maximum duration of the pause is δ seconds.[5]

a1. During the pause, every bidder still active in the auction has three alternative actions: staying, jump-bidding, or dropping out unless the bidder is a *highest jump-bidder*, whose current jump bid is the highest among all jump bids.

[5] The duration of the pause triggered by a jump bid is assumed to be exogenous just for simplicity. Our results can be extended to allow for the following case of endogenous duration of a pause. The maximum duration of the pause is equal to the time it takes for the price clock to reach the jump bid level had there not been the jump bid; i.e., say the price of item k jumps by Δ_k and the speed for the price clock is $\dot{p}_k$, then the maximum duration of the pause is equal to $\Delta_k/\dot{p}_k$. The endogeneity of the duration of a pause prevents bidders from slowing down an auction by submitting smaller and smaller "jump" bids, though bidders have no incentive to do that in the equilibrium constructed here.

a2. If all but the highest jump-bidder have dropped out during the pause, the auction ends without finishing the δ seconds and the highest jump-bidder buys the good at a price equal to his jump bid.
a3. All jump bids are commonly observed even if they are submitted simultaneously.
a4. If there are multiple active bidders at the end of the pause, the price clock resumes from the highest jump bid.

A6. A *tie* occurs if a bidder drops out from an auction and all the other currently active bidders drop out from the auction either at the same instant or during the pause triggered by the dropout action. The rule to break such ties is:

a1. Each bidder involved in a tie chooses whether to *concede*.
a2. If all but one bidders concede, then sell the object to the one who does not concede at the current price; if at least two bidders do not concede, then resume the price clock and let those who do not concede become active again.
a3. If all bidders concede, then one of them is selected randomly with equal probabilities and the selected one buys the good at its current price.

6. Avoiding exposure via jump bidding

In the simultaneous auctions reformulated in Section 5, a *decisive period* starts when: (i) in one of the auctions, all the remaining active local bidders have just dropped out, (ii) at least one local bidder is still active in the other auction, and (iii) the global bidder is still active in both auctions. Then the auction described in (i) is in a *paused phase* as its price clock pauses according to amendment A4 defined in Section 5; hence we call it *paused auction*. The auction described in (ii), by contrast, is still in an *active phase*, hence called *active auction*.

Proposition 3. *Assume that* (i) *no local bidder bids for an item above its value for the bidder and* (ii) *at the start of a decisive period with current price vector (p_A, p_B), the posterior support of any x-local bidder i's type $u_i(x)$ has infimum p_x $(\forall x \in \{A, B\})$ and the posterior support of global bidder's $u_\gamma(AB)$ has infimum $p_A + p_B$. Then in this decisive period there exists a continuation equilibrium in which it occurs almost surely that the global bidder knows whether he can profitably acquire both items before he buys any item.*

The idea is that the global bidder γ can, before committing to buy the item in the paused auction, find out the eventual price of its complement by making a jump bid in the active auction. Like an equilibrium bid in a first-price auction, bidder γ's jump bid fully reveals his maximum willingness to pay in the active auction given that he is to win the paused auction at the paused price. Then the local bidders with lower valuations immediately drop out, and those with higher valuations immediately respond with jump bids. They prefer to signal their types through jump bids because γ's maximum willingness to pay would jump if he has made a purchase commitment in the paused auction. If all local bidders drop out, bidder γ wins in the active auction and pays his jump bid. Else bidder γ learns that the price will be too high for him to acquire both items profitably, hence he immediately stops bidding for both items and, if he wants, drops out from the paused auction and concedes the good to the local bidder whose dropout triggered the pause. Hence the global bidder suffers no exposure problem.

6.1. The interim types and jump bids during the pause

Consider a decisive period. Without loss of generality, let the paused auction be the auction of item A (briefly *auction A*), with the price paused at p_A. Let us calculate the global bidder γ's valuation of winning in the other auction, the auction for item B (briefly *auction B*), during the pause of auction A. The lemmas in this subsection are proved in Appendix B.

If $u_\gamma(A) \geqslant p_A$, bidder γ's decision is straightforward. He would immediately resume auction A, thereby ending the pause and buying item A. The profit from buying A at this price is nonnegative whether γ will be able to acquire B or not, due to inequality (1). Having bought A, bidder γ's valuation of winning B becomes $u_\gamma(AB) - u_\gamma(A)$, which will be his dropout price in auction B.

The case of $u_\gamma(A) < p_A$ is more complicated, which includes the following two subcases: If $u_\gamma(B) \geqslant u_\gamma(AB) - p_A$, bidder γ's optimal action is to bid for item B alone:

Lemma 1. *If $u_\gamma(B) \geqslant u_\gamma(AB) - p_A$, then global bidder γ prefers buying item B alone to buying both items or buying A alone.*

If $u_\gamma(B) < u_\gamma(AB) - p_A$, bidder γ wants to acquire both items up to a certain point:

Lemma 2. *If $u_\gamma(B) < u_\gamma(AB) - p_A$ and $u_\gamma(A) < p_A$, then it is dominated for global bidder γ to buy item B alone and, during the pause, it is profitable for him to buy item B if and only if it is profitable for him to buy both items, i.e., if and only if $u_\gamma(AB) - p_A > p_B$.*

Thus, the only nontrivial case for bidder γ during the pause of auction A is "$u_\gamma(B) < u_\gamma(AB) - p_A$ and $u_\gamma(A) < p_A$." In this case, the global bidder's maximum willingness to pay for item B during the pause, by Lemma 2, is equal to

$$t_\gamma := u_\gamma(AB) - p_A. \tag{4}$$

Also denote

$$t_i := u_i(B) \quad \forall i \neq \gamma. \tag{5}$$

Call t_i the *interim type* of bidder i for any i still active during the pause of auction A.[6]

For every bidder i active at the start of the pause, initialize G_i to be the distribution function of t_i, derived from F_{iA}, F_{iB} and $F_{\gamma C}$, conditional on the history of the game up to the start of the pause. Denote T_i for the support of G_i, $G_{-i} := (G_j)_{j \neq i}$,

$$t_{-i}^{(1)} := \max\{t_j \colon\ j \neq i\},$$

and $T_{-i}^{(1)}$ for the support of $t_{-i}^{(1)}$. For any $t_i \in T_i$, define

$$\beta_{i,G_{-i}}(t_i) := \mathbb{E}_{t_{-i}^{(1)}}\big[t_{-i}^{(1)} \bigm| t_{-i}^{(1)} \leqslant t_i; G_{-i}\big], \tag{6}$$

i.e., the expected value of the highest rival's interim type conditional on its not exceeding i's interim type, given the distributions G_{-i}. Let

$$\beta_{i,G_{-i}}^{-1}(x_i) := \big\{t_i \in T_i \colon\ \beta_{i,G_{-i}}(t_i) = x_i\big\}.$$

Lemma 3. *The function $\beta_{i,G_{-i}}$ is weakly increasing. Furthermore, if $\inf T_i \geqslant \inf T_{-i}^{(1)}$ and $\beta_{i,G_{-i}}^{-1}(x_i) \neq \emptyset$, then for any $j \neq i$ and almost every t_j (relative to G_j),*

$$t_j \leqslant \inf \beta_{i,G_{-i}}^{-1}(x_i) \quad or \quad t_j > \sup \beta_{i,G_{-i}}^{-1}(x_i). \tag{7}$$

6.2. The proposed jump-bidding equilibrium

Starting from the beginning of the pause of auction A at the paused price p_A:

1. If (global) bidder γ drops out from auction A during the pause, the local bidder i whose dropout triggered the pause does not concede item A to γ.
2. The strategy of bidder γ is:
   a. if $u_\gamma(A) \geqslant p_A$,
      i. immediately resume auction A, thereby buying A and ending the pause,
      ii. and bid for item B if and only if the price p_B of B is below $u_\gamma(AB) - u_\gamma(A)$;
   b. if $u_\gamma(A) < p_A$ and $u_\gamma(B) \geqslant u_\gamma(AB) - p_A$,
      i. drop out from auction A immediately,
      ii. concede A to the local bidder(s) whose dropout triggered the pause,
      iii. continue in auction B as long as $u_\gamma(B) > p_B$;
   c. if $u_\gamma(A) < p_A$ and $u_\gamma(B) < u_\gamma(AB) - p_A$,
      i. immediately submit a jump bid equal to $\beta_{\gamma,G_{-\gamma}}(t_\gamma)$ for item B,
      ii. during the pause of auction B triggered by the jump bid,
         A. if every local bidder i either drops out or does not respond with a jump bid x_i such that $\sup \beta_{i,G_{-i}}^{-1}(x_i) > t_\gamma$, then bidder γ immediately takes actions 2(a)i and 2(a)ii,
         B. if some local bidder i submits a jump bid x_i such that $\inf \beta_{i,G_{-i}}^{-1}(x_i) \geqslant t_\gamma$, then bidder γ i mmediately drops out of auction B and takes actions 2(b)i and 2(b)ii,
         C. if some local bidder i makes a jump bid x_i such that $\inf \beta_{i,G_{-i}}^{-1}(x_i) < t_\gamma < \sup \beta_{i,G_{-i}}^{-1}(x_i)$, then bidder γ plays strategy b in the equilibrium presented in Proposition 1.
3. The strategy of any active local bidder i (in the active auction B) is:
   a. unless bidder γ has made a jump bid, stay in auction B without jump bidding and drop out at $p_B = u_i(B)$,
   b. if bidder γ has made a jump bid x_γ during the pause of auction A,
      i. if $t_i \leqslant \inf \beta_{\gamma,G_{-\gamma}}^{-1}(x_\gamma)$, drop out immediately,
      ii. else immediately make a jump bid equal to $\beta_{i,G_{-i}}(t_i)$, with G_{-i} being the posterior distributions updated by bidder γ's jump bid,
         A. if $t_i \leqslant \inf \beta_{j,G_{-j}}^{-1}(x_j)$ given the jump bid x_j from some local bidder $j \neq i$, drop out immediately unless $t_i = \beta_{j,G_{-j}}^{-1}(x_j) = \max_k \beta_{k,G_{-k}}^{-1}(x_k)$, in which case i drops out if and only if $i < j$ (so that some local bidder stays),
         B. else stay in auction B without jump bidding and drop out at $p_B = u_i(B)$.

[6] Although global bidder γ's private information has three dimensions, $u_\gamma(A)$, $u_\gamma(B)$, and $u_\gamma(AB)$, his behavior is tractable because, given one auction pausing and the other auction expected to finish within the pause, his private information is reduced to only one dimension, $u_\gamma(AB) - p_A$.

6.3. The proof of Proposition 3

First, we show that if everyone abides by the proposed equilibrium then global bider γ almost surely resolves his price uncertainty.

By assumption (ii) of the proposition and Eqs. (4)–(5), $\inf T_i = p_B = \inf T_\gamma$ for any active B-local bidder i. Hence the condition $\inf T_i \geqslant \inf T_{-i}^{(1)}$ of Lemma 3 holds. Thus, from bidder γ's jump bid $\beta_{\gamma,G_{-\gamma}}(t_\gamma)$, every local bidder i can almost surely tell (i) $t_\gamma \geqslant t_i$ apart from (ii) $t_\gamma < t_i$. In case (i), bidder i immediately drops out (plan 3(b)i). In case (ii), bidder i immediately makes a jump bid $\beta_{i,G_{-i}}(t_i)$ based on the updated posteriors G_{-i} (plan 3(b)ii).

If all local bidders belong to case (i), bidder γ wins item B at the known price $\beta_{\gamma,G_{-\gamma}}(t_\gamma)$ before the pause of auction A ends. Since $\beta_{\gamma,G_{-\gamma}}(t_\gamma) \leqslant t_\gamma$ by Eq. (6), the bidder knows, by Eq. (4) that defines t_γ, that it is profitable for him to buy both items.

If some local bidder i belongs to case (ii), upon seeing i's jump bid, bidder γ knows that almost surely $t_\gamma < t_i \leqslant t_{-\gamma}^{(1)}$ and hence the price for B will be greater than t_γ if bidder γ does not drop out (plan 3(b)iiB). Thus, by Eq. (4), bidder γ knows that it is almost surely unprofitable for him to buy both items. Again he learns that during the pause of auction A. (Note that contingency 2(c)iiC occurs with zero probability on the path.)

6.3.1. The incentive for contingency plans 1, 2a, and 2b

Plan 1 follows from the fact that each local bidder's profit from buying item A is nonnegative, as his dropout price does not exceed his standalone value of the item, according to assumption (i) of the proposition.

Plan 2a has been justified by the second paragraph of Section 6.1.

In the contingency for plan 2b, Lemma 1 applies, hence bidder γ would take actions 2(b)i and 2(b)ii, dropping out from auction A and conceding A to the local bidders. Since the local bidders do not concede (plan 1), bidder γ frees himself from any obligation of buying A. With only item B to consider, the optimality of plan 2(b)iii is obvious.

6.3.2. The global bidder's incentive for contingency plan 2c

Under the contingency of plan 2c, $u_\gamma(B) < u_\gamma(AB) - p_A$ and $u_\gamma(A) < p_A$, so Lemma 2 applies, and bidder γ either buys both items if the payoff from doing so is positive, or buys neither of them if the payoff is negative.

Plan 2(c)iiA: In this case, bidder γ learns that for every active B-local bidder i, $t_i \leqslant \sup \beta_{i,G_{-i}}^{-1}(x_i) \leqslant t_\gamma$, where the first inequality follows from γ's expectation that i abides by plan 3(b)ii. Then Eq. (4) implies $u_\gamma(AB) - p_A \leqslant \max_i t_i$ and so γ's payoff from buying both items is positive. Thus, it is optimal for bidder γ to buy both items by taking actions 2(a)i and 2(a)ii according to plan 2(c)iiA.

Plan 2(c)iiB: In this case, $\inf \beta_{i,G_{-i}}^{-1}(x_i) \geqslant t_\gamma$. Expecting that other players abide by the jump-bidding function (plan 3(b)ii), bidder γ learns that $t_\gamma \leqslant t_{-\gamma}^{(1)}$. Then p_B will be greater than or equal to t_γ if bidder γ does not drop out (plan 3(b)iiB). Hence it is optimal for bidder γ to drop out of both auctions by following plan 2(c)iiB.

Plan 2(c)iiC: In this case, bidder γ is uncertain about whether $u_\gamma(AB) \geqslant p_A + \max_i t_i$ holds or not, as in the case analyzed in Section 3. With B-local bidders staying in the auction up to their values (3(b)iiB), it is a best response for γ to follow the equilibrium strategy in Section 3.

Claim. Under the contingency of plan 2c, bidder γ prefers making a jump bid to not doing so. If he does not jump bid, the local bidders will stay without jump bidding until the price reaches their values (plan 3a), hence the continuation play is the equilibrium presented in Section 3. His expected payment upon winning in this continuation equilibrium is the same as the one if he jump-bids and wins. That is because in both cases p_A has been fixed and the price for item B will be equal to $t_{-i}^{(1)}$ in expectation. The events in which bidder γ wins, however, are different in the two cases. If bidder γ makes a jump bid equal to $\beta_{\gamma,G_{-\gamma}}(t_\gamma)$, the event where he wins is exactly the event where his profit is positive conditional on winning. By contrast, if γ does not jump-bid and hence follows the equilibrium in Section 3, the event where he wins is not aligned with the event where his profit is positive conditional on winning, because overdiffusion and overconcentration are both probable (Proposition 2). Thus, bidder γ would rather jump-bid according to the proposed equilibrium.

Lemma 4. *If bidder γ is to make a jump bid, his optimal jump bid is equal to $\beta_{\gamma,G_{-\gamma}}(t_\gamma)$.*

Proof. First observe, by definition of interim types, $t_\gamma - p_B = u_\gamma(AB) - p_A - p_B$ is equal to bidder γ's payoff if he wins in auction B at price p_B during the pause of auction A.

Second, observe that making a jump bid is equivalent to the action of picking a point $\hat{t}_\gamma \in T_\gamma$ and submitting it to a proxy, who on γ's behalf submits the jump bid $\beta_{\gamma,G_{-\gamma}}^{-1}(\hat{t}_\gamma)$, which almost surely fully reveals $\hat{t}_\gamma$. We claim that it is optimal for bidder γ to pick $\hat{t}_\gamma = t_\gamma$.

If $t_{-\gamma}^{(1)} \leqslant \hat{t}_\gamma$, then bidder γ wins immediately (plan 3(b)i) and pays the jump bid.

Else, $t_{-\gamma}^{(1)} > \hat{t}_\gamma$. Then bidder γ cannot win immediately. Given local bidders' response 3(b)iiB, if γ does not drop out, the best he can hope for is that he wins if and only if $t_{-\gamma}^{(1)} \leqslant t_\gamma$ and he pays $t_{-\gamma}^{(1)}$ upon winning. (He cannot do better than that if the price uncertainty is not resolved within the pause of auction A.)

Both cases combined, bidder γ's expected payoff from $\hat{t}_\gamma$ is less than or equal to

$$\begin{aligned}
&\text{Prob}\{t^{(1)}_{-\gamma} \leqslant \hat{t}_\gamma\}\big(t_\gamma - \beta_{\gamma,G_{-\gamma}}(\hat{t}_\gamma)\big) + \mathbb{E}_{t^{(1)}_{-\gamma}}\big[\big(t_\gamma - t^{(1)}_{-\gamma}\big)\mathbf{1}_{\hat{t}_\gamma < t^{(1)}_{-\gamma} \leqslant t_\gamma}\big] \\
&\overset{(6)}{=} \text{Prob}\{t^{(1)}_{-\gamma} \leqslant \hat{t}_\gamma\}\big(t_\gamma - \mathbb{E}_{t^{(1)}_{-\gamma}}\big[t^{(1)}_{-\gamma} \bigm| t^{(1)}_{-\gamma} \leqslant \hat{t}_\gamma\big]\big) + \mathbb{E}_{t^{(1)}_{-\gamma}}\big[\big(t_\gamma - t^{(1)}_{-\gamma}\big)\mathbf{1}_{\hat{t}_\gamma < t^{(1)}_{-\gamma} \leqslant t_\gamma}\big] \\
&= t_\gamma\,\text{Prob}\{t^{(1)}_{-\gamma} \leqslant t_\gamma\} - \mathbb{E}_{t^{(1)}_{-\gamma}}\big[t^{(1)}_{-\gamma}\big(\mathbf{1}_{t^{(1)}_{-\gamma} \leqslant \hat{t}_\gamma} + \mathbf{1}_{\hat{t}_\gamma < t^{(1)}_{-\gamma} \leqslant t_\gamma}\big)\big] \\
&\overset{(6)}{=} \text{Prob}\{t^{(1)}_{-\gamma} \leqslant t_\gamma\}\big(t_\gamma - \beta_{\gamma,G_{-\gamma}}(t_\gamma)\big),
\end{aligned}$$

which is equal to bidder γ's expected payoff from picking $\hat{t}_\gamma = t_\gamma$. Thus, it is optimal for bidder γ to reveal his interim type t_γ truthfully by submitting the jump bid $\beta_{\gamma,G_{-\gamma}}(t_\gamma)$. □

Therefore, we have shown bidder γ's incentive to follow plan 2c.

6.3.3. A local bidder's incentive for contingency plan 3

The optimality of plan 3a is obvious, since the global bidder does not jump bid. Plan 3(b)iiB is always optimal. Plan 3(b)iiA is a best reply because bidder i has learned that some other local bidder $j \neq i$ has a higher value than he does; if i does not drop out, then local bidder j follows plan 3(b)iiB and the price for item B will be higher than the value for i.

Suppose that we are in the contingency of plan 3b, i.e., bidder γ has made a jump bid x_γ. We shall prove the optimality of the plan.

First, plan 3(b)i is a best reply for any active B-local bidder i, because under the contingency of 3(b)i bidder γ's jump bid x_γ has revealed to i that $t_i \leqslant \inf\beta^{-1}_{\gamma,G_{-\gamma}}(x_\gamma) \leqslant t_\gamma$. If bidder i deviates by not dropping out immediately, either some other local bidder j makes a jump bid signaling that $t_j > \inf\beta^{-1}_{\gamma,G_{-\gamma}}(x_\gamma)$ and hence $t_j > t_i$, or global bidder γ buys item A immediately (plan 2(c)iiA) thereby raising γ's highest bid for item B from $u_\gamma(AB) - p_A$ to $u_\gamma(AB) - u_\gamma(A)$ (since $u_\gamma(A) < p_A$). In either case, bidder i cannot win with positive profits.

Second, consider the contingency under plan 3(b)ii, i.e., bidder γ's jump bid reveals to local bidder i that $t_i > \inf\beta^{-1}_{\gamma,G_{-\gamma}}(x_\gamma)$. Then local bidder i does not want to drop out, because there is a positive probability with which i wins with positive profits. We shall verify plan 3(b)ii by establishing the following two claims.

Claim 1. Conditional on not dropping out, a local bidder i prefers making a jump bid that signals $t_i > \inf\beta^{-1}_{\gamma,G_{-\gamma}}(x_\gamma)$ to not doing so. If none of the local bidders do so, the global bidder buys item A immediately (plan 2(c)iiA); consequently, γ's maximum willingness to pay for item B jumps from $u_\gamma(AB) - p_A$ to $u_\gamma(AB) - u_\gamma(A)$ (since the fact that γ has made a jump bid implies that γ is in the contingency of plan 2c, so $u_\gamma(A) < p_A$) and hence reducing bidder i's winning probability, which by the envelope formula reduces i's expected payoff. Thus, bidder i prefers to submit a jump bid that signals $t_i > \inf\beta^{-1}_{\gamma,G_{-\gamma}}(x_\gamma)$.

Claim 2. If a local bidder i is to respond with a jump bid, it is optimal for i to make a jump bid equal to $\beta_{i,G_{-i}}(t_i)$ given the updated posterior G_{-i}. The proof parallels that of Lemma 4 with the substitutions $\hat{t}_\gamma \to \hat{t}_i$, $t^{(1)}_{-\gamma} \to t^{(1)}_{-i}$, and $G_{-\gamma} \to G_{-i}$. If $t^{(1)}_{-i} \leqslant \hat{t}_i$, then bidder i wins immediately because the global bidder immediately drops out by plan 2(c)iiB and the other local bidders immediately drop out by plan 3(b)iiA. Else, $t^{(1)}_{-i} > \hat{t}_i$. Then bidder i cannot win immediately. If he does not drop out, the price he needs to pay in order to win will be at least $t^{(1)}_{-i}$: if $t^{(1)}_{-i} = t_j$ for some local bidder j, j stays (plan 3(b)iiB); if $t^{(1)}_{-i} = t_\gamma$, global bidder γ buys item A and bids even higher for B (plan 2(c)iiA). Thus, the calculation in the proof of Lemma 4 follows.

By Claims 1 and 2, plan 3(b)ii is a best reply for any active B-local bidder i under the contingency of 3(b)ii.

7. Overconcentration

If the continuation equilibrium in Proposition 3 is expected, the global bidder in the simultaneous auctions can bid for both items without the risk of negative profits, and he has no price uncertainty when buying any item. That allows the global bidder to continue bidding for the package $\{A, B\}$ until he knows that its total price will exceed the combined value $u_\gamma(AB)$.

The local bidders, by contrast, may choose to stop bidding for an item before its price reaches the bidder's valuation. For instance, when an A-local bidder becomes the only one competing with the global bidder for item A while the auction for item B is still going on, if the local bidder drops out at a price p_A, he may still get to buy item A at price p_A because the B-local bidders may outbid the global bidder, who would then concede item A to this A-local bidder. Hence a local bidder may free ride the local bidders in the other auction. (This is the *threshold problem* typically attached to package auctions.) While the local bidders cannot overcome the incentive constraint for them to cooperate fully in their competition against the global bidder, the global bidder can overcome the exposure problem due to the jump-bidding signals. That suggests an overcentration prediction.

While the intuition for this prediction is compelling, its proof is complicated because the simultaneous-auction game may admit multiple equilibria. Some could involve jump bidding and signaling prior to any decisive period. There are also tacitly collusive equilibria where some bidder submits a high bid regardless of his valuation and the others drop out at zero price. Hence we restrict attention to those equilibria that satisfy the following conditions.

Condition 1. The continuation equilibrium presented in Proposition 3 is played once any decisive period starts unless the assumption in Proposition 3 is not satisfied.

According to the auction rules defined in Section 5, we say a bidder i active in an auction *quits* if and only if i drops out from the auction and, in the event that all other bidders active in the auction also drop out during the pause triggered by the dropout, concedes the item to other bidders. By rule A6 in Section 5, quitting results in losing in the auction with a strictly positive probability; once i has quit an auction, the bidder either loses in the auction or buys the item at its current price, with the latter happening only if all other bidders active in the auction also quit during the pause and bidder i wins the tie-breaking lottery.

Condition 2. At any node of the game with current price vector (p_A, p_B), for each $x \in \{A, B\}$, if $p_x > u_i(x)$ for x-local bidder i then i quits auction x; if $p_A + p_B > u_\gamma(AB)$ then bidder γ quits at least one auction.

Condition 3. At any node of the game when a decisive period has not started and the current price vector is (p_A, p_B), for any bidder i (local or global), if $S \subseteq \{A, B\}$ is the set of items for which bidder i is still active then the posterior about u_i is the prior distribution conditional on the event $u_i(S) \geqslant \sum_{x \in S} p_x$.

Condition 1 is clearly needed in our context. Conditions 2 and 3 guarantee respectively the no-overbidding and identical-infimum assumptions in Proposition 3. Thus, by Condition 1, the jump-bidding continuation equilibrium is played during any decisive period. Condition 3 also shuts down the signaling effect of jump bids unless they are made during a decisive period, while leaving intact their effect of manipulating the pace of the auction. Conditions 2 and 3 together help to bypass part of the multiple-equilibrium complication.

Lemmas 5–7 characterize the perfect Bayesian equilibria (PBE) of the simultaneous-auction game that satisfy the above conditions. They are proved in Appendix C.

Lemma 5. *In any PBE satisfying Conditions 2–3, at any node when a decisive period has not started, with current price vector (p_A, p_B), the following claims are true*:

a. *if every active bidder i is active in only one auction, denoted $x(i)$, then any active bidder i continues if $p_{x(i)} < u_i(x(i))$ and quits if $p_{x(i)} > u_i(x(i))$;*
b. *if Condition 1 is also satisfied, then in any auction $x \in \{A, B\}$ where there are at least two active local bidders, an active local bidder i quits auction x only if $p_x \geqslant u_i(x)$.*

Lemma 6. *In any PBE satisfying Conditions 1–3, at any node when a decisive period has not started and the current price vector is (p_A, p_B), the global bidder γ continues in both auctions if $p_A + p_B < u_\gamma(AB)$ and quits at least one auction if $p_A + p_B > u_\gamma(AB)$.*

Lemma 7. *In any PBE satisfying Conditions 1–3, overdiffusion occurs with zero probability and, if the equilibrium allocation is not for sure ex post efficient, then overconcentration occurs with a strictly positive probability.*

The overconcentration prediction could be explained with the assumption that every active bidder stays passively in an auction without jump bidding as long as a decisive period has not started. Suppose that only one A-local bidder i remains active while the global bidder γ is competing with him and several active B-local bidders. Since bidder γ stays in both auctions until the total price reaches $u_\gamma(AB)$ (Lemma 6), to rule out overconcentration we need each local bidder to stay until the price reaches his valuation. But when the current price p_A is sufficiently close to i's valuation $u_i(A)$, bidder i strictly prefers dropping out to continuing. If i drops out, thereby triggering a decisive period, the jump-bidding continuation equilibrium is played (Condition 1) and there is a strictly positive probability say π with which some B-local bidder outbids global bidder γ, who then concedes item A to bidder i at its current price p_A, giving i an expected payoff $(u_i(A) - p_A)\pi$. By contrast, if bidder i continues until the price reaches $u_i(A)$, he gets a nonzero payoff only if the following event happens during the interval when the price of item A ascends from p_A to $u_i(A)$: either global bidder γ quits, or all the B-local bidders stop thereby triggering a decisive period. Since prices ascend continuously according to the clock, the probability of this event is in the order of $O(u_i(A) - p_A)$, hence bidder i's expected payoff from abiding by the equilibrium is in the order of only $o(u_i(A) - p_A) < (u_i(A) - p_A)\pi$, a contradiction.

The problem of this heuristic argument is that an equilibrium price trajectory does not necessarily follow the pace of the price clocks passively. With possible jump bids, big changes in p_B may happen even when the increment of p_A is tiny. Our proof uses a mechanism-design method that is robust to such dynamic details.

Proposition 4. *In any PBE satisfying Conditions 1–3, overconcentration occurs with a strictly positive probability and overdiffusion occurs with zero probability.*

Proof. Pick any equilibrium specified by the hypothesis of the proposition. By Lemma 7, overdiffusion occurs with zero probability, and it suffices the overconcentration claim to prove that the equilibrium is not for sure ex post efficient.

Suppose, to the contrary, that the equilibrium is for sure ex post efficient. Then by the payoff equivalence theorem (Milgrom, 2004), every local bidder's expected payment at the equilibrium is equal to his expected payment in the corresponding Vickrey mechanism where his ex post payment is equal to the *spillover* due to the bidder, i.e., the total payoff difference for the other bidders (global and local) due to his presence and type. To check that the theorem is applicable, first note that the ex post efficiency of the equilibrium implies that its allocation is the same as in the Vickrey mechanism. Second, note that the payoff for any x-local bidder ($x \in \{A, B\}$) is zero if $u_i(x) = 0$, whether in this equilibrium or in the Vickrey mechanism with the spillover payment scheme: In this equilibrium, with zero valuation the bidder has zero winning probability and pays zero according to the English auction format; in the Vickrey mechanism, with zero valuation the bidder's presence makes no difference to other bidders' payoffs and hence generates zero spillover.

Let I_A denote the set of all A-local bidders, and I_B that of all B-local bidders, so that $I = I_A \cup I_B \cup \{\gamma\}$. Given any profile of realized valuations, $u := (u_i)_{i \in I}$, let $P_i^{\mathrm{e}}(u)$ denote local bidder i's payment in the equilibrium and $P_i^{\mathrm{vic}}(u)$ the bidder's Vickrey payment defined above. Denote $P^{\mathrm{e}} := \sum_{i \in I_A \cup I_B} P_i^{\mathrm{e}}$ and $P^{\mathrm{vic}} := \sum_{i \in I_A \cup I_B} P_i^{\mathrm{vic}}$. We shall prove a contradiction by showing

$$\mathbb{E}_u\big[P^{\mathrm{vic}}(u)\big] < \mathbb{E}_u\big[P^{\mathrm{e}}(u)\big]. \tag{8}$$

The ex post efficiency supposition of the equilibrium also implies that a local bidder does not quit when the current price is still less than his valuation. Otherwise, with the global bidder continuing in both auctions up to the level $p_A + p_B = u_\gamma(AB)$ (Lemma 6), overconcentration occurs with a strictly positive probability.

Any realized profile u of valuations across bidders determines a trajectory T_u of the vector (p_A, p_B) of the prices in the two auctions when they run their courses according to the equilibrium. According to the bidding behavior characterized in the previous paragraph, both auctions continue until the price trajectory hits one of these hyperplanes:

$$H_A := \Big\{(p_A, p_B)\colon\ p_A = \max_{I_A} u_i(A)\Big\};$$

$$H_B := \Big\{(p_A, p_B)\colon\ p_B = \max_{I_B} u_i(B)\Big\};$$

$$H_C := \big\{(p_A, p_B)\colon\ p_A + p_B = u_\gamma(AB)\big\}.$$

Next we compare $P^{\mathrm{vic}}(u)$ with $P^{\mathrm{e}}(u)$ in every possible case of the equilibrium price trajectory.

Case 1. T_u hits hyperplane H_A, with current price vector (p_A, p_B). Then the last A-local bidder say i_* drops out, auction A pauses at price $p_A = \max_{I_A} u_i(A)$, and $u_{i_*}(A) = \max_{I_A} u_i(A)$ by the efficiency supposition. The dropout triggers a decisive period and the jump-bidding continuation equilibrium defined in Section 6.2 (Condition 1). We analyze each possible subcase in the equilibrium, parallel to the contingencies for the global bidder's strategy (plan 2) defined there (except the zero-probability contingency 2(c)iiC).

Subcase 1(a). Without making any jump bid, global bidder γ buys A and continues in auction B. This corresponds to contingency 2a in the jump-bidding equilibrium: $u_\gamma(A) \geqslant p_A$ and γ bids for B up to the price $u_\gamma(AB) - u_\gamma(A)$. Thus, in equilibrium, A-local bidders pay zero as they do not win A, and a B-local bidder j wins if any only if $u_j(B)$ is above

$$\max\Big\{u_\gamma(AB) - u_\gamma(A), \max_{I_B \setminus \{j\}} u_j(B)\Big\}, \tag{9}$$

which is also his payment if he wins. The Vickrey payments are exactly the same. A-local bidders' Vickrey payment is zero because none of them is pivotal to the outcome of the efficient allocation. B-local bidders' Vickrey payment is nonzero if and only if the highest-value B-local bidder say j has higher value than $u_\gamma(AB) - u_\gamma(A)$ and in that case the spillover due to j is equal to (9). Thus, $P^{\mathrm{vic}}(u) = P^{\mathrm{e}}(u)$.

Subcase 1(b). Without making any jump bid, global bidder γ quits auction A and continues in auction B. This corresponds to contingency 2b in the jump-bidding equilibrium: $u_\gamma(B) \geqslant u_\gamma(AB) - p_A$ and $u_\gamma(A) < p_A$, so bidder γ bids for B up to the price $u_\gamma(B)$. Thus, in equilibrium, the A-local bidder i_* wins item A at the price p_A and a B-local bidder j wins if and only if $u_j(B)$ is greater than $\max_{I_B\cup\{\gamma\}\setminus\{j\}} u_j(B)$, which is also his payment if he wins. The Vickrey payment from the B-local bidders is exactly the same as their equilibrium payment. However, the case for the A-local bidders is different. The A-local bidder i_* is pivotal to the outcome of the efficient allocation, and his spillover is equal to

$$\begin{aligned}
&\max\Big\{\max_{i\in I_A\setminus\{i_*\}} u_i(A), u_\gamma(AB) - \max_{I_B\cup\{\gamma\}} u_j(B), u_\gamma(A)\Big\} \\
&\quad\leqslant \max\Big\{\max_{i\in I_A\setminus\{i_*\}} u_i(A), u_\gamma(AB) - u_\gamma(B), u_\gamma(A)\Big\} \\
&\quad\leqslant \max_{I_A} u_i(A) = p_A,
\end{aligned}$$

where the last inequality is due to the fact $u_\gamma(B) \geqslant u_\gamma(AB) - p_A$ and $u_\gamma(A) < p_A$. Thus,

$$\mathbb{E}_u\big[P^{\text{vic}}(u) \bigm| \text{case 1(b)}\big] \leqslant \mathbb{E}_u\big[P^{\text{e}}(u) \bigm| \text{case 1(b)}\big].$$

Subcase 1(c). After jump bidding, global bidder γ buys item A and continues in auction B. This corresponds to contingency 2(c)iiA in the jump-bidding equilibrium: after making a jump bid, bidder γ learns from the B-local bidders' reactions that he can acquire both items profitably; so in equilibrium both items go to γ and $P^{\text{e}}(u) = 0$. Since the equilibrium is assumed efficient, no local bidder is supposed to win in the allocation, so none is pivotal to the outcome of the allocation, hence $P^{\text{vic}}(u) = 0 = P^{\text{e}}(u)$.

Subcase 1(d). After jump bidding, global bidder γ quits in both auctions A and B. This is contingency 2(c)iiB in the jump-bidding equilibrium. In that contingency,

$$u_\gamma(A) < p_A = \max_{I_A} u_i(A), \tag{10}$$

$$u_\gamma(B) < u_\gamma(AB) - p_A = u_\gamma(AB) - \max_{I_A} u_i(A). \tag{11}$$

In equilibrium, bidder γ's jump bid fully reveals his interim type $u_\gamma(AB) - p_A$ (Lemma 3); any B-local bidder j with valuation greater than this interim type responds with a jump bid, which by Eq. (6) is equal to the expected value of $t^{(1)}_{-j}$, the highest interim type of j's rivals,

$$t^{(1)}_{-j} = \max\Big\{u_\gamma(AB) - p_A, \max_{k\in I_B\setminus\{j\}} u_k(B)\Big\}, \tag{12}$$

conditional on the event that $t^{(1)}_{-j} \leqslant u_j(B)$. Since bidder γ quits both auctions in this subcase, such a B-local bidder j exists and his jump bid signals to bidder γ that it is unprofitable for γ to acquire both items; then (10)–(11) imply that it is unprofitable for γ to acquire any item. Thus, in equilibrium, the A-local bidder i_* wins item A at the price

$$p_A = \max_{I_A} u_i(A) = P^{\text{e}}_{i_*}(u), \tag{13}$$

and a B-local bidder j with $t^{(1)}_{-j} \leqslant u_j(B)$ wins item B at j's jump bid

$$\mathbb{E}_{t^{(1)}_{-j}}\big[t^{(1)}_{-j} \bigm| t^{(1)}_{-j} \leqslant u_j(B)\big] = P^{\text{e}}_j(u). \tag{14}$$

Now we calculate the Vickrey payments. Since bidder i_* is the only remaining A-local bidder when the price trajectory hits the hyperplane H_A, i_* is pivotal to the outcome of the efficient allocation and his spillover is equal to

$$P^{\text{vic}}_{i_*}(u) = \max\Big\{u_\gamma(AB) - \max_{I_B\cup\{\gamma\}} u_j(B), u_\gamma(A)\Big\}. \tag{15}$$

For any B-local bidder j, if $t^{(1)}_{-j} > u_j(B)$ then j is not pivotal to the outcome of the efficient allocation and hence his spillover equals zero; if $t^{(1)}_{-j} < u_j(B)$ then j's spillover is equal to

$$\begin{aligned}
P^{\text{vic}}_j(u) &= \max\Big\{u_\gamma(AB) - \max_{I_A\cup\{\gamma\}} u_i(A), \max_{k\in I_B\setminus\{j\}} u_k(B), u_\gamma(B)\Big\} \\
&\overset{(10)}{=} \max\Big\{u_\gamma(AB) - \max_{I_A} u_i(A), \max_{k\in I_B\setminus\{j\}} u_k(B), u_\gamma(B)\Big\} \\
&\overset{(11)}{=} \max\Big\{u_\gamma(AB) - \max_{I_A} u_i(A), \max_{k\in I_B\setminus\{j\}} u_k(B)\Big\} \\
&\overset{(12)}{=} t^{(1)}_{-j}.
\end{aligned} \tag{16}$$

The expected value of B-local bidders' total equilibrium payment is equal to that of their total Vickrey payment:

$$\begin{aligned}
\mathbb{E}_u\left[\sum_{j\in I_B} P_j^{\mathrm{e}}(u) \,\middle|\, \text{case 1(d)}\right] &= \sum_{j\in I_B} \mathbb{E}_{t_{-j}^{(1)}}\left[P_j^{\mathrm{e}}(u)\mathbf{1}_{t_{-j}^{(1)}\leqslant u_j(B)} \,\middle|\, \text{case 1(d)}\right] \\
&\overset{(14)}{=} \sum_{j\in I_B} \mathbb{E}_{t_{-j}^{(1)}}\left[\mathbb{E}_{t_{-j}^{(1)}}\left[t_{-j}^{(1)} \,\middle|\, t_{-j}^{(1)} \leqslant u_j(B)\right]\mathbf{1}_{t_{-j}^{(1)}\leqslant u_j(B)} \,\middle|\, \text{case 1(d)}\right] \\
&= \sum_{j\in I_B} \mathbb{E}_{t_{-j}^{(1)}}\left[t_{-j}^{(1)}\mathbf{1}_{t_{-j}^{(1)}\leqslant u_j(B)} \,\middle|\, \text{case 1(d)}\right] \\
&\overset{(16)}{=} \sum_{j\in I_B} \mathbb{E}_{t_{-j}^{(1)}}\left[P_j^{\mathrm{vic}}(u)\mathbf{1}_{t_{-j}^{(1)}\leqslant u_j(B)} \,\middle|\, \text{case 1(d)}\right] \\
&= \mathbb{E}_u\left[\sum_{j\in I_B} P_j^{\mathrm{vic}}(u)\mathbf{1}_{t_{-j}^{(1)}\leqslant u_j(B)} \,\middle|\, \text{case 1(d)}\right] \\
&= \mathbb{E}_u\left[\sum_{j\in I_B} P_j^{\mathrm{vic}}(u)\mathbf{1}_{j \text{ is pivotal}} \,\middle|\, \text{case 1(d)}\right]. \qquad (17)
\end{aligned}$$

Whereas, A-local bidders' total equilibrium payment is greater than their total Vickrey payment. That is because for almost every profile u of realized valuations in subcase 1(d)

$$\max\left\{u_\gamma(AB) - \max_{I_B\cup\{\gamma\}} u_j(B), u_\gamma(A)\right\} < \max_{I_A} u_i(A).$$

The $u_\gamma(A)$ on the left-hand side is strictly less than $\max_{I_A} u_i(A)$ by inequality (10). The $u_\gamma(AB) - \max_{I_B\cup\{\gamma\}} u_j(B)$ on the left-hand side is less than $\max_{I_A} u_i(A)$, otherwise both items should have gone to global bidder γ in the efficient allocation. The inequality is strict unless $u_\gamma(AB) - \max_{I_B\cup\{\gamma\}} u_j(B) = \max_{I_A} u_i(A)$, which is of zero measure and hence zero probability due to the no-gap no-atom property of the posteriors assured by Condition 3 and the no-atom assumption of the priors. Thus, Eqs. (13) and (15) imply $P_{i_*}^{\mathrm{vic}}(u) < P_{i_*}^{\mathrm{e}}(u)$ almost surely in subcase 1(d). This combined with Eq. (17) implies

$$\mathbb{E}_u\left[P^{\mathrm{vic}}(u) \,\middle|\, \text{case 1(d)}\right] < \mathbb{E}_u\left[P^{\mathrm{e}}(u) \,\middle|\, \text{case 1(d)}\right].$$

Case 2. T_u hits hyperplane H_B. This is symmetric to Case 1, with A and B switching roles.

Case 3. T_u hits hyperplane H_C, with current price vector (p_A, p_B). Now that $p_A + p_B = u_\gamma(AB)$, in equilibrium global bidder γ quits in at least one auction immediately (Lemma 6). If bidder γ continues in auction x with $x \in \{A, B\}$, then since he has quit the other auction, bidder γ bids for x up to its standalone valuation, hence the total Vickrey and equilibrium payments made by x-local bidders are identical; if $u_\gamma(x)$ is not a maximum of

$$\{u_k(x)\colon k \in I_x \cup \{\gamma\}\}$$

then the x-local bidders' Vickrey/equilibrium payment is equal to the second highest in the set; else the payment is equal to zero. Thus, we need only to compare the payments in the auction where bidder γ quits immediately. Without loss of generality, let it be the auction for item A. There are two possibilities:

Subcase 3(a). Only one A-local bidder say i_* remains active when T_u hits hyperplane H_C. Then bidder i_* is pivotal to the outcome of the efficient allocation, and his spillover is equal to

$$\begin{aligned}
P_{i_*}^{\mathrm{vic}}(u) &= \max\left\{u_\gamma(AB) - \max_{I_B\cup\{\gamma\}} u_j(B), u_\gamma(A)\right\} \\
&\leqslant \max\{u_\gamma(AB) - p_B, u_\gamma(A)\} \\
&= \max\{p_A, u_\gamma(A)\} \\
&\leqslant p_A,
\end{aligned}$$

where the first inequality follows from the fact that in equilibrium, as long as a decisive period has not started, B-local bidders quit unless p_B has not exceeded their valuations (Condition 2), and the second inequality is due to the fact $u_\gamma(A) \leqslant p_A$ (otherwise bidder γ would not have quit auction A immediately). Since bidder i_* wins item A and pays the price p_A at equilibrium, we have $P_{i_*}^{\mathrm{vic}}(u) \leqslant P_{i_*}^{\mathrm{e}}(u)$ and hence $P^{\mathrm{vic}}(u) \leqslant P^{\mathrm{e}}(u)$ in this subcase.

Subcase 3(b). At least two A-local bidders remain active when T_u hits hyperplane H_C. Then the total Vickrey and equilibrium payments made by the A-local bidders are identically equal to the second highest element in $\{u_i(A)\colon\ i \in I_A\}$.

Summing across Cases 1 to 3 yields inequality (8), a contradiction as desired. □

8. Conclusion

Due to the transparency of English auctions, the simultaneous auctions analyzed above may admit multiple equilibria. An open question is whether there exists an equilibrium where the exposure problem is not eliminated or mitigated. The answer may vary with the details of the model. For example, when the last A-local bidder drops out, the global bidder may somehow expect that the remaining B-local bidders would never jump bid, hence he would buy good A immediately without waiting for their signals. In that case, the exposure problem is not mitigated at all. However, if we slightly modify the model so that no one can buy the item in a paused auction until the exogenous δ-second pause expires, then one can show that this exposure-prone equilibrium cannot survive, because a remaining B-local bidder would have time to signal through jump bids.

If the model is modified to allow a local bidder to bid for the item commonly known to have zero value to him, a new aspect of the exposure problem may emerge. For instance, it may happen that only a B-local bidder remains to compete with the global bidder and the local bidder is bidding for *both* items. When the B-local bidder drops out from auction A, she has no incentive to not concede the item to the global bidder. Hence the global bidder may not be able to withdraw his obligation of buying item A when he knows that he cannot profitably acquire item B. While the global bidder may preempt such a risk by threatening to concede item A to the B-local bidder when its price reaches a certain threshold, the construction of such an equilibrium is outside the scope of this paper.

The main point of this paper is not that the exposure problem can be mitigated if the decentralized simultaneous auctions are modified to allow jump bidding, but rather that models capturing dynamic details of English auctions may allow us to incorporate new self-enforcing arrangements in which economic agents signal and forecast prices without relying on any central coordination.

Appendix A. The proof of Proposition 2

Without loss of generality, suppose there are only three distinct bidders, an A-local bidder α, a B-local bidder β, and the global bidder γ. Consider the trajectory of the prices determined by the price clock. Since the price of each item is assumed to ascend from zero in a constant speed, every point (p_A, p_B) on the trajectory follows the equation

$$p_B = \lambda p_A$$

for some constant $\lambda > 0$. Consider the event defined by the following inequalities:

$$\lambda u_\gamma(A) > u_\gamma(B), \tag{18}$$

$$u_\gamma(AB) - u_\gamma(A) > \lambda u_\gamma(A), \tag{19}$$

$$v_\gamma\big(B, u_\gamma(A)\big) > \lambda u_\gamma(A). \tag{20}$$

This event is nonempty since it contains u_γ such that $u_\gamma(B) = 0$, $u_\gamma(A) = \epsilon \approx 0$ and $\mathbb{E}_{\tilde{u}_\alpha(A)}[(u_\gamma(AB) - \tilde{u}_\alpha(A))^+] > \lambda\epsilon$. Furthermore, the event is of strictly positive probability because the above strict inequalities define an open subset in the support of the no-gap distribution of type profiles (the subset is open because $v_\gamma(B, u_\gamma(A))$, by Eq. (3), is continuous in $u_\gamma(AB)$ and $u_\gamma(A)$). We claim that, in this event, overconcentration and overdiffusion are each possible with strictly positive probability.

By (2)–(3) and (18)–(20), we can prove the following facts, depicted in Fig. 1.

$$u_\gamma(AB) - u_\gamma(A) > v_\gamma\big(B, u_\gamma(A)\big) > \lambda u_\gamma(A) > u_\gamma(B), \tag{21}$$

$$u_\gamma(AB) - u_\gamma(B) > v_\gamma\big(A, u_\gamma(B)\big) > u_\gamma(A), \tag{22}$$

$$v_\gamma\big(A, u_\gamma(AB) - u_\gamma(A)\big) = u_\gamma(A), \tag{23}$$

$$v_\gamma\big(B, u_\gamma(AB) - u_\gamma(B)\big) = u_\gamma(B). \tag{24}$$

By the continuity of $v_\gamma(A, \cdot)$ and $v_\gamma(B, \cdot)$, (21)–(24) imply that, for any u_γ satisfying (18)–(20), there is a unique (p'_A, p'_B), with $p'_A > u_\gamma(A)$ and $p'_B > \lambda u_\gamma(A)$, that solves

$$p'_B = \lambda p'_A \quad \text{and} \quad v_\gamma\big(A, p'_B\big) = p'_A$$

and there exists a unique (p''_A, p''_B), with $p''_A > u_\gamma(A)$ and $p''_B > \lambda u_\gamma(A)$, that solves

$$p''_B = \lambda p''_A \quad \text{and} \quad v_\gamma\big(B, p''_A\big) = p''_B.$$

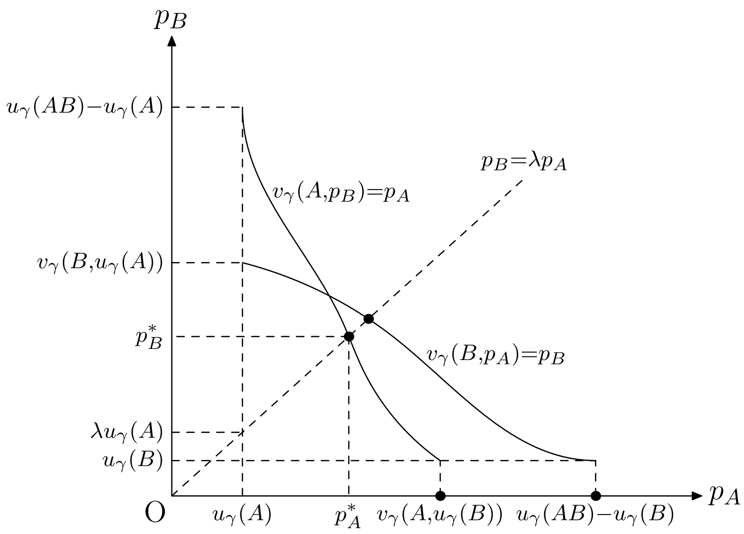

Fig. 1. Bidding in the clock model.

By the above definition of (p'_A, p'_B) and (p''_A, p''_B) and (21)–(24),

$$u_\gamma(AB) > \max\{p'_A + p'_B, p''_A + p''_B\}, \tag{25}$$

$$\min\{p'_A, p''_A\} > u_\gamma(A) \quad \text{and} \quad \min\{p'_B, p''_B\} > u_\gamma(B). \tag{26}$$

Define

$$(p^*_A, p^*_B) := (\min\{p'_A, p''_A\}, \lambda \min\{p'_A, p''_A\}).$$

According to provision (b) of the equilibrium strategy, global bidder γ drops out from at least one auction when the price vector (p_A, p_B) reaches (p^*_A, p^*_B). Thus, as local bidders continue up to their values $u_\alpha(A)$ and $u_\beta(B)$, global bidder γ quits at least one auction if

$$u_\alpha(A) > p^*_A \quad \text{and} \quad u_\beta(B) > p^*_B; \tag{27}$$

if (27) is not satisfied then γ wins in at least one auction and bids for the other item up to its marginal value. Inequalities (25) and (26) respectively imply

$$u_\gamma(AB) > p^*_A + p^*_B, \tag{28}$$

$$p^*_A > u_\gamma(A) \quad \text{and} \quad p^*_B > u_\gamma(B). \tag{29}$$

Overdiffusion. By (28), there is an event of strictly positive probability where both

$$u_\gamma(AB) > u_\alpha(A) + u_\beta(B)$$

and (27) hold. This event projected onto the $(u_\alpha(A), u_\beta(B))$-plane corresponds to the dark area in Fig. 2. Due to (27), the global bidder fails to acquire both items at equilibrium, yet $u_\gamma(AB) > u_\alpha(A) + u_\beta(B)$ implies that he should acquire both items according to the efficiency criterion. Hence overdiffusion occurs in this event.

Overconcentration. By (29), there is an event of strictly positive probability where

$$u_\alpha(A) < p^*_A, \tag{30}$$

$$u_\beta(B) < u_\gamma(AB) - u_\gamma(A), \tag{31}$$

$$u_\gamma(AB) < u_\alpha(A) + u_\beta(B). \tag{32}$$

This event projected onto the $(u_\alpha(A), u_\beta(B))$-plane corresponds to the gray area in Fig. 2. Inequality (30) means that at equilibrium the A-local bidder quits before the price trajectory reaches (p^*_A, p^*_B), so item A is acquired by the global bidder, who subsequently bids for item B up to the price $u_\gamma(AB) - u_\gamma(A)$ according to the provision (d) of the equilibrium strategy. Then (31) implies that he eventually outbids the B-local bidder thereby acquiring both items at equilibrium. Inequality (32) means that the global bidder should not have both items according to the efficiency criterion. Hence overconcentration occurs in this event.

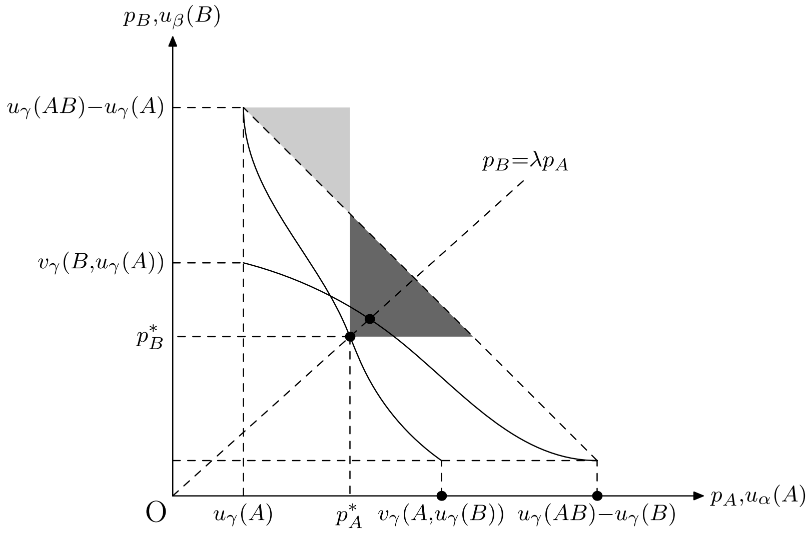

Fig. 2. Grey area: overconcentration; dark area: overdiffusion.

Appendix B. The proofs of Lemmas 1–3

Proof of Lemma 1. Let p_B denote the eventual price for item B. By $u_\gamma(B) \geqslant u_\gamma(AB) - p_A$,

$$u_\gamma(B) - p_B \geqslant u_\gamma(AB) - p_A - p_B. \tag{33}$$

Then bidder γ would rather bid only for item B than bid for both items. Furthermore, coupled with (1), inequality (33) implies that $u_\gamma(B) - p_B \geqslant u_\gamma(A) + u_\gamma(B) - p_A - p_B$, hence $0 \geqslant u_\gamma(A) - p_A$, so bidder γ does not want to buy item A alone. □

Proof of Lemma 2. By the hypothesis $u_\gamma(B) < u_\gamma(AB) - p_A$,

$$u_\gamma(B) - p_B < u_\gamma(AB) - p_A - p_B,$$

hence the payoff from buying both items is greater than the payoff from buying item B alone. Also, it is unprofitable to buy A alone since $u_\gamma(A) < p_A$. Thus, during the pause, bidder γ chooses between two goals: (i) to buy both items, which if realized would yield a payoff equal to $u_\gamma(AB) - p_A - p_B$; (ii) to buy none, which if realized would yield zero payoff. Thus, during the pause, the valuation of winning item B is equal to the valuation of buying both items, i.e., $u_\gamma(AB) - p_A - p_B$, as asserted. □

Proof of Lemma 3. By definition (6), the function $\beta_{i,G_{-i}}$ is weakly increasing. Suppose it is not strictly increasing, then for some $t_i, t_i' \in T_i$ with $t_i < t_i'$, $\beta_{i,G_{-i}}(t_i) = \beta_{i,G_{-i}}(t_i')$. Then (6) implies that $t_{-i}^{(1)}$ has zero mass in $(t_i, t_i']$, i.e.,

$$\prod_{j \neq i} G_j(t_i') = \prod_{j \neq i} G_j(t_i). \tag{34}$$

Note $t_i \geqslant \inf T_i \geqslant \inf T_{-i}^{(1)}$. Hence for any $x > t_i$, $\prod_{j\neq i} G_j(x) > 0$. Thus,

$$\forall x > t_i \; \forall j \neq i \quad G_j(x) > 0. \tag{35}$$

Pick any $j \neq i$. If $t_j \in (t_i, t_i']$, then (35), applied to $x = t_j$, implies that there is a positive probability with which t_j is the realized $t_{-i}^{(1)}$. Then (34) implies $\text{Prob}\{t_j\colon\ t_i < t_j \leqslant t_i'\} = 0$, hence (7). □

Appendix C. The proofs of Lemmas 5–7

Proof of Lemma 5. Claim (a): Since every active bidder is active in only one auction, the game at this node has degenerated into two separate single-object English auctions. By Condition 2, x-local bidder i quits when $p_x > u_i(x)$. We show that bidder i does not quit when $p_x < u_i(x)$. To show that, first observe that in equilibrium the following event occurs with zero probability: if an active bidder quits then all other bidders active in the auction also quit during the pause triggered by his dropout. If the event happens in equilibrium, then a bidder say j for whom $u_j(x) > p_x$ would rather stay to win and obtain a positive payoff $u_j(x) - p_x$ than abide by the equilibrium that gives him only a fraction of this positive payoff (rule A6.a3 in Section 5). Thus, if bidder i quits now then he almost surely loses the auction and gets zero payoff, because all other bidders active in the auction have valuations higher than the current price almost surely, by Condition 3. By contrast, if he continues, bidder i has a strictly positive probability to win with a strictly positive expected payoff because the posterior distributions of his active rivals are nondegenerate by Condition 3.

Claim (b): If bidder i loses now, he gets zero payoff. By contrast, continuing gives him a strictly positive expected payoff due to nondegenerate posterior beliefs (Condition 3). Thus, quitting at this node is strictly suboptimal if quitting results in losing for sure. It does result in losing for sure, because not all the other active local bidders in the auction drop out during the pause triggered by i's dropout. Suppose by negation that they all drop out. Then a decisive period starts and the jump-bidding continuation equilibrium is played by Condition 1 (coupled with Conditions 2 and 3, which guarantee the assumptions for Proposition 3). If the global bidder buys the good then they all get zero payoff; if the global bidder concedes the item, then these local bidders have to split the winning probability. Hence one of the other local bidders strictly prefers to deviate by continuing thereby increasing his probability of winning by a strictly positive quantity. □

Proof of Lemma 6. Pick any perfect Bayesian equilibrium satisfying Conditions 1, 2 and 3. Consider any node of the game such that a decisive period has not started. If $p_A + p_B > u_\gamma(AB)$ then global bidder γ quits at least one auction, by Condition 2.

Suppose that $p_A + p_B < u_\gamma(AB)$. Let us compare global bidder γ's alternatives and show that the unique best response for γ is to continue in both auctions.

If he drops out from both auctions then bidder γ gets zero payoff. That is because the active local bidders almost surely have valuations above the current prices (Condition 3) and hence they do not quit now (Lemma 5.a).

For any profile $u_{-\gamma}$ of realized valuations across bidders other than γ and for each item $x \in \{A, B\}$, denote $u^{(1)}(x)$ for the highest valuation among all x-local bidders, i.e.,

$$u^{(1)}(x) := \max_{I_x} u_i(x).$$

By Lemma 5.a, if global bidder γ drops out from auction $x \in \{A, B\}$ and continues in the other auction $-x$, then his expected payoff is

$$\mathbb{E}_{u^{(1)}(-x)}\big[\big(u_\gamma(-x) - u^{(1)}(-x)\big)^+ \mid p_A, p_B\big]. \tag{36}$$

Suppose bidder γ continues for both items until either $p'_A + p'_B \geqslant u_\gamma(AB)$ or a decisive period begins. If a decisive period starts at price vector (p'_A, p'_B) with auction x paused at some price $p'_x \geqslant p_x$, then the jump-bidding continuation equilibrium constructed in Proposition 3 is played (Condition 1), and bidder γ almost surely knows the eventual price $u^{(1)}(-x)$ of the other item $-x$ before he commits to buying any item, hence his ex post payoff is

$$\max\big\{\big(u_\gamma(x) - p'_x\big)^+, \big(u_\gamma(-x) - u^{(1)}(-x)\big)^+, \mathbb{E}_{\tilde{u}^{(1)}(-x)}\big[\big(u_\gamma(AB) - p'_x - \tilde{u}^{(1)}(-x)\big)^+ \mid p'_{-x}\big]\big\},$$

where the third item is due to the fact that bidder γ submits the jump bid

$$\mathbb{E}_{\tilde{u}^{(1)}(-x)}\big[\tilde{u}^{(1)}(-x) \bigm| u_\gamma(AB) - p'_x \geqslant \tilde{u}^{(1)}(-x);\, p'_{-x}\big]$$

according to the jump-bidding continuation equilibrium. If $p'_A + p'_B \geqslant u_\gamma(AB)$ before any decisive period starts, bidder γ has to quit from at least one of the auctions (Condition 2) and his payoff from that standpoint is equal to

$$\max\big\{\mathbb{E}_{u_{-\gamma}}\big[\big(u_\gamma(A) - u^{(1)}(A)\big)^+ \mid p'_A\big], \mathbb{E}_{u_{-\gamma}}\big[\big(u_\gamma(B) - u^{(1)}(B)\big)^+ \mid p'_B\big]\big\}$$

according to Lemma 5.a. Thus, bidder γ's expected payoff from continuing at the current node is equal to a convex combination between

$$\mathbb{E}_{u_{-\gamma},p'}\left[\max\left\{\begin{array}{c}\big(u_\gamma(x) - p'_x\big)^+, \big(u_\gamma(-x) - u^{(1)}(-x)\big)^+, \\ \mathbb{E}_{\tilde{u}^{(1)}(-x)}\big[\big(u_\gamma(AB) - p'_x - \tilde{u}^{(1)}(-x)\big)^+ \mid p'_{-x}\big]\end{array}\right\} \,\middle|\, p_A, p_B\right] \tag{37}$$

and

$$\mathbb{E}_{u_{-\gamma},p'}\big[\max\big\{\mathbb{E}_{u_{-\gamma}}\big[\big(u_\gamma(A) - u^{(1)}(A)\big)^+ \mid p'_A\big], \mathbb{E}_{u_{-\gamma}}\big[\big(u_\gamma(B) - u^{(1)}(B)\big)^+ \mid p'_B\big]\big\} \mid p_A, p_B\big]. \tag{38}$$

Since the posterior distributions of the active local bidders at the current node are all nondegenerate (Condition 3), by Jensen's inequality and the fact that $\max\{x, y, z\}$ is a convex function of (x, y, z), (36) is strictly less than (37) and strictly less than (38). □

Proof of Lemma 7. Take any equilibrium satisfying Conditions 1, 2 and 3. As long as a decisive period has not started, the bidding behavior is: Every A-local bidder continues bidding for A as long as the current price is less than its value to the bidder and there is at least one other active A-local bidder, and likewise for the B-auction (Lemma 5); the global bidder γ continues in both auctions if and only if $p_A + p_B < u_\gamma(AB)$ (Lemma 6). Thus, any item sold before any decisive period starts is allocated efficiently.

Once a decisive period has started, say auction A has paused at price p_A, there are only three possibilities, parallel to the contingencies for the global bidder's strategy (plan 2 defined in Section 6.2) in the jump-bidding continuation equilibrium that is played according to Condition 1: (i) without any jump bid, bidder γ buys item A and bids for B up to $u_\gamma(AB) - u_\gamma(A)$; (ii) without any jump bid, γ quits from auction A and bids for B up to $u_\gamma(B)$; or (iii) γ makes a jump bid and either acquires both items or quits from at least one auction. According to the jump-bidding equilibrium, cases (i) and (ii) each end with an efficient outcome.

Consider case (iii). If γ has quit only one auction say auction A, then in the continuation equilibrium he bids for item B up to $u_\gamma(B)$ and so item B is allocated efficiently. If γ quits both auctions, the two items, auctioned separately to single-item bidders, are again allocated efficiently in the continuation equilibrium; furthermore, in the jump-bidding continuation equilibrium, bidder γ quits both auctions because he has learned that

$$u_\gamma(AB) \leqslant p_A + \max_{I_B} u_j(B) \leqslant \max_{I_A} u_i(A) + \max_{I_B} u_j(B)$$

where the second inequality is due to the fact that an A-local bidder has been active up to the price p_A and hence his valuation is no less than p_A (Condition 2); hence overdiffusion is impossible in this subcase. If γ acquires both items in the jump-bidding continuation equilibrium, then overdiffusion is again impossible.

Thus, the only possible kind of inefficiency is overconcentration. □

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