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High Bids and Broke Winners[1]

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This paper analyzes auctions where budget-constrained bidders have options to declare bankruptcy. It predicts a bidding equilibrium that changes discontinuously in a borrowing rate available to bidders. When the borrowing rate is above a threshold, high-budget bidders win, and the likelihood of bankruptcy is low. When the borrowing rate is below the threshold, the winner is the most budget-constrained bidder and is most likely to declare bankruptcy. This result explains the "high bids and broke winners" anomaly in the C-Block FCC spectrum auction. Based on its equilibrium analysis, the paper proves that a seller can profit from offering to finance the highest bidder at a below-market interest rate, even with default risk. *Journal of Economic Literature* Classification Numbers: D44, D45, D82, G33, L96.

Key Words: auction; budget; default; spectrum auction; C-block auction; FCC; PCS.

1. INTRODUCTION

Auction theory is mostly confined to cash-only auctions with "deep-pocket" bidders who cannot default. Such a restriction makes it difficult to analyze auctions of high-stake objects. In these auctions, the worth of a good is large compared to the wealth of a bidder, so paying bids through financing is prevalent, and defaults often occur. Shortly after the $19 billion self-off of Brazil's telecommunication giant Telebras (July 29, 1998, Mehta [21]), one of its winners has already begun renegotiating about the down-payment (August 4, 1998, Bloomberg [4]). The C-block disaster, where the U.S. government sold a block of radio frequencies for $10 billion (Spring 1996) to firms who later defaulted, has yet to close its final chapter in

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bankruptcy courts (detailed later). In auctions of high-stake projects, bidders' liquidity constraints cannot be ignored. This aspect, however, is what most of auction theory has neglected.

This paper therefore analyzes a first-price sealed-bid auction where budget-constrained bidders can default on their bids. The equilibrium analysis shows that budget constraints and default risk together have a deep impact on bidding behavior, seller's profit, and the likelihood of bankruptcy.

Disasters in spectrum auctions. This paper was inspired by the 1996 C-block radio frequency spectrum auction conducted by the United States Federal Communications Commission (FCC). In this auction, the FCC auctioned off the licenses for using the radio frequencies within a C-block spectrum. Unlike in previous spectrum auctions, the FCC in this auction allowed winning bidders to delay their payments at a below-market borrowing rate. The winning bids of the C-block auctions thus totaled $10.2 billion. On a per-consumer basis, this was almost three times as high as the prices in previous spectrum auctions of frequencies in the A- and B-blocks. While Congress, occupied by the balanced budget issue at that time, quickly counted that amount as a source of income, few of the winners of the C-block auction made their payments. Many of them declared bankruptcy, including NextWave, Pocket Communications, General Wireless, and Airadigm Communications. Others lobbied for lighter payment terms. The FCC has collected almost none of the payment pledged by the C-block bidders. In March 1999, the FCC raised only about $400 million in a re-auction of the licenses returned by the C-block licensees. However, most licenses of the defaulted firms remain wrapped up in the litigation of bankruptcy and appeals. Frustrated, the FCC has lobbied Congress to change bankruptcy laws.[2]

Point of departure. It is clear from the above example that financial constraints and default risk are important in major privatization auctions.

[2] Up to April 1998, the FCC had offered four options to the failing C-block licensees to partially or fully return the licenses to FCC for re-auction. These options, however, attracted criticisms from both failing licensees and bidders who dropped out of the auction. In early 1999, several bankruptcy courts reduced some defaulters' payment obligations. NextWave's obligation, for example, was cut from its bid of $4.74 billion to $1.02 billion. This ruling regarding NextWave, however, was overturned by an appeals court in December 1999. In January 2000, the FCC revoked the licenses won by NextWave and planned to re-auction them in July 2000, but afterwards a bankruptcy court ruled to nullify the revocation. A new round of appeals is inevitable.

See Atlas [1] for the C-block events up to the fall of 1997. For events up to the fall of 1999, see Cunard [11], Harbert [14], Pitch [23], and *Communications Today* [12]. For the 1999 re-auction, see Chen [9]. For the FCC's attempts to affect bankruptcy legislation, see Silva [27] and Weaver [29]. For the ups and downs in the bankruptcy proceedings up to the spring of 2000, see Luna [18] and Sill and Lin [26].

In order to focus on the impact of these two elements, we formulate bidder-heterogeneity in terms of wealth instead of value.[3] Thus, we consider a single-object common-value auction environment. Each bidder has his own *budget*, which can be viewed as an amount of funds already available to him. If the bidder makes a payment p, then his's cost is equal to p plus a cost of outside financing, and this cost of outside financing is equal to a constant r times the amount by which the payment p exceeds his budget. This constant r we shall call *borrowing rate*, and we assume that it is the same to all bidders. A bidder's budget is private information to the bidder and is assumed to be independently and identically distributed across bidders.

Default risk enters as follows. The object being auctioned has uncertain value. After winning the object, the winning bidder gets to know its true value. If the bidder makes the payment, then he pays the bid and bears the cost of outside financing if the bid exceeds his budget. But the bidder may avoid the payment by declaring bankruptcy. If the winner chooses to do so, then his entire budget is taken away. For tractability, this paper assumes that the value of the auctioned object is either zero with probability θ, or a positive number v, and that both θ and v are common knowledge to the bidders and the seller.

The solution of the auction game. This paper obtains a closed-form solution of the above auction game. One novel result is that bidders with the *lowest* budgets may bid the most. More precisely, the symmetric Bayes–Nash equilibrium bidding strategy "flips" (Theorems 3.1 and 3.2). That is, a bid, as a function of the bidder's budget, is upward-sloping when the borrowing rate is above a threshold (curve *ABCD* in Fig. 1), and is downward-sloping when the borrowing rate is below the threshold (curve *EFG* in Fig. 1). In the downward-sloping case, low-budget bidders bid high and high-budget bidders bid low, the winning bid is higher than the expected value of the object, and the winner is the most budget-constrained bidder, who is most likely to declare bankruptcy. Looking at this equilibrium of "high bids and broke winners," one cannot help recalling the C-block disaster in the FCC spectrum auctions.

The intuition of this equilibrium is the following. Due to the default option, the bidders are in fact bidding an option instead of a cash payment. That is why bids may go beyond the expected value of the auctioned item. Because the penalty of bankruptcy is assumed to be proportional to a bidder's budget, low-budget bidders have less to lose from bankruptcy than high-budget bidders. That is why bidders with the lowest budgets may bid the highest. Specifically, imagine a bidder who wins by submitting a bid

[3] Such a formulation of bidder-heterogeneity is due to Che and Gale [7].

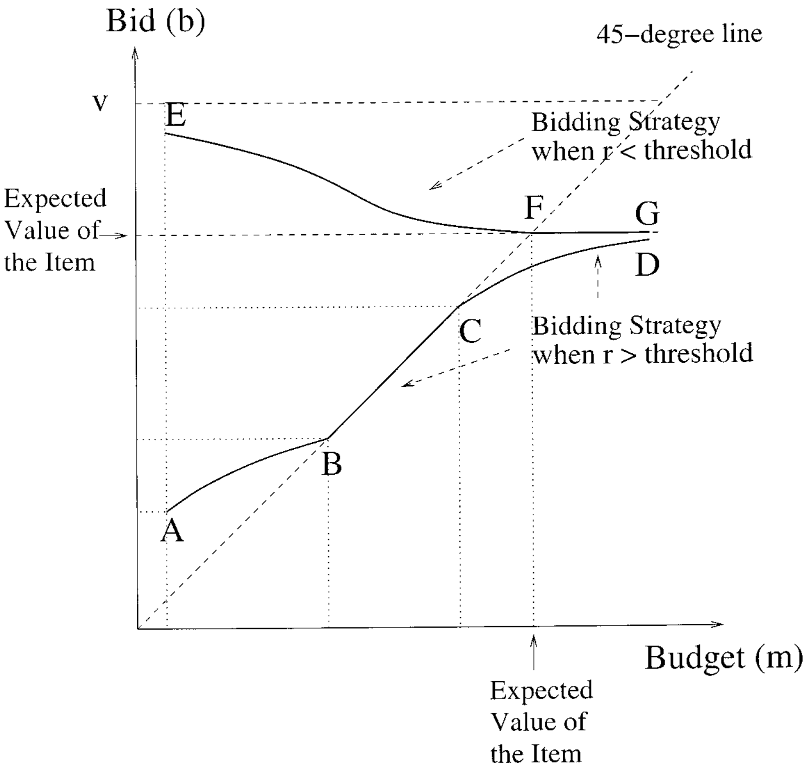

FIG. 1. The "flipping" equilibrium bidding strategy.

above his budget. The object the bidder wins has value either v or zero. In the former case, the winner may honor his bid, bearing the cost of outside financing at the borrowing rate r. In the latter case, the winner may declare bankruptcy, losing his budget. The expected cost of such a contingency plan is therefore a weighted average of the financing cost and the bankruptcy penalty. When the borrowing rate r is sufficiently low, one may neglect the financing cost and consider only the bankruptcy penalty, which is proportional to a bidder's budget by assumption. Consequently, with the borrowing rate r sufficiently low, low-budget bidders bid high while high-budget bidders bid low.

The main driving force behind this equilibrium is the dual role of the notion of budget formulated in this paper. On one hand, a bidder's budget represents his "money in the pocket"; on the other hand, the budget represents his liability. As will be clear in Sections 4.1 and 4.2, such a notion can be generated by a model that incorporates imperfect capital markets and immediate bankruptcy liquidation.

The equilibrium outcome in the other case, with borrowing rates above the threshold, is also new. It turns out that there are multiple equilibria where bids are non-monotone in budgets. In such an equilibrium, bids rise with budgets along the curve *ABC* in Fig. 1 and then go up and down instead of moving along the curve *CD*. These non-monotone bids are bounded between the bids at points *C* and *G* (Proposition 3.1 and Remark 3.2). The reason for such non-monotonicity is that the private information is a budget constraint instead of a valuation or cost. If one bids below his budget, then the constraint is non-binding and hence unable to determine the bid.

Can the seller profit from offering subsidies? The seller could alleviate bidders' financing costs, thereby intensifying their competition and raising bids. The question is whether such a manipulation is profitable to the seller. More precisely, assume that there is an *exogenous* borrowing rate q; the borrowing rate r described above is chosen by the seller. Before the auction begins, the seller commits to financing the extra fund for the highest bidder at the borrowing rate r lower than the exogenous rate q. The question is: can such a subsidy raise the seller's expected profit?

This paper proves that the answer is "yes" in some cases (Proposition 4.2). The intuition is that a lower borrowing rate makes it less costly to bid above budgets, so low-budget bidders would bid higher; to keep up with the competition, high-budget bidders have to bid higher. As long as the winner is the bidder with the highest budget, the seller bears little cost from offering such an interest subsidy. To guarantee that the winner is such a bidder, the seller needs only to offer a borrowing rate above the threshold, so that the equilibrium bidding strategy is upward-sloping (curve *ABCD* in Fig. 1).

Although the seller may profit from offering subsidies, an excessive subsidy can hurt her (Proposition 4.1). The reason is that an excessively low borrowing rate leads to the equilibrium of "high bids and broke winners." At that equilibrium, high bidders have low budgets. The winner would either declare bankruptcy (if the object is valueless) or would burden the seller with a large amount of borrowing.

Although this paper focuses on first-price sealed-bid auctions, the above result of "flipping equilibrium" (Fig. 1) also holds for second-price sealed-bid auctions. Indeed, one can easily prove that the dominant-strategy equilibrium of a second-price auction has a graph similar to that in Fig. 1. This paper focuses on first-price auctions because of their analytical challenge and their prevalence in both literature and practice. Even within the class of first-price auctions, financial constraints and default risk bring into light a rich set of auction-design problems, such as financing packages, security deposits, and bankruptcy arrangements.

Related literature. An early paper about budget-constrained bidders is by Pitchik and Schotter [22], who provide an experimental study of budget constraints in sequential auctions with perfect information. More papers on the impact of financial constraints emerged with the FCC spectrum auctions, where bids could be too large for the perfect-capital-market assumption to apply.[4] Analytical works in this field include Che and Gale [6–8], Laffont and Robert [17], and Benoît and Krishna [3]. Che and Gale [7] are the first who outline a framework with costly outside financing, from which this paper develops. Benoît and Krishna, expanding upon the multi-unit complete-information framework of Pitchik and Schotter, study bidders' strategic choice of budget constraints. Default risk is not addressed in these papers.

The impact of default risk on auctions was first noticed when researchers conducting auction experiments debated the effects of limited liability on their results (Hansen and Lott [13] and Kagel and Levin [16]). Considering a two-bidder and two-type model, Hansen and Lott [13] have noticed that overbidding can be an equilibrium behavior due to bidders' limited liability. This idea is formalized and proved in Section 3.3 of this paper. Waehrer [28] analyzes the effect of default risk, here bidders' limited liabilities are identically a security deposit chosen by the seller. In that paper, Waehrer shows that a lower deposit makes bidders more aggressive. Waehrer's model also treats the case of affiliated values, and he considers renegotiation. Another paper related to default risk is by Harstad and Rothkopf [15]. They analyze a common-value auction where the winning bidder can withdraw his bid after learning about the bids of others. Financial constraints are not addressed in these papers.

The benefit of subsidizing disadvantaged bidders is discussed by Ayres and Cramton [2]. They provide an empirical argument that the FCC substantially raised its revenue in the "regional narrowband" spectrum auction in 1994 by subsidizing minority-owned firms. The work by Rothkopf, Harstad, and Fu [24] and that by Corns and Schotter [10] show that a seller can benefit from offering discounts to low-value bidders. Che and Gale [7] notice that a seller in second-price auctions may benefit from offering below-market interest rates. A paper by Sen [25], while not about auctions, explains why sellers of consumer durables often provide below-market interest rates. My paper focuses on the cost and benefit of subsidizing financially constrained bidders, and it differs from the above papers by allowing default.

[4] For example, in their introduction to the spectrum-auction literature, McMillan, Rothschild, and Wilson [20] recognize the need for an auction theory with budget constraints.

Organization. Section 2 spells out the model. Section 3 solves the auction game. We split the solution into two cases. Section 3.3 solves the auction game when the borrowing rate is below a threshold; Section 3.4 solves the game when the borrowing rate is above that threshold. Section 3.5 compares the probabilities of bankruptcy across borrowing rates. Section 4 uses those results to analyze the seller's choice of interest subsidies.

2. A MODEL OF THE AUCTION GAME

Our analysis is based on the following auction game. One can view the auction as a subgame generated by an environment of capital markets and bankruptcy arrangements. We will delay the specification of such an environment until Section 4.1.

A seller is to auction off an indivisible good to one of n competing bidders ($n \geqslant 2$) through a first-price sealed-bid auction without a reserve price. Each bidder i ($\forall i = 1, ..., n$) has a *budget* of $m_i \geqslant 0$. Here the notion of budget has a two-fold meaning. First, it means a bidder's liquidity constraint: if bidder i makes a payment greater than m_i, then he must borrow the extra amount at a *borrowing rate* $r \geqslant 0$; in other words, the bidder's cost of making a payment of p is

$$C(p, m_i, r) := \begin{cases} p & \text{if} \quad p \leqslant m_i \\ p + r(p - m_i) & \text{otherwise.} \end{cases} \tag{1}$$

Second, a budget represents the penalty of default: if bidder i defaults after winning the auction, then the budget m_i is taken away from him. A bidder's private information is his budget. Bidders' budgets are independently and identically distributed according to a commonly known distribution function F, with density function f.

The good to be auctioned has zero value for the seller. For each bidder, the good's value is either v with probability $1 - \theta$ or zero with probability θ, where v and θ are commonly known parameters such that $v > 0$ and $\theta \in [0, 1)$. The actual value of the good is not revealed until the good is sold to a bidder. Knowing the realized value, the winner of the good decides whether to default on his bid. Specifically, the auction game proceeds as follows:

1. Each bidder submits a bid independently. The highest bidder ("winner") wins the good. Ties are resolved by a random draw with equal probability. The other bidders get zero payoff.

2. The value of the good is revealed.
3. The winner chooses whether to pay his bid b_w or to default.
   a. If he defaults, then the winner loses his entire budget m_w and gets zero in return, so his payoff is $-m_w$.
   b. Otherwise, the winner pays his bid b_w to the seller for the object. In doing so, the winner bears a cost $C(b_w, m_w, r)$ defined by Eq. (1). The winner's payoff is then $\mathbf{v} - C(b_w, m_w, r)$, where $\mathbf{v}$ is the actual value of the auctioned item.

The auction game is then over.

For convenience, we assume that the support of the distribution function F of budgets is $[\underline{m}, \bar{m}]$. Here $\bar{m}$ can be a nonnegative real number or infinity. (When $\bar{m}$ equals infinity, we abuse the notation $[\underline{m}, \bar{m}]$ to mean $[\underline{m}, \infty)$.) We further assume the following properties of the distribution. Due to Assumption 1, a bidder's budget matters. Assumption 2 is a typical technical assumption in auction literature. Assumption 3 is weaker than the standard monotone-hazard-rate assumption in the auction literature. We will use it to obtain the budget-revealing equilibrium of the auction game (Section 3.4).

Assumption 1. $\underline{m} < (1-\theta)\, v < \bar{m}$.

Assumption 2. F has a probability density function f that is continuous and positive at every point in $[\underline{m}, \bar{m}]$.

Assumption 3. For all $x \in (\underline{m}, \bar{m})$, $\frac{d}{dx}\left[\frac{F(x)}{f(x)}\right] > 1 - n$.

Assumption 3 is satisfied by any uniform or exponential distributions (including any truncation of exponential distributions). Notice that this assumption implies that every expression $x + \frac{(1-\theta)(1+r)}{n-1}\frac{F(x)}{f(x)}$, with $\theta \in [0, 1)$ and $r \geqslant 0$, is strictly increasing in $x \in (\underline{m}, \bar{m})$.

2.1. *The Interpretation of the Model*

The auction game modeled above roughly corresponds to the actual timing of the C-block auction. Before the auction began, the FCC announced a ten-year installment payment plan for winning bidders. The plan offered a generous borrowing rate pegged to the 30-year Treasury bond. The auction took place in the spring of 1996. Bidders were mainly start-ups planning to launch their business in the digital wireless industry, an industry mostly unprecedented at that time. After the auction, the firms that won licenses signed contracts with the FCC according to the installment payment plan, and they began to build up their projects by hiring labor and equipment.

Afterwards, these firms began to realize the profitability of their projects. Since the spring of 1997, bankruptcies occurred among many of these firms.

The most important concept in the model is the notion of budget, which represents both a bidder's liquidity constraint and his liability. As will be clear in Section 4.1, such a notion can be generated by a model that incorporates imperfect capital markets and immediate bankruptcy liquidation. Before that section, we need only to notice that the imperfection of capital markets is reflected by Eq. (1), which implies that a bidder's borrowing rate r is greater than his lending rate, normalized to zero.

This paper treats budget as an exogenous variable to a bidder. Implicitly ruled out are the cases where a bidder hides his assets from bankruptcy courts or forms a subsidiary and has the subsidiary bid instead. However, some results of this paper can be shown to be applicable to these cases, if a bidder must commit to a chosen budget before bidding.

Section 4.1 will also model the destination of the good in case of default and the payment arrangement of an insolvent bidder. Until that section, we abstract away these aspects for the ease of exposition.

2.2. *Solution Concept*

A bidder's strategy is a mapping from budgets to bids. Given any borrowing rate $r \geqslant 0$, a (Bayes–Nash) equilibrium of the auction game (induced by r) is a profile $(\beta_i)_{i=1}^n$, with $\beta_i\colon [\underline{m}, \bar{m}] \to [0, \infty)$ being the strategy of bidder i, such that, for each bidder i and each possible budget $m_i \in [\underline{m}, \bar{m}]$, the bid $\beta_i(m_i)$ maximizes bidder i's (von Neumann–Morgenstern) expected payoff given his budget m_i and others bidding according to β_{-i}. We restrict our attention to *symmetric equilibria*, those equilibria whose bidding strategies β_i are identical across bidder i. We further restrict our attention to those symmetric equilibria whose bidding strategies are continuous functions of budgets. We will use β to denote the bidding function in such an equilibrium. For any $A \subseteq [\underline{m}, \bar{m}]$, the function β restricted to the set A will be denoted by $\beta|_A$.

3. THE SOLUTION OF THE AUCTION GAME

This section solves the auction game for each level of the borrowing rate r. Due to the option of default, a distinctive equilibrium result is that bids need not increase with budgets. If the borrowing rate is below a threshold, low-budget bidders would bid high and high-budget bidders would bid low, and the winning bid would be higher than the expected value of the good being auctioned.

We shall begin the analysis at the winner's bankruptcy decision (Section 3.1). Section 3.2 then proves that the bids which exceed budgets may increase or decrease in budgets, depending on whether the borrowing rate is above or below a threshold. Accordingly, Section 3.3 obtains the equilibrium for borrowing rates below the threshold (curve *EFG* in Fig. 1), and Section 3.4 obtains the bidding equilibrium for those above the threshold (curve *ABCD* in Fig. 1). The solution in this case is technical. Section 3.5 contrasts the probabilities of bankruptcy between the two cases.

3.1. *The Bankruptcy Decision*

Assume from now on that a bidder does not take strictly dominated actions. Let b_w denote a winning bid and m_w denote the winner's budget. Notice that *a winner does not declare bankruptcy unless the value of the good is zero*. The reason is that a winner would not have bid b_w such that $C(b_w, m_w, r) > v$; thus, if the good has value v, the winner would get $-m_w$ if he goes bankrupt, while he could have received a nonnegative payoff if he honors the bid. Consequently, a winner's expected payoff (before the revelation of the value) is

$$(1-\theta)[v - C(b_w, m_w, r)] + \theta \max\{-m_w, -C(b_w, m_w, r)\},$$

and his bankruptcy decision results from the comparison between the two arguments within the $\max\{\cdots\}$. Rewriting the above expression according to the definition of the cost function (Eq. (1)), we derive that *a winner's expected payoff* (*before the revelation of the value*) *is* $(1-\theta)\, u(b_w, m_w, r)$, *where*

$$u(b_w, m_w, r) = \begin{cases} v - \dfrac{1}{1-\theta} b_w & \text{if } \; b_w \leqslant m_w \\ v - (1+r)\, b_w + \left(r - \dfrac{\theta}{1-\theta} \right) m_w & \text{otherwise.} \end{cases} \tag{2}$$

Since $b_w < m_w$ is equivalent to $C(b_w, m_w, r) < m_w$ (by Eq. (1)), *a winner who gets a valueless good declares bankruptcy if* $b_w > m_w$, *does not declare bankruptcy if* $b_w < m_w$, *and is indifferent about the decision if* $b_w = m_w$.

Let $\text{Prob}_s[\text{win} \mid b]$ denote the probability that a bid b is the winning bid, provided that others bid according to a bidding strategy s. Define

$$V_s(b, m, r) := u(b, m, r)\, \text{Prob}_s[\text{win} \mid b], \qquad \forall b \geqslant 0, \quad \forall m \in [\underline{m}, \bar{m}], \quad \forall r \geqslant 0. \tag{3}$$

By Eq. (2), a bidder gets an expected payoff $(1-\theta)\, V_s(b, m, r)$ from bidding b, if he has budget m and others play the strategy s.

Because the penalty for bankruptcy is proportional to one's budget, a winner with a sufficiently high budget would not declare bankruptcy in any event:

LEMMA 3.1. *For a bidder with a budget above* $(1-\theta)\,v$, *it is a dominated strategy to bid strictly above* $(1-\theta)\,v$ *or to declare bankruptcy if the bidder wins.*

Proof. Consider a bidder with budget $m > (1-\theta)\,v$. Let $\mathrm{E}[\mathbf{v}]$ denote the expected value of the auctioned item, which is equal to $(1-\theta)\,v$. If this bidder declares bankruptcy, with such a high budget $m > \mathrm{E}[\mathbf{v}]$, his payoff will fall below $-\mathrm{E}[\mathbf{v}]$. We first claim that the bidder would not declare bankruptcy if he wins. To see that, suppose to the contrary that the bidder did declare bankruptcy. Then by the statement below Eq. (2), the bidder must have bid $b \geqslant m$, so

$$u(b, m, r) < \theta(-\mathrm{E}[\mathbf{v}]) + (1-\theta)(v-b) \leqslant -\theta\mathrm{E}[\mathbf{v}] + (1-\theta)(v-\mathrm{E}[\mathbf{v}]) = 0$$

and the bid b is dominated. Thus, the bidder would not declare bankruptcy in any event, and so his gain conditional on winning the auction is either v with probability $1-\theta$ or 0 with probability θ. Therefore, the bidder would not bid more than $\mathrm{E}[\mathbf{v}]$, which is $(1-\theta)\,v$. Q.E.D

3.2. *Why the Poor Might Bid Higher Than the Rich*

Because the penalty of bankruptcy is assumed to be proportional to a bidder's budget, low-budget bidders have less to lose from bankruptcy than high-budget bidders. Thus, bidders with the lowest budgets may bid the highest. Specifically, consider a bidder's plan of bidding above his budget, honoring the bid if the good has the positive value, and declaring bankruptcy if otherwise. The ex ante cost of this contingency plan is a convex combination of financing cost and bankruptcy penalty. While the first part is high for low-budget bidders, the second part is low for them. When the borrowing rate r is sufficiently low, the first part may become negligible, so the contingency plan would be more costly to high-budget bidders than to low-budget ones. Thus, bids may increase or decrease in budgets, depending on the level of the borrowing rate. The following lemma obtains the threshold at which the equilibrium bidding strategy "flips."

LEMMA 3.2 (Strict Monotonicity). *Let* $r \geqslant 0$ *and let* $\beta\colon [\underline{m}, \bar{m}] \to R$ *be the bidding strategy of a symmetric equilibrium of the auction game induced by* r. *Suppose that* β *is continuous over some open interval* $N \subset [\underline{m}, \bar{m}]$ *and*

$\beta(m) > m$ for each $m \in N$. Then $\beta|_N$ is strictly decreasing if $r < \frac{\theta}{1-\theta}$ and strictly increasing if $r > \frac{\theta}{1-\theta}$.

Proof. Let us temporarily assume the following:

1. (*Weak Monotonicity*) Assume all the hypotheses of Lemma 3.2. Then $\beta|_N$ is weakly decreasing if $r < \frac{\theta}{1-\theta}$ and weakly increasing if $r > \frac{\theta}{1-\theta}$.
2. (*Atomless Bids*) If $\beta\colon [\underline{m}, \bar{m}] \to R$ is the bidding strategy of a symmetric equilibrium of the auction game induced by r, then there is no subset $E \subseteq [\underline{m}, \bar{m}]$ of positive probability measure such that $\beta|_E \equiv b$ for some b and $u(b, m, r) > 0$ for some $m \in E$.

Let $r < \frac{\theta}{1-\theta}$. We shall prove that $\beta|_N$ is strictly decreasing. (The proof for the case $r > \frac{\theta}{1-\theta}$ is analogous.) Suppose that $\beta|_N$ is not strictly decreasing. Then by the claim of weak monotonicity there is a nondegenerate interval $[c, d] \subseteq N$ on which β is constant. This, by the claim of atomless bids and the fact that $[c, d]$ has a positive probability measure (the distribution function F is strictly increasing), would be impossible unless $u(\beta(t), t, r) \leqslant 0$ for all $t \in [c, d]$. With β being an equilibrium strategy, $u(\beta(t), t, r) < 0$ is impossible. Neither can $u(\beta(t), t, r) \equiv 0$, because that would imply, by Eq. (2) and $\beta(t) > t$, that $\beta(t)$ is strictly decreasing in t for all $t \in [c, d]$, while β is supposed to be constant on $[c, d]$ if it is not strictly decreasing. Thus, the supposition that $\beta|_N$ is not strictly decreasing has led to a contradiction. Therefore, the proposition will be proved if the above two claims are proved. We hence prove them in the following.

Proof of weak monotonicity. Pick any $x, x' \in N$. Since β is an equilibrium strategy, the expected payoff for a bidder with budget x from bidding $\beta(x)$ cannot be lower than the bidder's expected payoff if he bids $\beta(x')$ instead. A similar relation holds for a bidder with budget x'. If x and x' are sufficiently close to each other so that $\beta(x) > x'$ and $\beta(x') > x$, then the two previous relations imply that

$$\left(r - \frac{\theta}{1-\theta}\right)(x - x')(\mathrm{Prob}_\beta[\mathrm{win} \mid \beta(x)] - \mathrm{Prob}_\beta[\mathrm{win} \mid \beta(x')]) \geqslant 0. \tag{4}$$

This inequality shows why the equilibrium bidding strategy "flips": If $r > \frac{\theta}{1-\theta}$, then $x > x'$ implies that $\mathrm{Prob}_\beta[\mathrm{win} \mid \beta(x)] \geqslant \mathrm{Prob}_\beta[\mathrm{win} \mid \beta(x')]$; with $\mathrm{Prob}_\beta[\mathrm{win} \mid \cdot]$ strictly increasing on the range of β (because F is strictly increasing and β continuous), we then have $\beta(x) \geqslant \beta(x')$. In contrast, if $r < \frac{\theta}{1-\theta}$, then $x > x'$ implies that $\mathrm{Prob}_\beta[\mathrm{win} \mid \beta(x)] \leqslant \mathrm{Prob}_\beta[\mathrm{win} \mid \beta(x')]$ and hence $\beta(x) \leqslant \beta(x')$! A standard compactness argument extends this local result to cover those x and x' that are not sufficiently near each other. ∎

Proof of atomless bids. Recall that $u(b, m, r)$ is equivalent to the payoff to the winner of the auction (up to a positive constant). This payoff is continuous in bids b. Thus, if $u(b, m, r)$ is positive, then bidding slightly higher than b would still give a positive payoff conditional on winning. At such a position, a bidder with budget m would not bid b if he or she believes that there is a positive probability for the event "others bid b." The reason is that bidding slightly higher than b would improve his or her chance to win by a positive number, while the sacrifice due to the slightly higher bid is negligible. The formal proof is merely an $\varepsilon - \delta$ version of the reasoning here. ▮

Thus, we have completed the proof for the proposition. Q.E.D

As a consequence of the above lemma, there is no symmetric Bayes–Nash equilibrium whose bidding function is continuous and strictly increasing when the borrowing rate is below the threshold $\frac{\theta}{1-\theta}$. The rest of this section therefore separately solves the auction game for two kinds of borrowing rates: those above the threshold and those below it. (The equilibrium at the threshold rate is trivial and will be discussed in Section 3.5.)

3.3. *High Bids and Broke Winners: The Equilibrium for* $r < \frac{\theta}{1-\theta}$

When the borrowing rate is $r < \theta/(1-\theta)$, Lemma 3.2 says that there is no hope to find a symmetric equilibrium with continuous and strictly increasing bidding strategy. Instead, the equilibrium obtained here exhibits the novel property that high-budget bidders bid low and low-budget bidders bid high, indeed, higher than the expected value of the good being auctioned. Furthermore, this equilibrium is the only solution when borrowing rates are below the threshold.

The Introduction has given the intuition behind this result. The main step in the derivation of this result is to find out who submits the lowest bid. First, recall the fact that bidders with sufficiently high budgets would not declare bankruptcy, since the penalty for bankruptcy is proportional to budgets. Consequently, these bidders would never bid above their budgets (Lemma 3.1). Bidders with low budgets, in contrast, would bid above their budgets due to the low borrowing rate and light bankruptcy penalty. Lemma 3.2 then implies that bids from these bidders strictly decrease in budgets. Thus, the lowest bid, which exists by the continuity of the bidding strategy, can only be submitted by high-budget bidders who bid below their budgets. Consequently, a zero-payoff argument deduces that the lowest bid is the expected value of the good. The rest of the derivation then becomes a simple task of solving differential equations.

If it is indeed true that bids decrease in budgets, then a bidder wins if his budget is the lowest among all bidders. Thus, let m^L_{-i} denote the *lowest* budget among a bidder's rivals, and let $\mathrm{E}_{m^L_{-i}}$ denote the expected-value operator on functions of the random variable m^L_{-i}. We now state the result.

THEOREM 3.1. *Let* $r \in [0, \frac{\theta}{1-\theta})$ *be the borrowing rate given to the bidders.*

1. (*Existence*) *It is an equilibrium of the auction game induced by* r *that each bidder bids according to the strategy* $\beta\colon [\underline{m}, \bar{m}] \to R$ *given by*

$$\beta(m) := \begin{cases} \mathrm{E}_{m^L_{-i}}\left[\dfrac{v + r' m^L_{-i}}{1+r} \,\middle|\, m^L_{-i} \geqslant m \right] & \text{if} \quad \underline{m} \leqslant m \leqslant (1-\theta)\, v \\ (1-\theta)\, v & \text{otherwise,} \end{cases} \tag{5}$$

where $r' := r - \frac{\theta}{1-\theta}$.

2. (*Uniqueness*) *If* $\bar{m} < \infty$, *then "every bidder plays* β*" is the only symmetric equilibrium of the auction game whose bidding strategy is continuous.*

Theorem 3.1 will be proved in Sections 3.3.1 (uniqueness) and 3.3.2 (existence). One readily sees from Eq. (5) that the equilibrium bidding strategy looks like the downward-sloping curve *EFG* in Fig. 1. It is bounded between the expected value $(1-\theta)\, v$ and the maximum value v of the good, strictly decreasing for all budgets in $[\underline{m}, (1-\theta)\, v)$, and constantly equal to $(1-\theta)\, v$ for all budgets greater than $(1-\theta)\, v$. The equilibrium bidding strategy is differentiable, with derivative

$$\beta'(m) = \begin{cases} \dfrac{r'}{1+r} \dfrac{(n-1)\, f(m)}{(1-F(m))^n} \displaystyle\int_m^{(1-\theta)\, v} (1-F(t))^{n-1}\, dt \\ \qquad \text{if} \quad \underline{m} < m < (1-\theta)\, v \\ 0 \qquad \text{if} \quad (1-\theta)\, v \leqslant m < \bar{m}. \end{cases} \tag{6}$$

One can easily prove that the bid $\beta(m)$ rises as r falls, for each budget $m \in [\underline{m}, (1-\theta)\, v)$.

3.3.1. *Deriving the "high bid" equilibrium* (*uniqueness proof*). Take any $r \in [0, \frac{\theta}{1-\theta})$ and let β be a symmetric equilibrium bidding strategy that is continuous. We will prove that β satisfies Eq. (5). That will accomplish the uniqueness proof.

LEMMA 3.3. *If* $\bar{m} < \infty$ *then* $\min \beta = (1-\theta)\, v$ *and* $\beta|_{[(1-\theta)\, v, \bar{m}]} \equiv (1-\theta)\, v$.

Proof. Let $\bar{m} < \infty$. With β a continuous function on the compact space $[\underline{m}, \bar{m}]$, there is a $z \in [\underline{m}, \bar{m}]$ such that $\beta(z) = \min \beta$. We first claim that

$\beta(z) \leqslant z$. To prove that, we need only consider two cases: either (i) $z = \bar{m}$ or (ii) $z < \bar{m}$. Lemma 3.1 has covered case (i), since $\bar{m} > (1-\theta)\, v$ by Assumption 1. In case (ii), $\beta(z) \leqslant z$ follows from Lemma 3.2 (recall that r is below the threshold) and the fact $\beta(z) = \min \beta$. Thus, $\beta(z) \leqslant z$. Since $\beta(z)$ is the lowest bid at equilibrium, a standard zero-payoff argument implies that $u(\beta(z), z, r) = 0$. (The argument uses the continuity of the functions $u(\cdot, \cdot, r)$ and β.) This, coupled with $\beta(z) \leqslant z$, implies that $\beta(z) = (1-\theta)\, v$ (Eq. (2)). Consequently, Lemma 3.1 implies that $\beta|_{[(1-\theta)\, v, \bar{m}]} \equiv (1-\theta)\, v$. The lemma is therefore proved. ▮

LEMMA 3.4. *The bidding strategy β is strictly decreasing over the interval $[\underline{m}, (1-\theta)\, v)$ and $\beta(m) > (1-\theta)\, v$ for any budget m in the interval.*

Proof. Since, at equilibrium, bidders with budgets $m < (1-\theta)\, v$ bid at least $(1-\theta)\, v$ (Lemma 3.3), they bid above their budgets. Consequently, Lemma 3.2 implies that the bidding strategy β is strictly decreasing over the interval $[\underline{m}, (1-\theta)\, v)$. Therefore, with β continuous, we have $\beta(m) > \beta((1-\theta)\, v) = (1-\theta)\, v$ for all $m \in [\underline{m}, (1-\theta)\, v)$. This proves the lemma. ▮

LEMMA 3.5. *For any $m \in [m, (1-\theta)\, v)$,*

$$\beta(m) = \frac{1}{1+r} \left[v + r'm + r' \int_m^{(1-\theta)\, v} \left(\frac{1-F(t)}{1-F(m)} \right)^{n-1} dt \right].$$

Proof. Define $\beta_1 := \beta|_{[\underline{m}, (1-\theta)\, v)}$. By Lemma 3.4, β_1 is strictly decreasing. Thus, for any b in the range of β_1, the event "b is the highest bid" is equivalent to "$\beta_1^{-1}(b)$ is lower than the lowed budget m^L_{-i} among other bidders." Let $G(\beta_1^{-1}(b))$ denote the probability of this event. Note that $G = (1-F)^{n-1}$. A bidder's objective (Eq. (3)) is thus

$$V_{\beta_1}(b, m, r) = (v - (1+r)\, b + r'm)\, G(\beta_1^{-1}(b)),$$
$$\forall b \in \text{Range}\, \beta_1, \quad \forall m \in [\underline{m}, (1-\theta)\, v). \tag{8}$$

To find the functional form of the bidding strategy β_1, we use a technique in Matthews [19]. For all m and m' in the range of β_1, the equilibrium condition implies that $V_{\beta_1}(\beta_1(m), m, r) \geqslant V_{\beta_1}(\beta_1(m'), m, r)$ and $V_{\beta_1}(\beta_1(m'), m', r) \geqslant V_{\beta_1}(\beta_1(m), m', r)$. It follows from (8) that

$$\frac{v + r'm'}{1+r} (G(m) - G(m')) \leqslant \frac{\beta_1(m)\, G(m) - \beta_1(m')\, G(m')}{m - m'}$$
$$\leqslant \frac{v + r'm}{1+r} (G(m) - G(m')).$$

Thus, $G(x)\,\beta_1(x)$ is a differentiable function of x for all $x \in (\underline{m}, (1-\theta)\,v)$, and

$$\frac{d}{dm}(G(m)\,\beta_1(m)) = -\frac{v + r'm}{1+r}\,G'(m).$$

This being true for all $m \in (\underline{m}, (1-\theta)\,v)$, we can solve this differential equation of β_1 with the boundary condition $\beta_1((1-\theta)\,v) = (1-\theta)\,v$ (Lemma 3.3). The lemma then follows. ▮

Note that Eq. (7) is equivalent to the upper branch of Eq. (5). Combining Lemmas 3.3 and 3.5, we have proved that the equilibrium bidding strategy β necessarily satisfies Eq. (5). This completes the uniqueness proof. Q.E.D

3.3.2. *Verifying the "high bid" equilibrium* (*existence proof*). This section verifies that it is an equilibrium of the auction game for each bidder to bid according to the function β defined in (5) when $r < \frac{\theta}{1-\theta}$. This proves the existence part of Theorem 3.1.

We first claim that bidding $(1-\theta)\,v$ is optimal for any $m \geqslant (1-\theta)\,v$ given that others play the strategy β. The reason for the claim is that bidding below $(1-\theta)\,v$, thereby getting zero probability of winning, is not better than bidding exactly $(1-\theta)\,v$, and bidding above $(1-\theta)\,v$ is strictly dominated by the zero bid, as Lemma 3.1 shows.

We next prove that $\beta(m)$ is optimal for each $m < (1-\theta)\,v$. Thus, pick any such m. First, any bid $b \leqslant (1-\theta)\,v$ is dominated for such an m. The reason is that (i) bidding below $(1-\theta)\,v$ is strictly dominated by bidding $(1-\theta)\,v$ and (ii) bidding $(1-\theta)\,v$ is in turn strictly dominated. The reason for claim (i) is that bidding $b < (1-\theta)\,v$ gives zero expected payoff since $(1-\theta)\,v = \min \beta$, while bidding $(1-\theta)\,v$ yields a positive expected payoff, because

$$u((1-\theta)\,v, m, r) = v - (1+r)(1-\theta)\,v + r'm = -r'[(1-\theta)\,v - m] > 0$$

by Assumption 1. The reason for claim (ii) is that the bid $(1-\theta)\,v$ is submitted by the rivals with positive probability; thus, the bidder with budget $m < (1-\theta)\,v$ does better by bidding slightly above $(1-\theta)\,v$, since $u((1-\theta)\,v, m, r) > 0$ as calculated above. Thus, an optimal bid for m, if it exists, must be greater than $(1-\theta)\,v$.

To complete the proof for the claim that $\beta(m)$ is optimal for m, therefore, we need only to prove that $\beta(m)$ maximizes $V_{\beta_1}(\cdot, m, r)$ (Eq. (8)) over $((1-\theta)\,v, \beta(\underline{m})]$, which is the range of β_1. At any interior point b of this range, the objective $V_{\beta_1}(\cdot, m, r)$ is differentiable. Letting $x := \beta_1^{-1}(b)$ (x is well defined since β_1 is strictly monotone), we have

$$D_1 V_{\beta_1}(b, m, r) = -(1+r)\, G(x) + [v - (1+r)\, \beta(x) + r'm]\, G'(x)/\beta_1'(x),$$

where the notation G is from the proof of Lemma 3.5. By Eqs. (6) and (7) and the fact $\beta_1'(x) < 0$, we obtain

$$D_1 V_{\beta_1}(b, m, r) = (\text{a positive term}) \times r'(m-x) \begin{cases} > 0 & \text{if} \quad b < \beta_1(m) \\ = 0 & \text{if} \quad b = \beta_1(m) \\ < 0 & \text{if} \quad b > \beta_1(m), \end{cases}$$

since $r' = r - \frac{\theta}{1-\theta} < 0$ and β_1 is strictly decreasing. It follows that $\beta_1(m)$ is the maximum of $V_{\beta_1}(\cdot, m, r)$ over the range of β_1. Thus, $\beta_1(m)$ is optimal for m, provided that others bid according to β. We have hence verified that β is an equilibrium bidding strategy. Q.E.D

3.4. *The Budget-Revealing Equilibrium for* $r > \frac{\theta}{1-\theta}$

We now turn to solve the auction game when the borrowing rate r is above the threshold $\theta/(1-\theta)$. It turns out that in this case there is a symmetric equilibrium where bids are continuous in budgets. Different from the case of "high bids and broke winners," bids at this equilibrium strictly increase with budgets. Furthermore, this is the only symmetric equilibrium where bids are strictly increasing and continuous in budgets. The technicalities in this case are more complicated than in the previous case. In particular, there are other symmetric equilibria where bids are non-monotone in budgets. Interestingly, the difference between these equilibria and the one with monotone bids shrinks when the number of bidders increases.

Let us start with a heuristic derivation of the solution. Imagine that all bidders use an equilibrium bidding strategy that is continuous, but not necessarily monotone, in budgets. By Lemma 3.1, bidders with sufficiently high budgets bid below their budgets. On the other hand, bidders with sufficiently low budgets bid above theirs, for otherwise the large difference between the expected value of the good and their budgets would attract such a bidder to bid slightly higher. Consequently, with bids continuous in budgets, the graph of the bidding function must cross the 45° line in the budget-bid plane (Fig. 1). Let $m_*(r)$ denote the lowest budget for such a crossing point. By Lemma 3.2, we know that bids go up on the interval $[\underline{m}, m_*(r)]$ and reach $m_*(r)$ at budget $m_*(r)$.

Let us move on to find out the bids for budgets above $m_*(r)$. Since we have not assumed monotonicity, such a bid could be as low as a bid

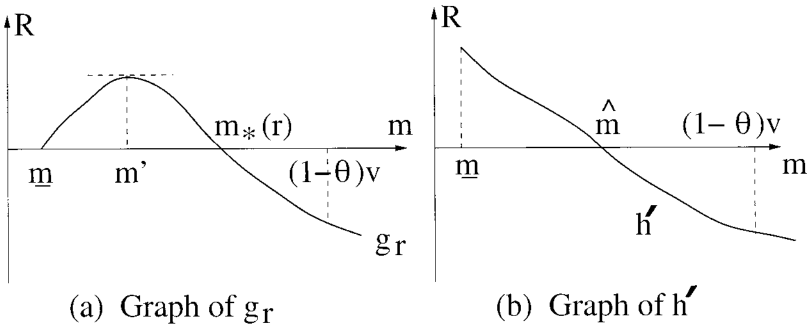

FIG. 2. The graphs of (a) g_r and (b) h'.

for budgets below $m_*(r)$. That is impossible, however, due to the financing cost. Specifically, if a bidder with budget above $m_*(r)$ bids the same as a bidder with budget below $m_*(r)$, then the former would be bidding below his budget and the latter bidding above hers. Not burdened by financing cost, the former would have less marginal cost in raising bids than the latter. Thus, one can prove that bidders with budgets above $m_*(r)$ would bid at least $m_*(r)$.

It then follows that a bidder with a budget below $m_*(r)$ wins if his budget exceeds the highest budget among his rivals, whether bids beyond $m_*(r)$ are monotone or not. Consequently, pinning down the bids for the budgets in $[\underline{m}, m_*(r)]$ becomes a relatively easy task of solving a differential equation. The solution of this equation also locates the crossing point $m_*(r)$. It turns out that $m_*(r)$ is a root of an equation $g_r(x)=0$, with g_r defined by:

$$g_r(x) := \left(v - \frac{x}{1-\theta}\right) F(x)^{n-1} - \left(r - \frac{\theta}{1-\theta}\right) \int_{\underline{m}}^{x} F(t)^{n-1}\, dt, \qquad \forall x \in [\underline{m}, \bar{m}]. \tag{9}$$

From Assumption 3 and the fact $r > \frac{\theta}{1-\theta}$, one can easily prove that (i) g_r is positive over $(\underline{m}, m_*(r))$, zero at $m_*(r)$, and negative for $m > m_*(r)$ and (ii) g_r strictly increases up to a point $m' \in (\underline{m}, m_*(r))$ and then strictly decreases. Part (a) of Fig. 2 illustrates these facts.

Due to these properties of the function g_r, it turns out that bidders with budgets above $m_*(r)$ would never bid above their budgets. The next question is whether these bidders bid below their budgets. This question leads us to a function h:

$$h(x) := \left(v - \frac{x}{1-\theta}\right) F(x)^{n-1}, \qquad \forall x \in [\underline{m}, \bar{m}]. \tag{10}$$

Notice from Eq. (2) that $(1-\theta)\,h(x)$ is the expected payoff for a bidder from bidding x within his budget, if *exactly* those bidders with budgets below x bid below x. From Assumption 3 and the intermediate- value theorem, one can easily prove that h has a unique maximum $\hat{m}$ strictly between $\underline{m}$ and $(1-\theta)\,v$. Furthermore, h is strictly increasing over $(\underline{m}, \hat{m})$ and strictly decreasing for $m > \hat{m}$. Part (b) of Fig. 2 illustrates these facts.

Owing to the properties of the function h, it turns out that, for budgets above $m_*(r)$, bids are equal to budgets at least up to the budget $\max\{\hat{m}, m_*(r)\}$. The proof for this fact is complicated. Roughly speaking, since the function h is increasing up to the budget $\hat{m}$, it is better for a bidder with a budget below $\hat{m}$ to bid his budget than bid below it.

We will then obtain a picture of any symmetric equilibrium whose bidding function is continuous: bidders with budgets below $m_*(r)$ bid above their budgets, those with budgets between $m_*(r)$ and $\max\{\hat{m}, m_*(r)\}$ bid their budgets, and those with budgets above $\max\{\hat{m}, m_*(r)\}$ bid between $\max\{\hat{m}, m_*(r)\}$ and their budgets.

That is the farthest we can go without any additional assumption about the bidding function. It turns out that the bidding function need not be monotone for budgets above $\max\{\hat{m}, m_*(r)\}$ (Remark 3.2). In particular, there is a continuum of symmetric equilibria whose bidding functions are non-monotone at the budgets above $\max\{\hat{m}, m_*(r)\}$. The reason for such non-monotonicity is that the bidder-type "budget" is a constraint instead of a valuation or cost. If bids are below budgets, the budget constraint is non-binding and hence unable to determine the bid.

If we add the restriction that a bidding function is strictly increasing, then the number of solutions shrinks to exactly one. We present this unique solution now. Denote m^H_{-i} for the highest budget among a bidder's rivals and $E_{m^H_{-i}}$ for the corresponding expected-value operator. Recall the notations g_r and h defined above.

THEOREM 3.2. *Let $r > \frac{\theta}{1-\theta}$ be a borrowing rate given to the bidders.*

1. (*Existence*) *It is an equilibrium of the auction game induced by r that each bidder bids according to the strategy β: $[\underline{m}, \bar{m}] \to R$ defined by*

$$\beta(m) = \begin{cases} \mathrm{E}_{m^H_{-i}}\left[\dfrac{v + r' m^H_{-i}}{1+r} \,\middle|\, m^H_{-i} \leqslant m\right] & \text{if} \quad \underline{m} \leqslant m < m_*(r) \\ m & \text{if} \quad m_*(r) \leqslant m \leqslant m^*(r) \\ (1-\theta)\left[v - \dfrac{h(m^*(r))}{F(m)^{n-1}}\right] & \text{otherwise,} \end{cases} \tag{11}$$

where $r' := r - \frac{\theta}{1-\theta}$, $m_*(r)$ *is the unique non-*$\underline{m}$ *root of the equation* $g_r(x) = 0$, *and* $m^*(r) = \max\{m_*(r), \hat{m}\}$, *with* $\hat{m}$ *being the unique maximum of the function* h.

2. (*Uniqueness*) "*Every bidder plays* β" *is the only symmetric equilibrium such that bids are strictly increasing and continuous in budgets.*

Theorem 3.2 will be proved in Sections 3.4.1 (uniqueness) and 3.4.2. (existence). The following remark highlights the properties of the equilibrium bidding strategy, whose graph looks like the upward-sloping curve *ABCD* in Fig. 1. Specifically, bidders with low budgets bid above their budgets, those in the middle bid their budgets, while those with higher budgets bid below theirs.

Remark 3.1. Given any borrowing rate $r > \frac{\theta}{1-\theta}$, the equilibrium bidding strategy β of the auction game induced by r has the following properties:

(a) The function is strictly increasing, continuous, and bounded from above by $(1-\theta)\,v$.

(b) $\beta(m) > m$ if $m < m_*(r)$, $\beta(m) = m$ if $m_*(r) \leqslant m \leqslant m^*(r)$, and $\beta(m) < m$ otherwise.

(c) $m_*(\cdot)$ is a one-to-one function over the domain $(\frac{\theta}{1-\theta}, \infty)$, and its derivative is negative.

(d) β is piecewise differentiable and

$$\beta'(m) = \begin{cases} \dfrac{r'}{1+r} \dfrac{(n-1)\,f(m)}{F(m)^n} \displaystyle\int_{\underline{m}}^{m} F(t)^{n-1}\,dt \\ \qquad \text{if} \quad \underline{m} < m < m_*(r) \\ 1 \qquad \text{if} \quad m_*(r) < m < m^*(r) \\ (1-\theta)\left(v - \dfrac{m^*(r)}{1-\theta}\right) F(m^*(r))^{n-1} \dfrac{(n-1)\,f(m)}{F(m)^n} \\ \qquad \text{if} \quad m^*(r) < m < \bar{m}. \end{cases} \tag{12}$$

(e) As $r \to \frac{\theta}{1-\theta}$, the function β increases and uniformly converges to $(1-\theta)\,v$.

(f) $m_*(r) \to (1-\theta)\,v$ as $n \to \infty$.

Proof. Parts (a), (d), and (e) are trivial. To prove Part (f), recall from the definition of $m_*(r)$ that $g_r(m_*(r)) = 0$, so

$$v - \frac{m_*(r)}{1-\theta} = r' \int_{\underline{m}}^{m_*(r)} \left(\frac{F(t)}{F(m_*(r))}\right)^{n-1} dt.$$

This equation, together with the assumption $(1-\theta)\, v > \underline{m}$ and the fact $m_*(r) < (1-\theta)\, v$, gives Part (f). To prove the other parts, (b) and (c), recall the geometric shapes of functions g_r and h illustrated in Fig. 2. Part (b) follows from the fact that "$\beta(m) > m$ for $m < m_*(r)$" is equivalent to "$g_r(m) > 0$ for $m < m_*(r)$" and that "$\beta(m) < m$ for $m > m^*(r)$" is equivalent to "$h(m) < h(m^*(r))$ for $m > m^*(r)$." For part (c), one easily calculates that

$$m'_*(r) = \frac{\int_{\underline{m}}^{m_*(r)} F(t)^{n-1}\, dt}{g'_r(m_*(r))}, \qquad \forall r \in \left(\frac{\theta}{1-\theta}, \infty\right), \tag{13}$$

and the derivative is negative due to fact (ii) of function g_r. Thus, all parts of this remark are proved. Q.E.D

Let us remark on a bidder's expected payoff at this equilibrium. From Eqs. (2) and (11), and the strict monotonicity of the equilibrium bidding strategy β, one can easily calculate the equilibrium expected payoff $U(m, r)$ of a bidder with budget m as

$$U(m, r) = \begin{cases} (1-\theta)[h(m) - g_r(m)] & \text{if } \ \underline{m} \leqslant m \leqslant m_*(r) \\ (1-\theta)\, h(m) & \text{if } \ m_*(r) \leqslant m \leqslant m^*(r) \\ (1-\theta)\, h(m^*(r)) & \text{if } \ m^*(r) \leqslant m \leqslant \bar{m}, \end{cases}$$

where functions g_r and h are defined in Eqs. (9) and (10). Figure 3 depicts the graph of $U(\cdot, r)$. In this figure, the graph over the interval $[\underline{m}, m_*(r)]$ is convex because the second derivative of $h(m) - g_r(m)$ is $r'(n-1)\, F(m)^{n-2}\, f(m) > 0$; the graph over $[m_*(r), m^*(r)]$ is concave because the derivative h' is positive and decreasing over that interval (Fig. 2b).

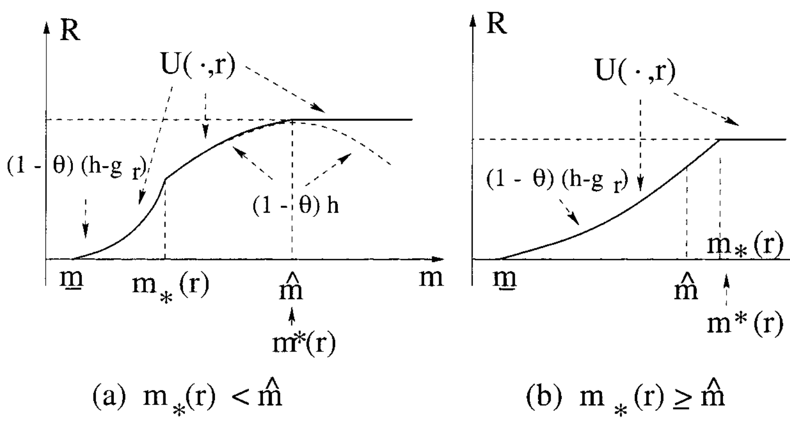

FIG. 3. Bidders' equilibrium expected payoffs.

3.4.1. *Deriving the budget-revealing equilibrium (uniqueness proof).* Take any $r > \frac{\theta}{1-\theta}$. Let β be a symmetric equilibrium bidding strategy of the auction game induced by the borrowing rate r. We will derive the solution for β in two steps. Assuming that the bidding function β is continuous, Step 1 (Proposition 3.1) proves that the graph of β for low and middle budgets looks like the curve ABC in Fig. 1. Adding the assumption that β is strictly increasing, Step 2 (Lemma 3.6) proves that the graph of β for high budgets looks like the curve CD in the same figure, i.e., β satisfies Eq. (11). That will accomplish the uniqueness proof.

PROPOSITION 3.1. *Suppose $\bar{m} < \infty$ and β is continuous. Then,* (i) *for each $m \in [\underline{m}, m_*(r))$, the bid $\beta(m)$ is above budget m and equal to the first branch of the right-hand side of Eq.* (11), (ii) *for each $m \in [m_*(r), \max\{m_*(r), \hat{m}\}]$, the bid $\beta(m) = m$, and* (iii) *for each $m > \max\{m_*(r), \hat{m}\}$, the bid $\beta(m) \in [\max\{m_*(r), \hat{m}\}, m]$.*

Proof. We first establish that $\beta(\underline{m})$ is the lowest bid. To prove that, note that the lowest bid exists, since β is continuous on the compact space $[\underline{m}, \bar{m}]$. Let z be a budget such that $\beta(z)$ is the lowest bid. By a standard zero-payoff argument, which uses the continuity of $u(\beta(\cdot), \cdot, r)$, a bidder with budget z would get zero expected payoff conditional on winning, i.e., $u(\beta(z), z, r) = 0$. We claim that $\beta(z) > z$. Suppose not; then the fact $u(\beta(z), z, r) = 0$ would imply that the lowest bid is $(1-\theta)\, v$ (Eq. (2)); consequently, $\beta(\underline{m}) \geqslant (1-\theta)\, v$, so $u(\beta(\underline{m}), \underline{m}, r) < 0$, and a type-$\underline{m}$ bidder would rather bid zero: a contradiction. Thus, the bidder submitting the lowest bid must bidding above his budget, i.e., $\beta(z) > z$. Since the equilibrium bidding function must be strictly increasing over the regions where bids exceed budgets (Lemma 3.2), $z = \underline{m}$, i.e., $\beta(\underline{m})$ is the lowest bid.

As $\beta(\underline{m}) = \min \beta$, we have $u(\beta(\underline{m}), \underline{m}, r) = 0$. By Eq. (2) and Assumption 1,

$$\beta(\underline{m}) = \frac{(1-\theta)\, v + r'\underline{m}}{1+r} > \underline{m}. \tag{14}$$

Consequently, with β continuous, we have established the fact that bidders with sufficiently low budgets bid above their budgets. Since bidders with sufficiently high budgets bid below their budgets (Lemma 3.1), there is a unique $m_*(r) \in (\underline{m}, (1-\theta)\, v]$ such that the bid function β is strictly increasing and $\beta(m) > m$ for all $m \in [\underline{m}, m_*(r))$, and $\beta(m_*(r)) = m_*(r)$.

We next derive the behavior of β beyond the point $m_*(r)$. We first claim that

$$\beta(m) \geqslant m_*(r), \qquad \forall m > m_*(r). \tag{15}$$

Suppose not; then, without loss of generality and by the continuity of β, we can pick some m and m' such that $m > m_*(r) > m'$, $\beta(m) = \beta(m') > \min \beta$. Since the monotone function $\text{Prob}_\beta[\text{win} \mid \cdot]$ is differentiable almost everywhere, we can choose m and m' such that $\text{prob}_\beta[\text{win} \mid \cdot]$ is differentiable at $\beta(m)$. We then know that $\beta(m') > m'$ (since $m_*(r) > m'$) and $\beta(m) < m$ (since β is strictly increasing up to $m_*(r)$). With β continuous, we can choose an open interval N of $\beta(m)$ such that $m > b > m'$ for all $b \in N$. Thus, the objective (Eq. (3)) of a bidder with budget m and that with budget m' are, respectively,

$$V_\beta(b, m, r) = \left(v - \frac{b}{1-\theta} \right) \text{Prob}_\beta[\text{win} \mid b] \qquad \text{and}$$

$$V_\beta(b, m', r) = (v - (1-r)\, b + r'm')\, \text{Prob}_\beta[\text{win} \mid b]$$

for all $b \in N$. Since $\beta(m)$ maximizes both objectives and $\text{Prob}_\beta[\text{win} \mid \cdot]$ is differentiable at $\beta(m)$, we have the first-order necessary condition

$$D_1 V_\beta(\beta(m), m, r) = 0 = D_1 V_\beta(\beta(m), m', r).$$

But then one can easily deduce from this equation that $(1-\theta)\, v = m'$, which is a contradiction because $m' < m_*(r) < (1-\theta)\, v$. Thus, Eq. (15) is true.

We are now ready to find the functional form of β over the interval $[\underline{m}, m_*(r)]$. Let $\beta_1 := \beta|_{[\underline{m}, m_*(r)]}$. Pick any $m \in (\underline{m}, m_*(r))$. From the standpoint of a bidder with budget m, by Eq. (15) and the fact that β_1 is strictly increasing, the bidder wins if m exceeds the highest budget m^H_{-i} among his rivals. Note that the cumulative distribution function of m^H_{-i} if $F(\cdot)^{n-1}$, so the bidder's objective $V_{\beta_1}(b, m, r)$ is $(v - (1+r)\, b + r'm)\, F(\beta_1^{-1}(b))^{n-1}$ for those b in the range of β_1. By the same technique in the proof of Lemma 3.5, one can find the derivative of the function $F(x)^{n-1}\, \beta_1(x)$ for all $x \in (\underline{m}, m_*(r))$. As in that proof, this gives a linear differential equation of β_1. Coupled with the boundary condition Eq. (14), this equation gives

$$\beta(m) = \frac{1}{1+r} \left[v + r'm - r' \int_{\underline{m}}^{m} \left(\frac{F(t)}{F(m)} \right)^{n-1} dt \right], \qquad \forall m \in [\underline{m}, m_*(r)]. \tag{16}$$

One can easily show that this equation is the same as the first branch of Eq. (11).

By Eq. (16), "$\beta(x) = x$" is equivalent to $g_r(x) = 0$. Thus, we have obtained $m_*(r)$ as the root of the equation $g_r(x) = 0$ other than $\underline{m}$.

We next claim that

$$\beta(m) \leqslant m, \qquad \forall m > m_*(r). \tag{17}$$

Suppose not; then by the continuity of β there would be a nondegenerate interval $[m_1, m_2]$ contained in $[m_*(r), \bar{m}]$ such that $\beta(m) > m$ for all $m \in (m_1, m_2)$ and $\beta(m) = m$ for $m = m_1$ and m_2. By the same proof as Eq. (15), we would have $\beta(m) \geqslant m_2$ if $m \in \{m_1, m_2\}$. Consequently, mimicking the derivation of Eq. (16), we get, for each $m \in (m_1, m_2)$, the bid $\beta(m)$ is equal to the right-hand side of that equation plus $c/F(m)^{n-1}$ for some constant c. Since $\beta(m_1) = m_1$ and $\beta(m_2) = m_2$, we then have $g_r(m_1) = -c = g_r(m_2)$; but that is impossible because g_r is strictly decreasing on $[m_*(r), \bar{m}]$ (Fig. 2a) and $m_2 > m_1 \geqslant m_*(r)$. Thus, Eq. (17) is true.

For the rest of the proof, define

$$\zeta := \sup\{x \in [\underline{m}, \bar{m}] : m' \in [\underline{m}, x] \Rightarrow \beta(m') \geqslant m'\}.$$

One readily sees that ζ exists, $\beta(\zeta) = \zeta$, and $\zeta \geqslant m_*(r)$. We first claim that

$$x \geqslant \min\{\zeta, \hat{m}\} \Rightarrow \beta(x) \geqslant \min\{\zeta, \hat{m}\}. \tag{18}$$

To prove Eq. (18), suppose to the contrary that $\xi := \min_{x \geqslant \zeta} \beta(x) < \min\{\zeta, \hat{m}\}$. By Eq. (17), we have $m_*(r) \leqslant \xi < \min\{\zeta, \hat{m}\}$ and $\xi = \beta(y)$ for some $y > \min\{\xi, \hat{m}\}$. By the definition of ζ, we have $\beta(\xi) = \xi$. By the choice of ξ, bidders with budgets above ξ do not bid below ξ, so $F(\xi)^{n-1} = \text{Prob}_\beta[\text{win} \mid \xi]$. Thus,

$$V_\beta(\xi, \xi, r) = h(\xi) < h(\min\{\zeta, \hat{m}\}),$$

where the inequality holds because h is strictly increasing up to $\hat{m}$. By Eq. (17) and the fact that β is strictly increasing up to $m_*(r)$,

$$F(x)^{n-1} \leqslant \text{Prob}_\beta[\text{win} \mid x], \qquad \forall x \in [m_*(r), \bar{m}]. \tag{19}$$

Consequently,

$$h(\min\{\zeta, \hat{m}\}) \leqslant V_\beta(\min\{\zeta, \hat{m}\}, \min\{\zeta, \hat{m}\}, r) \leqslant V_\beta(\beta(y), y, r) = V(\xi, \xi, r),$$

where the second inequality follows from the fact that $y \geqslant \min\{\zeta, \hat{m}\}$, so a bid of $\min\{\zeta, \hat{m}\}$ is available to a bidder with budget y. But then we reach a contradiction that $V(\xi, \xi, r)$ is less than itself. Thus, Eq. (18) is true.

To complete the proof of the proposition, we need only to prove that

$$\beta(m) \begin{cases} = m & \text{if} \quad m \in [m_*(r), \max\{m_*(r), \hat{m}\}] \\ \geqslant \max\{m_*(r), \hat{m}\} & \text{if} \quad m > \max\{m_*(r), \hat{m}\}. \end{cases} \tag{20}$$

By the definition of ζ and Eqs. (17) and (18), it suffices to show $\zeta \geqslant \hat{m}$. Thus, suppose to the contrary that $\zeta < \hat{m}$. Then $m_*(r) \leqslant \zeta < \hat{m}$ and there would be an $m \in (\zeta, \hat{m}]$ such that $\beta(m') < m'$ for all $m' \in (\zeta, m]$. By

Lemma 3.7 to be proved at the end of this Section, this implies that $V_\beta(\beta(m), m, r) = V_\beta(\beta(\zeta), \zeta, r)$. By Eq. (18), we have $\text{Prob}_\beta[\text{win} \mid \zeta] = F(\zeta)^{n-1}$, since the bid ζ is atomless. The definition of h in Eq. (10) then implies

$$V_\beta(\beta(m), m, r) = V_\beta(\beta(\zeta), \zeta, r) = V_\beta(\zeta, \zeta, r) = h(\zeta) < h(m),$$

where the inequality holds because h is strictly increasing up to $\hat{m}$ and $\zeta < m \leqslant \hat{m}$. By (19),

$$h(m) \leqslant V_\beta(m, m, r) \leqslant V_\beta(\beta(m), m, r).$$

But then we reach a contradiction that $V_\beta(\beta(m), m, r)$ is less than itself. Therefore, the claim $\zeta \geqslant \hat{m}$, and hence Eq. (20), is proved.

Combining Eqs. (16), (17), and (20), we have proved the proposition.

Q.E.D

The above proposition gives us the unique functional form of the bidding strategy β up to $\max\{m_*(r), \hat{m}\}$. In order to obtain the functional form of β for higher budgets, we need to add the assumption that β is strictly increasing.

LEMMA 3.6. *If β is strictly increasing, as well as continuous, then β satisfies Eq.* (11).

Proof. Although this lemma does not assume $\bar{m} < \infty$, the conclusion of Proposition 3.1 still follows. The reason is that the only role played by $\bar{m} < \infty$ is to ensure that a lowest bid exists; here its existence is guaranteed since β is now strictly increasing. Thus, we need only to prove that β satisfies the third branch of Eq. (11) for budgets above $\max\{m_*(r), \hat{m}\}$.

Since bidders with budgets sufficiently large bid below their budgets (Lemma 3.1), there is an $m^*(r) \in [\max\{m_*(r), \hat{m}\}, (1-\theta)\, v]$ such that $\beta(m) < m$ for all $m > m^*(r)$ and $\beta(m^*(r)) = m^*(r)$. We claim that

$$m^*(r) = \max\{m_*(r), \hat{m}\}. \tag{21}$$

Suppose not; then $m^*(r) > \max\{m_*(r), \hat{m}\}$. (The reverse "$<$" cannot hold, by the conclusion (ii) of Proposition 3.1.) Consequently, a bidder with budget $m^*(r)$ would rather bid $\max\{m_*(r), \hat{m}\}$ instead of $m^*(r)$. To see that, note that $F(x)^{n-1}$ is equal to $\text{Prob}_\beta[\text{win} \mid x]$ for each budget x such that $\beta(x) = x$, since β is strictly increasing. Thus, bidding $\max\{m_*(r), \hat{m}\}$ gives the type-$m^*(r)$ bidder an expected payoff $(1-\theta)\, h(\max\{m_*(r), \hat{m}\})$, while bidding $m^*(r)$ gives only $(1-\theta)\, h(m^*(r))$. The former is greater than the latter because the function h is strictly decreasing over $[\hat{m}, \bar{m}]$. Thus, Eq. (21) is proved.

We are therefore ready to pin down the behavior of β over its entire domain. Let $\beta_2 := \beta|_{[m^*(r), \bar{m}]}$. If a bidder has budget $m \in [m^*(r), \bar{m}]$, with β_2 strictly increasing, his objective $V_{\beta_2}(b, m, r)$ is $(v - b/(1-\theta))\, F(\beta_2^{-1}(b))^{n-1}$ for those b in the range of β_2. Mimicking the derivation of Eq. (16), one can derive a differential equation of β_2 from the symmetric equilibrium condition. Coupled with the boundary condition $\beta_2(m^*(r)) = m^*(r)$, this differential equation gives

$$\beta(m) = (1-\theta)\left[v - \left(v - \frac{m^*(r)}{1-\theta} \right) \left(\frac{F(m^*(r))}{F(m)} \right)^{n-1} \right], \qquad \forall m \in [m^*(r), \bar{m}]. \tag{22}$$

This equation is obviously the same as the third branch of Eq. (11). Combined with the conclusion of Proposition 3.1, this proves that β satisfies Eq. (11). We have hence proved the lemma. Q.E.D

To complete the uniqueness proof, we prove a lemma used in the proof of Eq. (20):

LEMMA 3.7. *Let β be a symmetric equilibrium bidding strategy that is continuous in budgets and let N be a bounded open interval of $[\underline{m}, \bar{m}]$. If $\beta(m) < m$ for all $m \in N$ then there is a constant c such that $V_\beta(\beta(m), m, r) = c$ for all $m \in N$.*

Proof. For each $x \in N$, the continuity of β allows us to choose a sufficiently small neighborhood $N(x) \subseteq N$ of x such that "$\beta(x') < x'$, $\beta(x') < x$, and $\beta(x) < x'$" holds for all $x' \in N(x)$. Since β is an equilibrium strategy, we have

$$V_\beta(\beta(x), x, r) = V_\beta(\beta(x'), x', r), \qquad \forall x' \in N(x).$$

Suppose not; say "$>$" holds instead. Then a bidder with budget x' would do better by bidding $\beta(x)$ rather than $\beta(x')$. A standard compactness argument extends this local result of identical equilibrium payoff to the entire interval N: For any $m, m' \in N$ with $m < m'$, the open cover $\{N(x): x \in [m, m']\}$ of the compact set $[m, m']$ has a finite subcover; one can then easily prove that $V_\beta(\beta(m), m, r) = V_\beta(\beta(m'), m', r)$. This proves the lemma. Q.E.D

The above lemmas imply that the bidding strategy β must obey Eq. (11). Thus, we have completed the uniqueness proof.

3.4.2. *Verifying the budget-revealing equilibrium* (*existence proof*). Let $r > 0$ and let β be the corresponding function given by Eq. (11). With β strictly increasing, a bidder's objective is

$$V_\beta(b, m, r) = u(b, m, r)[F(\beta^{-1}(b))]^{n-1}, \qquad \forall b \in \text{Range}\, \beta, \quad \forall m \in [\underline{m}, \bar{m}]. \tag{23}$$

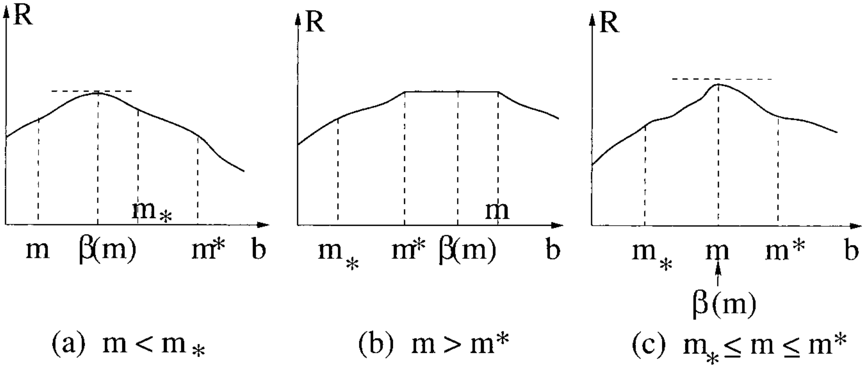

FIG. 4. The function $V_\beta(\cdot, m, r)$.

Pick any $m \in [\underline{m}, \bar{m}]$. We shall prove that the bid $\beta(m)$ maximizes $V_\beta(\cdot, m, r)$. Since the function $V_\beta(\cdot, m, r)$ is not differentiable at points $m_*(r)$, $m^*(r)$, and m, we cannot check some second-order condition once and for all. Instead, we need to partition the range of possible bids and verify optimal bids interval-by-interval. The analysis amounts to proving that the graph of the function $V_\beta(\cdot, m, r)$ is one of the three in Fig. 4, depending on the position of m relative to $m_*(r)$ and $m^*(r)$. Denote $\beta_1 := \beta|_{[\underline{m}, m_*(r)]}$, $\beta_2 := \beta|_{[m^*(r), \bar{m}]}$, and $\beta_3 := \beta|_{[m_*(r), m^*(r)]}$.

Case 1. $m < m_*(r)$. We partition the range of bids into the intervals $(-\infty, m)$, $[m, m_*(r)]$, $(m_*(r), m^*(r)]$, and $(m^*(r), \infty)$. We shall compare the bid $\beta(m)$ with the bids in each interval.

Step (a). *$\beta(m)$ maximizes $V_\beta(\cdot, m, r)$ over* $[m, m_*(r)]$. Take any $b \in (m, m_*(r))$. Then b belongs to the range β_1, and $V_\beta(b, m, r) = (v - (1+r)\, b + r'm)\, F(\beta_1^{-1}(b))^{n-1}$. By Eqs. (11) and (12), the derivative is

$$D_1 V_\beta(b, m, r) = (\text{a positive term}) \times r'(m - x) \begin{cases} > 0 & \text{if} \quad b < \beta_1(m) \\ = 0 & \text{if} \quad b = \beta_1(m) \\ < 0 & \text{if} \quad b > \beta_1(m), \end{cases}$$

since $r' = r - \frac{\theta}{1-\theta} > 0$. It follows that $\beta_1(m)$ maximizes $V_\beta(\cdot, m, r)$ over $(m, m_*(r))$. By the continuity of $V_\beta(\cdot, \cdot, r)$, this result extends to the end points m and $m_*(r)$.

Step (b). *$\beta(m)$ strictly dominates any bid $b \in (m_*(r), m^*(r)]$.* Take any $b \in (m_*(r), m^*(r)]$. Then b belongs to the range of β_3, so $\beta^{-1}(b) = b$. Thus,

$$\begin{aligned} V_\beta(b, m, r) &= \left[v - \frac{b}{1-\theta} - r'(b-m) \right] F(b)^{n-1} \\ &= \left(v - \frac{b}{1-\theta} \right) F(b)^{n-1} - r'(b-m)\, F(b)^{n-1} \\ &< r' \int_{\underline{m}}^{b} F(t)^{n-1}\, dt - r'(b-m)\, F(b)^{n-1} \\ &< r' \int_{\underline{m}}^{m} F(t)^{n-1}\, dt \\ &= V_\beta(\beta(m), m, r), \end{aligned}$$

where the first inequality comes from the fact that $g_r(b) < 0$ $(b > m_*(r))$, and the second inequality from the fact that $\int_m^b F(t)^{n-1}\, dt < (b-m)\, F(b)^{n-1}$, which is true since F is strictly increasing (Assumption 2). (Again $r' > 0$ is used.) We have hence shown that $V_\beta(b, m, r) < V_\beta(\beta(m), m, r)$.

Step (c). *$\beta(m)$ strictly dominates any bid $b > m^*(r)$.* It suffices to show that $V_\beta(\cdot, m, r)$ is strictly decreasing on $[m^*(r), (1-\theta)\, v]$. Hence we need only to prove that the derivative $D_1 V_\beta(\cdot, m, r) < 0$ over $(m^*(r), (1-\theta)\, v)$. Thus, take any $b \in (m^*(r), (1-\theta)\, v)$. In a manner similar to Step (a), one can easily derive that $D_1 V_\beta(\cdot, m, r)$ equals a positive term multiplied by $r'(m - (1-\theta)\, v)$, which is negative because $r' > 0$ and $m < m_*(r) < (1-\theta)\, v$ (the function β is bounded from above by $(1-\theta)\, v$). The step is hence achieved.

Step (d). *$\beta(m)$ strictly dominates any bid $b < m$.* One needs only to prove that $V_\beta(\cdot, m, r)$ is strictly increasing over $[\beta(\underline{m}), m)$, which can be done by mimicking Step (c).

Case 2. $m > m^*(r)$. We partition the range of possible bids as in Case 1. Mimicking Step (a) of that case, one can prove that $V_\beta(\cdot, m, r)$ is constant on $[m^*(r), m]$ and that the bids in this interval ($\beta(m)$ is one of them) strictly dominate any bid $b < m_*(r)$ and any bid $b > m$. The only case different from Case 1 is the claim that *any bid $b \in [m_*(r), m^*(r))$ is strictly dominated by $m^*(r)$*. This claim is vacuously true if $m_*(r) = m^*(r)$. Thus, to prove this claim, suppose that $m_*(r) \neq m^*(r)$. Then by the definition of $m^*(r)$ we have $m^*(r) = \hat{m} > m_*(r)$. The function h is hence strictly increasing on $[\underline{m}, m^*(r)]$ (Fig. 2b). Thus, for any $b \in [m_*(r), m^*(r))$, with b and $m^*(r)$ lying in the range of β_3,

$$V_\beta(b, m, r) = h(b) < h(m^*(r)) = V_\beta(m^*(r), m),$$

as claimed.

Case 3. $m \in [m_*(r), m^*(r)]$. One can deal with this case in the same way as Case 1, except for the claim that *any bid* $b \in (m, m^*(r)]$ *is strictly dominated by* m $(=\beta(m))$. To prove this claim, it suffices to show that $V_\beta(\cdot, m, r)$ is strictly decreasing over $(m, m^*(r)]$. We hence need only to prove that $D_1 V_\beta(\cdot, m, r) < 0$ over $(m, m^*(r))$. Take any $b \in (m, m^*(r))$. Then b belongs to the range of β_3 and $V_\beta(b, m, r) = [v - (1+r)\, b + r'm]\, F(b)^{n-1}$. Thus,

$$
\begin{aligned}
\frac{D_1 V_\beta(b, m, r)}{\text{a positive term}} &= v + r'm - (1+r)\left[b + \frac{F(b)}{(n-1)\, f(b)} \right] \\
&< v + r'm - (1+r)\left[m + \frac{F(m)}{(n-1)\, f(m)} \right] \\
&= g_r'(m)/((n-1)\, f(m)) \\
&\leqslant 0,
\end{aligned}
$$

where the first inequality comes from the fact $b > m$ and Assumption 3, the second equality comes from the definition of r' and g_r (Eq. (9)), and the second inequality comes from the facts that $g_r'(x) \leqslant 0$ for all $x \geqslant m_*(r)$ (Fig. 2a).

With all possibilities exhausted, we have verified that β is a symmetric equilibrium bidding strategy of the auction game induced by r. Q.E.D

3.4.3. *Multiple equilibria with non-monotone bids.* With the borrowing rate above the threshold, the previous sections have proved that there is only one symmetric equilibrium with a continuous and strictly increasing bidding strategy. If we allow a bidding strategy to be non-monotone, however, there are multiple solutions, as the following remark asserts. As mentioned before, the reason why there are multiple equilibria is that a bidder's type is a constraint instead of a valuation or cost. Thus, a bid is not determined if the budget constraint is non-binding.

Remark 3.2. If $r > \theta/(1-\theta)$ and $\bar{m} < \infty$, then there is a continuum of symmetric equilibria whose bidding strategies are continuous, piecewise differentiable, and non-monotone over the interval $(m^*(r), \bar{m}]$.

Proof. We will construct a continuum of symmetric equilibria from the equilibrium given by Eq. (11). Recall the definition of $m^*(r)$ in Theorem 3.2. Notice that $m^*(r) < \bar{m}$ and $h(m^*(r)) > h(\bar{m})$ (since h is strictly decreasing starting from $\hat{m}$). Thus, there is a continuum of points $x \in (m^*(r), \bar{m})$ such that

$$(1-\theta)(v - h(m^*(r))) \leqslant x < \bar{m}. \tag{24}$$

Pick such an x. Consider the mapping ϕ that "folds" the interval $[x, \bar{m}]$: $\phi(t) := t$ if $x \leqslant t \leqslant (x+\bar{m})/2$, and $\phi(t) := x+\bar{m}-t$ if $(x+\bar{m})/2 \leqslant t \leqslant \bar{m}$. We now construct a symmetric equilibrium where a bidder with budget $m \in [(x+\bar{m})/2, \bar{m}]$ bids the same as the one with budget $\phi(m)$. Specifically, we want to have a bidding strategy β_x to be the same as β up to the budget x, strictly increasing from x to $(x+\bar{m})/2$, strictly decreasing from $(x+\bar{m})/2$ to $\bar{m}$, and $\beta_x(m) = \beta_x(\phi(m))$ for all $m \in [x, \bar{m}]$.

From the viewpoint of a bidder, with his rivals expected to play the strategy β_x, the situation is the same as the case where his rivals' bids strictly increase in their budgets and their budgets are independent random draws from a distribution Φ_x with support $[\underline{m}, (x+\bar{m})/2]$:

$$\Phi_x(m) := \begin{cases} F(m) & \text{if} \quad m \leqslant x \\ F(m) + [1 - F(\bar{m} - (m-x))] & \text{if} \quad x \leqslant m \leqslant (x+\bar{m})/2. \end{cases}$$

Mimicking Eq. (22), one derives

$$\beta_x(m) = (1-\theta)\left[v - \left(v - \frac{\beta(x)}{1-\theta} \right) \left(\frac{\Phi_x(x)}{\Phi_x(m)} \right)^{n-1} \right] = (1-\theta)\left(v - \frac{h(m^*(r))}{\Phi_x(m)^{n-1}} \right),$$

for all $m \in [x, (x+\bar{m})/2]$, where the second equality is due to the fact that

$$(v - \beta(x)/(1-\theta))\, \Phi_x(x)^{n-1} = V_\beta(\beta(x), x, r) = h(m^*(r))$$

by Lemma 3.7. We now extend the bidding strategy β_x over the entire domain $[\underline{m}, \bar{m}]$:

$$\beta_x(m) := \begin{cases} \beta(m) & \text{if} \quad m \leqslant x \\ \beta_x|_{[x, (x+\bar{m})/2]}(m) & \text{if} \quad x \leqslant m \leqslant (x+\bar{m})/2 \\ \beta_x|_{[x, (x+\bar{m})/2]}(\phi(m)) & \text{if} \quad (x+\bar{m})/2 \leqslant m \leqslant \bar{m}. \end{cases}$$

One can verify that this bidding strategy β_x comprises a symmetric equilibrium of the auction game. The reason is that the graph of $\beta_x|_{[x, \bar{m}]}$, where β_x differs from β, is bounded between the constants $\beta(x)$ and x, due to the choice of x (Eq. (24)). Therefore, a bidder with budget below x is unaffected by the switch from β to β_x, and a bidder with budget above x can view others' budgets as distributed according to Φ_x. Also notice that β_x is continuous (since $\beta_x(x) = \beta(x)$) and piecewise differentiable.

Thus, we have constructed a symmetric equilibrium with bidding strategy β_x non-monotone on the interval $[x, \bar{m}]$. Since there is a continuum of such x, we have proved the remark. Q.E.D

A reader bothered by the multiple equilibria may find comfort from Proposition 3.1, which implies that these equilibria all agree with Eq. (11) up to the point $m^*(r)$ and that the non-monotone part of the bidding strategies is bounded between $m^*(r)$ and $(1-\theta)\,v$. Consequently, when the number n of bidders is large, the gap between $m^*(r)$ and $(1-\theta)\,v$ shrinks (Remark 3.1 (f)), so the non-monotonicity of a bidding strategy becomes insignificant.

3.5. *The Likelihood of Bankruptcy*

From the solution of the auction game, we can calculate the probability of bankruptcy as a function of the borrowing rate r. Recall from Section 3.1 a winner's bankruptcy decision. When the borrowing rate is blow the threshold $\theta/(1-\theta)$, the auction game has the "high bids and broke winners" equilibrium (Theorem 3.1), so bankruptcy occurs if and only if the lowest budget among all bidders is less than $(1-\theta)\,v$ and the auctioned item has zero value. Thus, the probability of bankruptcy (i.e., a winner declaring bankruptcy) with a borrowing rate below the threshold is $\theta(1-(1-F((1-\theta)\,v))^n)$.

When the borrowing rate r is above the threshold, the equilibria are characterized by Theorem 3.2 and Proposition 3.1, so bidders with budget above $m_*(r)$ bid within their budgets. For notational convenience, assume that a winner would honor his bid when he is indifferent between whether to default or not.[5] Consequently, bankruptcy occurs if and only if the highest budget among all bidders is less than $m_*(r)$ and the auctioned item has zero value. Thus, the probability of bankruptcy with borrowing rate $r>\theta/(1-\theta)$ is $\theta F(m_*(r))^n$.

When the borrowing rate $r=\theta/(1-\theta)$, any bidder's expected payoff conditional on winning is $(1-\theta)\,v-b$ (Eq. (2)). It is then obvious that at equilibrium every bidder bids the expected value $(1-\theta)\,v$ of the good. With bid independent of budget, a winner's budget is a random draw from F. Thus, the probability of bankruptcy with borrowing rate $\theta/(1-\theta)$ is $\theta F((1-\theta)\,v)$.

Therefore, the probability of bankruptcy is the highest when the borrowing rate r is below the threshold, drops discontinuously when r rises to the threshold, and falls continuously as r rises. (The last part is due to the fact that $m_*(\cdot)$ is strictly decreasing by Remark 3.1(c).) Furthermore, the gap at the threshold rate widens as the number n of bidders increases. Figure 5 illustrates these facts.

[5] Dropping this assumption would not change our conclusion qualitatively. We would simply need to use the equilibrium given by Eq. (11) and to use $m^*(r)$ instead of $m_*(r)$ in this section.

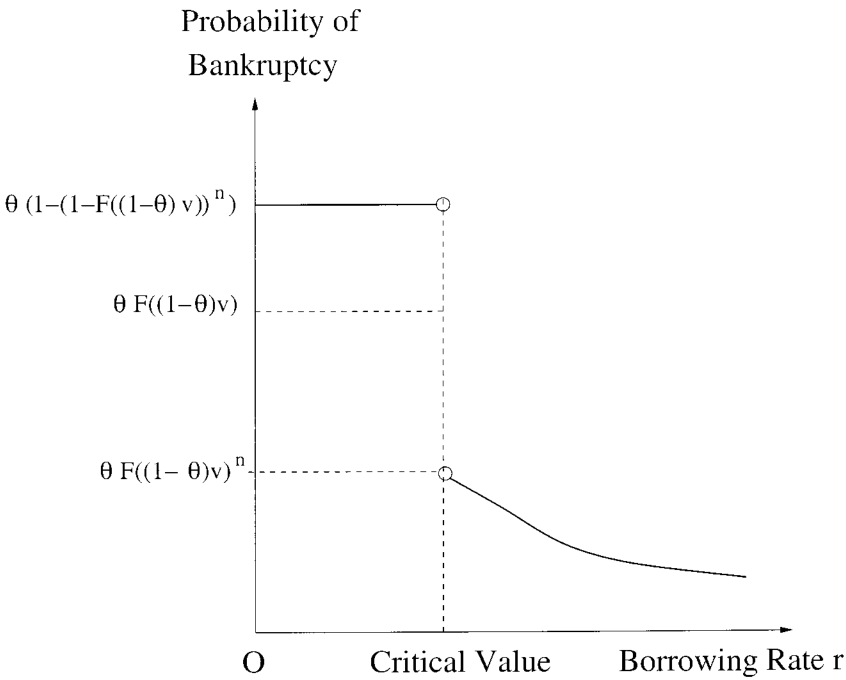

FIG. 5. The probability of bankruptcy determined by the borrowing rate.

4. HOW THE SELLER PROFITS FROM GIVING SUBSIDIES

From the solution of the auction game, comparative statics shows that bids at equilibrium rise as the borrowing rate falls (Eqs. (5) and (11)). Consequently, the seller can raise bids if she can lower the borrowing rate. In particular, the seller may offer to finance the winning bidder at a lower interest rate. The question is: can such a subsidy raise the seller's expected profit? This section proves that the answer is "yes" in some cases (Proposition 4.2), but it also cautions that an excessive use of such a subsidy can lead to the equilibrium of "high bids and broke winners," thereby hurting the seller (Proposition 4.1).

4.1. *A Model of Seller-Provided Financing*

Let us first specify the capital markets and bankruptcy arrangements. We assume immediate liquidation for bankruptcy. Specifically, if a bidder owes debts of the amount D and has assets of the amount A, and if he declares bankruptcy or is insolvent ($D > A$), then all of that bidder's assets are transferred to the creditors up to the amount A; i.e., the bidder returns $\min\{A, D\}$ and keeps only $A - \min\{A, D\}$ for himself. Implicitly assumed here is that a bidder cannot hide his budget from a bankruptcy court.

For the capital markets, we assume that a bidder can borrow from only two sources: either from the capital market at a given borrowing rate $q \geqslant 0$, common to all bidders, or from the seller at a rate $r \geqslant 0$, publicly chosen by the seller before the bidding. After a bidder wins the auction and before the value of the good is realized, his financing proceeds as follows.

1. If the winner borrows from the capital market, then he first borrows an amount d from a lender such that d plus the winner's budget m_w is sufficient to cover the bid b_w. Then the winner pays b_w to the seller and gets the good in return. The value $\mathbf{v}$ of the good is realized. The winner's total amount of assets is thus $m_w + d - b_w + \mathbf{v}$, with liability $(1+q)\,d$. He therefore must return $\min\{m_w + d - b_w + \mathbf{v}, (1+q)\,d\}$ to the lender. The winner's payoff is hence

$$d - b_w + \mathbf{v} - \min\{m_w + d - b_w + \mathbf{v}, (1+q)\,d\}. \tag{25}$$

2. If the winner borrows from the seller, then the winner first delivers an amount $\hat{m}_w \leqslant m_w$ to the seller, who allows him to delay the payment of the rest $b_w - \hat{m}_w$. We assume that the seller bears a financing cost q per unit of the postponed amount. The winner receives the good from the seller, and then the value of the good $\mathbf{v}$ is realized. The winner's total amount of assets is thus $m_w - \hat{m}_w + \mathbf{v}$, with liability $(1+r)(b_w - \hat{m}_w)$. Therefore, he must return $\min\{m_w - \hat{m}_w + \mathbf{v}, (1+r)(b_w - \hat{m}_w)\}$ to the seller. The winner's payoff is thus

$$-\hat{m}_w + \mathbf{v} - \min\{m_w - \hat{m}_w + \mathbf{v}, (1+r)(b_w - \hat{m}_w)\}, \tag{26}$$

and the seller's profit is

$$\hat{m}_w - q(b_w - \hat{m}_w) + \min\{m_w - \hat{m}_w + \mathbf{v}, (1+r)(b_w - \hat{m}_w)\}. \tag{27}$$

Implicitly assumed above is that a bidder can lend only at an interest rate of zero. We further assume that $q > \theta/(1-\theta)$. As will be clear soon, this reflects the assumption of imperfect capital markets.

4.2. *The Seller's Expected Profit Function*

The above model implies the following fact: If a bidder wins by bidding above his budget, then he borrows exactly the amount by which the bid exceeds the budget, whichever financing option the bidder takes. Borrowing less than that amount is infeasible, since the bidder needs sufficient funds to cover his bid. Borrowing more than that amount is dominated, because the bidder has to pay a non-negative interest on the debt and, confined by a zero lending rate, cannot profit from lending what he borrows.

Consequently, we can substitute $b_w - m_w$ for the debt d in (25) and substitute the winner's true budget m_w for his reported budget $\hat{m}_w$ in (26). Since it is dominated for a bidder to bid at a cost higher than the maximum value v of the good, we know

$$v \geqslant (1 + \min\{r, q\})(b_w - m_w).$$

Thus, the winner's payoff becomes

$$\begin{cases} \mathbf{v} - b_w & \text{if} \quad b_w \leqslant m_w \\ \mathbf{v} - b_w - \min\{r, q\}(b_w - m_w) & \text{if} \quad b_w > m_w \quad \text{and} \quad \mathbf{v} = v \\ -m_w & \text{if} \quad b_w > m_w \quad \text{and} \quad \mathbf{v} = 0. \end{cases} \tag{28}$$

In other words, the winner declares bankruptcy if $b_w > m_w$ and $\mathbf{v} = 0$ and otherwise pays back the debt, including its interest. Thus, the winner's bankruptcy decision is the same as that in the auction game analyzed in Section 3. Specifically, a winner's payoff function here is the same as the one in Eq. (2). Thus, the equilibrium analysis in previous sections applies.

Notice that a bankrupt winner has zero assets, since he has borrowed just enough to cover his immediate payment to the seller. It follows that a lender will receive from her loan neither principal nor interest if $\mathbf{v} = 0$, and otherwise a net gain of interest. Thus, the borrowing rate where an outside lender makes zero profit is $\theta/(1-\theta)$. Our assumption that the borrowing rate $q > \theta/(1-\theta)$ hence reflects the imperfection of capital markets.

By (27) and (28), the seller's ex post profit from offering a loan to the winner at a rate $r < q$ is

$$\begin{cases} b_w & \text{if} \quad b_w \leqslant m_w \\ b_w - (q - r)(b_w - m_w) & \text{if} \quad b_w > m_w \quad \text{and} \quad \mathbf{v} = v \\ m_w - q(b_w - m_w) & \text{if} \quad b_w > m_w \quad \text{and} \quad \mathbf{v} = 0. \end{cases}$$

Denote $\chi_{[b_w \leqslant m_w]}$ for the indicator function of the event "$b_w \leqslant m_w$," and x^+ for $\max\{0, x\}$. By the above calculation, the seller's interim profit from offering a loan at rate r, after the selection of the winner (hence knowing b_w and m_w) and before the good reveals its value, is

$$(1-\theta)\left[b_w - \left(\frac{q}{1-\theta} - r \right)(b_w - m_w)^+ \right] + \theta[\chi_{[b_w \leqslant m_w]} b_w + (1 - \chi_{[b_w \leqslant m_w]})\, m_w]. \tag{29}$$

For each borrowing rate r, let β_r denote the symmetric equilibrium bidding strategy of the auction game induced by r, given by Eq. (5) if $r < \theta/(1-\theta)$ or Eq. (11) if $r > \theta/(1-\theta)$. From now on, we select the bid function β_r as the equilibrium played at each rate r. If the seller offers lending to a winner at a rate $r \in [0, q)$, then her expected profit $\pi(r)$ (from the viewpoint before the selection of the winner) is equal to the expected value of (29), with m_w being the random variable and $b_w = \beta_r(m_w)$.

If the seller does not offer any loan to the winner, then according to Section 4.1 the seller receives the winner's bid without bearing any financing cost, so her expected profit π_0 is the expected value of the winning bid $b_w = \beta_q(m_w)$.

4.3. *Guideline* 1: *Offer Borrowing Rates above* $\frac{\theta}{1-\theta}$

Does an interest subsidy raise the seller's expected profit? To answer this question, we need to keep in mind the fact that the equilibrium outcome of the auction game changes discontinuously at the threshold interest rate $\theta/(1-\theta)$.

Our first result (Proposition 4.1) is that the seller's expected profit is higher with a borrowing rate above the threshold than with a rate below, when the number of bidders is sufficiently large. The reason is that a winner in the first case is the bidder with the highest budget, while a winner in the second case has the lowest budget. The budget gap between the two cases widens when the number of bidders is large, with the winning bidder near the top of the budget support in one case and near the bottom in the other. With a higher budget, a winner in the first case is less likely to default and needs less interest subsidy from the seller.

PROPOSITION 4.1. *For any borrowing rates $r_1, r_2 \in [0, q)$ such that $r_1 > \theta/(1-\theta) > r_2$, the seller's expected profit $\pi(r_1)$ at r_1 is higher than the expected profit $\pi(r_2)$ at r_2 when the number n of bidders is sufficiently large. Specifically,*

$$\lim_{n \to \infty} (\pi(r_1) - \pi(r_2)) > q[(1-\theta)\, v - \underline{m}] > 0.$$

Proof. We will calculate the limit of an expected profit $\pi(r)$ through the limit of the corresponding equilibrium bidding strategy β_r. This is valid by the Lebesgue convergence theorem and the fact that the equilibrium bidding strategies are all bounded between zero and v.

Notice from Eqs. (5) and (11) that

$$\lim_{n \to \infty} \beta_r(m) = \begin{cases} \dfrac{v + r'm}{1+r} & \text{if } \; m \leqslant (1-\theta)\, v \\ (1-\theta)\, v & \text{if } \; m \geqslant (1-\theta)\, v. \end{cases}$$

As $n \to \infty$, the highest budget among all bidders converges in probability to $\bar{m}$, and the lowest budget converges in probability to $\underline{m}$. Since a winner's budget is the highest at the borrowing rate r_1 and is the lowest at r_2, one can easily calculate from the above equation and (29) that

$$\lim_{n \to \infty} \pi(r_1) = (1-\theta)\, v \qquad \text{and}$$

$$\lim_{n \to \infty} \pi(r_2) = \left(1 - \frac{q}{(1-\theta)(1+r_2)}\right)(1-\theta)\, v + \frac{q}{(1-\theta)(1+r_2)}\, \underline{m}.$$

Therefore, one easily calculates that

$$\lim_{n \to \infty} (\pi(r_1) - \pi(r_2)) = \frac{q}{1+r_2}\left[v - \frac{\underline{m}}{1-\theta}\right] > q[(1-\theta)\, v - \underline{m}],$$

which is positive by Assumption 1. (The last inequality follows from $r_2 < \theta/(1-\theta)$ and Assumption 1.) We have hence proved the proposition.[6]

Q.E.D

If the seller insists on offering a borrowing rate below the threshold, she can avoid the "high bids and broke winners" equilibrium by banning bids above the expected value $(1-\theta)\, v$ of the good. with such a ban, one can show that an equilibrium is that bids are identically $(1-\theta)\, v$, so the winner is randomly selected. Such a policy, however, is dominated by simply offering a borrowing rate at the threshold; the latter induces the same equilibrium bids and yet reduces the interest subsidy offered by the seller.

The threshold borrowing rate is also dominated. Raising the borrowing rate from the threshold to a rate slightly above it, the seller can expect to bear significantly less financing cost and still keep the winning bid almost unchanged. That is due to the different ways of winner selection. While a winner of the auction game at the threshold rate can be anywhere in the budget support (Section 3.5), a winner at higher borrowing rates is likely to be near the top of the budget support.

4.4. *Guideline* 2: *Boost Bids through Suitable Subsidy*

The previous Section has shown us the hazard of excessive subsidies. Should the seller then stay away from offering below-market interest rates?

[6] The proof remains valid even if we allow the equilibria of the auction game at the borrowing rate r_1 to be different from the one given by Eq. (11). See the comment at the end of Section 3.4.3.

This subsection shows that the seller indeed can raise expected profit by offering below-market rates.

Our approach to this result is simple. By the solution of the auction game, we know that bids at equilibrium rise as the borrowing rate falls. Thus, a below-market interest rate would definitely raise the expected winning bid. The drawback is that the seller would have to shoulder some financing cost for the winner. The net benefit from the interest subsidy therefore depends on a comparison between the two effects. To do that, we will calculate the difference between the seller's expected profit $\pi(r)$ from offering a loan at the interest rate r and her expected profit π_0 from offering no loan at all.

The following lemma gives a sufficient condition for an interest subsidy to raise the expected profit. As expected, this condition says that the effect of higher bids (the first term on the right-hand side of (30)) dominates the effect of financing cost (the second term on the right-hand side of (30)).

LEMMA 4.1. *For any interest rate $r \in (\theta/(1-\theta), q)$ such that $r \geqslant q/(1-\theta)-1$,*

$$\pi(r)-\pi_0 \geqslant \int_{m_*(r)}^{\bar{m}} (\beta_r(m)-\beta_q(m))\, dF_{(1)}(m) - \int_{\underline{m}}^{m_*(q)} [\beta_q(m)-m]\, dF_{(1)}(m), \tag{30}$$

where $F_{(1)}(\cdot) := F(\cdot)^n$ denotes the distribution function of the highest budget among all bidders.

Proof. Pick any borrowing rate $r \in (\theta/(1-\theta), q)$. By (11) and (29), the seller's expected profit from offering a rate r is

$$\pi(r) = \int_{m_*(r)}^{\bar{m}} \beta_r\, dF_{(1)} + \int_{\underline{m}}^{m_*(r)} \left[(1-\theta)\left(\beta_r(m) - \left(\frac{q}{1-\theta}-r\right)(\beta_r(m)-m)\right) + \theta m \right] dF_{(1)}(m)$$

and that from offering no loan is

$$\pi_0 = \int_{\underline{m}}^{\bar{m}} \beta_q(m)\, dF_{(1)}(m).$$

Since $r \leqslant q$ and the function $m_*(\cdot)$ is decreasing (Remark 3.1(c)), $m_*(r) \geqslant m_*(q)$. Partition the integral π_0 into the sum of the integrals $\int_{\underline{m}}^{m_*(q)}$, $\int_{m_*(q)}^{m_*(r)}$, and $\int_{m_*(r)}^{\bar{m}}$. Note that $\beta_q(m) \leqslant m$ for all $m \geqslant m_*(q)$. Consequently,

$$\pi_0 \leqslant \int_{\underline{m}}^{m_*(q)} [\beta_q(m) - m] \, dF_{(1)}(m) + \int_{\underline{m}}^{m_*(r)} m \, dF_{(1)}(m)$$

$$+ \int_{m_*(r)}^{\bar{m}} \beta_q(m) \, dF_{(1)}(m).$$

Thus, the difference $\pi(r) - \pi_0$ is at least as large as the right-hand side of Eq. (30) plus

$$\int_{\underline{m}}^{m_*(r)} \left[(1-\theta) \left(\beta_r(m) - \left(\frac{q}{1-\theta} - r \right) (\beta_r(m) - m) \right) + \theta m \right] dF_{(1)}(m)$$

$$- \int_{\underline{m}}^{m_*(r)} m \, dF_{(1)}(m).$$

Here the integrand of the first integral is greater than m if $r \geqslant q/(1-\theta) - 1$, since $\beta_r(m) > m$ for all $m < m_*(r)$. The inequality (30) hence follows. Q.E.D

Based on Lemma 4.1, the next proposition finds a set of environments in which offering a loan at a below-market rate yields a higher expected profit for the seller than not offering any loan at all. The functions g_q, h, and m^* in the proposition have been defined in (9), (10), and Theorem 3.2.

PROPOSITION 4.2. *If the exogenous rate $q \in (\theta/(1-\theta), 1]$ and*

$$(1-\theta)(1 - F((1-\theta)\, v))(1+q)\, h(m^*(q)) > \int_{\underline{m}}^{m^*(q)} g_q \, dF, \tag{31}$$

and if the equilibrium selected at each borrowing rate r is to play the bid function β_r, then there exists an $r \in (\theta/(1-\theta), q)$ sufficiently close to $\theta/(1-\theta)$ such that offering a loan to a winner at the interest rate r yields more expected profit than not offering any loan to the winner.

Proof. By Lemma 4.1, we need only to find an interest rate $r \in (\theta/(1-\theta), q)$ such that $r \geqslant q/(1-\theta) - 1$ and the right-hand side of (30) is positive. Recall the points $\hat{m}$, $m_*(r)$, and $m^*(r)$ from the equilibrium bidding function in Theorem 3.2. Consider two cases:

First, consider the case $m_*(q) \geqslant \hat{m}$. For any $r \in (\theta/(1-\theta), q)$, with the function $m_*(\cdot)$ decreasing, $m_*(r) > m_*(q) \geqslant \hat{m}$. Thus, $m^*(q) = m_*(q)$ and $m^*(r) = m_*(r)$. By Eqs. (9), (11), and (16), we have:

$$\int_{m_*(r)}^{\bar{m}} (\beta_r - \beta_q)\, dF_{(1)} = n(1-\theta)(1 - F(m^*(r)))(h(m^*(q)) - h(m^*(r)));$$

$$\int_{\underline{m}}^{m_*(q)} [\beta_q(m) - m]\, dF_{(1)}(m) = \frac{n}{1+q} \int_{\underline{m}}^{m_*(q)} g_q\, dF.$$

Since $r > \theta/(1-\theta)$ and $q \leqslant 1$, we have $r > q/(1-\theta) - 1$. Thus, Lemma 4.1 applies. By the fact $F(m^*(r)) < F((1-\theta)\, v)$, the profit difference $\pi(r) - \pi_0$ is positive if

$$(1-\theta)(1 - F((1-\theta)\, v))(1+q)(h(m^*(q)) - h(m^*(r))) \geqslant \int_{\underline{m}}^{m^*(q)} g_q\, dF. \tag{32}$$

Recall the fact that $m^*(r) \to (1-\theta)\, v$, and hence $h(m^*(r)) \to 0$, as $r \to \theta/(1-\theta)$ (Remark 3.1(e)). Consequently, if (31) is satisfied, then (32) holds for r sufficiently close to the threshold $\theta/(1-\theta)$. The proposition hence follows in the case $m_*(q) \geqslant \hat{m}$.

Second, consider the other case $m_*(q) < \hat{m}$. Then $m^*(q) = \hat{m}$. By the fact that $\hat{m} < (1-\theta)\, v$ and $m^*(r) \to (1-\theta)\, v$ as $r \to \theta/(1-\theta)$ (Remark 3.1(e)), we can choose $r > \theta/(1-\theta)$ sufficiently close to $\theta/(1-\theta)$ such that $m_*(r) > \hat{m}$ and so $m^*(r) = m_*(r)$. The same calculation in the previous case can therefore be carried out here. Having exhausted all cases, we have proved the proposition. Q.E.D

The above proposition provides a sufficiency test for whether it is profitable to offer an interest subsidy. As the examples in Section 4.4.1 will show, the set of parameter values satisfying the sufficient condition of the proposition is nonempty. Thus, offering an interest subsidy can be profitable in some cases.

In the context of the FCC spectrum auctions, Proposition 4.2 offers a partial rationalization for the policy of subsidizing winners at a below-market interest rate: If the rate is above the threshold $\theta/(1-\theta)$, such an interest subsidy may raise the seller's expected profits. The "high bids and broke winners" outcome in the C-block auction, however, indicates that the interest rate offered by the government is below the threshold, thereby hurting the seller (Proposition 4.1).

4.4.1. *Numerical examples.* Suppose that the budget of each bidder is uniformly distributed on [0, 1]. The following remark says that an interest subsidy raises the seller's expected profit if the number of bidders is above a lower bound.

Remark 4.1. When the budgets are uniformly distributed on $[0, 1]$ and $q \leqslant 1$, Inequality (31) is satisfied if the number n of bidders is so large that

$$n > \frac{v}{(1+q)(1-(1-\theta)\,v)(q(1-\theta)-\theta)} - 1. \qquad (33)$$

Proof. First, we calculate the functions h and g_r $(\forall r \in (\frac{\theta}{1-\theta}, q])$ by Eqs. (9) and (10):

$$h(x) = \left(v - \frac{x}{1-\theta}\right) x^{n-1};$$

$$g_r(x) = x^{n-1}\left(v - \left(\frac{1}{1-\theta} + \frac{r}{n} - \frac{\theta}{n(1-\theta)}\right) x\right).$$

Thus, we calculate $\hat{m}$ and $m_*(r)$ by their definitions in Theorem 3.2:

$$\hat{m} = \frac{n-1}{n}(1-\theta)\,v;$$

$$m_*(r) = v\left(\frac{1}{1-\theta} + \frac{r}{n} - \frac{\theta}{n(1-\theta)}\right)^{-1}.$$

Therefore, $m_*(r) > \hat{m}$ iff $r < (\frac{n}{n-1} + \theta)/(1-\theta)$. Since $q \leqslant 1$ by assumption, we have $m_*(q) > \hat{m}$, so $m^*(q) = m_*(q)$. One can then prove that (31) is equivalent to (33). Q.E.D

Condition (33) has an obvious economic meaning. A sufficiently large number of bidders shrinks the effect of an interest subsidy on the seller's financing cost. As long as the interest rate is above the threshold so that the winner is the richest bidder, a large number of bidders would "push" the winner toward the top of the budget support, making it probable that the seller bears no financing cost at all.

Notice that Condition (33) does not mean that we are considering the effects of an interest subsidy in the limit as the number n of bidders becomes arbitrarily large. Instead, the explicit lower bound for n required by the condition can be quite small. For example, suppose that $\theta = 1/4$, $v = 1/2$, and $q = 2/3$. Then (33) becomes $n \geqslant 23/25$, which is vacuously true.

5. CONCLUDING COMMENTS

By considering bidders' budget constraints and default options, this paper has obtained equilibrium results novel to auction theory. On one hand, the equilibrium of "high bids and broke winners," where the auction is won by the most budget-constrained bidder who would most likely declare bankruptcy, shows the impact of default options. On the other hand, the multiple equilibria at above-threshold borrowing rates, where high-budget bidders submit non-monotone bids, indicate that budget constrains as private types lead to a mathematical structure different from the one in standard auction theory, where bidders' types are preferences or costs.

The paper also demonstrates the seller's incentive to provide financing. This incentive arises because an interest subsidy to the highest bidder would intensify bidders' competition. Even with default risk, the seller can in some cases raise her expected profits by offering to finance the winning bidder at a below-market interest rate. In doing so, the seller needs to charge a rate above the threshold to avoid the equilibrium of "high bids and broke winners."

As noted previously, this paper has made a few simplifying assumptions. Further extensions may include other arrangements of default, bidding through subsidiaries, and pre-bidding financing. A related challenging problem is to design an optimal auction in our environment. Except for the work by Che and Gale [5], who consider one-bidder no-default cases, little is known about this problem.

APPENDIX: NOMENCLATURE

p	Payment
r	Borrowing rate
θ	Probability of zero value
v	Maximum value of the good
q	Exogenous borrowing rate
n	Number of bidders
i	Index for bidders
m_i	budget of bidder i
C	Cost function
F (resp. f)	Distribution (resp. density) function of budgets
b_w	Winner's bid
m_w	Winner's budget
$\mathbf{v}$	The (random) value of the good
$[\underline{m}, \bar{m}]$	The support of F

β	Symmetric equilibrium bidding strategy
$\beta\|_A$	Function β restricted to a set A
u	Bidder's payoff conditional on winning
s	bidding strategy
V_s	Expected payoff given others playing s
b	A bid
Prob	Probability
m	A budget
E	Expected-value operator
N	A neighborhood
m^L_{-i}	The lowest budget among one's rivals
r'	$\theta/(1-\theta)$
G	$(1-F)^{n-1}$
$\beta_1, \beta_2, \beta_3$	Function β restricted to some intervals
D_1	Partial derivative operator w.r.t. the first variable
$m_*(r)$	A cutoff point determined by r
g_r	A function determined by r
$\hat{m}$	A cutoff point independent of r
h (resp. h')	A function (resp. its derivative) independent of r
m^H_{-i}	The highest budget among one's rivals
$m^*(r)$	A cutoff point determined by r
U	A bidder's surplus
ζ, ξ	Temporary symbols
ϕ	A function
Φ_x	A distribution function determined by x
β_x	A bidding function indexed by x
d	Debt
$\hat{m}_w$	Winner's reported budget
x^+	$\max\{0, x\}$
χ	Indicator function
β_r	The equilibrium bid function at borrowing rate r
$\pi(r)$	The expected profit at borrowing rate r
π_0	The expected profit without offering a loan
$F_{(1)}$	F^n
$:=$	Is defined as
$\equiv$	Is constantly equal to

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