Microeconomics Core Exam Question

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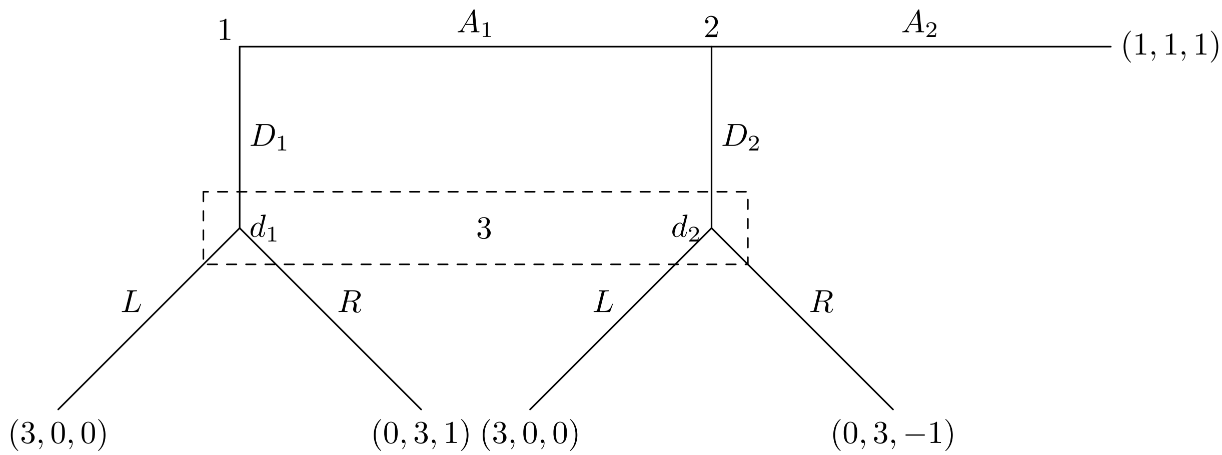
Question Consider the following extensive-form game



Denote mixed strategies by

$$\begin{aligned} \sigma_1 &:= xD_1 + (1-x)A_1, \\ \sigma_2 &:= yD_2 + (1-y)A_2, \\ \sigma_3 &:= zL + (1-z)R, \\ \sigma &:= (\sigma_1, \sigma_2, \sigma_3), \end{aligned}$$

with $x, y, z \in [0, 1]$. Use these notations in solving the following problems.

a. Calculate the necessary and sufficient condition for L to be a best response to σ. (Note the article "a"—rather than "the"—before "best response.")

b. Calculate the necessary and sufficient condition for A_2 to be a best response to σ.

c. Calculate the necessary and sufficient condition for A_1 to be a best response to σ.

d. Find the set of all the Nash equilibria where A_1 is played with probability one.

e. Find the set of all the Nash equilibria where A_1 is played with probability strictly between zero and one.

Solution

a. Calculate the necessary and sufficient condition for L to be a best response to σ.

L being *a* best response for player 3 is equivalent to L is *weakly* preferred to R given σ:

$$\begin{aligned} L \succeq_3 R &\iff 0 \geq 1 \cdot x + (-1) \cdot (1-x)y \\ &\iff (1-x)y \geq x. \end{aligned}$$

Equivalently, the answer can be $y/(1+y) \geq x$. Just do not write them as *strict* inequalities, as the question does not ask for L being *the* best response.

b. Calculate the necessary and sufficient condition for A_2 to be a best response to σ.

A_2 being a best response for player 2 is equivalent to either $x = 1$ (so A_2 is just as good as D_2 for player 2) or (non-exclusively) $A_2 \succeq_2 D_2$, with

$$\begin{aligned} A_2 \succeq_2 D_2 &\iff 1 \geq 0 \cdot z + 3 \cdot (1-z) \\ &\iff z \geq 2/3. \end{aligned}$$

c. Calculate the necessary and sufficient condition for A_1 to be a best response to σ.

$$\begin{aligned} A_1 \succeq_1 D_1 &\iff 3 \cdot yz + 1 \cdot (1-y) \geq 3 \cdot z \\ &\iff 1 - y \geq 3z(1-y) \\ &\iff [y = 1 \text{ or } z \leq 1/3]\,. \end{aligned}$$

Note that it is incorrect to write the third line as "$\iff z \leq 1/3$."

d. Find the set of all the Nash equilibria where A_1 is played with probability one.

Claim: This set is empty.

Proof: With A_1 played for sure, $x = 0$. Note $y = 1$, otherwise $z \leq 1/3$ (by Part (c) and $A_1 \succeq_1 D_1$) and $A_2 \succeq_2 D_2$ (by definition of the notation y), which in turn implies, by Part (b) and $x < 1$, that $z \geq 2/3$, contradiction. Now that $y = 1$, Part (c) and A_1 being played at all together imply that $z \leq 1/3$. Furthermore, since $x = 0$ and $y = 1$, we have $(1-x)y = 1 > 0 = x$. Then Part (a) implies $L \succ_3 R$, i.e., $z = 1$, contradiction. ■

(If a student misses the "or" nuance in parts (b) and (c) and gets only "$z \geq 2/3$" and "$z \leq 1/3$" for those parts, then the correct conclusion of empty set in part (d) would be easier to obtain than the above proof.)

e. Find the set of all the Nash equilibria where A_1 is played with probability strictly between zero and one.

Claim: This set is

$$\{(xD_1 + (1-x)A_1, D_2, zL + (1-z)R) : 1/2 \leq x < 1; z \leq 2/3; z > 0 \Rightarrow x = 1/2\}.$$

Proof: By the first half of the proof in Part (d), $y = 1$, which, by Part (c), suffices to guarantee that $A_1 \sim_1 D_1$. To guarantee that D_2 $(y = 1)$ is a best response, Part (b) requires that $z \leq 2/3$. With $z \leq 2/3$, R needs to be a best response, hence Part (a) requires $(1-x)y \leq x$, which, due to $y = 1$, is equivalent to $1 - x \leq x$, i.e., $x \geq 1/2$. Furthermore, if $z > 0$, then $0 < z \leq 2/3$ means that $L \sim_3 R$, which in turn requires (by Part (a)) that $(1-x)y = x$ and hence (since $y = 1$) $1 - x = x$, i.e., $x = 1/2$; in other words, $z > 0 \Rightarrow x = 1/2$. ■

Note: If one misses the "or $y = 1$" clause in Part (c), then the partial condition about $z \leq 1/3$ obtained there, coupled with $A_1 \sim_1 D_1$, would force $z = 1/3$; that means $L \sim_3 R$ and hence $(1-x)y = x$, which coupled with $y = 1$ would mean $x = 1/2$; then the answer for (e) would be only the singleton containing

$$\left(\frac{1}{2}D_1 + \frac{1}{2}A_1, D_2, \frac{1}{3}L + \frac{2}{3}R\right).$$