Qualifier Exam Question
Economics of Contracts
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The primitives Consider an auction environment with a single good for sale and two bidders. Each bidder's type, which is his private information, is independently drawn from $[0,1]$ according to a commonly known cumulative distribution function F whose density f is defined by

$$f(t) := \begin{cases} 2 & \text{if } 0 \le t < 1/3 \\ 1/2 & \text{if } 1/3 < t \le 1. \end{cases}$$

Given any realized type t_i, bidder i's ex post payoff is equal to $1+t_i-p$ if he wins the good and pays an amount p of money, equal to $-p'$ if he does not win the good and pays an amount p' of money, and equal to zero if he does not participate at all. (Thus, to a bidder of type t_i, the valuation of the good is equal to t_i+1.) The seller's valuation of the good is zero. Everyone is risk neutral.

Notations A mechanism means a list $(q_i, p_i)_{i=1}^2$ of functions defined on $M_1 \times M_2$ for some sets M_1 and M_2 such that if bidder 1 chooses an element m_1 from M_1, and bidder 2 chooses an element m_2 from M_2, then bidder i ($i = 1, 2$) gets the good with probability $q_i(m_1, m_2)$ and pays the seller an amount $p_i(m_1, m_2)$ of money. The seller's decision problem is to maximize his expected revenue among all mechanisms, each assessed by a Bayesian Nash equilibrium (BNE) admitted by the mechanism. If (s_1, s_2) is a BNE of a mechanism $(q_i, p_i)_{i=1}^2$ such that $s_i : [0,1] \to M_i$ is bidder i's equilibrium strategy, denote for any $i \in \{1,2\}$ and any $t_i \in [0,1]$

$$Q_i(t_i \mid s) := \int_0^1 q_i\left(s_i(t_i), s_{-i}(t_{-i})\right) dF(t_{-i}),$$

$$P_i(t_i \mid s) := \int_0^1 p_i\left(s_i(t_i), s_{-i}(t_{-i})\right) dF(t_{-i}).$$

1. Provide a solution—in terms of the primitives—for M_1 and M_2 that is as simple as possible without causing any loss of generality to the seller's decision. Explain your answer with no more than one sentence.

2. Given any mechanism $(q_i, p_i)_{i=1}^2$ and its associated BNE (s_1, s_2), use the notations provided above to do the following:

   a. Write down a bidder i's expected payoff from responding to (s_1, s_2) by choosing $s_i(t_i')$ while his true type is t_i.

   b. Write down a necessary and sufficient condition for (s_1, s_2) to constitute a BNE of $(q_i, p_i)_{i=1}^2$.

   c. How to reduce the seller's choice variable to the dimensions of (q_1, q_2) (i.e., how to eliminate the payment dimensions (p_1, p_2) of the choice variable)?

3. Derive from the primitives a function $V : [0,1] \to \mathbb{R}$ such that there is no loss of generality to assume that the seller's expected revenue from any mechanism $(q_i, p_i)_{i=1}^2$ and its associated BNE (s_1, s_2) is equal to

$$\int_0^1 \int_0^1 \sum_{i=1}^{2} V(t_i) q_i \left(s_1(t_1), s_2(t_2)\right) dF(t_1) dF(t_2).$$

4. Consider a mechanism that uses a *hierarchical allocation with respect to* V: $M_1 = M_2 = [0,1]$ and, for any $(t_1, t_2) \in M_1 \times M_2$ ($= [0,1]^2$ in this case), $q_i(t_1, t_2) = 1$ if $V(t_i) > V(t_{-i}) \geq 0$, $q_i(t_1, t_2) = 0$ if $V(t_i) < \max\{0, V(t_{-i})\}$, and $q_i(t_1, t_2) = 1/2$ if $V(t_i) = V(t_{-i}) \geq 0$. Does this mechanism admit a BNE (s_1, s_2) such that $s_i(t_i) = t_i$ for each i and each $t_i \in [0,1]$? Explain your answer briefly.

5. Characterize an optimal mechanism for the seller through the following steps:

   a. Calculate explicitly the inverse F^{-1} of F, the composite function $h := V \circ F^{-1}$ (i.e., $h(s) := V\left(F^{-1}(s)\right)$ for all $s \in [0,1]$), and the function H defined by $H(s) := \int_0^s h(r) dr$ for all $s \in [0,1]$.

   b. Denote $\widehat{H}$ for the convex hull of H. Let $0 < x < y < 1$ such that $\widehat{H} < H$ on (x, y), $\widehat{H}(x) = H(x)$, and $\widehat{H}(y) = H(y)$.

      i. Which of the following is true?

         A. $x < 1/3 < y$

         B. $2/3 < x < y$

         C. $x < 2/3 < y$

         D. $x < y < 2/3$

         E. none of the above

      ii. Write down an equation system that determines the values of x and y. Make these equations as explicit as possible according to the formulas of h, H, etc., without solving the equations.

   c. Treating x and y as known numbers, do the following:

      i. Provide the formula of a function $\widehat{h} : [0,1] \to \mathbb{R}$ that is equal to the derivative of $\widehat{H}$ almost everywhere on $[0,1]$.

      ii. Provide the formula of a function $\overline{V} : [0,1] \to \mathbb{R}$ such that the hierarchical allocation with respect to $\overline{V}$ (Step 4 with $\overline{V}$ replacing V) is optimal for the seller. Briefly explain why this allocation admits a BNE (s_1, s_2) such that $s_i(t_i) = t_i$ for each i and each $t_i \in [0,1]$.

Solution

1. $M_1 = M_2 = [0,1]$ by the revelation principle.

2. a. $(t_i + 1)Q_i(t_i' \mid s) - P_i(t_i' \mid s)$. (Note that the coefficient is $t_i + 1$ rather than t_i, as the valuation of the good is $t_i + 1$, not t_i.)

   b. Simultaneous satisfaction of three conditions:

      i. for each $i \in \{1,2\}$, $Q_i(\cdot \mid s)$ is weakly increasing on $[0,1]$;

      ii. for each $i \in \{1,2\}$ and any $t_i, t_i' \in [0,1]$,

$$(t_i + 1)Q_i(t_i \mid s) - P_i(t_i \mid s) = (t_i' + 1)Q_i(t_i' \mid s) - P_i(t_i' \mid s) + \int_{t_i'}^{t_i} Q_i(r \mid s)dr; \quad (1)$$

      iii. for each $i \in \{1,2\}$ and any $t_i \in [0,1]$, $(t_i + 1)Q_i(t_i \mid s) - P_i(t_i \mid s) \geq 0$.

   c. Plug $t_i' = 0$ into Eq. (1) to see that $P_i(\cdot \mid s)$ is determined by $Q_i(\cdot \mid s)$ up to a constant $P_i(0 \mid s)$, and the constant is pinned down by $Q_i(0 \mid s) - P_i(0 \mid s) = 0$ due to the participation constraint, or Condition 2(b.)iii.

3. By the given f, we have

$$F(t_i) = \begin{cases} 2t_i & \text{if } 0 \leq t_i \leq 1/3 \\ 2/3 + (t_i - 1/3)/2 = (t_i + 1)/2 & \text{if } 1/3 \leq t_i \leq 1. \end{cases} \quad (2)$$

   Thus, since the seller's valuation is zero and a type-t_i bidder's valuation is $t_i + 1$,

$$V(t_i) = t_i + 1 - \frac{1 - F(t_i)}{f(t_i)} = \begin{cases} 2t_i + 1/2 & \text{if } 0 \leq t_i < 1/3 \\ 2t_i & \text{if } 1/3 < t_i \leq 1. \end{cases} \quad (3)$$

   (If a student mistakes t_i for the valuation, then his answer is $V(t_i) = t_i - \frac{1-F(t_i)}{f(t_i)}$, which is $2t_i - 1/2$ if $0 \leq t_i < 1/3$, and $2t_i - 1$ if $1/3 < t_i \leq 1$.)

4. No, because the V in Eq. (3) drops at $t_i = 1/3$, from $\lim_{t\uparrow 1/3} V(t) = 7/6$ to $\lim_{t\downarrow 1/3} V(t) = 2/3$. Since both levels are positive, the hierarchical allocation $Q_i(\cdot \mid s)$ with respect to V is not weakly increasing on $[0,1]$, violating the monotonicity condition in Step 2(b.)i.

   (If a student mistakes t_i for the valuation, then the drop of V, consistent with the mistake, would be from $1/6$ to $-1/3$. Thus it is uncertain at this point whether the ironed level of V around the drop is positive or negative. If it is negative, the non-monotonicity of V would not hurt the monotonicity of Q_i, as types with negative (ironed) virtual utilities have zero probability of winning. Most students are unlikely to see this subtlety though. I would give almost full credit to an answer that recognizes the connection from the non-monotonicity of V to the non-monotonicity of Q_i and from the non-monotonicity of Q_i to the failure of its incentive compatibility.)

5. a. By Eq. (2),

$$F^{-1}(s) = \begin{cases} s/2 & \text{if } 0 \le s \le 2/3 \\ 2s-1 & \text{if } 2/3 \le s \le 1. \end{cases} \tag{4}$$

Thus, by Eq. (3),

$$h(s) = V\left(F^{-1}(s)\right) = \begin{cases} 2(s/2)+1/2 = s+1/2 & \text{if } 0 \le s < 2/3 \\ 2(2s-1) = 4s-2 & \text{if } 2/3 < s \le 1. \end{cases} \tag{5}$$

(The answer consistent with mistaking t_i for the valuation is $h(s) = 2(s/2)-1/2 = s-1/2$ if $0 \le s < 2/3$, and $h(s) = 2(2s-1)-1 = 4s-3$ if $2/3 < s \le 1$.)

If $s \in [0, 2/3]$, by the upper branch of (5),

$$H(s) = \int_0^s h(r)dr = \int_0^s (r+1/2)dr = s(s+1)/2.$$

Note $H(2/3) = 5/9$. Thus, if $s \in [2/3, 1]$, by the lower branch of (5),

$$H(s) = \int_0^{2/3} h(r)dr + \int_{2/3}^s h(r)dr = \frac{5}{9} + \int_{2/3}^s (4r-2)dr = 2s^2 - 2s + 1.$$

In sum,

$$H(s) = \begin{cases} s(s+1)/2 & \text{if } 0 \le s \le 2/3 \\ 2s^2-2s+1 & \text{if } 2/3 \le s \le 1. \end{cases} \tag{6}$$

(The answer consistent with mistaking t_i for the valuation is $H(s) = s(s-1)/2$ if $0 \le s \le 2/3$, and $H(s) = \frac{1}{9} + \int_{2/3}^s (4r-3)dr = 2s^2 - 3s + 11/9$ if $2/3 \le s \le 1$.)

b. i. C. $x < 2/3 < y$. That is because $s = 2/3$ is the point where h drops, according to Eq. (5), and hence its antiderivative H is non-convex around $s = 2/3$.

ii. By definition of convex hull, $\widehat{H}$ is affine on $[x, y]$ and its graph is tangent to that of H at x and at y. Thus the equation system is

$$h(x) = h(y),$$
$$H(y) - H(x) = h(x)(y-x).$$

To make these equations explicit, rewrite them according to Eqs. (5) and (6) and the fact $x < 2/3 < y$ (Step 5(b.)i):

$$x + 1/2 = 4y - 2,$$
$$2y^2 - 2y + 1 - x(x+1)/2 = (x+1/2)(y-x).$$

(The answer consistent with mistaking t_i for the valuation is: $x - 1/2 = 4y - 3$ and $2y^2 - 3y + 11/9 - x(x-1)/2 = (x-1/2)(y-x)$.)

c. i. Since h is increasing on $[0, 2/3)$ and on $(2/3, 1]$, $[x, y]$ is the only region on which the convex hull $\widehat{H}$ differs from the antiderivative H of h. Thus

$$\widehat{h}(s) = \begin{cases} h(s) & \text{if } s \notin [x, y] \\ h(x) & \text{if } s \in [x, y]. \end{cases}$$

Since $x < 2/3 < y$ (Step 5(b.)i), Eq. (5) implies

$$\widehat{h}(s) = \begin{cases} s + 1/2 & \text{if } 0 \leq s \leq x \\ x + 1/2 & \text{if } x \leq s \leq y \\ 4s - 2 & \text{if } y \leq s \leq 1. \end{cases} \tag{7}$$

(The answer consistent with mistaking t_i for the valuation is: $\widehat{h}(s) = s - 1/2$ if $0 \leq s \leq x$, $\widehat{h}(s) = x - 1/2$ if $x \leq s \leq y$, and $\widehat{h}(s) = 4s - 3$ if $y \leq s \leq 1$.)

ii. By the ironing procedure, $\overline{V} = \widehat{h} \circ F$. Combine this with Eqs. (2) and (7) and with the fact that

$$x/2 = F^{-1}(x) < F^{-1}(2/3) = 1/3 < F^{-1}(y) = 2y - 1,$$

which is due to $x < 2/3 < y$ and Eq. (4). Then

$$\begin{aligned} \overline{V}(t_i) &= \widehat{h}\left(F(t_i)\right) = \begin{cases} F(t_i) + 1/2 & \text{if } 0 \leq F(t_i) < x \\ x + 1/2 & \text{if } x \leq F(t_i) < y \\ 4F(t_i) - 2 & \text{if } y \leq F(t_i) \leq 1 \end{cases} \\ &= \begin{cases} 2t_i + 1/2 & \text{if } 0 \leq t_i < F^{-1}(x) \\ x + 1/2 & \text{if } F^{-1}(x) \leq t_i < F^{-1}(y) \\ 4\left(t_i + 1\right)/2 - 2 & \text{if } F^{-1}(y) \leq t_i \leq 1 \end{cases} \\ &= \begin{cases} 2t_i + 1/2 & \text{if } 0 \leq t_i < x/2 \\ x + 1/2 & \text{if } x/2 \leq t_i < 2y - 1 \\ 2t_i & \text{if } 2y - 1 \leq t_i \leq 1. \end{cases} \end{aligned}$$

(The answer consistent with mistaking t_i for the valuation is: $\overline{V}(t_i) = 2t_i - 1/2$ if $0 \leq t_i < x/2$, $\overline{V}(t_i) = x - 1/2$ if $x/2 \leq t_i < 2y - 1$, and $\overline{V}(t_i) = 2t_i - 1$ if $2y - 1 \leq t_i \leq 1$.)

Since $\overline{V}$ is weakly increasing on $[0, 1]$, so is the Q_i derived from the hierarchical allocation with respect to $\overline{V}$. Hence Q_i is incentive compatible.