Chapter 1: Firm's Supply

Elements of Decision: Lecture Notes of Intermediate Microeconomics

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1 Introduction

Economics is a science on relations among individuals. Such relations include trades, contracts, conflicts, households, markets, networks, societies, etc. Any such relation can be modeled into a game, where one tries to pick a move as a best response to the move he expects his counterpart may pick, knowing that the counterpart's move also results from her attempt to best respond to his move that she expects him to pick. Calculations of such *interactive reasoning* constitute *game theory*, the mathematical foundation of modern economics. The fundamental building block of such calculations is the mathematical method for an individual to decide on his best move. To introduce such *decision-theoretic* method in the simplest possible setting, we start by pretending that the counterpart of the individual is not a strategic player trying to game him but rather a dummy such as a purely competitive market, which the individual does not need to haggle with. Given this assumption of pure competition, learning the basic decision-theoretic techniques will constitute two thirds of this course. Then, for the remaining third, we will apply such techniques to basic concepts in game theory.

2 A firm's output decision problem

To illustrate a decision problem in its simplest form, suppose that a firm acts as an individual deciding how much of its output to sell.[1] Suppose that every unit of the output is sold for p dollars in the market (the aforementioned pure competition assumpiton), and that the firm incurs a cost $C(q)$ dollars if it sells q units of the output to the market. The question is How many units of the output should the firm supply to maximize its profit? As *profit* means revenue minus cost, the firm's profit from supplying q units is equal to $pq - C(q)$ dollars. Thus the firm's decision problem becomes choosing a nonnegative quantity q to maximize $pq - C(q)$, i.e.,

$$\max_{q \in \mathbb{R}_+} pq - C(q). \tag{1}$$

Problem (1) exemplifies an optimization problem: (i) There is an *objective*, the expression $pq - C(q)$ following the operator max (shorthand for maximization), which the decision maker is to maximize (or minimize if max is replaced by min). (ii) There is a *choice variable* and its *domain*, written underneath max to indicate that the decision maker is to choose an element from the domain to maximize his objective; here the choice variable is denoted q, its domain $\mathbb{R}_+$ (the set of nonnegative real numbers),[2] with the symbol $\in$ meaning "belongs to" or "is an element of."

[1] For a real-world example, think of a small dairy farm that sells a single kind of milk to a competitive market.

[2] Interchangeably $\mathbb{R}_+$ is also denoted by $[0, \infty)$, meaning the set of real numbers between zero and infinity ∞, with the [on the left signifying that the boundary point zero is included, and the) on the right that ∞ is excluded. The notation $(0, \infty)$, with the left [replaced by (, excludes zero and denotes the set of positive real numbers.

(iii) There is a *parameter*, which, p in this example, is assumed constant by the decision maker; it is important to keep in mind the distinction between a choice variable and a parameter: the former is up to the decision maker to choose, while the latter is not.

3 Cost function

To make Problem (1) more tractable, let us add two usual assumptions, and figure out their implications, of the above cost function C. It is usually *assumed* that C is of the form

$$C(q) = C_v(q) + c_0 \tag{2}$$

for any $q \geq 0$, where $C_v(q)$ varies with q with $C_v(0) = 0$, hence called *variable cost*, and c_0 is positive and constant to q, hence called *fixed cost*. Correspondingly, the *average cost*, defined to be $C(q)/q$ and denoted by $\mathrm{AC}(q)$, is decomposed by

$$\mathrm{AC}(q) = \frac{C(q)}{q} = \frac{C_v(q)}{q} + \frac{c_0}{q}, \tag{3}$$

with $C_v(q)/q$ called *average variable cost* (AVC), and c_0/q *average fixed cost*.

It is also *assumed* that C is differentiable and that its derivative $\frac{d}{dq}C$, called *marginal cost* and denoted by $\mathrm{MC}(q)$, is continuously decreasing in q when q rises from zero and, once q rises to a threshold level, becomes continuously increasing in q thereafter without upper bound. By Eq. (2),

$$\mathrm{MC}(q) = \frac{d}{dq}C(q) = \frac{d}{dq}C_v(q). \tag{4}$$

Draw a coordinate system whose horizontal axis stands for q, and vertical axis for the marginal cost and the various kinds of average costs. In this coordinate system the marginal cost function corresponds to a curve, U-shape because of the assumption stated before Eq. (4). Note the vertical intercept $\mathrm{MC}(0)$ of this curve. A fact is that $\mathrm{MC}(0)$ is also equal to the vertical intercept of the AVC curve, i.e.,

$$\mathrm{MC}(0) = \lim_{q\to 0} \frac{C_v(q)}{q}. \tag{5}$$

The right-hand side of this equation might look complicated from an elementary arithmetic viewpoint, as $C_v(q)/q$ would become zero divided by zero if $q = 0$. This issue is resolved by basic calculus, where L'Hôpital's rule implies

$$\lim_{q\to 0} \frac{C_v(q)}{q} = \lim_{q\to 0} \left(\frac{d}{dq}C_v(q) \Big/ \frac{d}{dq}q \right) \overset{(4)}{=} \lim_{q\to 0} \left(\mathrm{MC}(q)/\, 1\right) = \mathrm{MC}(0),$$

hence Eq. (5) is true. Thus, at the point $q = 0$, the MC and AVC curves coincide.

Starting from $q = 0$, and from the same vertical intercept, draw the graphs of both MC and AVC for those $q > 0$ that are nearby zero. Assumed U-shape, the MC curve must go downward for such near-zero q's. What about the AVC curve? To figure that out, recall the definition of AVC, $\mathrm{AVC}(q) = C_v(q)/q$, and calculate its derivative by the quotient rule:

$$\frac{d}{dq}\mathrm{AVC}(q) = \frac{d}{dq}\frac{C_v(q)}{q} = \frac{1}{q^2}\left(q\frac{d}{dq}C_v(q) - C_v(q) \right) = \frac{1}{q}\left(\mathrm{MC}(q) - \mathrm{AVC}(q)\right).$$

Thus, whenever $q > 0$,

$$\text{MC}(q) > (\text{resp.} <) \ \text{AVC}(q) \iff \frac{d}{dq}\text{AVC}(q) > (\text{resp.} <) \ 0, \tag{6}$$

where the double arrow $\Leftrightarrow$ is read as "if and only if," also written as iff, meaning that the statements on the two sides of the arrow are logically equivalent. Eq. (6) says that whenever the MC curve is above the AVC curve ($\text{MC}(q) > \text{AVC}(q)$), the AVC curve is upward sloping ($\frac{d}{dq}\text{AVC}(q) > 0$), and whenever MC is below AVC ($\text{MC}(q) < \text{AVC}(q)$), AVC is downward sloping ($\frac{d}{dq}\text{AVC}(q) < 0$). Consequently, at the positive q's nearby zero, the AVC curve must lie above MC. Why? Suppose, to the contrary, that AVC lies below MC at such q's; bounded from above by the downward-sloping MC curve, the AVC curve must be forced to slope downward at some of such q's, but then Eq. (6) implies that the AVC curve jumps above the MC curve at such q's, contradiction.

Thus, the AVC curve is above the MC curve and is downward sloping when q increases slightly from zero, and remains so as q further increases until the AVC curve crosses the MC curve. Why must such intersection happen? That is because the MC curve, assumed U-shape without upper bound, must become upward sloping eventually and then rise high enough to surpass AVC, which by Eq. (6) is downward sloping as long as MC is still below it. Once q is above the threshold at which the two curves intersect, we have $\text{MC}(q) > \text{AVC}(q)$ and then Eq. (6) implies that AVC slopes upward from now on. Hence—

> the AVC curve is also U-shape, and the point at which it turns from downward- to upward-sloping is exactly the intersection between the AVC and MC curves.

To plot the graph for the average cost AC, note from Eq. (3) that $\text{AC}(0) = c_0/0 = \infty$. Hence when q increases slightly from zero, the AC curve lies above the MC curve. Thus, by the same method as in the case of AVC, one can demonstrate that *the average cost curve is also U-shape, and the point at which it turns from downward- to upward-sloping is exactly the intersection between the AC and the MC curves*. Note again from Eq. (3) that the AC curve is always above the AVC curve and that the gap between them, which equals c_0/q, shrinks to zero as q goes to infinity.

4 The profit-maximizing output

To solve Problem (1), note that the objective $pq - C(q)$ is a differentiable function of the choice variable q. Thus, by basic calculus, if q_* is an *interior solution* of the problem in the sense that q_* solves Problem (1) and $q_* \neq 0$ (as zero is the boundary of the domain $\mathbb{R}_+$ for q), then q_* satisfies the *first-order condition*, i.e., the first-order derivative of the objective with respect to q is equal to zero when $q = q_*$. That is,

$$\frac{d}{dq}\left(pq - C(q)\right)\Big|_{q=q_*} = p - \frac{d}{dq}C(q)\Big|_{q=q_*} = 0.$$

Here the notation $|_{q=q_*}$ signifies the operation of calculating the derivative at the point q_*: first, take the derivative of $C(q)$ with respect to q for general q's; then plug $q = q_*$ into what you have just obtained. The above-displayed equation is equivalent to

$$p = \text{MC}(q_*). \tag{7}$$

Thus, if q_* is an interior solution then it is an output quantity at which the MC curve intersects the horizontal line of height p. Since MC is assumed U-shaped, if it intersects the horizontal line at all then in general there are two intersection points, one in the range where MC is downward sloping, the other upward sloping. Are they both solutions to Problem (1)? If not then how to tell which one is? To answer these questions, recall from basic calculus that if an interior point q_* maximizes the objective then it satisfies the *second-order condition*: the second-order derivative of the objective at q_* is nonpositive. To write down the second-order condition in our case, calculate the second-order derivative of the objective $pq - C(q)$ with respect to q:

$$\frac{d^2}{dq^2}\left(pq - C(q)\right) = -\frac{d}{dq}\mathrm{MC}(q).$$

Thus, the second-order condition is $-\frac{d}{dq}\mathrm{MC}(q) \leq 0$, i.e.,

$$\left.\frac{d}{dq}\mathrm{MC}(q)\right|_{q=q_*} \geq 0.$$

This condition eliminates the intersection point where MC slopes downward (where $\frac{d}{dq}\mathrm{MC}(q) < 0$), leaving the other intersection point the only possible candidate.

Thus, the only candidate for an interior solution of Problem (1) is the output level q_* at which the upward sloping portion of the MC curve intersects the horizontal line of height p. Satisfying both the first- and second-order conditions, q_* is a *local* maximum of the objective, local in the sense that the objective does not get higher when q is slightly above or below q_*. In general, satisfaction of both conditions does not suffice being the *global* maximum among all interior points, as there may be other local maximums. In our case, however, since q_* is the only local maximum, it is also the global maximum among all interior points.

Consequently, to solve Problem (1) we need only to compare the q_* obtained above with the corner point zero. The former generates profit $pq_* - C(q_*)$, while the latter generates profit $-c_0$ by Eq. (2) and the fact $C_v(0) = 0$. The profit-maximizing output level is q_* if $pq_* - C(q_*) > -c_0$, and zero if the inequality is reversed. This inequality, by Eq. (2), is equivalent to

$$pq_* - C_v(q_*) > 0, \quad \text{i.e.,} \quad p > \frac{C_v(q_*)}{q_*},$$

which, due to Eq. (7), is equivalent to

$$\mathrm{MC}(q_*) > \mathrm{AVC}(q_*).$$

Thus, q_**, the output level at which the upward-sloped portion of MC intersects the price line* p*, is the profit-maximum if the intersection lies above the intersection between MC and AVC; if it lies below the latter intersection then the profit-maximum is zero, i.e., supplying no output at all.*

5 An example

Suppose that the cost function is

$$C(q) := 2q^3 - 12q^2 + 30q + 100$$

for any $q \geq 0$ (where the symbol $:=$ means "is defined to be equal to"). One readily verifies that

$$\begin{aligned} c_0 &= 100, \\ C_v(q) &= 2q^3 - 12q^2 + 30q, \\ \mathrm{AC}(q) &= 2q^2 - 12q + 30 + 100/q, \\ \mathrm{AVC}(q) &= 2q^2 - 12q + 30, \\ \mathrm{MC}(q) &= 6q^2 - 24q + 30. \end{aligned} \tag{8}$$

Note, as in the general case demonstrated previously, $\mathrm{AVC}(0) = 30 = \mathrm{MC}(0)$ and $\mathrm{AC}(0) = \infty$. To find the intersection between MC and AVC, solve the equation $\mathrm{AVC}(q) = \mathrm{MC}(q)$, i.e.,

$$2q^2 - 12q + 30 = 6q^2 - 24q + 30,$$

which gives $q = 3$. Alternatively, we can find this intersection through the established fact that the intersection is the minimum of AVC. Take the derivative of AVC and find the minimum via the first-order condition:

$$\frac{d}{dq}\mathrm{AVC}(q) = \frac{d}{dq}\left(2q^2 - 12q + 30\right) = 4q - 12 = 0,$$

which gives $q = 3$, same as the previous method. (Note that $q = 3$ is the global minimum of AVC because AVC is U-shaped: its second-order derivative

$$\frac{d^2}{dq^2}\mathrm{AVC}(q) = \frac{d}{dq}(4q - 12) = 4 > 0$$

for all $q \geq 0$.) Analogously, to find the intersection between AC and MC we solve the equation

$$4q^2 - 12q = \frac{100}{q}, \quad \text{i.e.,} \quad q^3 - 3q^2 - 25 = 0.$$

Plugging the coefficients of this cubic equation into a cubic equation calculator,[3] we find $q \approx 4.3$.

Suppose that the market price for the output is \$18 per unit, i.e., $p = 18$. Let us calculate the profit maximum, denoted by q_*. By Eqs. (7) and (8),

$$18 = 6q_*^2 - 24q_* + 30, \tag{9}$$

which is equivalent to $(q_* - 2)^2 = 2$, implying that $q_* = 2 + \sqrt{2}$ or $q_* = 2 - \sqrt{2}$. To figure out which one is the solution, calculate the slope of the MC curve:

$$\frac{d}{dq}\mathrm{MC}(q) = 12q - 24,$$

which is nonnegative iff $q \geq 2$. Since $2 + \sqrt{2} > 2 > 2 - \sqrt{2}$, the MC curve is upward sloping at $2 + \sqrt{2}$, and downward sloping at $2 - \sqrt{2}$. Thus the latter fails the second-order condition, and $q_* = 2 + \sqrt{2}$ is the global maximum among all interior points.

We still need to compare the profit generated by $q_* = 2 + \sqrt{2}$ and that by $q = 0$. To do that, recall that $q = 3$ is where the MC curve intersects with the AVC curve, so that the former lies above the latter for all $q > 3$. Thus, since $2 + \sqrt{2} \approx 3.414 > 3$, we have $\mathrm{MC}(2 + \sqrt{2}) > \mathrm{AVC}(2 + \sqrt{2})$, i.e., $2 + \sqrt{2}$ is *the* profit-maximizing output level in this example.

[3] For example, http://www.1728.org/cubic.htm.

6 A firm's supply curve

How does a change in the market price p (due to some exogenous shock to which the firm has no influence) affect the firm's profit maximizing output? To answer this question, let us start with the above example, with the market price \$18 there replaced by the general symbol p. Following the same reasoning thereof, we obtain, as Eq (9), that

$$p = 6q_*^2 - 24q_* + 30.$$

This equation is the same as $6\left(q_*^2 - 4q_* + 4\right) - 24 + 30 = p$, i.e.,

$$6(q_* - 2)^2 = p - 6.$$

Hence q_* equals either $2 + \sqrt{(p-6)/6}$ or $2 - \sqrt{(p-6)/6}$. Note that the former is greater than 2 and the latter less than 2. As shown previously, the MC curve in this example is downward sloping iff $q < 2$. Hence it is downward sloping when $q = 2 - \sqrt{(p-6)/6}$. Thus

$$q_* = 2 + \sqrt{(p-6)/6}$$

is the global maximum of the profit among interior points. To compare q_* with the corner point $q = 0$, recall the fact that q_* is the profit maximum iff the point (q_*, p) belongs to the portion of the MC curve above the AVC curve, i.e., iff p is above the minimum level of AVC. As calculated previously, the AVC in this example attains its minimum at $q = 3$, hence

$$\min \text{AVC} = \text{AVC}(3) = 2 \times 3^2 - 12 \times 3 + 30 = 12.$$

Thus, the profit-maximizing output level is equal to $2 + \sqrt{(p-6)/6}$ if $p > 12$, and zero if $p < 12$. That gives us a mapping from any market price p to $S(p)$, the amount of output that the firm chooses to supply in order to maximize its profit given p:

$$S(p) = \begin{cases} 2 + \sqrt{(p-6)/6} & \text{if } p \geq 12 \\ 0 & \text{if } p \leq 12. \end{cases}$$

The function S obtained above is called *supply function* of the firm in this example. The graph of this function, with q on the horizontal axis and p on the vertical axis, is the corresponding *supply curve*. Note that, other than the lower portion of the curve where $S(p) = 0$, the curve (above the price 12) is upward sloping, like the ones assumed in Introductory Microeconomics.

The general case is similar to the above example. Given a market price p and a cost function C satisfying the assumptions in Section 3, the global maximum of the firm's profit among all interior points is the solution for q_* in Eq. (7) at which the MC curve is upward sloping. This solution is unique because an upward sloping curve can cross a horizontal line only once. Hence we can denote this unique solution for q_* by $\text{MC}^{-1}(p)$, read as "the inverse image of p via the upward sloping portion of MC." As in the example, $q_* = \text{MC}^{-1}(p)$ is the profit maximum if it is above the output at which the AVC attains its minimum, i.e., if p is above the minimum level of AVC; else zero is the profit maximum. In other words, in the general case,

$$S(p) = \begin{cases} \text{MC}^{-1}(p) & \text{if } p \geq \min \text{AVC} \\ 0 & \text{if } p \leq \min \text{AVC}, \end{cases}$$

which gives the firm's supply curve. Again, the upward portion of the supply curve, $\left(\mathrm{MC}^{-1}(p), p\right)$, belonging to the upward sloping part of the MC curve, is upward sloping as assumed in the Introductory Microeconomics (Figure 1).

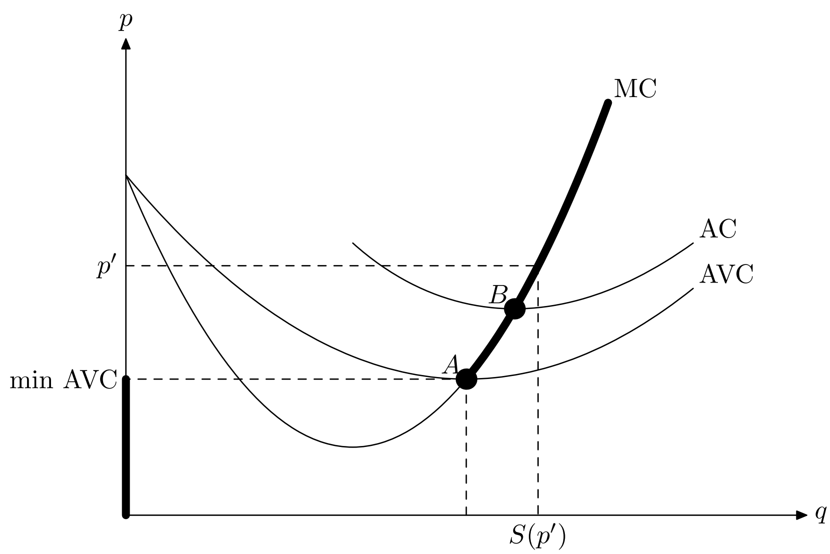

Figure 1: The thick vertical segment and thick curve starting from point A: Supply curve

7 What if the firm is a monopolist?

We have been assuming that the firm takes the market price as given. Suppose the opposite case where the firm sets the market price. That is the case where the firm is a monopolist, cornering the entire market. If the monopolist sets the price to be p dollars per unit of the good, consumers in order to buy the good have to pay this price because no other firm is offering the good. For a consumer to be willing to buy a unit at price p, his valuation of that unit must be no less than p. Thus, if $Q(p)$ denotes the number of units such that each unit is valued at least as high as p by some consumer, then $Q(p)$ is the quantity of the output supplied by the monopolist. Here Q is called *demand function* for the good supplied by the monopolist.[4]

To make this case easy to compare with our previous, pure competition case, let us convert the monopolist's decision on its market price p to the decision on its output quantity q. Suppose that the demand function Q is *one-to-one* in the sense that for any price level p there is at most one quantity q for which $q = Q(p)$. That means the *inverse* of the function Q exists in the sense that, for any total quantity q of the output supplied by the monopolist, there is a unique price level, denoted $P(q)$, for which $q = Q\left(P(q)\right)$. Here P is called *inverse demand function*. For example, if $Q(p) = 3 - 2p$ for each p, then solve for p from the equation $q = 3 - 2p$ to get $p = 3/2 - q/2$ and hence obtain $P(q) = 3/2 - q/2$.

In sum, a monopolist's decision problem is: Given cost function C, and inverse demand function P that maps any output quantity to the market price for the output, choose an output

[4] For notation convenience, we assume for the rest of this section, as in previous sections, that the output quantity can be any nonnegative real numbers, not only integers.

quantity to maximize its profit:

$$\max_{q \in \mathbb{R}_+} P(q)q - C(q). \tag{10}$$

To solve Problem (10), assume that P is a differentiable function. This, coupled with our previous assumption that C is differentiable, implies that the objective function in (10) is differentiable. Thus we can calculate the derivative

$$\frac{d}{dq}\left(P(q)q - C(q)\right) = \frac{d}{dq}P(q)q + P(q) - \frac{d}{dq}C(q),$$

where we use the product rule for derivatives. Denote $P'(q) := \frac{d}{dq}(P(q)$, $P''(q) := \frac{d^2}{dq^2}P(q)$ $(= \frac{d}{dq}P'(q))$ and, as in previous sections, $\mathrm{MC}(q) := \frac{d}{dq}C(q)$. Then

$$\begin{aligned} \frac{d}{dq}\left(P(q)q - C(q)\right) &= P'(q)q + P(q) - \mathrm{MC}(q), \\ \frac{d^2}{dq^2}\left(P(q)q - C(q)\right) &= P''(q)q + 2P'(q) - \frac{d}{dq}\mathrm{MC}(q), \end{aligned}$$

with the second equation again due to the product rule. Thus, the first-order condition is $P'(q_m)q_m + P(q_m) - \mathrm{MC}(q_m) = 0$, i.e.,

$$P'(q_m)q_m + P(q_m) = \mathrm{MC}(q_m); \tag{11}$$

and the second-order condition is

$$P''(q_m)q_m + 2P'(q_m) - \frac{d}{dq}\mathrm{MC}(q_m) \leq 0. \tag{12}$$

Thus, *if $q_m > 0$ is a profit-maximizing output quantity for the monopolist, then both (11) and (12) hold. Conversely, if (12) does not hold and $q_m > 0$ then q_m is not profit-maximizing.*

To illustrate how Eq. (11) pins down a monopolist's decision, consider the case where

$$P(q) = 354 - 3q$$

for all $q \geq 0$ and $C(q)$ is the same as that in Section 5 so that $\mathrm{MC}(q)$ is given by (8). Then $P'(q) = -3$ and Eq. (11) becomes

$$-3q_m + 354 - 3q_m = 6q_m^2 - 24q_m + 30,$$

which is reduced to $6q_m^2 - 18q_m - 324 = 0$ and further reduced to

$$q_m^2 - 3q_m - 54 = 0.$$

Solve this equation to obtain two solutions: $q_m = -6$ or $q_m = 9$. Since -6 does not belong to the choice domain, $q_m = 9$ is the only candidate. Then check $q_m = 9$ by the second-order condition. Note, in our example, that $P''(q) = 0$ and the left-hand side of (12) is equal to

$$2(-3) - (12q_m - 24) = -12(q_m - 3/2),$$

which is negative if $q_m = 9$. Thus, $q_m = 9$ satisfies both the first- and second-order conditions, and hence is the unique optimum among all $q > 0$.

Finally, to check whether $q_m = 9$ is the global maximum, compare it with $q = 0$. When $q = 0$ the profit is equal to $-C(0) = -100$. By contrast, when $q_m = 9$ the profit is equal to

$$(354 - 3 \cdot 9)9 - (2 \cdot 9^3 - 12 \cdot 9^2 + 30 \cdot 9 + 100) = 2943 - (1458 - 972 + 370) > -100.$$

Thus $q_m = 9$ is the global (profit-)maximum for the monopolist. That is, it would set the market price as $354 - 3 \times 9 = 327$ dollars per unit and sell 9 units of the output.

In general, note that $P(q)q$ is equal to the monopolist's revenue from the supply quantity q. Hence by the product rule the left-hand side of Eq. (11) is equal to the derivative of the revenue with respect to q, or *marginal revenue*, denoted by $\mathrm{MR}(q)$. Thus Eq. (11) becomes

$$\mathrm{MR}(q_m) = \mathrm{MC}(q_m). \tag{13}$$

Note that $\mathrm{MR}(q_m) = P'(q_m)q_m + P(q_m) < P(q_m)$, because $P'(q_m) < 0$ as the inverse demand curve is downward sloping. Thus,

$$P(q_m) > \mathrm{MC}(q_m). \tag{14}$$

In other words, *the monopolist sells its output at a price higher than its marginal cost*.

Figure 2 illustrates a monopolist's decision in the case where the inverse demand is $P(q) = a - bq$ for some positive parameters a and b, so the demand curve is a straight line with slope $-b$. Then $\mathrm{MR}(q) = a - 2bq$. That is, the graph of marginal revenue is a straight line that starts with the same vertical intercept as the demand curve and goes downward twice as fast as the demand curve does. Now locate the intersection point F between the marginal revenue and the marginal cost curves, and extend the vertical line passing through F to intersect the demand curve at point G. By Eq. (13), the horizontal coordinate of G is q_m; by definition of the demand curve, the vertical coordinate of G is the market price. Note, consistent to (14), point G is above F.

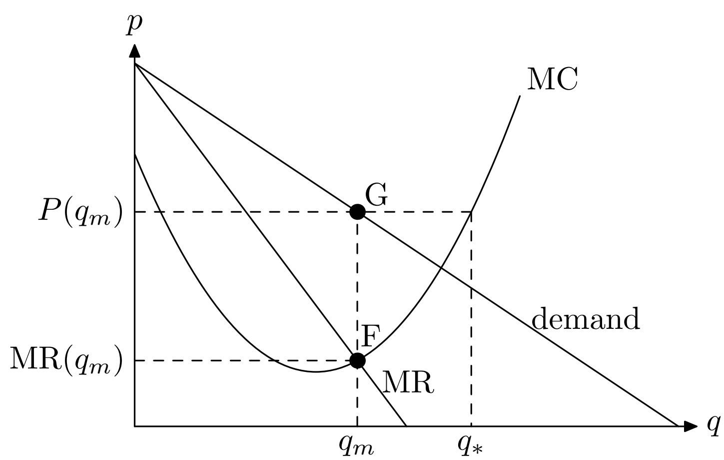

Figure 2: A monopolist's optimum

Let us contrast the monopolist's behavior with a firm that takes the market price $P(q_m)$ as given, and denote q_* for the latter's profit maximum. Then the first order condition for q_* is (7) where $p = P(q_m)$. Combine (7) with (14) to observe that

$$\mathrm{MC}(q_m) < \mathrm{MC}(q_*).$$

Since the MC curve is U-shape and, by the second-order condition, both q_m and q_* belong to the region where the MC curve is upward-sloping. Thus the above inequality implies

$$q_m < q_*.$$

That is, other things equal, *a monopolist supplies less than a price-taking firm*.

In addition to output quantity, the choice variable q can also be interpreted as quality in those cases where the quality of a product can be quantified precisely. Then the above inequality can be interpreted as that a monopolist's output is of lower quality than a price-taking firm's.

8 Exercises

1. Use L'Hôpital's rule to calculate $\lim_{x\to 0} \frac{\sin x}{x}$.

2. Consider the following decision problem,

$$\min_{x\in\{0,1,2,3\}} (x-\alpha)^2,$$

   where $\{0,1,2,3\}$ denotes the set whose elements are exactly those written inside the brace bracket $\{\cdot\}$, and α a positive constant belonging to the interval $(0,3)$ (the set of real numbers strictly between zero and three).

   a. What are the objective, the choice variable and its domain, and the parameter?

   b. Rewrite the above problem into its equivalent form ("equivalent" in the sense that the set of solutions remains the same) such that the operator min is replaced by max.

3. Demonstrate the last italicized statement of Section 3 by deriving a relationship, analogous to (6), between the slope of the AC curve and the relative position between the AC and MC curves.

4. Consider a function defined by $f(x) := 2x^2 - 3x + 1$ for all $x \in \mathbb{R}$ (i.e., for all real number x). Find the value of x that satisfies the first-order condition. Calculate the second-order derivative of f at this value of x. Is this value a local maximum of f?

5. For each of the following statements, plot a graph to demonstrate:

   a. A maximum of a differentiable function need not satisfy the first-order condition

   b. An interior point that satisfies the first-order condition need not be a local maximum

   c. An interior point that satisfies both the first- and second-order conditions need not be a global maximum

6. Which of the following statements is (are) true (where A and B denote statements, a, b and c represent real numbers, and $\Rightarrow$ denotes implication, e.g., "$A \Rightarrow B$" means "if A then B"):

   a. "if A then B" iff "if B is not true, then A is not true"

   b. "if A then B" $\Leftrightarrow$ "either B is true or A is not true"

c. $\frac{a-b}{c} > 0 \Rightarrow a > b$

7. Calculate the derivatives of the following functions, defined for all real numbers x:

   a. $f(x) = 3\sqrt{x} + 100$
   b. $f(x) = 6 \ln x$
   c. $f(x) = (2x + 1) \ln(x/3)$
   d. $f(x) = (x^{3/2} + x)/(5x - 7)$
   e. $f(x) = 2^x$

8. Consider a firm in a purely competitive market with cost function given by, for any $q \geq 0$,

$$C(q) := q^3 - 6q^2 + 30q + 50.$$

   a. Calculate the firm's average variable cost (AVC), marginal cost (MC) and average cost (AC) as functions of the output quantity q.
   b. Calculate the output quantity at which AVC is equal to MC.
   c. When the market price is \$25, what is the firm's profit-maximizing output quantity? What if the market price becomes \$20? What if the market price becomes \$10?
   d. Calculate the firm's supply function; graph the supply curve.

9. Consider a monopolist with inverse demand function P and cost function C given by

$$\begin{aligned} P(q) &:= a - bq, \\ C(q) &:= cq, \end{aligned}$$

   for all $q \geq 0$, with parameters $a > 0$, $b > 0$ and $c > 0$ such that $a > c$.

   a. Calculate the marginal revenue and marginal cost for any $q \geq 0$.
   b. Write down an equation as the first-order condition for any interior solution of the monopolist's profit maximization problem.
   c. Demonstrate that the second-order condition is always satisfied by any $q > 0$.
   d. Calculate the profit-maximizing output quantity for the monopolist. What is the corresponding market price?
   e. On a coordinate system like Figure 2, draw the graphs of the inverse demand function, the marginal revenue (MR) function, and the marginal cost (MC) function. Locate the intersection between the MR and MC graphs. Draw a vertical line through the intersection until it crosses the inverse demand curve. Verify that the coordinates of these intersections are consistent with Step 9d.

10. This exercise gives you a toy model to contrast pure competition with monopoly. Suppose that in the market for a particular kind of surgeries (e.g., root canal) the market price is $P(q)$ thousand dollars per surgery if totally q surgeries are supplied, with

$$P(q) := 100 - q/2$$

for any $q \in \{0, 1, 2, \ldots, 200\}$. For each surgeon, the total cost to provide/perform q surgeries ($q \in \{0, 1, 2, \ldots, 200\}$) is equal to $C(q)$ thousand dollars such that

$$C(q) := \begin{cases} 0 & \text{if } q = 0 \\ 3q & \text{if } q = 1 \\ 200q & \text{if } q > 1. \end{cases}$$

(Note the difference between this cost function from the smooth ones postulated in previous sections. You can think of such jumpy cost functions as a reflection of the capacity/equipment constraint that each surgeon faces.)

a. Suppose that each surgeon maximizes his own profit and takes as a parameter the market price p (thousand of dollars per surgery).
   i. Write down any surgeon's decision problem. Note that the domain of the choice variable is discrete.
   ii. Find each surgeon's profit-maximizing quantity of surgeries to perform when—
      A. the market price is 2 thousand dollars per surgery;
      B. the market price is 4 thousand dollars per surgery.
   iii. Suppose that there are totally 200 surgeons, all maximizing their own profits.
      A. If the market price is *above* 3—and below 100—thousand dollars per surgery, totally how many surgeries are supplied? Consequently, what is the new market price equal to?
      B. If the market price is *below* 3 thousand dollars per surgery, totally how many surgeries are supplied? Consequently, what is the new market price equal to?
      C. Suppose that the market price is equal to 3 thousand dollars per surgery, find all the profit-maxima for each surgeon. What is the total quantity of surgeries demanded by the market given this price? Denote this quantity by q_* and suppose that exactly q_* surgeons provide the surgeons, each providing one, and the other surgeons provide none. Consequently, does the market price change to a different level? Why doesn't any surgeon want to deviate from this arrangement (that exactly q_* among them perform surgeries and the rest perform none)?
      D. Conclude from the above that the stable (equilibrium) outcome is that the market price is equal to 3 thousand dollars per surgery and totally ________________ surgeries are provided.

b. Suppose that all the surgeons form a single guild to maximize total profit together (and to maximize profit they jointly decide how many surgeries to supply and who among them to perform the surgeries and how to internally compensate one another).
   i. Explain why the total cost function for the guild becomes

$$C(q) := 3q$$

   thousand dollars for any $q \in \{0, 1, 2, \ldots, 200\}$.
   ii. Write down the guild's decision problem.

iii. Find the guild's profit-maximizing quantity of surgeries and the corresponding market price per surgery. What is the total profit for the guild equal to?
(Hint: While the domain of the choice variable is discrete in this problem, you can start with the assumption that q can be any nonnegative real number to find the tentative profit-maximizing quantity. It will turn out that this quantity is an integer, hence it belongs to the discrete domain and thus is indeed the profit maximum.)