Chapter 2: Deployment of Inputs

Elements of Decision: Lecture Notes of Intermediate Microeconomics

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1 Cost minimization

The choice variable in the decision problem considered in Chapter 1 has only one dimension, the firm's quantity of output. Now we relax this assumption and consider a case with multidimensional choice variables. Specifically, suppose that our firm, instead of choosing an output quantity, chooses among multiple kinds of inputs to deliver a fixed quantity of its output. Suppose that there are two kinds of inputs, called Input 1 and Input 2. The firm's input deployment corresponds to a vector (x_1, x_2) in which x_1 denotes the quantity of input 1, and x_2 the quantity of input 2, that the firm is employing. Suppose that the technology given to the firm is characterized by a production function f of two independent variables, x_1 and x_2, so that $f(x_1, x_2)$ is the quantity of output that the firm supplies if it employs x_1 units of input 1 and x_2 units of input 2. Suppose that the market prices of the two inputs, denoted by w_1 and w_2, are taken as given by the firm. Then an input bundle (x_1, x_2) would cost the firm $w_1x_1 + w_2x_2$ dollars. Also suppose that the firm is to supply a fixed quantity y of its output (say Boeing having signed a contract to supply a certain quantity of weapons of mass destruction to the US military). Which input bundle can deliver this output quantity in the least costly manner? Put formally, our firm's decision problem is

$$\begin{aligned} \min_{(x_1,x_2)\in\mathbb{R}^2_+} \quad & w_1x_1 + w_2x_2 \\ \text{subject to} \quad & f(x_1, x_2) = y, \end{aligned} \tag{1}$$

where $\mathbb{R}^2_+$ denotes the set of all pairs (x, y) such that both x and y are elements of the set $\mathbb{R}_+$.[1]

Like the decision problem in Chapter 1, Problem (1) has an objective $w_1x_1 + w_2x_2$ and a choice variable (x_1, x_2), which is specified, underneath the minimization operator min, to belong to the domain $\mathbb{R}^2_+$. Unlike the previous problem, however, Problem (1) has a constraint, the equation $f(x_1, x_2) = y$ on the second line. This is an example for *constrained optimization problems*.

2 Marginal products and partial derivatives

Since the function f in the constraint of Problem (1) is multivariate, before solving the problem let us specify some useful structures of f. It is usually assumed that f is increasing in each of its arguments x_1 and x_2. When f is also differentiable, we define the *marginal product* (MP) of input 1 to be the partial derivative of f with respect to x_1, i.e.,

$$\text{MP}_1(x_1, x_2) := \frac{\partial}{\partial x_1} f(x_1, x_2),$$

[1] Recall that $\mathbb{R}_+ = [0, \infty)$. Thus we can write $\mathbb{R}^2_+$ equivalently as $[0, \infty)^2$.

which is the rate of increase in output with an infinitesimal increase in x_1, while x_2 is held constant. Likewise the marginal product of input 2 is

$$\mathrm{MP}_2(x_1, x_2) := \frac{\partial}{\partial x_2} f(x_1, x_2).$$

To calculate $\frac{\partial}{\partial x_1} f(x_1, x_2)$, simply take the derivative of f by treating x_1 (the variable indicated by $\frac{\partial}{\partial x_1}$) as the variable and every other variable (x_2 in this case) as a constant. For example, if $f(x_1, x_2) = x_1^3 x_2^{1/2}$ for all positive numbers x_1 and x_2, then

$$\mathrm{MP}_1(x_1, x_2) = \frac{\partial}{\partial x_1} f(x_1, x_2) = \frac{\partial}{\partial x_1} x_1^3 x_2^{1/2} = x_2^{1/2} \frac{\partial}{\partial x_1} x_1^3 = 3x_1^2 x_2^{1/2},$$

with the second equality due to x_2 being constant in the operation $\frac{\partial}{\partial x_1}$. Note that the marginal product of input 1 depends not only on the quantity x_1 of input 1 but also on x_2 of input 2.

3 Isoquants and their slopes

In general, if $g(x_1, x_2)$ is a function of two variables $(x_1, x_2) \in \mathbb{R}_+^2$, and if y is any number that belongs to the range of g, the set of all $(x_1, x_2) \in \mathbb{R}_+^2$ for which $g(x_1, x_2) = y$ is called *level surface* of g for the constant level y. Level surfaces are useful for us to visualize multi-variable functions such as our production function f and the objective function in Problem (1). In intermediate microeconomics, any level surface of the production function f is called *isoquant*. That is, the isoquant for any given output quantity y is the set of nonnegative 2-vectors (x_1, x_2) satisfying

$$f(x_1, x_2) = y. \tag{2}$$

Likewise, any level surface of the objective function in Problem (1), corresponding to $w_1 x_1 + w_2 x_2 = c$ for some constant expense c, is called called *isocost*.

The constraint $f(x_1, x_2) = y$ in Problem (1) says that whatever the firm does it must supply the fixed quantity y of outputs. In other words, whatever the firm chooses should correspond to a point that belongs to the isoquant for the fixed output quantity y.

To see the shape of the isoquant, pick any nonnegative x_1. There is at most one value for x_2 such that (x_1, x_2) belongs to the isoquant. Otherwise, say (x_1, x_2) and (x_1, x_2') both belong to the isoquant while $x_2' \neq x_2$, then we have $f(x_1, x_2) = y = f(x_1, x_2')$, contradicting the assumption that f is increasing in x_2. It follows that the isoquant is a curve in the x_1-x_2-plane. In other words, Eq. (2) implies a functional relationship between x_1 and x_2: for any x_1 such that (x_1, x_2) belongs to the isoquant for some quantity x_2, this quantity of x_2 is unique and hence we may denote it by $\tilde{x}_2(x_1)$, a function of x_1. Thus Eq. (2) becomes

$$f\left(x_1, \tilde{x}_2(x_1)\right) = y. \tag{3}$$

Note that the left-hand side of this equation is a function of only one variable, x_1. If, in addition, f is differentiable then the isoquant is a smooth curve, whose slope is calculated by taking the derivative with respect to x_1 on both sides of Eq. (3):

$$\frac{d}{dx_1} f\left(x_1, \tilde{x}_2(x_1)\right) = \frac{d}{dx_1} y = 0,$$

with the second equality due to y being constant. For the left-hand side, use the chain rule:

$$\frac{d}{dx_1} f\left(x_1, \tilde{x}_2(x_1)\right) = \frac{\partial}{\partial x_1} f\left(x_1, \tilde{x}_2(x_1)\right) + \frac{\partial}{\partial x_2} f\left(x_1, \tilde{x}_2(x_1)\right) \frac{d}{dx_1} \tilde{x}_2(x_1).$$

Combining these two equations we have

$$\frac{\partial}{\partial x_1} f\left(x_1, \tilde{x}_2(x_1)\right) + \frac{\partial}{\partial x_2} f\left(x_1, \tilde{x}_2(x_1)\right) \frac{d}{dx_1} \tilde{x}_2(x_1) = 0,$$

i.e.,

$$\frac{d}{dx_1} \tilde{x}_2(x_1) = -\frac{\frac{\partial}{\partial x_1} f\left(x_1, \tilde{x}_2(x_1)\right)}{\frac{\partial}{\partial x_2} f\left(x_1, \tilde{x}_2(x_1)\right)}, \tag{4}$$

which is the formula for the slope of the isoquant at any positive x_1, with $\tilde{x}_2(x_1)$ obtained from solving Eq. (2) for x_2.[2] The left-hand side of (4) is denoted by $\mathrm{TRS}(x_1)$, called *technical rate of substitution*. With the shorthands for marginal products, Eq. (2) can be written briefly as

$$\mathrm{TRS} = -\frac{\mathrm{MP}_1}{\mathrm{MP}_2}. \tag{5}$$

The negative sign here signifies that the isoquant curve is downward sloping: If the firm reduces the quantity of input 1, to stay on the same quantity of output the firm needs to increase the quantity of input 2. It is usually assumed that, when x_1 increases, the absolute value $|\mathrm{TRS}(x_1)|$ of the slope is decreasing, i.e., the downward sloping isoquant gets less steep: The more input 1 has the firm been using, the less quantity of input 2 is needed to substitute a tiny decrease of x_1 in order to maintain the same output quantity.

For example, take the previous $f(x_1, x_2) = x_1^3 x_2^{1/2}$. An isoquant corresponds to the equation

$$x_1^3 x_2^{1/2} = y \tag{6}$$

for some positive constant y. Solve this equation for x_2 to obtain

$$x_2 = \left(y x_1^{-3}\right)^2 = y^2 x_1^{-6},$$

hence

$$\tilde{x}_2(x_1) = y^2 x_1^{-6}.$$

Taking the derivative of $\tilde{x}_2$ we obtain the slope of the isoquant:

$$\mathrm{TRS} = \frac{d}{dx_1} \tilde{x}_2(x_1) = \frac{d}{dx_1}\left(y^2 x_1^{-6}\right) = -6 y^2 x_1^{-7} \overset{(6)}{=} -6\left(x_1^3 x_2^{1/2}\right)^2 x_1^{-7} = -\frac{6x_2}{x_1}.$$

Alternatively, and more simply, use Eq. (5). We have calculated MP_1 previously. Analogously,

$$\mathrm{MP}_2 = \frac{\partial}{\partial x_2}\left(x_1^3 x_2^{1/2}\right) = x_1^3 \frac{\partial}{\partial x_2} x_2^{1/2} = \frac{1}{2} x_1^3 x_2^{-1/2}.$$

Hence the slope of the isoquant at x_1 is equal to

$$\mathrm{TRS}(x_1) = -\frac{\mathrm{MP}_1}{\mathrm{MP}_2} = -\frac{3x_1^2 x_2^{1/2}}{(1/2) x_1^3 x_2^{-1/2}} = -6 x_1^{2-3} x_2^{1/2-(-1/2)} = -\frac{6x_2}{x_1}, \tag{7}$$

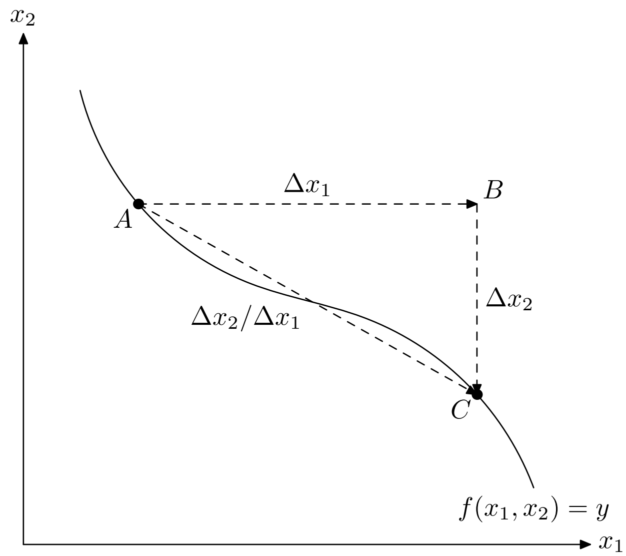

Figure 1: $\Delta x_2/\Delta x_1$ is the slope of AC

same as the result from the previous method. Since $\frac{6x_2}{x_1}$ is decreasing in x_1, the diminishing TRS assumption is satisfied.

The intuition for the derivation of Eq. (5) is illustrated by Figure 1, where the curve represents an isoquant (which does not satisfied the diminishing TRS assumption, by the way). Start with the input bundle A on the curve. Increase its quantity of input 1 by a tiny amount Δx_1; i.e., change the input bundle from A to B in the figure. With the production function assumed increasing, this increase in x_1 increases the output quantity; with the partial derivative MP_1 being the rate of change between output and input 1 when the change in x_1 is infinitesimal, this increase in the output quantity is approximately equal to $\mathrm{MP}_1 \cdot \Delta x_1$. To stay on the same isoquant as A, we need to decrease the quantity of input 2 by some amount such that the input bundle is changed from B down to the point C, back on the curve; denote this change in x_2 by Δx_2, which is a negative number, as it signifies a decrease. The change in the output quantity rendered by this change Δx_2, analogous to the change in x_1, is approximately equal to $\mathrm{MP}_2 \cdot \Delta x_2$. Thus, the total change in the output quantity, due to the changes from A to B and from B to C, is approximately equal to $\mathrm{MP}_1 \cdot \Delta x_1 + \mathrm{MP}_2 \cdot \Delta x_2$. But since A and C belong to the same isoquant, this total change in the output quantity is equal to zero. Hence we obtain

$$\mathrm{MP}_1 \cdot \Delta x_1 + \mathrm{MP}_2 \cdot \Delta x_2 \approx 0,$$

which is equivalent to

$$\frac{\Delta x_2}{\Delta x_1} \approx -\frac{\mathrm{MP}_1}{\mathrm{MP}_2}.$$

Note that the left-hand side, $\frac{\Delta x_2}{\Delta x_1}$, is simply the slope of the straight line AC in Figure 1. When Δx_1 converges to zero (denoted by $\Delta x_1 \to 0$), line AC becomes arbitrarily close to the tangent line of the curve at point A, and $\frac{\Delta x_2}{\Delta x_1}$ converges to the slope of this tangent line, which is the slope of the curve at point A, i.e., the TRS at A. Hence the above equation converges to Eq. (5).

[2] A student versed in calculus would recognize this paragraph as an instance of the implicit function theorem.

4 Cost-minimizing input bundle

To solve Problem (1), let us examine the objective on the x_1-x_2-plane. Given any two points on the plane, which one corresponds to the less costly input bundle? Pick any constant c and consider all input bundles that each cost c dollars given the market prices (w_1, w_2), i.e., the (x_1, x_2) such that

$$w_1 x_1 + w_2 x_2 = c. \tag{8}$$

In other words, consider the level surface of the objective function for the constant c. Eq. (8) is the same as

$$x_2 = \frac{c}{w_2} - \frac{w_1}{w_2} x_1,$$

which corresponds to the straight line on the x_1-x_2-plane with a negative slope $-w_1/w_2$ and vertical intercept c/w_2. The set of all (x_1, x_2) satisfying Eq. (8) is called *isocost line* corresponding to the input expense c. Now pick any other number $c' > c$, and consider the input bundles (x_1, x_2) that would cost the firm c' dollars:

$$w_1 x_1 + w_2 x_2 = c', \quad \text{i.e.,} \quad x_2 = \frac{c'}{w_2} - \frac{w_1}{w_2} x_1,$$

which is a line of the same slope as before but with a higher vertical intercept c'/w_2. Thus, the lower is an isocost line (in terms of its x_2-intercept), the less does any input bundle on that line would cost the firm.

It follows that solving Problem (1) amounts to finding a point on the isoquant that belongs to the lowest possible isocost. Clearly such lowest possible isocost is the isocost line that happens to touch a point on the isoquant and keeps the entire isoquant curve *above* (including the possibility of touching) the line. Such a straight line is called *supporting hyperplane*.[3] Any common point between this supporting hyperplane and the isoquant is a solution to Problem (1), i.e., a cost-minimizing input bundle that delivers the output quantity y. Note that the diminishing TRS assumption guarantees that such a supporting hyperplane exists.

When f is differentiable, the isoquant is a smooth curve and the supporting hyperplane becomes its tangent line, so the two have the same slope at their common point. The slope of the isoquant is given by Eq. (5), and that of the supporting hyperplane, itself an isocost line, is simply $-w_1/w_2$. Thus $-\frac{\text{MP}_1}{\text{MP}_2} = -\frac{w_1}{w_2}$ at any common point between the isoquant and its supporting hyperplane. In other words, at any cost-minimizing input bundle,

$$\frac{\text{MP}_1}{\text{MP}_2} = \frac{w_1}{w_2}. \tag{9}$$

Let us illustrate with the previous example, where $f(x_1, x_2) = x_1^3 x_2^{1/2}$. Suppose that input 1 costs \$48 per unit, and input 2, \$2 per unit. Then $w_1/w_2 = 48/2 = 24$. Plug this and Eq. (7) into Eq. (9) to obtain

$$\frac{6x_2}{x_1} = 24,$$

[3] "Hyperplane" is the general counterpart to a straight line in certain spaces with possible more than two dimensions. Our method can be generalized to cases with more than two inputs.

i.e., $x_2 = 4x_1$. Since the cost-minimizing input bundle belongs to the isoquant corresponding to y units of output, it must satisfy $f(x_1, x_2) = y$, i.e., $x_1^3 x_2^{1/2} = y$. Thus plug $x_2 = 4x_1$ into this equation to obtain

$$x_1^3 (4x_1)^{1/2} = y, \quad \text{i.e.,} \quad x_1^{7/2} = y/2.$$

Hence $x_1 = (y/2)^{2/7}$ and $x_2 = 4x_1 = 4(y/2)^{2/7}$. Thus, the cost-minimizing input bundle to deliver y units of output is

$$\left((y/2)^{2/7}, 4(y/2)^{2/7}\right).$$

For some production functions, the isoquants are not smooth curves (e.g., when the production function is not differentiable), hence Eq. (9) is not applicable. Nevertheless, we can solve Problem (1) by locating the common point between the isoquant and its supporting hyperplane that supports it from below.

For example, consider a production function defined by

$$f(x_1, x_2) := \min\{3x_1, x_2\},$$

i.e., $f(x_1, x_2)$ equals either $3x_1$ or x_2, whichever is smaller. The interpretation is that the two inputs are *perfect complements*. For instance, with one unit of input 1 and 3 units of input 2 the firm can produce up to $\min\{3 \times 1, 3\} = 3$ units of output; the firm cannot produce more even if it increases input 1 to 10 unless it also increases input 2, as $\min\{3 \times 10, 3\}$ is still 3. Pick any fixed output quantity y, so the isoquant corresponds to

$$\min\{3x_1, x_2\} = y.$$

To find its shape, set the two items inside $\min\{\cdot\}$ equal to each other to obtain $x_2 = 3x_1 = y$, which gives us the point $(y/3, y)$ in the x_1-x_2-plane. Note that this point belongs to the isoquant for y units of output. Starting from $(y/3, y)$ and moving horizontally to the right, we increase x_1 while holding x_2 fixed at y, and $f(x_1, x_2)$ by definition remains unchanged from the level y. Thus any point horizontally to the right of $(y/3, y)$ belongs to the same isoquant. Likewise any point vertically above $(y/3, y)$ also belongs to the same isoquant. Hence the isoquant is the L-shape path with its corner being $(y/3, y)$. Suppose as in the previous example that $w_1 = 48$ and $w_2 = 2$. Then the isocosts are straight lines of slope -24, one among which is the supporting hyperplane of the isoquant. The common point between the two, by the L-shape of the isoquant, is the corner $(y/3, y)$ of the L-shape path. Thus $(y/3, y)$ is the cost-minimizing input bundle to deliver output y.

For another example, consider a production function

$$f(x_1, x_2) := 3x_1 + x_2.$$

The interpretation is that the two inputs are *perfect substitutes*: Every unit of input 1 can be substituted by three units of input 2 without changing the output quantity. The isoquant in this example is the set of all nonnegative (x_1, x_2) for which

$$3x_1 + x_2 = y,$$

which is the straight segment with slope -3 and vertical intercept y. If the prices are $w_1 = 48$ and $w_2 = 2$ as in previous examples, the slope of the isocosts is -24, steeper than the isoquant at all points. Thus, the supporting hyperplane that supports the isoquant from below is the isocost line intersecting the isoquant at the vertical intercept. In this example, therefore, the cost-minimizing input bundle is $(0, y)$, meaning that the firm employs exclusively input 2 and none of input 1 to deliver y. While the marginal product of input 1 is always greater than that of input 2, with the former equal to 3 and the latter equal to 1, the firm opts for none of input 1 because its wage rate is too high.

5 Derivation of the cost function

In Chapter 1 we considered a firm's supply decision taking its cost function as given. Now we are ready to provide a foundation for the cost function: For a firm that has no influence on the market prices (w_1, w_2) of its inputs, the cost $C(y)$ of supplying y units output is equal to the minimum expense in supplying y, i.e.,

$$\begin{array}{rcll} C(y) & := & \min_{(x_1,x_2)\in\mathbb{R}^2_+} & w_1x_1 + w_2x_2 \\ & & \text{subject to} & f(x_1, x_2) = y. \end{array} \tag{10}$$

For example, when $f(x_1, x_2) = x_1^3 x_2^{1/2}$, we have found the cost-minimizing input bundle as $\left((y/2)^{2/7}, 4(y/2)^{2/7}\right)$, hence the cost function is given by

$$C(y) = w_1(y/2)^{2/7} + w_2 \cdot 4(y/2)^{2/7} = (y/2)^{2/7}(w_1 + 4w_2)$$

and the average cost

$$\mathrm{AC}(y) = \frac{C(y)}{y} = (2)^{-2/7}y^{-5/7}(w_1 + 4w_2).$$

In the example where $f(x_1, x_2) := \min\{3x_1, x_2\}$, $(y/3, y)$ is the cost-minimizing input bundle, so

$$\begin{aligned} C(y) &= w_1y/3 + w_2y = \frac{1}{3}y(w_1 + 3w_2), \\ \mathrm{AC}(y) &= \frac{1}{3}(w_1 + 3w_2). \end{aligned}$$

In the example where $f(x_1, x_2) := 3x_1 + x_2$, the cost-minimizing input bundle is $(0, y)$ and hence

$$\begin{aligned} C(y) &= w_2y, \\ \mathrm{AC}(y) &= w_2. \end{aligned}$$

6 Returns to scale

In the previous section, the firm's average cost of supplying y units of output is a decreasing function of y in the first example, and constant to y in the second and third. What determines whether the average cost is increasing, decreasing or constant in the output level? The answer depends on what type of returns to scale that the firm's production function exhibits.

A production function f exhibits *constant returns to scale* (CRS) iff

$$f(tx_1, tx_2) = tf(x_1, x_2) \tag{11}$$

for any $t > 1$ and any nonnegative input bundle (x_1, x_2). That is, when the quantity of every input is scaled up to t times its previous quantity, the maximum output of the firm is also scaled up to exactly t times its previous level. An interpretation of a CRS technology is that it is replicable in the sense that the same recipe of the inputs produces exactly the same output.[4]

For example, the previous $f(x_1, x_2) := \min\{3x_1, x_2\}$ is CRS: for any $t > 1$,

$$f(tx_1, tx_2) = \min\{3tx_1, tx_2\} = t \min\{3x_1, x_2\} = tf(x_1, x_2).$$

One can easily verify that the production function $f(x_1, x_2) := 3x_1 + x_2$ is also CRS. A class of CRS production functions, beloved by macroeconomists, is the *Cobb-Douglas* functions:

$$f(x_1, x_2) := Ax_1^{\alpha} x_2^{1-\alpha},$$

where A and α are constants with $A > 0$ and $0 < \alpha < 1$. Any such an f is CRS: for any $t > 1$,

$$f(tx_1, tx_2) = A(tx_1)^{\alpha}(tx_2)^{1-\alpha} = At^{\alpha+1-\alpha}x_1^{\alpha}x_2^{1-\alpha} = Atx_1^{\alpha}x_2^{1-\alpha} = f(x_1, x_2).$$

Note: While we define CRS by requiring Eq. (11) for all $t > 1$, the definition thereof implies that Eq. (11) holds for all $t > 0$. The case when $t = 1$ is trivial. Let us demonstrate the case when $t < 1$. Hence let $t < 1$, which means $1/t > 1$. Then

$$f(x_1, x_2) \overset{(11)}{=} f\left(\frac{1}{t}tx_1, \frac{1}{t}tx_2\right) = \frac{1}{t}f(tx_1, tx_2),$$

where the second equality can apply Eq. (11) because the $1/t$ here, playing the role of t in (11), is bigger than one as Eq. (11) requires. The above-displayed formula says that, whenever $t < 1$, $f(x_1, x_2) = f(tx_1, tx_2)/t$, which is exactly Eq. (11). Hence Eq. (11) is extended to the case $t < 1$.

A production function f exhibits *increasing returns to scale* (IRS) iff

$$f(tx_1, tx_2) > tf(x_1, x_2) \tag{12}$$

for any $t > 1$ and any nonnegative input bundle (x_1, x_2). That is, when the firm doubles the quantities of its inputs, it can more than double its output. The production function $f(x_1, x_2) = x_1^3 x_2^{1/2}$ considered previously is IRS: for any $t > 1$,

$$f(tx_1, tx_2) = (tx_1)^3(tx_2)^{1/2} = t^{3+1/2}x_1^3x_2^{1/2} > tx_1^3x_2^{1/2} = tf(x_1, x_2),$$

with the inequality due to $t > 1$.

A production function f exhibits *decreasing returns to scale* (DRS) iff

$$f(tx_1, tx_2) < tf(x_1, x_2)$$

for any $t > 1$ and any nonnegative input bundle (x_1, x_2). That is, when the firm doubles the quantities of its inputs, it cannot double its output. The production function $f(x_1, x_2) = x_1^{\alpha}x_2^{\beta}$, with α and β positive constants such that $\alpha + \beta < 1$, is DRS: for any $t > 1$,

$$f(tx_1, tx_2) = (tx_1)^{\alpha}(tx_2)^{\beta} = t^{\alpha+\beta}x_1^{\alpha}x_2^{\beta} < tx_1^{\alpha}x_2^{\beta} = tf(x_1, x_2),$$

[4] Usual suspects for such replicable technologies are those of McDonald's, Starbucks, Tim Hortons, and even Hollywood blockbuster production—hire a team of CGI experts and a group of stars, as well as a bendable script writer, and you churn out another equally forgettable action thriller.

with the inequality due to $t > 1$ and $\alpha + \beta < 1$.

Fact: *(i) if the production function exhibits CRS then the average cost is constant to the output level; (ii) if IRS then the average cost is decreasing in the output level; (iii) if DRS then the average cost is increasing in the output level.*

To *prove* (i), let (x_1^*, x_2^*) be a cost-minimizing input bundle that delivers one unit of output. Hence $f(x_1^*, x_2^*) = 1$ and $C(1) = w_1 x_1^* + w_2 x_2^*$. Pick any $y > 0$. By CRS, we apply Eq. (11), which we have explained holds for all $t > 0$, and let the y here play the role of t there:

$$f(yx_1^*, yx_2^*) = yf(x_1^*, x_2^*) = y \cdot 1 = y.$$

Thus, the bundle (yx_1^*, yx_2^*) delivers the output quantity y. Hence by Eq. (10),

$$C(y) \leq w_1 y x_1^* + w_2 y x_2^* = y\left(w_1 x_1^* + w_2 x_2^*\right) = yC(1).$$

We claim also that $C(y) \geq yC(1)$. Suppose not, then there exist an input bundle (x_1, x_2) such that $w_1 x_1 + w_2 x_2 < yC(1)$ and $f(x_1, x_2) = y$. Then, applying Eq. (11) to $f(x_1, x_2) = y$, we have

$$\begin{aligned} w_1\left(x_1/y\right) + w_2\left(x_2/y\right) &< C(1), \\ f\left(x_1/y, x_2/y\right) &= 1. \end{aligned}$$

That means the bundle $(x_1/y, x_2/y)$ can deliver one unit of the output with less expense than $C(1)$, a contradiction. Thus, $C(y) \geq yC(1)$. Since we have already shown $C(y) \leq yC(1)$, it follows that $C(y) = yC(1)$, i.e., the average cost $\mathrm{AC}(y) = C(y)/y = C(1)$, a constant. This completes the proof.

Claims (ii) and (iii) can be demonstrated based on a similar idea. For (ii): IRS, coupled with continuity of the production function, means that to scale the output level from y up to ty, the firm does not need to scale its inputs up to t times the previous quantities; it needs only to scale its inputs up to t' times, for some $t' < t$. Thus, $C(ty) < tC(y)$. This being true for any $t > 1$ and any $y > 0$, we have, for any $y' > y$ (and hence $y'/y > 1$),

$$\mathrm{AC}(y') = C(y')/y' = C\left(\frac{y'}{y}y\right)\Big/ y' < \frac{y'}{y}C(y)\Big/ y' = C(y)/y = \mathrm{AC}(y).$$

The proof of (iii), similar to that for (ii), is sketched by Exercise 8 in Section 7.

7 Exercises

1. Calculate the partial derivatives of the following functions of two variables:

   a. $f(x_1, x_2) = 3x_1 + 7x_2$
   b. $f(x_1, x_2) = 2\sqrt{x_1} + 3x_2$
   c. $f(x_1, x_2) = 3x_1^7 x_2^{1/3}$
   d. $f(x_1, x_2) = 3x_1^7 + x_2^{1/3}$
   e. $f(x_1, x_2) = (2x_1 - 5x_2)(x_2 + 1)$

2. Suppose the production function is: $f(x_1, x_2) := x_1^2 + x_2^2$ for all nonnegative x_1, x_2.

a. Pick any constant $y > 0$. Given the isoquant equation $x_1^2 + x_2^2 = y$, solve x_2 as an expression of x_1. Then calculate the derivative of x_2 with respect to x_1.

b. Calculate the marginal products. Then calculate the slope of an isoquant by Eq. (5). Compare the result with that in the previous step.

c. Does this production function satisfy the diminishing TRS assumption?

3. Suppose that the price for input 1 is \$10 and that for input 2 is \$5. For each of the following production functions defined for all nonnegative x_1 and x_2, calculate the cost-minimizing input bundle and the cost function $C(y)$:

a. $f(x_1, x_2) := x_1^{\alpha} x_2^{1-\alpha}$, where α is a parameter such that $0 < \alpha < 1$

b. $f(x_1, x_2) := \min\{x_1, 4x_2\}$

c. $f(x_1, x_2) := x_1 + 2x_2$

d. $f(x_1, x_2) := x_1^2 + x_2^2$

4. Consider the following decision problem:

$$\max_{(x_1,x_2)\in\mathbb{R}_+^2} \quad px_2 - wx_1$$
$$\text{subject to} \quad x_2 \leq \sqrt{x_1},$$

where $p > 0$ and $w > 0$ are parameters. The interpretation of this problem is that a firm produces its output with only a single kind of inputs, with x_1 denoting the input quantity and x_2 the output quantity, so that the firm needs to choose an input-output plan, $(x_1, x_2) \in \mathbb{R}_+^2$, to maximize its profit $px_2 - wx_1$, with p being the market price for the output, and w the wage rate for the input. The constraint $x_2 \leq \sqrt{x_1}$ says that, if it employs a quantity x_1 of the input, the firm can produce up to $\sqrt{x_1}$ units of the output but nothing beyond.

a. On a diagram of the x_1-x_2 plane, draw the set of all $(x_1, x_2) \in \mathbb{R}_+^2$ for which $x_2 \leq \sqrt{x_1}$. This is the entire *choice set* in this decision problem.

b. Given any $x_1 > 0$, calculate the slope of the graph of the equation $x_2 = \sqrt{x_1}$.

c. Pick any positive constant c. Draw the level surface of the objective function for the constant c. Calculate the vertical (x_2-) intercept of this level surface. Does the decision maker prefer to be on a level surface with higher vertical intercept or lower vertical intercept? If this level surface is a straight line, calculate its slope (in terms of w and p).

d. Draw the level surface of the objective function that is a supporting hyperplane of the choice set such that the choice set is contained in the lower side of the hyperplane.

e. Denote (x_1^*, x_2^*) for the point at which the above-specified supporting hyperplane touches the boundary of the choice set (i.e., the curve of $x_2 = \sqrt{x_1}$). Why is (x_1^*, x_2^*) the solution of our decision problem?

f. What is the relationship between the slope of the hyperplane and the slope of the choice set boundary at (x_1^*, x_2^*)? Calculate (x_1^*, x_2^*) in terms of the parameters p and w.

g. Use the formula of x_1^* obtained above to predict what a firm would do (increase or decrease its input employment x_1^*) if the wage rate w of the input becomes higher (say due to a new law that raises the minimum wage for labor, labor being the firm's input).

5. For each of the following production functions, determine whether it exhibits constant, increasing, or decreasing returns to scale:

   a. $f(x_1, x_2) = x_1 + \sqrt{x_2}$
   b. $f(x_1, x_2) = (x_1^{1/4} + x_2^{1/4})^4$
   c. $f(x_1, x_2) = \sqrt{x_1 + 3x_2}$

6. Recall that Section 6 defines CRS by requiring Eq. (11) for all $t > 1$ and later proves that for any CRS production function f Eq. (11) remains unchanged for all $0 < t < 1$. Now consider production functions f that exhibit increasing returns to scale (IRS), which Section 6 defines by requiring Ineq. (12) for all $t > 1$. Given any such IRS production function f, if $0 < t < 1$, does the inequality in (12) remain unchanged or turn to the reverse direction?

7. Consider the special case where our firm uses only one kind of input to produce its output, with production function f defined by

$$f(x) := (\alpha x)^{\beta}$$

   for any nonnegative input quantity x, where α and β are positive parameters. Denote the market price of the input by w, another positive parameter to the firm.

   a. For any $y > 0$, calculate:
      i. the cost-minimizing input quantity for this firm to supply output quantity y;
      ii. the cost $C(y)$ that the firm incurs in supplying output quantity y;
      iii. the average cost $\mathrm{AC}(y)$ that the firm incurs in supplying output quantity y.
   b. Does the production function exhibit constant, increasing, or decreasing, returns to scale, and is the firm's average cost constant, increasing, or decreasing, in its output quantity y,
      i. when $\beta = 1$?
      ii. when $\beta > 1$?
      iii. when $\beta < 1$?

8. To prove Claim (iii) in Section 6, that decreasing returns to scale implies increasing average cost, work out the following steps:

   a. Denote f for the production function that exhibits DRS; let w_1 and w_2 be the prices of inputs 1 and 2, respectively. Pick any $y > 0$ and any $t > 1$; let (x_1^*, x_2^*) be a cost-minimizing input bundle to produce the output quantity ty.
   b. Is $C(ty)$ greater than, equal to, or less than $w_1 x_1^* + w_2 x_2^*$?
   c. Why is $f(x_1^*/t, x_2^*/t) > y$?
   d. Assuming differentiability of f and $f(0,0) = 0$, use a theorem in calculus to explain why there exists an s such that $0 < s < 1$ and $f(s x_1^*/t, s x_2^*/t) = y$.

e. Use the definition of $C(y)$ and the equality in (d) to explain why $C(y) < w_1 x_1^*/t + w_2 x_2^*/t$.

f. Based on (b) and (e), is $C(y)$ greater than, equal to, or less than $C(ty)/t$?

g. Using the conclusion of (f), mimic the last displayed formula in Section 6 to show that $\text{AC}(y') > \text{AC}(y)$ whenever $y' > y$.