Chapter 4: Preference and Utility

Elements of Decision: Lecture Notes of Intermediate Microeconomics

Charles Z. Zheng

Tepper School of Business, Carnegie Mellon University

Last update: April 1, 2020

1 Preference relation

The decision problems considered in previous chapters each have an explicit, quantitative objective such as the profit or the cost of a firm. When the decision maker is a household or a consumer, whereas, an explicit objective, and especially a quantitative one, need not be readily available. To establish a mathematical method for such decisions, therefore, we need to come up with a formal representation of a decision-maker's objective based on some primitives that are reasonable to assume about him.

Such primitives are the decision-maker's preference relation that tells us, between any two alternatives, which one he likes better or whether he is indifferent. For any alternatives say x and y, denote $x \succ y$ for "x is preferred to y" to the decision maker, $x \sim y$ for "x is indifferent to y" to him, and $x \succeq y$ "x is at least as good as y" (i.e., "$x \succ y$ or $x \sim y$") to him.

Abstractly speaking, a structure that compares between two elements of a set is called *binary relation* on the set. For example, "is preferred to" is a binary relation on the set of alternatives for a decision maker, as when he says "x is preferred to y" he is comparing two alternatives, x and y. Clearly, "is indifferent to," "is at least as good as," "is as tall as," "is warmer than," "is equal to," "loves" and "hates" are each such binary relations. Here are three basic notions, defined below given any binary relation $\rhd$ on a set S of alternatives:

1. $\rhd$ is said *reflexive* iff for any alternative x in S, it is true that $x \rhd x$;
2. $\rhd$ is said *complete* iff for any two alternatives x and y in S, "$x \rhd y$ or $y \rhd x$" is true;
3. $\rhd$ is said *transitive* iff for any three alternatives x, y and z in S, if $x \rhd y$ and $y \rhd z$, then $x \rhd z$.

To illustrate these notions, imagine a cookie consumer who cares only about the amount of sugar in a cookie but cannot tell small differences. Specifically, she prefers any cookie A to B iff A contains more than one gram of sugar than B does; if the difference in sugar contents between A and B does not exceed one gram, she is indifferent between the two. To lighten the notation, let us identify the letter denoting a cookie with the letter denoting the amount of sugar in the cookie. Then the consumer's preference relation is formalized as, for any alternatives A and B, $A \succ B$ iff $A - B > 1$, $A \sim B$ iff $-1 \le A - B \le 1$, and $A \succeq B$ iff $A \succ B$ or $A \sim B$. Note from these formulas that $A \succeq B$ iff $A - B \ge -1$.

In this example, the weak relation $\succeq$ is reflexive because $A - A = 0$, hence $A \sim A$, which implies $A \succeq A$. The relation is also complete: for any A and B, either $A - B > 1$ or $A - B \le 1$. If $A - B > 1$ then $A \succ B$ and hence $A \succeq B$. Otherwise ($A - B \le 1$), either $A - B < -1$ or $A - B \ge -1$; if $A - B < -1$ then $B \succ A$ and hence $B \succeq A$; if $A - B < -1$ then, with $A - B \le 1$, we have $A \sim B$ and hence $A \succeq B$. Thus, "$A \succeq B$ or $B \succeq A$" is true.

Whereas, $\succeq$ in this example is not transitive. To prove that, we need only to construct a counterexample. To that end, consider the case where $A := 1$, $B := 1.6$ and $C := 2.1$. Then $A \sim B$, so $A \succeq B$; and $B \succeq C$, so $B \succeq C$. But since $A - C = -1.1 < -1$, "$A \succeq C$" is not true. Thus we have a case where $A \succeq B$, $B \succeq C$ and yet $A \succeq C$ is not true. Hence the $\succeq$ is not transitive.

By contrast, $\succ$ in this example is transitive. To prove that, pick any three alternatives A, B and C. Suppose $A \succ B$ and $B \succ C$; we just need to prove $A \succ C$. By $A \succ B$, we have $A - B > 1$; by $B \succ C$ we have $B - C > 1$. Sum the two inequalities to obtain $A - C = A - B + B - C > 2 > 1$, which implies $A \succ C$, as asserted. That completes the proof.

Now we come back to a subcategory of binary relations, a decision-maker's preference relations that we introduce at the beginning. To assume as little about a decision maker as possible, we take only the weak preference relation $\succeq$ as the primitives and derive the strict $\succ$ and indifference $\sim$ relations by the following definition:

$$\begin{aligned} x \succ y &\quad\Leftrightarrow\quad [x \succeq y \text{ and not } [y \succeq x]]\,, \\ x \sim y &\quad\Leftrightarrow\quad [x \succeq y \text{ and } y \succeq x]\,. \end{aligned} \tag{1}$$

It is usually assumed that the weak preference relation $\succeq$ are reflexive, complete and transitive. In the context of weak preference relations, these notions are called axioms, as they sound quite self-evident. A weak preference that is not reflexive, say "x is not at least as good as x," would sound absurd. If the completeness axiom fails, say for some alternatives x and y neither $x \succeq y$ nor $y \succeq x$ holds for the decision maker, then he would not be able to decide what he would choose between x and y. (Note that "$x \succeq y$ is not true" is not sufficient to imply "$y \succ x$" unless $\succeq$ is complete, c.f. Exercise 3.) If transitivity fails, say $x \succeq y$, $y \succeq z$ and not $z \succeq x$, the decision-maker would be trapped in an infinite loop if he were to rank the three alternatives (c.f. Exercise 4).

2 Utility function

A *utility function* for a decision maker is a numerical representation of his preference relation: to each alternative x the function assigns a number $u(x)$ such that, for any two alternatives x and x', $u(x) \geq u(x')$ iff $x \succeq x'$, and $u(x) > u(x')$ iff $x \succ x'$. (Note that the definition implies $u(x) = u(x')$ iff $x \sim x'$.) If a decision maker has a utility function, his decision has a quantitative objective as the decision problems in previous chapters. The question is Under what conditions does his preference relation representable by a utility function?

The answer is straightforward if there are at most *countably* many alternatives in the sense that we can label ("count") all alternatives by integers $1, 2, 3, \ldots$ so that all alternatives can be arranged into a list

$$A_1, A_2, A_3, \ldots.$$

In this case, a utility function exists if $\succeq$ satisfies the three axioms: To every alternative say B in this list, we assign a real number $u(B)$ in decimal format

$$0.a_1 a_2 a_3 \cdots \tag{2}$$

such that, for every k, the kth digit $a_k = 1$ if $B \succ A_k$, and $a_k = 0$ if "$B \succ A_k$" is not true. By the completeness axiom, every digit a_k is assigned the value either one or zero. Hence $u(B)$ is well-defined for every alternative B. By the transitivity axiom, $B \succeq B'$ iff "the set $\{k = 1, 2, \ldots : B' \succ A_k\}$ is contained in the set $\{k = 1, 2, \ldots : B \succ A_k\}$," which in turn is equivalent to $u(B) \geq u(B')$, because any nonzero digit of the decimal expression of $u(B')$ is also a nonzero digit of the decimal expression of $u(B)$. By the same token, $B \sim B'$ is equivalent to $u(B) = u(B')$. Thus, u is a utility function.

When there are uncountably many alternatives, whereas, the three axioms are not enough to guarantee existence of a utility function. For example, suppose that the set of alternatives is $\mathbb{R}^2_+$, the nonnegative quadrant of the plane, which is uncountable as a mathematical fact. Consider a preference relation called *lexicographic preference*: for any two points (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^2_+$, $(x_1, x_2) \succ (y_1, y_2)$ iff either $x_1 > y_1$ or "$x_1 = y_1$ and $x_2 > y_2$." Hence the points on the plane are ordered first according to their first coordinates and then, if they have the same first coordinates, their second coordinate.[1] For instance, $(2, 1) \succ (1, 100)$ and $(2, 1.1) \succ (2, 1)$. Now define the weak-preference counterpart $\succeq$ of the lexicographic ordering $\succ$ by "$(x_1, x_2) \succeq (y_1, y_2)$ iff $[(x_1, x_2) \succ (y_1, y_2)$ or $(x_1, x_2) = (y_1, y_2)]$." It is not hard to prove that this $\succeq$ satisfies all the three axioms.

However, it is impossible to have a utility function for the lexicographic preference. The reason is a little bit involving: Suppose that there were such a utility function. Pick any nonnegative real number a and consider the vertical line such that $x_1 = a$. By definition of the lexicographic preference, any two different points on this line would be assigned different numbers, with the higher point assigned the higher number. Thus, the set of numbers assigned to the points on this vertical line span an interval that is nondegenerate in the sense that the interval does not collapse to a single point. Likewise, for any other nonnegative real number b such that $a > b$, the numbers assigned to the vertical line with $x_1 = b$ also span a nondegenerate interval. Furthermore, the former interval is entirely above the latter, because any point with $x_1 = a$ is preferred to any point with $x_1 = b$ by definition of the preference, and hence the former would be assigned a higher number than the latter (Figure 1). Consequently, since there are uncountably many nonnegative real numbers such

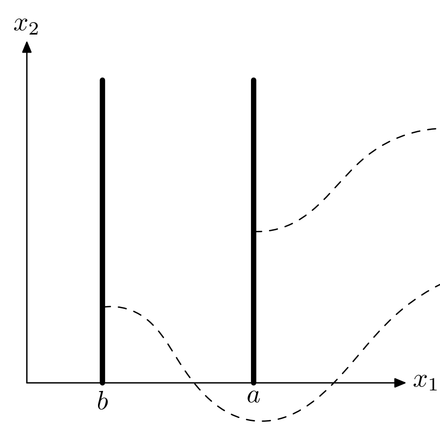

Figure 1: The real line has no room to stack up all the vertical lines in the plane

as a and b, there are uncountably many such mutually disjoint nondegenerate intervals that are lined up one on the top of the other. But that is impossible due to a mathematical fact that the real line cannot be partitioned into uncountably many non-overlapping nondegenerate intervals.

[1] A literary articulation of the lexicographic preference on three-dimension alternatives is arguably a poem by the Hungarian poet Sándor Petöfi, well-known to those Chinese who grew up in a period when the Chinese government used to legitimize itself by the past revolution: "Liberty and love / These two I must have. / For my love I'll sacrifice / My life. / For liberty I'll sacrifice / My love."

To guarantee existence of numerical representations for preferences, therefore, we need to add an assumption about the preferences when the set of alternatives is uncountable such as $\mathbb{R}^2_+$. A preference relation on $\mathbb{R}^2_+$ is said *continuous* iff, for any two points x and y in $\mathbb{R}^2_+$ such that $x \succ y$, there is a circle around x, and a circle around y, such that any point inside the former circle (as well as in $\mathbb{R}^2_+$) is preferred to any point inside the latter circle. In other words, if x is preferred to y then such a preference is unchanged when x and y are each perturbed slightly. The lexicographic preference is not continuous. To see that, consider $(1,1)$ and $(1,2)$. By definition of the preference, $(1,2) \succ (1,1)$. But if you perturb $(1,1)$ slightly to $(1+\epsilon,1)$ for some $\epsilon > 0$, $(1+\epsilon,1) \succ (1,2)$ no matter how small ϵ is.

The continuity assumption, together with the three axioms, is sufficient to guarantee existence of utility functions. A theorem attributed to Debreu states that any continuous preference relation on $\mathbb{R}^2_+$ can be represented by a utility function if its weak preference $\succeq$ is reflexive, complete and transitive. The proof of the theorem is outside the scope of this course.

3 Indifference curves and marginal rate of subsitution

From now on we focus on $\mathbb{R}^2_+$ as the set of alternatives. Any point (x_1, x_2) in $\mathbb{R}^2_+$ represents a *consumption bundle* consisting of a quantity x_1 of good 1 and a quantity x_2 of good 2. With a utility function u on $\mathbb{R}^2_+$, a preference relation can be graphically represented by *indifference curves*. An indifference curve means the set of all consumption bundles that are indifferent to one another, i.e., the set of (x_1, x_2) such that $u(x_1, x_2) = c$ for some constant c.

To clarify the shapes of the indifference curves, we add two more assumptions. A preference relation is said *strongly monotone* iff, for any two consumption bundles (x_1, x_2) and (y_1, y_2), $(x_1, x_2) \succ (y_1, y_2)$ if $x_1 \geq y_1$, $x_2 \geq y_2$, and at least one of the two inequalities are strict. If the preference has a utility function u, then strong monotonicity means that $u(x_1, x_2) > u(y_1, y_2)$ for all (x_1, x_2) and (y_1, y_2) such that $x_1 \geq y_1$, $x_2 \geq y_2$ and at least one of these inequalities are "$>$." Examples of utility functions for strongly monotone preference relations are: $u(x_1, x_2) := x_1^3 x_2^{1/2}$ and $u(x_1, x_2) := 2x_1 + 3x_2$.[2] The way to figure out the indifference curves given a utility function is the same as how we figured out isoquants given a production function in Chapter 2.

Given an indifference curve, pick any point (x_1, x_2) on the curve. With the preference relation strongly monotone, any point lying to the northeast of (x_1, x_2) is preferred to (x_1, x_2). Thus, no indifference curve has an upward-sloping portion. In other words, indifference curves are downward sloping. The slope of the indifference curve at point (x_1, x_2) is called *marginal rate of substitution* (MRS) at the corresponding consumption bundle (x_1, x_2). With indifference curves downward sloping, MRS is a negative number (negative infinity if the curve is vertical at (x_1, x_2)). The negativity signifies the intuition that if a consumer reduces her consumption of good 1 she needs to increase her consumption of good 2 to keep herself as well off as before. The absolute value of MRS measures the maximum amount of good 2 she is willing to give up for a tiny amount of increase in good 1.

Last is the assumption of *diminishing MRS*: the slope of every indifference curve, in absolute

[2] Note that the utility function of a strongly monotone preference relation is necessarily increasing in each dimension while the other dimension is held constant. For example, $u(x_1, x_2) := \min\{4x_1, x_2\}$ is constant when x_1 is held constant, even if x_2 becomes larger. Thus the preference relation it represents is not strongly monotone.

value, is decreasing when x_1 increases. That is, every indifference curve, downward sloping due to monotonicity, gets flatter as x_1 increases. Its intuitive meaning is that the more a person is consuming a good, the less she is willing to give up, in terms of the other good, for another tiny increase of the former good.

When the utility function is differentiable and increasing in each dimension, there is a convenient formula to calculate MRS. The method is the same as the formula for TRS in Chapter 3. Consider any indifference curve $u(x_1, x_2) = c$ for some constant c. This equation implies a functional relationship $\tilde{x}_2$ between the horizontal coordinate x_1 and the coordinate x_2 of any point on the curve. Hence the equation becomes

$$u(x_1, \tilde{x}_2(x_1)) = c.$$

Differentiate both sides of this equation with respect to x_1 to obtain

$$\frac{\partial}{\partial x_1}u(x_1, \tilde{x}_2(x_1)) + \frac{\partial}{\partial x_2}u(x_1, \tilde{x}_2(x_1))\frac{d}{dx_1}\tilde{x}_2(x_1) = 0,$$

i.e.,

$$\frac{d}{dx_1}\tilde{x}_2(x_1) = -\frac{\frac{\partial}{\partial x_1}u(x_1, \tilde{x}_2(x_1))}{\frac{\partial}{\partial x_2}u(x_1, \tilde{x}_2(x_1))}.$$

Note that the left-hand side is just the slope of the indifference curve at x_1. For the right-hand side, denote the partial derivative $\frac{\partial}{\partial x_1}u(x_1, \tilde{x}$ by MU_1 (marginal utility of good 1), and $\frac{\partial}{\partial x_2}u(x_1, \tilde{x}_2(x_1))$ by MU_2 (marginal utility of good 2). Then the equation becomes

$$\mathrm{MRS} = -\frac{\mathrm{MU}_1}{\mathrm{MU}_2}. \tag{3}$$

For example, given the previous utility function $u(x_1, x_2) := x_1^3 x_2^{1/2}$, an indifference curve of bundles with positive quantities corresponds to the equation

$$x_1^3 x_2^{1/2} = c \tag{4}$$

for some positive constant c. Solving this equation for x_2, we have

$$x_2 = \left(c x_1^{-3}\right)^2 = c^2 x_1^{-6},$$

hence

$$\tilde{x}_2(x_1) = c^2 x_1^{-6}.$$

Taking the derivative of $\tilde{x}_2$ we obtain the slope of the indifference curve:

$$\mathrm{MRS} = \frac{d}{dx_1}\tilde{x}_2(x_1) = \frac{d}{dx_1}\tilde{x}_2\left(c^2 x_1^{-6}\right) = -6c^2 x_1^{-7} \overset{(4)}{=} -6\left(x_1^3 x_2^{1/2}\right)^2 x_1^{-7} = -\frac{6x_2}{x_1}.$$

Alternatively, we start by calculating the marginal utilities:

$$\begin{aligned}
\mathrm{MU}_1(x_1, x_2) &= \frac{\partial}{\partial x_1}\left(x_1^3 x_2^{1/2}\right) = x_2^{1/2}\frac{\partial}{\partial x_1}\left(x_1^3\right) = 3x_2^{1/2}x_1^2; \\
\mathrm{MU}_1(x_1, x_2) &= \frac{\partial}{\partial x_2}\left(x_1^3 x_2^{1/2}\right) = x_1^3\frac{\partial}{\partial x_2}\left(x_2^{1/2}\right) = \frac{1}{2}x_1^3 x_2^{-1/2}.
\end{aligned}$$

Then Eq. (3) implies

$$\mathrm{MRS}(x_1, x_2) = -\frac{3x_2^{1/2}x_1^2}{\frac{1}{2}x_1^3x_2^{-1/2}} = -\frac{6x_2}{x_1}.$$

Note that $|\mathrm{MRS}| = 6x_2/x_1$ is decreasing when x_1 increases. Hence the the preference relation represented by u exhibits diminishing MRS.[3]

By contrast, the preference relation represented by $u(x_1, x_2) := 2x_1 + 3x_2$, an instance of a "perfect substitutes" preference, does not exhibit diminishing MRS at all. One readily sees that any indifference curve is of the form $2x_1 + 3x_2 = c$ for some constant c, i.e., a straight line of slope $-2/3$. Hence $|\mathrm{MRS}|$ is equal to the constant $2/3$, not diminishing in x_1.

4 Risky decisions and expected utilities

A decision maker under risk faces uncertain outcomes, though he can calculate the probability of each possible outcome. The consequence of his decision depends on which outcome actually occurs. For instance, say there are n possible outcomes such that outcome k occurs with probability π_k (for all $k = 1, \ldots, n$, hence $\pi_1 + \ldots \pi_n = 1$); a decision chosen by him may result in a payoff contingent on which outcome occurs such that x_k is the dollar amount that he gets if outcome k occurs (for all $k = 1, \ldots, n$). The *probability measure* $(\pi_1, \ldots, \pi_n)$ and the *contingent payoff* $(x_1, \ldots, x_n)$ together constitute a *gamble*, or *lottery*, succinctly denoted by

$$\pi_1\overrightarrow{x_1} + \cdots + \pi_n\overrightarrow{x_n}. \tag{5}$$

For example, a lottery ticket that costs one dollar to purchase and pays off ten thousand dollars if you win, with the odds of winning being one out of a million, corresponds to

$$\frac{1}{10^6}\overrightarrow{(10^4 - 1)} + \frac{10^6 - 1}{10^6}\overrightarrow{(-1)}. \tag{6}$$

With such explicitly quantitative structure of gambles, quantitative representations of the decision-maker's objective are at hand. One approach is to measure his objective by the *expected value* of the monetary payoffs in the gamble, i.e., simply calculate the weighted sum

$$\pi_1x_1 + \cdots + \pi_nx_n$$

if the gamble is (5). For example, the expected value of the lottery ticket (6) is equal to

$$\frac{1}{10^6}(10^4 - 1) + \frac{10^6 - 1}{10^6}(-1) = 10^{-2} - 10^{-6} - 1 + 10^{-6} = -(1 - 10^{-2}) = -0.99. \tag{7}$$

Intuitively speaking, if a gamble is repeated for sufficiently many times, then its monetary payoff in average is approximately equal to its expected value.

However, measuring one's objective by the expected values of the gambles can be problematic. Just look at Eq. (7); if everyone evaluates lotteries according to their expected values, no one

[3] More precisely, this preference relation exhibits diminishing MRS for all consumption bundles with a positive quantity of each good. If either $x_1 = 0$ or $x_2 = 0$, whereas, the indifference curve is $x_1^3x_2 = 0$, which is the union of the nonnegative parts of the horizontal and vertical axes. Along this indifference curve, MRS is equal to $-\infty$ when $x_1 = 0$, and equal to zero for all $x_1 > 0$. Hence $|\mathrm{MRS}|$ is not decreasing in x_1.

would buy lottery tickets! A more suitable approach is to convert the monetary payoffs into utilities according some increasing utility function u and then calculate the expected value of such contingent utilities. That is, for the gamble (5), replace the monetary payoff x_k by a utility $u(x_k)$, for each possible outcome k, and then calculate the weighted sum

$$\pi_1 u(x_1) + \cdots + \pi_n u(x_n),$$

which is called *expected utility* of the gamble (5) given *vNM utility function* u.[4] For example, if $u(x) := x^3/10^5$ for all x, the expected utility of the lottery ticket (6) would be equal to

$$\frac{1}{10^6}(10^4-1)^3 10^{-5} + \frac{10^6-1}{10^6}(-1)^3 10^{-5} = 10^{-5}\left(\frac{1}{10^6}(10^4-1)^3 + \frac{10^6-1}{10^6}(-1)^3\right) \approx 9.997,$$

which would justify buying the lottery ticket.

A decision maker is said *risk averse* if his vNM utility function is differentiable and its derivative is a decreasing function of the monetary payoff.[5] To understand the term risk aversion, consider a simple case with only two possible outcomes, so a contingent payoff is a pair (x_1, x_2) of real numbers, corresponding to a point in the plane $\mathbb{R}^2$, and the expected utility of any gamble is equal to $\pi_1 u(x_1) + \pi_2 u(x_2)$. His preference quantified as such, the decision maker's indifference curve corresponds to an equation

$$\pi_1 u(x_1) + \pi_2 u(x_2) = c$$

for some constant c. Hence the slope of the indifference curve, by Eq. (3), is equal to

$$\text{MRS} = -\frac{\text{MU}_1}{\text{MU}_2} = -\frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)},$$

where u' denotes the derivative of u. Now suppose that u is increasing and its derivative u' is decreasing. Thus, when x_1 enlarges and x_2 shrinks, the derivative $u'(x_1)$ becomes small, and $u'(x_2)$ large, hence $\frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)}$ shrinks. That is, when we move (x_1, x_2) along any indifference curve $\pi_1 u(x_1) + \pi_2 u(x_2) = c$, which is downward sloping and hence making x_1 bigger and x_2 smaller, the slope of the curve diminishes in absolute value. Thus the preference exhibits diminishing MRS. Now pick any two distinct points, say A and B, on the indifference curve. With the curve getting flatter when x_1 enlarges, the segment between A and B, except these two endpoints, are above this indifference curve. It then follows from strong monotonicity of the preference that any point C on the straight segment between A and B belongs to a higher indifference curve, more preferred to A, and more preferred to B (c.f. Exercise 11a.). Note that A is riskier than C, and so is B, because the contingent payoff of A, and that of B, are each more extreme in the two possible outcomes than that of C. Thus, with such a kind of vNM utility functions, a decision maker prefers less risky gambles than more risky ones, hence called risk averse.

A decision maker is said *risk neutral* if his objective is measured by the expected value of the gambles. Risk neutrality corresponds to the special case of expected utilities where the vNM utility function is linear, e.g., $u(x) = x$ for all x.

[4] The acronym vNM stands for von Neumann and Morgenstern, who initiated game theory and built an axiomatic system to justify the usage of expected utilities.

[5] That is, the graph of the function has diminishing slope. Note that the slope can be positive or negative. When it is positive, the upward-sloping curve becomes flatter and flatter as the independent variable increases. When the slope is negative, by contrast, "diminishing slope" means that the downward-sloping curve becomes steeper and steeper. Inspect the graphs in both cases and you will see why functions of such properties are called *concave*.

5 Exercises

1. Suppose that a robot is given four principles to guide its actions: obey the commander (obedience); protect human lives (humanity); get any assigned task done (task); and protect itself from harm (self-preservation). To tell the robot how to prioritize these principles, a coder enters the following list of *ordered* pairs:

   (humanity, self-preservation),
   (humanity, task),
   (humanity, obedience),
   (self-preservation, task),
   (obedience, self-preservation),
   (task, obedience).

   Then the coder programs the robot so that it prioritizes principle A over principle B, denoted $A\mathscr{P}B$, if and only if (A, B) belongs to the above list.

   a. Is the binary relation $\mathscr{P}$ reflexive? complete? transitive?

   b. A binary relation $\triangleright$ on a set S is said to be *total* iff, for any two *distinct* elements x and y of S, "$x \triangleright y$ or $y \triangleright x$" is true. Is the $\mathscr{P}$ defined above a total binary relation on the four principles?

2. Suppose $\succeq$ is reflexive. Is the strict preference relation $\succ$ defined in (1) reflexive? If Yes, prove it; if No, provide a counterexample.

3. Prove: If $\succeq$ is complete then, for any alternatives x and y, "$x \succeq y$ is not true" implies "$y \succ x$."

4. Consider a complete preference relation such that, for some alternatives x, y and z, $x \succeq y$ and $y \succeq z$ and yet "$x \succeq z$" is not true. Prove that it is impossible to have a utility function to represent this preference relation. (Hint: Use Exercise 3.)

5. If u is the utility function of a reflexive, complete and transitive preference relation on a countable set of alternatives according to (2), is it possible to have two alternatives, B and B', such that $u(B) = 0.101\cdots$ and $u(B') = 0.011\cdots$?

6. Imagine a person who has preferences over his seat number, which is one of the nine numbers $1, 2, 3, \ldots, 9$. He prefers the number 8 to any other number; if 8 is not available he would like to have 3; and if neither 8 nor 3 is available, he would like to have 9. After 9, he would like to have 2. And he hates the number 4 the most. For the rest, he is indifferent.[6] Use the utility function described in (2) to assign numerical values to each of the seat numbers.

7. Prove that the lexicographic preference $\succ$ defined in Section 2 is transitive and strongly monotone.

[6]In Cantonese, this author's native tongue, the pronunciation of the number 8 resembles that of "growing"; number 3, "alive"; 9 "durable"; 2 "easy"; and 4 "dead."

8. Consider a voter who ranks politicians according the number of anti-abortion legislations, *measured in integers*, and the import tariff, measured in real numbers, that are proposed by the politician. Suppose that the voter's preference relation is lexicographic: between any two candidates say A and B, if A proposes more anti-abortion legislations than B then he likes A more than B; if the two candidates propose the same number of anti-abortion legislations, then the voter likes the one who proposes a lower tariff.

   a. Is it possible to construct a utility function for this voter's preferences according to (2)?
   b. Is this voter's preference relation continuous? Draw a graph to illustrate your answer.
   c. Does there exist a utility function that represents this voter's preference relation?

9. Consider a binary relation $\succ$ on $\mathbb{R}^2_+$ that is defined by

$$(x_1, x_2) \succ (x_1', x_2') \iff \big[\text{either } x_1 + x_2 > x_1' + x_2' \text{ or } \big[x_1 + x_2 = x_1' + x_2' \text{ and } x_1 > x_1'\big]\big]$$

   for any (x_1, x_2) and (x_1', x_2') in $\mathbb{R}^2_+$. Fill in the following blanks:

   a. $(2, 1) \succ (1, 2)$ is ________________ (true, false, undecided)
   b. For any sufficiently small $\epsilon > 0$, $(2, 1) \succ (1+\epsilon, 2)$ is ________________ (true, false, undecided)
   c. $\succ$ is ________________ (continuous, not continuous) on $\mathbb{R}^2_+$ because $(2, 1)$__________$(1, 2)$ ($\succ$, not $\succ$, $\sim$, not $\sim$, $\prec$, not $\prec$) and $(1 + \epsilon, 2)$__________$(2, 1)$ ($\succ$, not $\succ$, $\sim$, not $\sim$, $\prec$, not $\prec$) for ____________ (some, any) $\epsilon > 0$.
   d. "$\succ$ is reflexive" is _______________ (true, false, undecided)
   e. "If $(x_1', x_2') \neq (x_1'', x_2'')$, then either $(x_1', x_2') \succ (x_1'', x_2'')$ or $(x_1'', x_2'') \succ (x_1', x_2')$" is _______________ (true, false, undecided) because (fill in the following blanks with $<$, $\leq$, $=$, $\geq$, $>$), if neither $(x_1', x_2') \succ (x_1'', x_2'')$ nor $(x_1'', x_2'') \succ (x_1', x_2')$ is true, then $x_1' + x_2'$ __________ $x_1'' + x_2''$ and x_1' __________ x_1''; consequently, x_2'__________x_2'' and hence (x_1', x_2') __________ (x_1'', x_2'').

10. For each of the following utility functions, graph the indifference curve corresponding to the utility level equal to one, i.e., $u(x_1, x_2) = 1$, and the indifference curve corresponding to the utility level equal to two:

   a. $u(x_1, x_2) := x_1^3 x_2^{1/2}$
   b. $u(x_1, x_2) := 2x_1 + 3x_2$
   c. $u(x_1, x_2) := \min\{4x_1, x_2\}$
   d. $u(x_1, x_2) := \max\{4x_1, x_2\}$
   e. $u(x_1, x_2) := x_1^2 + x_2^2$

11. Suppose that a preference relation satisfies all axioms and assumptions in this chapter. Use x as the shorthand for a bundle (x_1, x_2), and likewise x' for (x_1', x_2'), etc.

   a. Informally demonstrate a fact that *if $x \sim x'$ then for any point x'' that belong to the straight segment between x and x' and is distinct from x and x', we have $x'' \succ x$ and $x'' \succ x'$*:[7]

[7] A preference relation that has the property described by this italic statement is said *strongly convex*.

i. Draw the points x and x' such that their positions are consistent to $x \sim x'$. Given the strong monotonicity assumption, is it possible for one of them to belong to the northeast of the other?

ii. Draw the indifference curve to which x and x' belong; be sure that it satisfies the diminishing MRS assumption.

iii. Note that the entire segment between x and x', with the two endpoints excluded, lie above the indifference curve.

iv. Use the strong monotonicity assumption to conclude the demonstration.

b. Informally demonstrate a fact that *if $x \succ x'$ then for any point x'' that belong to the straight segment between x and x' and is distinct from x and x', $x'' \succ x'$*:[8]

i. Draw x and x' such that x is to the northeast of x'. Note that this belongs to one of the cases where $x \succ x'$. Note also that in this case the "then" clause of the above italicized statement is true.

ii. Draw x and x' such that neither point belongs to the northeast of the other.

iii. Draw the indifference curve to which bundle x' belongs; be sure that it satisfies the monotonicity and diminishing MRS assumptions.

iv. One case is that the indifference curve lies below the segment between x and x' (except the point x'). Note that if it is this case then the above italicized assertion is true. Hence consider the other case, where the indifference curve crosses the segment at some point say x''. In that case, extend the indifference curve beyond x'' to illustrate that part of the curve lies above x; obtain a logical contradiction by the strong monotonicity assumption, thereby showing that the latter case is impossible.

12. For each gamble listed below, calculate its expected value and expected utility given vNM utility function $u(x) := \sqrt{x}$ for any nonnegative monetary payoff x.

a. Toss a fair coin, *fair* in the sense of getting Head with probability 1/2, and Tail with probability 1/2; the monetary payoff is 9 if Head, and 1 if Tail.

b. Randomly pick an integer in $\{0, 1, 2, \ldots, 9\}$ with equal probability (so each integer gets pick with probability 1/10); the monetary payoff is equal to the integer that gets picked.

c. A fair coin is tossed twice; the monetary payoff is equal to 2 if Head occurs on the first trial, equal to 4 if the first occurrence of Head is on the second trial, and zero if Head occurs on neither trial.

d. A fair coin is tossed repeatedly until Head appears; the monetary payoff is 2^n if the first occurrence of Head is on trial n (for any $n = 1, 2, \ldots$).[9]

[8] A preference relation that has the property described by this italic statement is said *convex*.

[9] This gamble is referred to as St. Petersburg paradox, due to D. Bernoulli. Calculate its expected value and you will see why it may sound paradoxical.