Chapter 6: Revealed Preference

Elements of Decision: Lecture Notes of Intermediate Microeconomics

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Last update: March 24, 2020

1 The main idea

If we do not know what a consumer's preference is but can observe his consumption choice under various prices, can we infer about his preference? For the question to be meaningful, suppose that the consumer's choice is consistent with the rational decision modeled in Chapter 5. For simplicity, assume that his preference relation, albeit unknown to us a priori, satisfies the three axioms, is monotone, smooth and exhibits diminishing MRS. Consequently, given any nonnegative prices and income, the consumer's optimal consumption bundle is unique and lies on the budget line. Now suppose that the consumer is observed to choose a bundle $(\bar{x}_1, \bar{x}_2)$ when the price is p_1 for good 1 and p_2 for good 2. From this we immediately know that the consumer's income at this instance is equal to $p_1\bar{x}_1 + p_2\bar{x}_2$, so his budget line follows the equation

$$p_1x_1 + p_2x_2 = p_1\bar{x}_1 + p_2\bar{x}_2$$

and his budget set is the shaded area in Figure 1, with the bundle he chose labeled A. Now let B

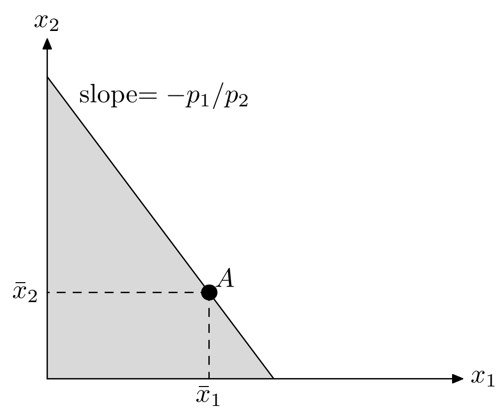

Figure 1: The grey area: the budget set when A is chosen

refer to any point in the grey area except A. By definition of a budget set, B is affordable to the consumer when he chooses A. The fact that he chooses A over B should reveal to us that he prefers A at least as much as B, otherwise he would not have chosen A. Furthermore, by unicity of the optimal bundle, we deduce that he prefers A to B. In sum, from an observation such as Figure 1 we can infer that the consumer likes A more than any bundle in the grey area except A.

Thus, we say that bundle A is *directly revealed preferred* to bundle B, denoted by $A \succ_{\mathscr{D}} B$, iff B belongs to the budget set given which the consumer chooses A. Note that this definition is based purely on what the consumer actually does rather than what he really thinks. But, armed with the notion of revealed preference, one can go a long way from even a single datapoint such as "the consumer chooses A given prices (p_1, p_2)." To illustrate, suppose that the prices and income are altered in such a way that the bundle A remains affordable to the consumer. That results in two implications through a revealed-preference argument, demonstrated below.

Prediction While there is no way for us to predict the consumer's new consumption quantity given the altered prices and income, as we do not know what his preferences are, we nonetheless can give qualitative predictions such as whether his consumption of the good rises or falls because of the budget shock. This is obvious by inspection of a graph such as Figure 2, where

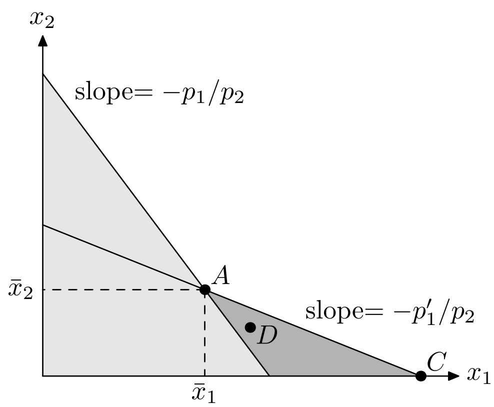

Figure 2: $p_1' < p_1$

the budget line after the price shock becomes flatter than before while the original chosen bundle A, due to a certain change in the income, remains on the new budget line. In the new budget set, the light grey area can be excluded because all the points there have been directly revealed worse than A while A is still affordable given the new budget set. That leaves the dark area as the only possibility. By monotonicity of the preference relation, we can further narrow the possibility down to the segment AC except A.

Welfare comparison Since the bundle A before the price and income shock is still affordable after the shock, the consumer is better-off. He would not be worse-off than before the shock because he can still choose A, the best bundle before the shock. Furthermore, the consumer is better-off: Note that the indifference curve passing through point A is tangent to the steep budget line at A; by diminishing MRS this indifference curve cuts into the interior of the dark area. Hence there is a point such as D in Figure 2 that belongs to both this interior and the curve; with monotone preferences, the consumer can do better than D by consuming slightly more of both goods, which he can afford because D is not on the budget line.

It should be kept in mind that both implications require the premise that the bundle chosen under the original budget is still affordable under the new budget (c.f. Exercise 6).

2 Prediction and the substitution effect

The idea behind Figure 2 is both intuitive and profound. Let us emphasize it through considering a price change opposite to that of the figure. Suppose that the price of good 1 increases from p_1 to $p_1 + \Delta p_1$ for some $\Delta p_1 > 0$, while that of good 2 stays constant, and that due to some change of the consumer's income he can afford the previous optimum bundle A before the price change. Thus, the budget line switches from l_A to l_B in Figure 3. Does point B, the optimum after the price change, belong to the left or the right of A? To figure that out, recall that l_B, by definition, is obtained by rotating l_A around point A until the rotated line has the steeper slope $-(p_1 + \Delta p_1)/p_2$. That means line l_B is steeper than line l_A, so Figure 3 is not lying. Since A is optimal given the original budget line l_A, it is uniquely so within the original budget set as MRS is diminishing. Hence every bundle in the grey (and not dark) region of Figure 3, other than A itself, is worse than A. Thus, none of such bundles would be chosen by the consumer when his budget line becomes the new one, because the better bundle A, belonging to the new budget line, is still affordable given the new price $p_1 + \Delta p_1$. That leaves the dark area in Figure 3 as the only possible bundles that the consumer would choose given price $p_1 + \Delta p_1$. Hence the optimal bundle given l_B as the budget line

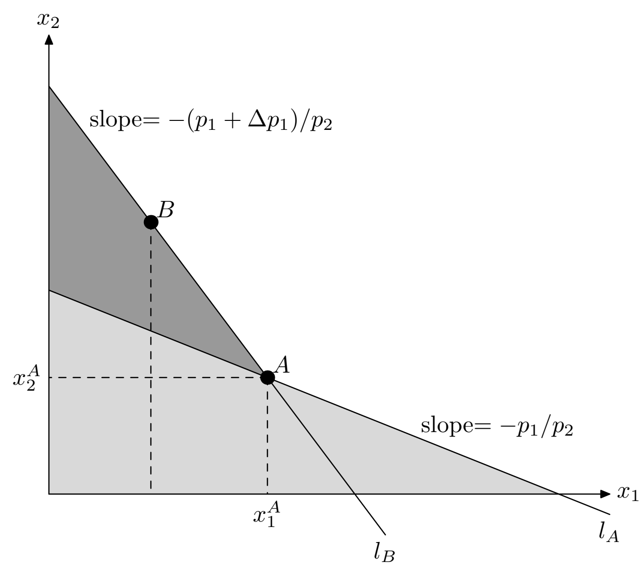

Figure 3: $A \to B$: the substitution effect of $p_1 \to p_1 + \Delta p_1$

is located to the left of A. Consequently,

$$x_1^B - x_1^A < 0,$$

which coupled with $\Delta p_1 > 0$ implies

$$\frac{x_1^B - x_1^A}{\Delta p_1} < 0. \tag{1}$$

Note that Ineq. (1) is true whether Δp_1 is positive or negative. In the case where $\Delta p_1 < 0$, which is the case of Figure 2, $x_1^B - x_1^A > 0$ as we have deduced there that the new optimum bundle belongs to the dark area in Figure 2, to the right of the previous optimum A. Then, since $\Delta p_1 < 0$ ($p_1' < p_1$), Ineq. (1) again follows.

The quotient $\frac{x_1^B - x_1^A}{\Delta p_1}$ is called *substitution effect* of a price change. We have established a fact, without relying on specific information about a consumer's preferences, that *Substitution effect is negative*.

3 Welfare comparison

3.1 Interpersonal comparison

Two individuals live in different economies in different currencies, given different incomes and facing different market prices. Would one of them rather switch identities with the other so as to swap their given market prices and incomes? This question might sound impossible to answer without information about their preferences or incomes, but the answer turns out to be Yes if one of them can still afford his current choice if he assumes the identity of the other. For example, imagine a world of only two commodities: coffee and sandwiches. Say Ian lives in a town where he has to pay 4 dollars per cup of coffee and 6 dollars per sandwich, and he consumes 3 cups of coffee and 2 sandwiches per day; whereas, Jing lives in another town where the prices are 10 yuans per cup of coffee and 12 yuans per sandwich, and she consumes 2 cups of coffee and 3 sandwich per day. Then, assuming monotone preferences, we can calculate that the income for Ian is equal to

$$4 \times 3 + 6 \times 2 = 24$$

dollars per day, and the income of Jing equal to

$$10 \times 2 + 12 \times 3 = 56$$

yuans per day. In Jing's town, the bundle $(3, 2)$ chosen by Ian costs

$$10 \times 3 + 12 \times 2 = 54$$

yuans, less than Jing's income of 56 yuans. Thus, if Ian lives in Jing's town and has Jing's income, he can still afford his original bundle. Then it follows from the revealed-preference reasoning explained in the previous section that Ian would be better-off swapping identities with Jing. Would Jing also like to switch? Her current consumption $(2, 3)$ in Ian's town costs

$$4 \times 2 + 6 \times 3 = 26$$

dollars, bigger than Ian's income. Thus, in switching with Ian, Jing cannot afford her current choice. Hence the revealed-preference reasoning is not applicable to her case. That, however, does not imply that Jing would reject the switch; we do not have enough information to predict one way or the other for Jing's case (see Exercise 5 for a precise treatment).

3.2 What we can infer from macroeconomic indices

By the same token, one could sometimes make welfare inferences from an aggregate economic index. Specifically, denote:

- $x_k^t :=$ the quantity of consumption of good k in the current year t
- $x_k^b :=$ the quantity of consumption of good k in the base year b
- $p_k^t :=$ the price of good k in the current year t
- $p_k^b :=$ the price of good k in the base year b

A quantity index based on the base-year quantities is defined to be

$$\mathscr{L}_q := \frac{p_1^b x_1^t + \ldots + p_n^b x_n^t}{p_1^b x_1^b + \ldots + p_n^b x_n^b}.$$

Now suppose the news is that $\mathscr{L}_q < 1$. That means, by definition of $\mathscr{L}_q$,

$$p_1^b x_1^t + \ldots + p_n^b x_n^t < p_1^b x_1^b + \ldots + p_n^b x_n^b.$$

If we make an assumption, routine in macroeconomics, that the entire economy were as if a single consumer, then the above inequality says that what this representative consumer chooses in the current year, $(x_1^t, \ldots, x_n^t)$, would have been affordable in the base year. Consequently, by the revealed-preference reasoning, the representative consumer would rather go back to the base year than stay in the current year, just like Ian would rather become Jing in the previous subsection. In other words, the representative consumer of the economy is worse-off.

There are also quantity indices based on base-year prices, price indices based on base-year quantities, and those based on base-year prices. The inference methods on them are analogous.

3.3 Sales tax rebate

It is straightforward to demonstrate, with a revealed-preference reasoning, that sales tax hurts consumers. A sales tax, say on good 1, increases the price that a consumer has to pay for good 1. That pivots the budget line around its x_2-intercept and hence shrinks the consumer's budget set. Thus, whatever the consumer chooses with the sales tax belongs to his budget set without the sales tax. Hence the consumer would do better without the sales tax than with the tax.[1]

In imperfect situations where a sales tax is deemed necessary, say consumption of the good may hurt the environment, is there a way to compensate the welfare loss caused by the tax? A natural idea is to return the tax revenue back to the consumer. One might think that such a tax rebate can restore the consumer's income and hence his welfare (while the sales tax reduces his consumption deemed harmful to the environment). But is that really true? To figure that out, let p_1 be the price of good 1 without the sales tax, and t the sales tax rate so that the price of good 1 with the sales tax becomes $p_1(1+t)$; fix the price of good 2 to some constant p_2, and the consumer's income without the sales tax to be m; and denote $(\bar{x}_1, \bar{x}_2)$ for the consumption bundle chosen by the consumer given the sales tax and the tax rebate. Hence the revenue raised by this sales tax is equal to $p_1 t \bar{x}_1$, which is returned to the consumer so that his income becomes $m + p_1 t \bar{x}_1$.[2] Thus, the consumer's budget constraint given the sales tax and the rebate policy is

$$p_1(1+t)x_1 + p_2 x_2 \leq m + p_1 t \bar{x}_1.$$

This inequality is satisfied when $(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$, as $(\bar{x}_1, \bar{x}_2)$ is the consumer's choice given the sales tax and the rebate policy. Hence

$$p_1(1+t)\bar{x}_1 + p_2 \bar{x}_2 \leq m + p_1 t \bar{x}_1,$$

[1] Real-world vindications of such revealed-preference reasoning can be easily torn from the headlines, e.g., the violent protest against the Macron gas tax in France not too long ago.

[2] If you wondered how that could be done in the real world, think of such rebate as an idealized model for the practice of using the tax revenue to pay college professors to teach intermediate microeconomics thereby enabling the consumers to earn back the money they paid to the tax agency. Anyway, we are simply picking the best case scenario of sales tax to argue that the tax hurts the consumer even in the best case scenario.

which, with $p_1 t\bar{x}_1$ canceled out on both sides, is equivalent to

$$p_1\bar{x}_1 + p_2\bar{x}_2 \le m.$$

That means $(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$ also satisfies

$$p_1 x_1 + p_2 x_2 \le m,$$

the budget constraint without the sales tax. In other words, whatever the consumer chooses under the sales tax and rebate policy would be affordable without the sales tax. Therefore the revealed-preference argument goes that the consumer is worse-off with the policy than without. That is, even an entire tax revenue is still not enough to make up for the welfare loss caused by a sales tax. The argument is illustrated by Figure 4.

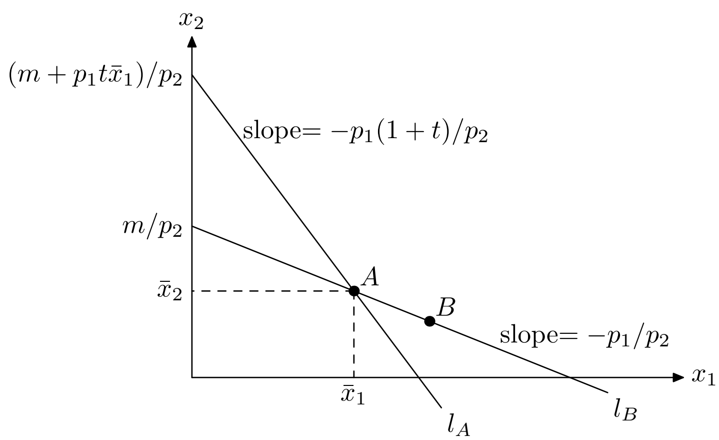

Figure 4: A: the choice given l_A (tax & rebate); B: the choice given l_B (no tax)

4 Indirectly revealed preference

With only one datapoint such as "the consumer chose bundle (x_1, x_2) given prices (p_1, p_2)," we have not enough information to infer the consumer's preference between (x_1, x_2) and those bundles (x_1', x_2') for which

$$p_1 x_1' + p_2 x_2' > p_1 x_1 + p_2 x_2.$$

To see why, note that the above inequality does not imply that (x_1', x_2') is affordable given prices (p_1, p_2). Thus we cannot conclude that $(x_1, x_2) \succ_{\mathscr{D}} (x_1', x_2')$; nor can we conclude the reverse, lacking other datapoints.

If we have multiple datapoints, however, we might be able to infer the consumer's preference between bundles neither of which is directly revealed preferred to the other. For instance, suppose that we have two observations: first, the consumer chose bundle (x_1, x_2) given prices (p_1, p_2) while bundle (x_1'', x_2'') was affordable, i.e.,

$$p_1 x_1'' + p_2 x_2'' \le p_1 x_1 + p_2 x_2;$$

second, he chose bundle (x_1'', x_2'') given prices (p_1'', p_2'') while (x_1', x_2') was affordable, i.e.,

$$p_1'' x_1' + p_2'' x_2' \le p_1'' x_1'' + p_2'' x_2''.$$

The first observation implies $(x_1, x_2) \succ_{\mathscr{D}} (x_1'', x_2'')$, and the second implies $(x_1'', x_2'') \succ_{\mathscr{D}} (x_1', x_2')$. If the consumer's behavior is consistent with a rational decision, his preference should be transitive, so the inferences $(x_1, x_2) \succ_{\mathscr{D}} (x_1'', x_2'')$ and $(x_1'', x_2'') \succ_{\mathscr{D}} (x_1', x_2')$ combined should imply that he prefers (x_1, x_2) to (x_1', x_2'). In this case we say (x_1, x_2) is *indirectly* revealed preferred to (x_1', x_2').

In general, we say that bundle A is *indirectly revealed preferred* to bundle B, denoted by $A \succ_{\mathscr{I}} B$, iff, for some integer $m \geq 2$ and some bundles $C_1, \ldots, C_{m-1}$, we have

$$A \succ_{\mathscr{D}} C_1 \succ_{\mathscr{D}} \cdots \succ_{\mathscr{D}} C_{m-1} \succ_{\mathscr{D}} B.$$

Figure 5 illustrates such an example with two datapoints: $A \succ_{\mathscr{D}} C$ and $C \succ_{\mathscr{D}} B$. Since bundle B lies above the budget line l_A given which A was chosen, and we have no observation where B was chosen, there is no direct revealed preference between A and B. But indirect revealed preference between them is available: since $A \succ_{\mathscr{D}} C$ and $C \succ_{\mathscr{D}} B$, we have $A \succ_{\mathscr{I}} B$.

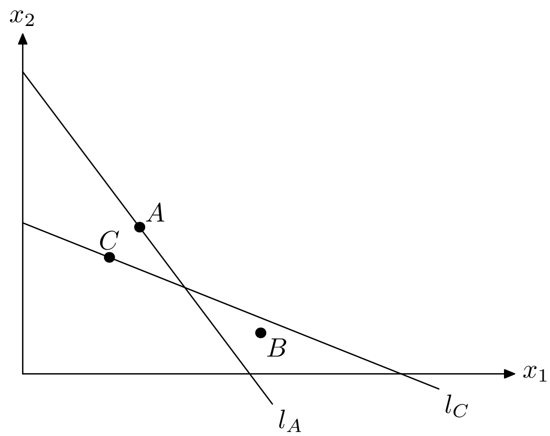

Figure 5: A: the choice given budget line l_A; C: the choice given budget line l_C

5 When can one's behavior be rationalized?

The exercise of inferring one's preferences from his choice relies on the premise that his choice is consistent with rational decision-making. But then how do we know if this premise holds or not? Asking our subject "Are you rational?" would be silly. Even if he meant to tell us the truth, our subject's perception about himself may differ from the reality of himself. Furthermore, even if his choice is consistent with a rational decision-making procedure, he need not be following that process consciously, calculating his MRS condition and watching out for the corner solutions in figuring out what to buy in grocery stores, nor would he necessarily know even if he followed the procedure subconsciously. Thus we take a behavioral viewpoint. Instead of asking whether a decision maker is rational or not, we ask whether his behavior is *rationalizable* or not, i.e., whether or not there exists some well-behaved preference relation such that the choices we have observed him make coincide with the solutions of the decision problem with the preference relation being the objective. If the answer is Yes then we can conclude that our subject behaves *as if* a rational decision maker maximizing his preference subject to his budget constraint. Now that rationalizability is purely about a person's observed actions rather than his mind, a condition for rationalizability should also be purely about his observed actions. Economists have found two such conditions. One is called the weak axiom, and the other strong axiom, of revealed preference.

To be brief in the rest of the chapter, we shall use *consumer* as the shorthand for the dataset that records all the choices that the consumer made in response to various market prices. Based on this dataset the relations $\succ_{\mathscr{D}}$ and $\succ_{\mathscr{I}}$ are defined as in previous sections.

5.1 The weak axiom

A consumer is said to satisfy the *weak axiom of revealed preference* (WARP) iff

$$A \succ_{\mathscr{D}} B \Longrightarrow \text{not}\,[B \succ_{\mathscr{D}} A] \tag{2}$$

for any bundles A and B. That is, if A is directly revealed preferred to B in one observation, then there is no other observation where B is directly revealed preferred to A, otherwise is a contradiction. Hence it is obvious that satisfaction of the weak axiom is a necessary condition for a consumer to be rationalizable.

Figure 6 shows an example where the weak axiom is violated. There, bundle A was chosen given budget line l_A, under which bundle C lies, hence $A \succ_{\mathscr{D}} C$. Whereas, A also lies below l_C, the budget line give which bundle C was chosen, hence $C \succ_{\mathscr{D}} A$.

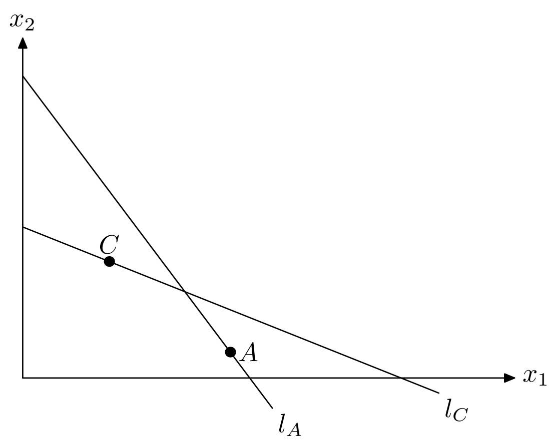

Figure 6: A violation of the weak axiom

To illustrate how to check the weak axiom numerically, consider for example the following dataset of a consumer in a world with only two goods:

1. when prices $p_1 = 2$ and $p_2 = 2$, the chosen bundle (x_1, x_2) was $(10, 1)$;
2. when $(p_1, p_2) = (2, 1)$, $(x_1, x_2) = (5, 5)$;
3. when $(p_1, p_2) = (1, 2)$, $(x_1, x_2) = (5, 4)$.

First, make the following table, where the three rows correspond to the price pairs in the three observations, and the three columns the chosen bundles respectively. In the cell on row i and column j ($i = 1, 2, 3$ and $j = 1, 2, 3$), enter the total expense of the chosen bundle in the jth observation under the prices in the ith observation. For example, in the row-2 column-3 cell, \$14=\$2 $\times$ 5 + \$1

$\times$ 4, signifying that the consumer would have spent \$14 had he chosen bundle $(5, 4)$ in observation 3 given the prices $(\$2, \$1)$ in observation 2.

	(10, 1)	(5, 5)	(5, 4)
(\$2, \$2)	\$22	\$20	\$18
(\$2, \$1)	\$21	\$15	\$14
(\$1, \$2)	\$12	\$15	\$13

Next, infer direct reveled preferences row by row. In the first row, $22 > 20$ and $22 > 18$, meaning that when $(10, 1)$ was chosen, bundles $(5, 5)$ and $(5, 4)$ are each affordable, hence $(10, 1) \succ_{\mathscr{D}} (5, 5)$ and $(10, 1) \succ_{\mathscr{D}} (5, 4)$. In the second row, $21 > 15 > 14$: when $(5, 5)$ was chosen, which costed \$15, bundle $(5, 4)$ was affordable while $(10, 1)$ was not. Hence $(5, 5) \succ_{\mathscr{D}} (5, 4)$. By the same token, the third row implies that $(5, 4) \succ_{\mathscr{D}} (10, 1)$. Collecting all the inferences, we find $(10, 1) \succ_{\mathscr{D}} (5, 4)$ and $(5, 4) \succ_{\mathscr{D}} (10, 1)$, which violates (2), hence the consumer does not satisfy the weak axiom.

5.2 The strong axiom

While the weak axiom is a necessary condition for rationalizability, it is not sufficient. To show the insufficiency, consider the following dataset of a consumer in a world with three goods:

1. when the prices $(p_1, p_2, p_3) = (1, 3, 10)$, the chosen bundle $(x_1, x_2, x_3) = (3, 1, 4)$;
2. when $(p_1, p_2, p_3) = (4, 3, 6)$, $(x_1, x_2, x_3) = (2, 5, 3)$;
3. when $(p_1, p_2, p_3) = (1, 1, 5)$, $(x_1, x_2, x_3) = (4, 4, 3)$.

As in the previous subsection, make the following table of expenses. For example, in the row-3 column-1 cell, \$24=\$1$\times$ 3+\$1 $\times$ 1+\$5 $\times$ 4, meaning that the bundle $(3, 1, 4)$ chosen in observation 1 would have costed \$24 had it been bought given the prices $(\$1, \$1, \$5)$ in observation 3.

	(3, 1, 4)	(2, 5, 3)	(4, 4, 3)
(\$1, \$3, \$10)	\$46	\$47	\$46
(\$4, \$3, \$6)	\$39	\$41	\$46
(\$1, \$1, \$5)	\$24	\$22	\$23

Now infer from the table direct revealed preferences. The first row means $(3, 1, 4) \succ_{\mathscr{D}} (4, 4, 3)$, the second row means $(2, 5, 3) \succ_{\mathscr{D}} (3, 1, 4)$, and the third row, $(4, 4, 3) \succ_{\mathscr{D}} (2, 5, 3)$. Note that there is no violation of the weak axiom; however, $(3, 1, 4) \succ_{\mathscr{D}} (4, 4, 3)$ and $(4, 4, 3) \succ_{\mathscr{D}} (2, 5, 3)$ together mean $(3, 1, 4) \succ_{\mathscr{I}} (2, 5, 3)$, reversing the inference $(2, 5, 3) \succ_{\mathscr{D}} (3, 1, 4)$. That is impossible if the consumer is rationalizable.

Thus, the weak axiom does not suffice rationalizability. A condition that suffices is the strong axiom. To introduce the axiom, define a relation $\succ_*$ by, for any bundles A and B,

$$A \succ_* B \Longleftrightarrow [A \succ_{\mathscr{D}} B \text{ or } A \succ_{\mathscr{I}} B]\,; \tag{3}$$

in other words, $A \succ_* B$ means bundle A is directly or indirectly revealed preferred to bundle B. A consumer is said to satisfy the *strong axiom of revealed preference* (SARP) iff

$$A \succ_* B \Longrightarrow \text{not}\,[B \succ_* A] \tag{4}$$

for any bundles A and B.

If you find Eq. (4) terse to penetrate, note that it is equivalent to

$$\text{not}\,[A \succ_* B \succ_* A]\,,$$

which, by definition of $\succ_*$ and $\succ_{\mathscr{I}}$, is equivalent to

$$\text{not}\,[A \succ_{\mathscr{D}} C_1 \succ_{\mathscr{D}} \cdots \succ_{\mathscr{D}} C_{m-1} \succ_{\mathscr{D}} A \text{ for some integer } m \geq 2]\,.$$

In other words, the strong axiom requires that there be no cycle of direct revealed preference of any length. The shortest cycles is of length 2 ($m = 2$), in the form of $A \succ_{\mathscr{D}} C_1 \succ_{\mathscr{D}} A$. One readily sees that the strong axiom implies the weak axiom, as the weak axiom, Eq. (2), is equivalent to

$$\text{not}\,[A \succ_{\mathscr{D}} B \succ_{\mathscr{D}} A]\,,$$

ruling out only 2-cycles, whereas the strong axiom rules out all cycles.

5.3 The strong axiom is necessary and sufficient for rationalizability

It is clear that the strong axiom is necessary for rationalizability, because violation of the axiom means that the dataset has a cycle $A \succ_{\mathscr{D}} C_1 \succ_{\mathscr{D}} \cdots \succ_{\mathscr{D}} C_{m-1} \succ_{\mathscr{D}} A$ of some length m, which by transitivity implies that the consumer prefers A to A itself, a contradiction. What is non-trivial, however, is that *the strong axiom is also a sufficient condition for rationalizability*. This mathematical fact was first proved in full generality by the late Marcel Ket Richter [1].[3]

Here is a sketch of the reasoning why the strong axiom guarantees rationalizability. To rationalize a consumer's observed behavior, or demand, we need to come up with a preference relation on the set of all possible consumption bundles such that, for any bundle X that we observe him choose given some budget set, X is strictly preferred to any other bundle that belongs to that budget set. Naturally, let us try the $\succ_*$ defined by (3) according to his revealed preference. First, note that $\succ_*$ is well-defined in the sense that if $A \succ_* B$ then it will never be contradicted by $B \succ_* A$. That is owing to the strong axiom: $A \succ_* B$, according to (3), means that $A \succ_{\mathscr{D}} B$ or $A \succ_{\mathscr{I}} B$; then the strong axiom implies that it is impossible to have $B \succ_{\mathscr{D}} A$ or $B \succ_{\mathscr{I}} A$, otherwise we would have a cycle. Second, to see why $\succ_*$ does work, pick any bundle X that we observe the consumer choose given some budget set say $\mathscr{B}$, and pick any other bundle Y that belongs to $\mathscr{B}$. The fact that Y belongs to $\mathscr{B}$ means that $X \succ_D Y$ and hence, by definition of $\succ_*$, $X \succ_* Y$. With Y an arbitrary alternative to X in $\mathscr{B}$, X is indeed the optimum in $\mathscr{B}$ if $\succ_*$ is the preference relation.

A rationalizing preference relation $\succ_*$ should also be *total* in the sense that, for any two distinct bundles A and B, either $A \succ_* B$ or $B \succ_* A$. That is not yet guaranteed by the previous construction. For instance, the consumer might have never chosen A or B in our dataset; then the most we can infer about A and B is that there might be some bundles C and D such that $C \succ_* A$

[3] He was this author's graduate school advisor. Richter was initially hired by the University of Minnesota to teach finance. The epiphany that uplifted him to abstract economics happened almost 60 years ago, when Richter, then a young assistant professor, stood in a hallway waiting to have a conversation with the late Leo Hurwicz, his senior colleague, who was in the middle of a conversation with a graduate student. Frustrated by the student's vague articulation, Hurwicz asked a question loudly, which Richter overheard: "Do you mean for all x there exists a y, or there exists a y for all x?" Even a simple transposition of quantifiers as such can make a world of difference! The revelation dawning on Richter, he soon started the journey to mathematical economics and became a pioneer thereof.

and $D \succ_* B$, but that does not imply a relation between A and B. To achieve totality, we need to extend the relation $\succ_*$ to the entire set of possible bundles in a consistent manner.

To illustrate such an extension, consider only the case where there are only finitely many possible bundles for the consumer to choose. Start with any pair of bundles such that $A \succ_* B$. For any third bundle C that is related to A or B via $\succ_*$, append C to the list according to its ordering: say $C \succ_* A$, then the list becomes $C \succ_* A \succ_* B$; if $B \succ_* C$, then $A \succ_* B \succ_* C$; else, C is revealed worse than A and better than B, then write $A \succ_* C \succ_* B$. There is no contradiction in constructing this list, because $\succ_*$ by the strong axiom is transitive. Hence we obtain a list of three bundles, ordered by $\succ_*$: $Y_1 \succ_* Y_2 \succ_* Y_3$. Continue this process, adding to the list any other bundle that is $\succ_*$-related to some members in the existing list, until there is no more distinct bundle to add to the list. Such a *maximal* list is in the form of

$$Y_1 \succ_* Y_* \succ_* \cdots \succ_* Y_k$$

for some integer k. Then start with another bundle outside the previous list and construct another maximal list analogously,

$$Z_1 \succ_* Z_2 \succ_* \cdots \succ_* Z_j$$

for some integer j ($j = 1$ if Z_1 is not $\succ_*$-related to any bundle other than itself). Note that no member of the second list is $\succ_*$-related to anyone of the first list, otherwise it would have been added to the first list already. Thus, we can simply declare $Y_k \succ_* Z_1$ without violating the transitivity of $\succ_*$ (or the strong axiom). With this declaration we merge the two list into one:

$$Y_1 \succ_* Y_* \succ_* \cdots \succ_* Y_k \succ_* Z_1 \succ_* Z_2 \succ_* \cdots \succ_* Z_j.$$

For any bundle outside this list, repeat the above procedure. Since there are only finitely many possible bundles, the procedure will eventually halt, and we obtain a list that exhausts all possible bundles. It follows that, with $\succ_*$ fully extended by the above procedure, for any distinct bundles A and B, either $A \succ_* B$ or $B \succ_* A$. Thus $\succ_*$ becomes total.[4]

6 Exercises

1. Suppose that a consumer's preference relation is monotone, smooth (c.f. §3, Chapter 5) and exhibiting diminishing MRS, and that the given market prices and income are each positive.

   a. Explain why the consumer's optimum is unique and belongs to the budget line.

   b. Explain why the bundle B in Figure 3 cannot be the same as A in that figure.

2. Draw a figure analogous to Figure 2 to illustrate the revealed-preference argument in the case where $p_1' > p_1$.

3. Nathan's preference relation is monotone, smooth and exhibiting diminishing MRS. He is given a fixed quantity of income (and nothing else). When the price is $2 for good 1 and

[4] When the number of possible bundles is infinite, the proof is similar, though the above procedure, based on mathematical induction, needs to be justified by an additional mathematical axiom (which one would learn in a course of set theory or real analysis).

\$6 for good 2, Nathan chooses the bundle $(3,3)$. Now the price of good 1 has increased to \$4 while the price of good 2 and Nathan's income remain unchanged, and Nathan ends up choosing a new bundle such that his consumption of good 1 is 2.5 units.

a. Write down the equation for Nathan's budget line before the price change, and that after the price change.

b. Suppose that, after the price change, Nathan's income is adjusted so that he has the exact amount of income to afford the previous bundle $(3,3)$ under the new price. Then, given such adjustment and the new price:

   i. write down the equation for Nathan's budget line;

   ii. if the quantity of good 1 optimally chosen by Nathan in this circumstance is one of the following: 2 units, 3 units, 4 units, or 9 units, then which one should it be? (hint: Figure 3)

4. With the notations in Section 3.2, define a quantity index $\mathscr{P}_q$ with the goods weighted by current year prices,
$$\mathscr{P}_q := \frac{p_1^t x_1^t + \cdots + p_n^t x_n^t}{p_1^t x_1^b + \cdots + p_n^t x_n^b},$$
and a price index $\mathscr{L}_p$ with weights determined by the base-year quantities,
$$\mathscr{L}_p := \frac{p_1^t x_1^b + \cdots + p_n^t x_n^b}{p_1^b x_1^b + \cdots + p_n^b x_n^b}.$$

a. If the news is that $\mathscr{P}_q < 1$ (e.g., "GDP has dropped!"), is the representative consumer better-off (than the base year), worse-off, or there is insufficient information to tell one way or the other?

b. If the news is that
$$\mathscr{L}_p < \frac{p_1^t x_1^t + \cdots + p_n^t x_n^t}{p_1^b x_1^b + \cdots + p_n^b x_n^b},$$
is the representative consumer better-off (than the base year), worse-off, or there is insufficient information to tell one way or the other?

5. Recall from Section 3.1 the data about Ian and Jing:

situation	p_c	p_s	x_c	x_s
Ian	\$4	\$6	3	2
Jing	¥10	¥12	2	3

Assume that both Ian and Jing have preference relations that satisfy the three axioms, are monotone, smooth and exhibit diminishing MRS. With specific numbers available, we can do more than in Problem 4:

a. On a coordinate system with the horizontal axis x_c standing for the quantity of coffee, and vertical axis x_s for the quantity of sandwich, draw the budget line given to Jing and label it l_J; and draw the consumption bundle chosen by Jing and label it J.

b. On the same diagram obtained in the previous step, draw the budget line l_I given to Ian and the consumption bundle I chosen by Ian.

c. Is bundle J directly revealed preferred to bundle I by Jing? Is bundle J directly revealed preferred to bundle I by Ian?

d. Would Ian be better-off in switching to the income and prices given to Jing? Do we know for sure that Jing is better-off in switching to the income and prices given to Ian?

e. Does the budget set given to Ian contain bundles outside the budget set given to Jing? Do we know for sure that Jing is *worse*-off in having the income and prices given to Ian?

f. Suppose that the data about Jing is not the last row in the above table, but rather

situation	p_c	p_s	x_c	x_s
Jing	¥10	¥14	2	3

On the above-described x_c-x_s diagram, redraw Ian's budget line l_I as before, and draw Jing's budget line l'_J according this data rather than the one in the previous table. Is Ian's budget set completely contained by Jing's budget set? Now do we know for sure that Jing is worse-off in having the income and prices given to Ian?

6. We have argued that the sales tax and rebate policy makes a consumer worse-off because, as in Figure 4, the optimal bundle A given the budget line l_A with the policy is still affordable given the budget line l_B without the policy. However, if the budget line switches from l_B without the policy to l_A with the policy, the bundle A remains affordable. Why then is it not true that the consumer is better-off if his budget line switches from l_B to l_A? (If it were true then the sales tax and rebate policy would make the consumer better-off, a contradiction to what we have demonstrated in the text.)

7. Here is a table of the prices and demands of a consumer Kate, whose behavior was observed in the following three different, independent situations.

situation	p_1	p_2	x_1	x_2
A	\$2	\$4	6	2
B	\$5	\$5	3	5
C	\$8	\$2	5	4

Assume that Kate's preference relation satisfies all three axioms, is continuous and monotone, and exhibits diminishing MRS.

a. Draw a diagram that contains, for each of the three observations, Kate's budget line. Label the bundle chosen in each case by the letters A, B and C, and the budget line in each case by l_A, l_B and l_C.

b. Does Kate's behavior satisfy the weak axiom? Does your answer suffice a conclusion on the rationalizability of Kate's behavior?

c. Does Kate's behavior satisfy the strong axiom? Does your answer suffice a conclusion on the rationalizability of Kate's behavior?

d. On the above diagram:

i. Shade with a light color all the points that are certain to be worse for Kate than bundle B.

ii. Shade with a dark color all the points that are certain to be better for Kate than B (use Exercise 10.b in Chapter 4).

iii. Which region does Kate's indifference curve passing through B belong to?

8. Here are three independent observations of the consumption bundle (x_1, x_2, x_3) chosen by Larry given the price vector (p_1, p_2, p_3) in a three-commodity economy.

situation	p_1	p_2	p_3	x_1	x_2	x_3
A	\$1	\$2	\$3	5	3	2
B	\$3	\$2	\$1	4	1	5
C	\$2	\$3	\$1	6	4	1

a. Based on the above data, produce a table, analogous to the ones in Sections 5.1 and 5.2, that shows the expenditures of the bundles given various price vectors.

b. Is Larry's behavior consistent to the weak axiom? The strong axiom? Rationalizable?

References

[1] Marcel K. Richter. Revealed preference theory. *Econometrica*, 34(3):635–645, July 1966. 5.3