Chapter 1: Firm's Supply

Elements of Decision: Lecture Notes of Intermediate Microeconomics 1

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1 Introduction

Economics is a science on relations among individuals. Such relations include trades, contracts, conflicts, households, markets, networks, societies, etc. Any such relation can be modeled into a game, where one tries to pick a move as a best response to the move he expects his counterpart may pick, knowing that the counterpart's move also results from her attempt to best respond to his move that she expects him to pick. Calculations of such *interactive reasoning* constitute *game theory*, the mathematical foundation of modern economics. The fundamental building block of such calculations is the mathematical method for an individual to decide on his best move. To introduce such *decision-theoretic* method in the simplest possible setting, we start by pretending that the counterpart of the individual is not a strategic player trying to game him but rather a dummy such as a purely competitive market, which the individual does not need to haggle with. Given this assumption of pure competition, learning the basic decision-theoretic techniques is what we will do throughout this course, the gateway to more serious trainings in economics.

2 A firm's output decision problem

To illustrate a decision problem in its simplest form, suppose that a firm acts as an individual deciding how much of its output to sell.[1] Suppose that every unit of the output is sold for p dollars in the market (the aforementioned pure competition assumpiton), and that the firm incurs a cost $C(q)$ dollars if it sells q units of the output to the market. The question is How many units of the output should the firm supply to maximize its profit? As *profit* means revenue minus cost, the firm's profit from supplying q units is equal to $pq - C(q)$ dollars. Thus the firm's decision problem becomes choosing a nonnegative quantity q to maximize $pq - C(q)$, i.e.,

$$\max_{q \in \mathbb{R}_+} pq - C(q). \tag{1}$$

Problem (1) exemplifies an optimization problem: (i) There is an *objective*, the expression $pq - C(q)$ following the operator max (shorthand for maximization), which the decision maker is to maximize (or minimize if max is replaced by min). (ii) There is a *choice variable* and its *domain*, written underneath max to indicate that the decision maker is to choose an element from the domain to maximize his objective; here the choice variable is denoted q, its domain $\mathbb{R}_+$ (the set of nonnegative real numbers),[2] with the symbol $\in$ meaning "belongs to" or "is an element of." (iii) There is a *parameter*, which, p in this example, is assumed constant by the decision maker; it

[1] For a real-world example, think of a small dairy farm that sells a single kind of milk to a competitive market.

[2] Interchangeably $\mathbb{R}_+$ is also denoted by $[0, \infty)$, meaning the set of real numbers between zero and infinity ∞, with the [on the left signifying that the boundary point zero is included, and the) on the right that ∞ is excluded. The notation $(0, \infty)$, with the left [replaced by (, excludes zero and denotes the set of positive real numbers.

is important to keep in mind the distinction between a choice variable and a parameter: the former is up to the decision maker to choose, while the latter is not.

3 Cost function

To make Problem (1) more tractable, let us add two usual assumptions, and figure out their implications, of the above cost function C. It is usually *assumed* that C is of the form

$$C(q) = C_v(q) + c_0 \tag{2}$$

for any $q \geq 0$, where $C_v(q)$ varies with q with $C_v(0) = 0$, hence called *variable cost*, and c_0 is positive and constant to q, hence called *fixed cost*. Correspondingly, the *average cost*, defined to be $C(q)/q$ and denoted by $\mathrm{AC}(q)$, is decomposed by

$$\mathrm{AC}(q) = \frac{C(q)}{q} = \frac{C_v(q)}{q} + \frac{c_0}{q}, \tag{3}$$

with $C_v(q)/q$ called *average variable cost* (AVC), and c_0/q *average fixed cost*.

It is also *assumed* that C is differentiable and that its derivative $\frac{d}{dq}C$, called *marginal cost* and denoted by $\mathrm{MC}(q)$, is continuously decreasing in q when q rises from zero and, once q rises to a threshold level, becomes continuously increasing in q thereafter without upper bound. By Eq. (2),

$$\mathrm{MC}(q) = \frac{d}{dq}C(q) = \frac{d}{dq}C_v(q). \tag{4}$$

Draw a coordinate system whose horizontal axis stands for q, and vertical axis for the marginal cost and the various kinds of average costs. In this coordinate system the marginal cost function corresponds to a curve, U-shape because of the assumption stated before Eq. (4). Note the vertical intercept $\mathrm{MC}(0)$ of this curve. A fact is that $\mathrm{MC}(0)$ is also equal to the vertical intercept of the AVC curve, i.e.,

$$\mathrm{MC}(0) = \lim_{q\to 0} \frac{C_v(q)}{q}. \tag{5}$$

The right-hand side of this equation might look complicated from an elementary arithmetic viewpoint, as $C_v(q)/q$ would become zero divided by zero if $q = 0$. This issue is resolved by basic calculus, where L'Hôpital's rule implies

$$\lim_{q\to 0} \frac{C_v(q)}{q} = \lim_{q\to 0} \left(\frac{d}{dq}C_v(q) \Big/ \frac{d}{dq}q \right) \overset{(4)}{=} \lim_{q\to 0} \left(\mathrm{MC}(q)/\, 1 \right) = \mathrm{MC}(0),$$

hence Eq. (5) is true. Thus, at the point $q = 0$, the MC and AVC curves coincide.

Starting from $q = 0$, and from the same vertical intercept, draw the graphs of both MC and AVC for those $q > 0$ that are nearby zero. Assumed U-shape, the MC curve must go downward for such near-zero q's. What about the AVC curve? To figure that out, recall the definition of AVC, $\mathrm{AVC}(q) = C_v(q)/q$, and calculate its derivative by the quotient rule:

$$\frac{d}{dq}\mathrm{AVC}(q) = \frac{d}{dq}\frac{C_v(q)}{q} = \frac{1}{q^2}\left(q\frac{d}{dq}C_v(q) - C_v(q) \right) = \frac{1}{q}\left(\mathrm{MC}(q) - \mathrm{AVC}(q) \right).$$

Thus, whenever $q > 0$,

$$\mathrm{MC}(q) > (\text{resp.} <)\ \mathrm{AVC}(q) \iff \frac{d}{dq}\mathrm{AVC}(q) > (\text{resp.} <)\ 0, \tag{6}$$

where the double arrow $\Leftrightarrow$ is read as "if and only if," also written as iff, meaning that the statements on the two sides of the arrow are logically equivalent. Eq. (6) says that whenever the MC curve is above the AVC curve ($\mathrm{MC}(q) > \mathrm{AVC}(q)$), the AVC curve is upward sloping ($\frac{d}{dq}\mathrm{AVC}(q) > 0$), and whenever MC is below AVC ($\mathrm{MC}(q) < \mathrm{AVC}(q)$), AVC is downward sloping ($\frac{d}{dq}\mathrm{AVC}(q) < 0$). Consequently, at the positive q's nearby zero, the AVC curve must lie above MC. Why? Suppose, to the contrary, that AVC lies below MC at such q's; bounded from above by the downward-sloping MC curve, the AVC curve must be forced to slope downward at some of such q's, but then Eq. (6) implies that the AVC curve jumps above the MC curve at such q's, contradiction.

Thus, the AVC curve is above the MC curve and is downward sloping when q increases slightly from zero, and remains so as q further increases until the AVC curve crosses the MC curve. Why must such intersection happen? That is because the MC curve, assumed U-shape without upper bound, must become upward sloping eventually and then rise high enough to surpass AVC, which by Eq. (6) is downward sloping as long as MC is still below it. Once q is above the threshold at which the two curves intersect, we have $\mathrm{MC}(q) > \mathrm{AVC}(q)$ and then Eq. (6) implies that AVC slopes upward from now on. Hence—

> the AVC curve is also U-shape, and the point at which it turns from downward- to upward-sloping is exactly the intersection between the AVC and MC curves.

To plot the graph for the average cost AC, note from Eq. (3) that $\mathrm{AC}(0) = c_0/0 = \infty$. Hence when q increases slightly from zero, the AC curve lies above the MC curve. Thus, by the same method as in the case of AVC, one can demonstrate that *the average cost curve is also U-shape, and the point at which it turns from downward- to upward-sloping is exactly the intersection between the AC and the MC curves*. Note again from Eq. (3) that the AC curve is always above the AVC curve and that the gap between them, which equals c_0/q, shrinks to zero as q goes to infinity.

4 The profit-maximizing output

To solve Problem (1), note that the objective $pq - C(q)$ is a differentiable function of the choice variable q. Thus, by basic calculus, if q_* is an *interior solution* of the problem in the sense that q_* solves Problem (1) and $q_* \neq 0$ (as zero is the boundary of the domain $\mathbb{R}_+$ for q), then q_* satisfies the *first-order condition*, i.e., the first-order derivative of the objective with respect to q is equal to zero when $q = q_*$. That is,

$$\frac{d}{dq}\left(pq - C(q)\right)\Big|_{q=q_*} = p - \frac{d}{dq}C(q)\Big|_{q=q_*} = 0.$$

Here the notation $|_{q=q_*}$ signifies the operation of calculating the derivative at the point q_*: first, take the derivative of $C(q)$ with respect to q for general q's; then plug $q = q_*$ into what you have just obtained. The above-displayed equation is equivalent to

$$p = \mathrm{MC}(q_*). \tag{7}$$

Thus, if q_* is an interior solution then it is an output quantity at which the MC curve intersects the horizontal line of height p. Since MC is assumed U-shaped, if it intersects the horizontal line at all then in general there are two intersection points, one in the range where MC is downward sloping, the other upward sloping. Are they both solutions to Problem (1)? If not then how to tell which one is? To answer these questions, recall from basic calculus that if an interior point q_* maximizes the objective then it satisfies the *second-order condition*: the second-order derivative of the objective at q_* is nonpositive. To write down the second-order condition in our case, calculate the second-order derivative of the objective $pq - C(q)$ with respect to q:

$$\frac{d^2}{dq^2}\left(pq - C(q)\right) = -\frac{d}{dq}\mathrm{MC}(q).$$

Thus, the second-order condition is $-\frac{d}{dq}\mathrm{MC}(q) \le 0$, i.e.,

$$\left.\frac{d}{dq}\mathrm{MC}(q)\right|_{q=q_*} \ge 0.$$

This condition eliminates the intersection point where MC slopes downward (where $\frac{d}{dq}\mathrm{MC}(q) < 0$), leaving the other intersection point the only possible candidate.

Thus, the only candidate for an interior solution of Problem (1) is the output level q_* at which the upward sloping portion of the MC curve intersects the horizontal line of height p. Satisfying both the first- and second-order conditions, q_* is a *local* maximum of the objective, local in the sense that the objective does not get higher when q is slightly above or below q_*. In general, satisfaction of both conditions does not suffice being the *global* maximum among all interior points, as there may be other local maximums. In our case, however, since q_* is the only local maximum, it is also the global maximum among all interior points.

Consequently, to solve Problem (1) we need only to compare the q_* obtained above with the corner point zero. The former generates profit $pq_* - C(q_*)$, while the latter generates profit $-c_0$ by Eq. (2) and the fact $C_v(0) = 0$. The profit-maximizing output level is q_* if $pq_* - C(q_*) > -c_0$, and zero if the inequality is reversed. This inequality, by Eq. (2), is equivalent to

$$pq_* - C_v(q_*) > 0, \quad \text{i.e.,} \quad p > \frac{C_v(q_*)}{q_*},$$

which, due to Eq. (7), is equivalent to

$$\mathrm{MC}(q_*) > \mathrm{AVC}(q_*).$$

Thus, q_**, the output level at which the upward-sloped portion of MC intersects the price line* p*, is the profit-maximum if the intersection lies above the intersection between MC and AVC; if it lies below the latter intersection then the profit-maximum is zero, i.e., supplying no output at all.*

5 An example

Suppose that the cost function is

$$C(q) := 2q^3 - 12q^2 + 30q + 100$$

for any $q \geq 0$ (where the symbol $:=$ means "is defined to be equal to"). One readily verifies that

$$\begin{aligned} c_0 &= 100, \\ C_v(q) &= 2q^3 - 12q^2 + 30q, \\ \mathrm{AC}(q) &= 2q^2 - 12q + 30 + 100/q, \\ \mathrm{AVC}(q) &= 2q^2 - 12q + 30, \\ \mathrm{MC}(q) &= 6q^2 - 24q + 30. \end{aligned} \tag{8}$$

Note, as in the general case demonstrated previously, $\mathrm{AVC}(0) = 30 = \mathrm{MC}(0)$ and $\mathrm{AC}(0) = \infty$. To find the intersection between MC and AVC, solve the equation $\mathrm{AVC}(q) = \mathrm{MC}(q)$, i.e.,

$$2q^2 - 12q + 30 = 6q^2 - 24q + 30,$$

which gives $q = 3$. Alternatively, we can find this intersection through the established fact that the intersection is the minimum of AVC. Take the derivative of AVC and find the minimum via the first-order condition:

$$\frac{d}{dq}\mathrm{AVC}(q) = \frac{d}{dq}\left(2q^2 - 12q + 30\right) = 4q - 12 = 0,$$

which gives $q = 3$, same as the previous method. (Note that $q = 3$ is the global minimum of AVC because AVC is U-shaped: its second-order derivative

$$\frac{d^2}{dq^2}\mathrm{AVC}(q) = \frac{d}{dq}(4q - 12) = 4 > 0$$

for all $q \geq 0$.) Analogously, to find the intersection between AC and MC we solve the equation

$$4q^2 - 12q = \frac{100}{q}, \quad \text{i.e.,} \quad q^3 - 3q^2 - 25 = 0.$$

Plugging the coefficients of this cubic equation into a cubic equation calculator,[3] we find $q \approx 4.3$.

Suppose that the market price for the output is \$18 per unit, i.e., $p = 18$. Let us calculate the profit maximum, denoted by q_*. By Eqs. (7) and (8),

$$18 = 6q_*^2 - 24q_* + 30, \tag{9}$$

which is equivalent to $(q_* - 2)^2 = 2$, implying that $q_* = 2 + \sqrt{2}$ or $q_* = 2 - \sqrt{2}$. To figure out which one is the solution, calculate the slope of the MC curve:

$$\frac{d}{dq}\mathrm{MC}(q) = 12q - 24,$$

which is nonnegative iff $q \geq 2$. Since $2 + \sqrt{2} > 2 > 2 - \sqrt{2}$, the MC curve is upward sloping at $2 + \sqrt{2}$, and downward sloping at $2 - \sqrt{2}$. Thus the latter fails the second-order condition, and $q_* = 2 + \sqrt{2}$ is the global maximum among all interior points.

We still need to compare the profit generated by $q_* = 2 + \sqrt{2}$ and that by $q = 0$. To do that, recall that $q = 3$ is where the MC curve intersects with the AVC curve, so that the former lies above the latter for all $q > 3$. Thus, since $2 + \sqrt{2} \approx 3.414 > 3$, we have $\mathrm{MC}(2 + \sqrt{2}) > \mathrm{AVC}(2 + \sqrt{2})$, i.e., $2 + \sqrt{2}$ is *the* profit-maximizing output level in this example.

[3] For example, http://www.1728.org/cubic.htm.

6 A firm's supply curve

How does a change in the market price p (due to some exogenous shock to which the firm has no influence) affect the firm's profit maximizing output? To answer this question, let us start with the above example, with the market price \$18 there replaced by the general symbol p. Following the same reasoning thereof, we obtain, as Eq (9), that

$$p = 6q_*^2 - 24q_* + 30.$$

This equation is the same as $6\left(q_*^2 - 4q_* + 4\right) - 24 + 30 = p$, i.e.,

$$6(q_* - 2)^2 = p - 6.$$

Hence q_* equals either $2 + \sqrt{(p-6)/6}$ or $2 - \sqrt{(p-6)/6}$. Note that the former is greater than 2 and the latter less than 2. As shown previously, the MC curve in this example is downward sloping iff $q < 2$. Hence it is downward sloping when $q = 2 - \sqrt{(p-6)/6}$. Thus

$$q_* = 2 + \sqrt{(p-6)/6}$$

is the global maximum of the profit among interior points. To compare q_* with the corner point $q = 0$, recall the fact that q_* is the profit maximum iff the point (q_*, p) belongs to the portion of the MC curve above the AVC curve, i.e., iff p is above the minimum level of AVC. As calculated previously, the AVC in this example attains its minimum at $q = 3$, hence

$$\min \mathrm{AVC} = \mathrm{AVC}(2) = 2 \times 3^2 - 12 \times 3 + 30 = 12.$$

Thus, the profit-maximizing output level is equal to $2 + \sqrt{(p-6)/6}$ if $p > 12$, and zero if $p < 12$. That gives us a mapping from any market price p to $S(p)$, the amount of output that the firm chooses to supply in order to maximize its profit given p:

$$S(p) = \begin{cases} 2 + \sqrt{(p-6)/6} & \text{if } p \geq 12 \\ 0 & \text{if } p \leq 12. \end{cases}$$

The function S obtained above is called *supply function* of the firm in this example. The graph of this function, with q on the horizontal axis and p on the vertical axis, is the corresponding *supply curve*. Note that, other than the lower portion of the curve where $S(p) = 0$, the curve (above the price 12) is upward sloping, like the ones assumed in Introductory Microeconomics.

The general case is similar to the above example. Given a market price p and a cost function C satisfying the assumptions in Section 3, the global maximum of the firm's profit among all interior points is the solution for q_* in Eq. (7) at which the MC curve is upward sloping. This solution is unique because an upward sloping curve can cross a horizontal line only once. Hence we can denote this unique solution for q_* by $\mathrm{MC}^{-1}(p)$, read as "the inverse image of p via the upward sloping portion of MC." As in the example, $q_* = \mathrm{MC}^{-1}(p)$ is the profit maximum if it is above the output at which the AVC attains its minimum, i.e., if p is above the minimum level of AVC; else zero is the profit maximum. In other words, in the general case,

$$S(p) = \begin{cases} \mathrm{MC}^{-1}(p) & \text{if } p \geq \min \mathrm{AVC} \\ 0 & \text{if } p \leq \min \mathrm{AVC}, \end{cases}$$

which gives the firm's supply curve. Again, the upward portion of the supply curve, $\left(\mathrm{MC}^{-1}(p), p\right)$, belonging to the upward sloping part of the MC curve, is upward sloping as assumed in the Introductory Microeconomics (Figure 1).

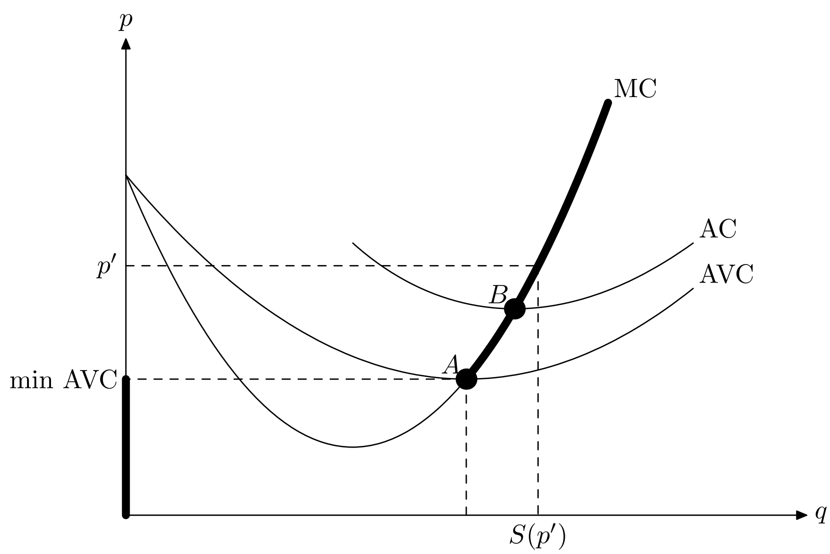

Figure 1: The thick vertical segment and thick curve starting from point A: Short-run supply curve

7 Long-run supply

We have been assuming so far that the firms is stuck with the fixed cost c_0 even when it supplies zero quantity of the output. The assumption is realistic in the short run, within which the firm might have made contractual commitment to the employment of some inputs, such as union workers. In the long run, however, the firm can shut down completely, or move overseas, thereby shedding off the entire fixed cost. Thus, in the long-run, in deciding whether to shut down or not, the firm compares not between $pq_* - C(q_*)$ and $-c_0$ as in Section 4, but rather between $pq_* - C(q_*)$ and zero, i.e., between p and $C(q_*)/q_*$. Thus, the firm's supply in the long run is zero if the market price falls below the average cost rather than below the average variable cost. In other words, the long-run supply curve is the portion of the upward-sloped part of the MC curve down to the minimum of AC, not further down the minimum of AVC. In Figure 1, the long-run supply curve is the thick curve down to the point B.

8 Exercises

1. Use L'Hôpital's rule to calculate $\lim_{x\to 0} \frac{\sin x}{x}$.

2. Consider the following decision problem,

$$\min_{x\in\{0,1,2,3\}} (x-\alpha)^2,$$

where $\{0, 1, 2, 3\}$ denotes the set whose elements are exactly those written inside the brace bracket $\{\cdot\}$, and α a positive constant belonging to the interval $(0, 3)$ (the set of real numbers strictly between zero and three).

a. What are the objective, the choice variable and its domain, and the parameter?

b. Rewrite the above problem into its equivalent form ("equivalent" in the sense that the set of solutions remains the same) such that the operator min is replaced by max.

3. Demonstrate the last italicized statement of Section 3 by deriving a relationship, analogous to (6), between the slope of the AC curve and the relative position between the AC and MC curves.

4. Consider a function defined by $f(x) := 2x^2 - 3x + 1$ for all $x \in \mathbb{R}$ (i.e., for all real number x). Find the value of x that satisfies the first-order condition. Calculate the second-order derivative of f at this value of x. Is this value a local maximum of f?

5. For each of the following statements, plot a graph to demonstrate:

   a. A maximum of a differentiable function need not satisfy the first-order condition

   b. An interior point that satisfies the first-order condition need not be a local maximum

   c. An interior point that satisfies both the first- and second-order conditions need not be a global maximum

6. To understand the role of the vertical line in $\frac{d}{dq}C(q)\Big|_{q=q_*}$, suppose $C(q) = \alpha q^2$ for all $q \geq 0$, where α denotes a parameter (i.e., constant to q).

   a. Calculate the derivative of $C(q)$ with respect to q.

   b. Plug $q = \alpha$ into the derivative obtained in the previous step. The result obtained thereof is $\frac{d}{dq}C(q)\Big|_{q=\alpha}$; without the $|_{q=\alpha}$ there, the notation $\frac{d}{dq}C(q)$ would mean the result of merely the first step.

   c. Calculate the derivative $\frac{d}{d\alpha}\left(\frac{d}{dq}C(q)\Big|_{q=\alpha}\right)$, which tells us how the marginal cost at $q = \alpha$ is affected by the parameter α.

   d. Note that $C(\alpha) = \alpha^3$. Calculate $\frac{d}{d\alpha}C(\alpha)$ and then calculate $\frac{d}{d\alpha}\left(\frac{d}{d\alpha}C(\alpha)\right)$. Note that $\frac{d}{d\alpha}\left(\frac{d}{d\alpha}C(\alpha)\right) \neq \frac{d}{d\alpha}\left(\frac{d}{dq}C(q)\Big|_{q=\alpha}\right)$.

7. Which of the following statements is (are) true (where A and B denote statements, a, b and c represent real numbers, and $\Rightarrow$ denotes implication, e.g., "$A \Rightarrow B$" means "if A then B"):

   a. "if A then B" iff "if B is not true, then A is not true"

   b. "if A then B" $\Leftrightarrow$ "either B is true or A is not true"

   c. $\frac{a-b}{c} > 0 \Rightarrow a > b$

8. Calculate the derivatives of the following functions, defined for all real numbers x:

   a. $f(x) = 3\sqrt{x} + 100$

b. $f(x) = 6 \ln x$

c. $f(x) = (2x + 1) \ln(x/3)$

d. $f(x) = (x^{3/2} + x)/(5x - 7)$

e. $f(x) = 2^x$

9. Suppose that the cost function is given by, for any $q \geq 0$,

$$C(q) := q^3 - 6q^2 + 30q + 50.$$

a. Calculate the firm's average variable cost (AVC), marginal cost (MC) and average cost (AC) as functions of the output quantity q.

b. Calculate the output quantity at which AVC is equal to MC.

c. When the market price is \$25, what is the firm's profit-maximizing output quantity? What if the market price becomes \$20? What if the market price becomes \$10?

d. Calculate the firm's short-run supply function; graph the short-run supply curve.

e. Calculate the firm's *long*-run supply function; graph the long-run supply curve.