Chapter 2: Input-Output Decision

Elements of Decision: Lecture Notes of Intermediate Microeconomics 1

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1 The production set

The choice variable in the decision problem considered in Chapter 1 has only one dimension, the firm's quantity of output. In general, a choice variable may have multiple dimensions. For instance, a firm may choose not only how much output to supply but also how much input to employ. Assume for simplicity that there is only one kind of input. Then a firm needs to choose a quantity x of the input to employ, and a quantity y of the output to supply. Hence what the firm chooses amounts to a two-dimension vector (x, y), called *production plan*. Clearly not all production plans are feasible given the technology that the firm has at the time when it is making the decision. The set of all production plans that are technologically feasible to the firm is called *production set*. A production set is usually assumed to be of the form

$$\left\{(x, y) \in \mathbb{R}^2_+ : y \leq f(x)\right\} \tag{1}$$

for some function f. The above formula says that this set contains all the points (x, y) in the nonnegative quadrant of the plane (or, briefly put, $(x, y) \in \mathbb{R}^2_+$)[1] such that the vertical coordinate of the point (x, y) is not above the $f(x)$.

Here the function f is called *production function*, and its graph *production possibility frontier* (PPF).[2] On the x-y-plane, the area between the PPF and the horizontal axis is the graph of the set (1). It is usually assumed that f is increasing, concave and differentiable. A function f is *increasing* iff $f(x)$ increases when x increases, so its graph is an upward-sloping curve if f is differentiable. An increasing function of x is *concave* if the slope of its graph gets less steep as x increases; its graph is also said *concave* in that case.[3]

2 Isoprofit lines

Suppose that the market price of the output is p dollars per unit, and the wage rate of the input is w dollars per unit, each taken as given by the firm. Then the firm's profit from carrying out a production plan (x, y) is equal to $py - wx$. Thus the firm's decision problem is to maximize $py - wx$

[1] The notation $\mathbb{R}^2_+$ denotes the set of all pairs (x, y) such that both x and y are elements of the set $\mathbb{R}_+$.

[2] To those who are confused by the upward sloping graph that we call PPF here because of the memory of the downward sloping PPF in Introductory textbooks, note that those textbooks (implicitly) follow a convention that a good is an input in a production plan iff its quantity in the plan is negative. Given that convention, our production plan (x, y) here would have been denoted $(-x, y)$. Thus, draw a coordinate system with vertical axis y and *horizontal axis being* $-x$; draw the graph of $y \leq f(x)$ to see a downward sloping PPF—on a thin sheet of paper with dark, heavy ink; then look at the graph from the back of the page to see the upward sloping PPF we have here.

[3] In general, a function $f(x)$ of x is *concave* iff its slope decreases when x increases. Note that when f is decreasing, its slope is negative, hence concavity means that the absolute value of its slope *increases* when x increases. In other words, the curve of a decreasing function $f(x)$ gets steeper as x increases.

by choosing a production plan from its production set (1), i.e.,

$$\max_{(x,y)\in\mathbb{R}^2_+} \quad py - wx \qquad (2)$$
$$\text{subject to} \quad y \leq f(x).$$

Like the decision problem considered in Chapter 1, Problem (2) has an objective $py - wx$ and a choice variable (x, y), which is specified, underneath the max operator, to belong to the domain $\mathbb{R}^2_+$, i.e., to be nonnegative in both coordinates. Unlike the previous problem, however, Problem (2) has a constraint, the inequality $y \leq f(x)$ on the second line. This is an example for *constrained optimization problems*.

While there is a trick to reduce Problem (2) to a problem with only a unidimensional choice variable, we do not use that trick here. Instead, the following is a technique that, albeit elementary, conveys the basic idea for a general approach to multidimensional choice variables.

Let us start with representing the objective on the x-y-plane. Given any two points on the plane, i.e., two production plans, how to tell which one generates larger profits than which? Pick any constant number π and consider all the production plans (x, y) that generate π dollars as the profit, i.e., the (x, y) such that

$$py - wx = \pi. \qquad (3)$$

This equation is equivalent to

$$y = \frac{\pi}{p} + \frac{w}{p}x,$$

which corresponds to the straight line on the x-y-plane with slope w/p and y-intercept π/p. The set of all the points (x, y) satisfying Eq. (3) is called an *iosprofit line*. The word isoprofit signifies that any two points on the line correspond to two production plans that would, if technologically feasible and carried out, generate the same amount of profit. Now pick any other number π' such that $\pi' > \pi$, and consider the production plans (x, y) that generate π' dollars as the profit. Analogous to the previous case, the isoprofit line in this case corresponds to the equation

$$y = \frac{\pi'}{p} + \frac{w}{p}x,$$

which is a straight line parallel to the previous one, as the slope w/p is the same, and positioned higher, as the y-intercept π'/p is bigger than the previous one, with $\pi' > \pi$. Thus, *the objective of Problem (2) corresponds to the family of isoprofit lines, each of the constant slope w/p, such that the higher is an isoprofit line positioned along the y-axis, the larger is the profit that any point of the line generates*.

3 Profit-maximizing production plans

Now that a firm's profit is represented by the isoprofit lines, we know that the firm would like to reach as high an isoprofit line as possible. However, not all isoprofit lines are reachable, as they need not be technologically feasible. To be reachable, an isoprofit line needs to contain a technologically feasible production plan, i.e., share a common point with the production set. Among such reachable isoprofit lines, the one with the highest y-intercept, hence the largest profit, is the *supporting* *hyperplane* of the PPF, i.e., the isoprofit line that keeps all elements of the production set on only one side of the line and touches the frontier of the set somewhere. The point where the supporting hyperplane touches the PPF, itself an element of the production set, is technologically feasible, hence this isoprofit line is reachable. Since the entire production set belongs to only one side of the hyperplane, no technologically feasible production plan belongs to an isoprofit line above the supporting hyperplane. Hence the supporting hyperplane is the highest isoprofit line that the firm can reach, and any point where it touches the PPF a profit-maximizing production plan. Thus, *a production plan solves Problem (2) iff it is a common point between the PPF and the isoprofit line that is the supporting hyperplane of the PPF.*

For example, suppose that the production function f is given by $f(x) := \sqrt{x}$ for all $x \geq 0$. Then the production set is $\{(x, y) \in \mathbb{R}^2_+ : y \leq \sqrt{x}\}$. Note that f is an increasing and differentiable function on $\mathbb{R}_+$. Furthermore, it is concave because its derivative

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2\sqrt{x}}$$

is decreasing in x. With f differentiable, the PPF is a smooth curve, hence it is tangent to its supporting hyperplane, so the PPF and its supporting hyperplane have the same slope at their common point. The slope of the former, by basic calculus, is the derivative of f; the slope of the latter, which is an isoprofit line, is equal to w/p by the previous section. Thus, if the supporting hyperplane touches the PPF when the input level equals x, then

$$\frac{1}{2\sqrt{x}} = \frac{w}{p},$$

i.e., $x = p^2/(4w^2)$. To find the y-coordinate of the common point between the supporting hyperplane and the PPF, plug $x = p^2/(4w^2)$ into $f(x)$ to obtain

$$f\left(p^2/(4w^2)\right) = \left(p^2/(4w^2)\right)^{1/2} = \frac{p}{2w}.$$

Hence the supporting hyperplane touches the PPF at the point $\left(p^2/(4w^2), p/(2w)\right)$, which constitutes the profit-maximizing production plan.[4]

4 Comparative statics

Comparative statics is the study of how changes in the parameters of a problem affect its solution. That means, in our input-output decision problem, how changes in the input or output prices affect the profit-maximizing production plan. In the above example, with $\left(p^2/(4w^2), p/(2w)\right)$ the profit-maximizing production plan, we can do comparative statics explicitly: Since $p^2/(4w^2)$ is increasing when p increases, and decreasing when w increases, we obtain a comparative statics conclusion that, within this example, higher wage rate for the input means a profit-maximizing firm would hire less input, and higher output price means it would hire more input.

[4] One may wonder why we do not check a second-order condition in this example while we did in Chapter 1. The reason is that the PPF is assumed concave here, hence any supporting hyperplane of the PPF necessarily bounds the entire production set *from above*. In Chapter 1, by contrast, the profit is not concave in the quantity of output (because the MC curve is assumed U-shape), hence a straight line tangent to the profit curve does not necessarily bound the set of possible profits from above.

The above observation goes way beyond the example. Consider any production function f that is increasing, differentiable and concave. Given any market price p and wage rate w, the profit-maximizing production plan is the common point between the PPF and its supporting hyperplane with slope w/p. Now suppose that w becomes higher (due to an exogenous shock to which the firm has no influence, say a minimum wage legislation, with the firm a small restaurant hiring minimum wage earners). That means w/p is bigger and hence the supporting hyperplane steeper. Since the PPF is concave, its slope gets steeper only with smaller x. Hence the common point between the PPF and its supporting hyperplane moves to the left, with the input x reduced. That means the output y is also reduced because the PPF is upward sloped, with f an increasing function.

The above reasoning relies on the assumption that the production function is increasing, differentiable and concave. What happens if these assumptions need not hold? Following is an elegant reasoning, called *revealed-preference argument*, mastering of which would help one go a long way. Let (x, y) be a profit-maximizing production plan given output price p and input wage rate w, and (x', y') a profit-maximizing production plan given output price p' and input wage rate w'. Since (x', y') is a profit-maximizing production plan, it is technologically feasible, hence it is an option that the firm could have chosen when it opts for (x, y) given the parameter (p, w). The fact that the firm does not opt for (x', y') means that, given output price p and input wage w, the profit generated by (x', y') does not exceed the profit generated by (x, y) (otherwise (x, y) could not have been a profit-maximizing plan given (p, w)). Thus

$$py - wx \geq py' - wx'.$$

Repeating the above argument by switching the roles of $(x, y; p, w)$ and $(x', y'; p', w')$, we obtain analogously

$$p'y' - w'x' \geq p'y - w'x.$$

Sum these two inequalities to get

$$py - wx + p'y' - w'x' \geq py' - wx' + p'y - w'x,$$

i.e.,

$$p(y - y') - p'(y - y') \geq w(x - x') - w'(x - x'),$$

i.e.,

$$(y - y')(p - p') \geq (x - x')(w - w'). \tag{4}$$

Consider the case where the output price is unchanged ($p' = p$) while the input wage rate is higher ($w' > w$). Then Ineq. (4) becomes

$$0 \geq (x - x')(w - w').$$

That implies, since $w - w' < 0$, $x - x' \geq 0$. Thus, *if the input wage rate is higher then the profit-maximizing input quantity is either unchanged or reduced*.

Likewise consider the case where $w = w'$ and $p' > p$. Then Ineq. (4) becomes

$$(y - y')(p - p') \geq 0.$$

With $p - p' < 0$, this inequality implies that $y - y' \leq 0$. Thus, if the output price is higher then the profit-maximizing output is either unchanged or higher.

The beauty of the above reasoning is that it uses none of the assumptions about the production function. It is based purely on the notion of profit maximization.

5 Revealed profitability

So far we have been analyzing the input-output decision from within, assuming that we know what the production set is for the firm. In the more realistic case where we do not know the firm's production set, can we still predict the firm's action? The answer, remarkably, is Yes.

The idea is that, although at the outset we do not know the firm's production set, we can infer about it from the firm's past actions. Suppose that when $p = 2$ and $w = 4$ the firm's action is $(1, 3)$, i.e., hiring one unit of the input and supplying three units of the output. Then we know that, when $p = 2$ and $w = 4$, the firm's profit is equal to $2 \times 3 - 4 \times 1 = 2$ and the isoprofit line passing through the firm's chosen production plan $(1, 3)$ corresponds to the equation

$$2y - 4x = 2, \quad \text{i.e.,} \quad y = 1 + 2x,$$

the straight line with slope 2 and y-intercept 1. It follows that no point above this line can belong to the production set for the firm, otherwise the firm, trying to maximize its profit, would not have opted for the production plan $(1, 3)$.

Suppose that after an exogenous price shock, the prices become $p = 6$ and $w = 3$, to which the firm responds with the production plan $(5, 6)$. Then we know that, given such prices, the firm's profit is equal to $6 \times 6 - 3 \times 5 = 21$, and the isoprofit line passing through the firm's action $(5, 6)$ is

$$6y - 3x = 21, \quad \text{i.e.,} \quad y = \frac{7}{2} + \frac{1}{2}x,$$

the line with slope $1/2$ and y-intercept $7/2$. By the same token as in the above paragraph, we deduce that no point above this straight line can belong to the firm's production set.

Thus, combining these two observations of the firm's past actions, we infer that the firm's production set is contained by the region jointly bounded from above by the two straight lines, $y = 1 + 2x$ and $y = \frac{7}{2} + \frac{1}{2}x$, and that both $(1, 3)$ and $(5, 6)$ belong to the production set (Figure 1).

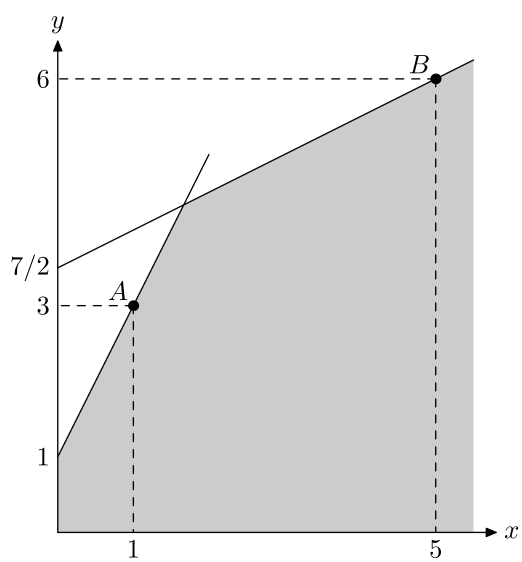

Figure 1: The grey area contains the entire production set, which contains A and B

Now suppose that another shock in the market changes the prices to $p = 3$ and $w = 3$. What would be our sharpest prediction of the firm's response? Given these prices, the slope of the isoprofit lines for the firm becomes $3/3 = 1$. On the x-y-plane, draw the isoprofit line passing through the point $(1, 3)$, and the isoprofit line passing through $(5, 6)$, each with the new slope 1. Each of the two lines can be reached by the firm, since both production plans $(1, 3)$ and $(5, 6)$ belong to its production set, as the firm has opted for them before. Note that the isoprofit line of $(1, 3)$ lies above that of $(5, 6)$. Thus, the former is the lowest possible isoprofit that the firm would go: any production plan below the line, even if technologically feasible, yields less profit than the plan $(1, 3)$ given $p = 3$ and $w = 3$. It follows that, given $p = 3$ and $w = 3$, the firm's profit-maximizing production plan belongs to the dark area in Figure 2. Furthermore, any point in the dark area may be the firm's profit-maximizing production plan, as it is possible for the point to lie on the frontier of the production set—such possibility we cannot rule out based on the two past actions of the firm.

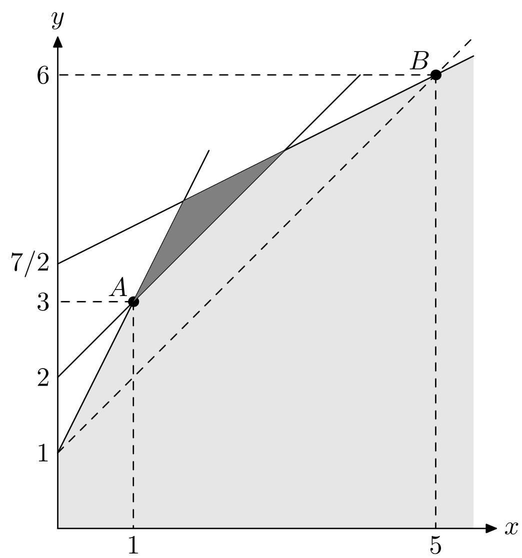

Figure 2: The dark area is the sharpest prediction of the firm's action given $p = w = 3$

Finally, note that the upper boundary of the grey area in Figure 1 is concave. This is no coincidence. Pick any number $x' > 5$ and suppose that after another price shock the output price becomes p', and input wage rate w', and that we observe the firm choosing a production plan (x', y') in response. As in the previous case, the slope becomes w'/p' for all the isoprofit lines, and the one passing through (x', y') (point C in Figure 3) becomes part of the upper boundary of the firm's production set. Note that this boundary line cannot be steeper than the one passing through B. Otherwise given the prices (p', w') point C would yield less profit than B and hence would not have been chosen by the firm. Thus, *whether a firm's production possibility frontier is concave or not, we know that the frontier is approximately concave*.

6 Exercises

1. Mimicking (1), express in set-theoretic notations the isoprofit line corresponding to Eq. (3).

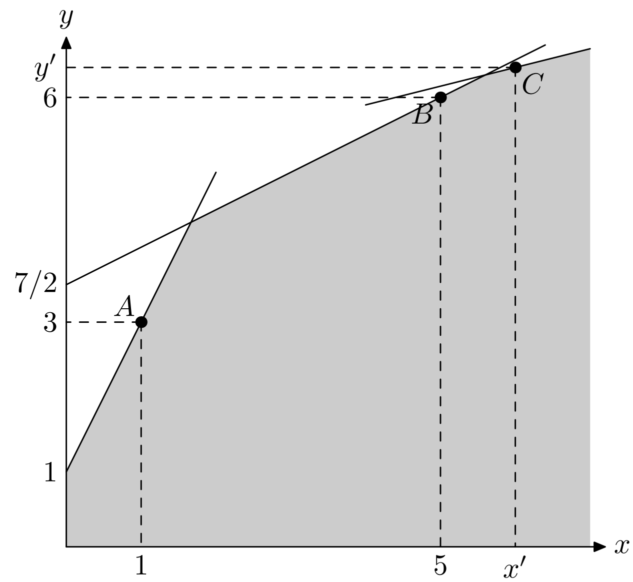

Figure 3: The revealed approximated boundary of the production set is concave

2. For each of the following functions, tell whether it is concave through graphing it or calculating its second-order derivative:

   a. $f(x) := \ln x$ for all $x > 0$

   b. $f(x) := 2x^2 + 3x - 1$ for all real numbers x

   c. $f(x) := \sqrt{x}$ for all $x \geq 0$

   d. $f(x) := \sqrt{4 - x^2}$ for all $0 \leq x \leq 2$

3. Suppose that a firm's production function is given by $f(x) := \sqrt{x+1} - 1$ for all $x \geq 0$.

   a. Calculate the slope of the PPF as a function of the input quantity x. What is the slope equal to at the production plan $(0, 0)$?

   b. Suppose that the (per-unit) price is equal to \$10 for the output, and \$4 for the input. Calculate the profit-maximizing production plan.

   c. Suppose that the input price rises to \$15 (say due to a minimum wage legislation), while the output price is unchanged. Calculate the profit-maximizing production plan.

   d. Let p denote the price of the output, and w that of the input, each a parameter to the firm.

      i. Mimic the procedure in Step 3b. to calculate the profit-maximizing production plan (which should be a pair of expressions of p and w).

      ii. When $p < 2w$, is the production plan obtained in Step 3(d.)i still feasible? If not feasible, what is the profit-maximizing production plan? (Hint: Graph the isoprofit lines given $p < 2w$ and compare their slopes with that of the PPF.)

4. Suppose that a firm's production function is given by $f(x) := x^2$ for all $x \geq 0$, and that the price is equal to \$8 for the output, and \$4 for the input.

a. Calculate the tangent point between the PPF and an isoprofit line.

b. Does this tangent point correspond to a profit-maximizing production plan? Explain your answer with a graph consisting of the production set and several isoprofit lines.

5. A profit-maximizing firm produces a single output from a single input. Let y denote the quantity of the output produced, and x the quantity of the input employed, by the firm. Denote w for the price of the input and p the price of the output. Two independent observations of the firm are:

obervation	x	y	w	p
A	8	5	\$2	\$8
B	1	2	\$9	\$9

a. On an x-y-plane, shade with dotted lines the set of all the production plans that the above data imply *cannot* be technologically feasible. On the diagram, specify the coordinates of the vertices that determine the boundary of this set.

b. In the third observation, $w = \$3$ and $p = \$9$. On the diagram described above, shade with solid lines the set to which the firm's profit-maximizing production plan belongs in this observation; make this prediction as precise as possible. Write down the slopes of all the edges that constitute the boundary of this set.

c. In the fourth observation, $x = 9$. Which of the following input-output price pairs is/are consistent to profit maximization in this observation?

   i. $w = 20, p = 20$
   ii. $w = 15, p = 5$
   iii. $w = 4, p = 6$
   iv. $w = 3, p = 9$
   v. none of the above

d. Suppose that w has increased while p remains unchanged. Then which of the following statement(s) is/are necessarily true?

   i. The firm increases y or leaves it unchanged
   ii. The firm reduces y or leaves it unchanged
   iii. The firm increases x or leaves it unchanged
   iv. The firm reduces x or leaves it unchanged
   v. None of the above, because the production frontier is not necessarily concave

6. Consider a firm that uses two kinds of inputs and produces two kinds of outputs. Hence a production plan is in the form of $(x_1, x_2; y_1, y_2)$, denoting the plan of hiring x_1 units of input 1 and x_2 units of input 2, and producing y_1 units of output 1 and y_2 units of output 2. Denote p_1 for the market price of output 1, p_2 that of output 2, w_1 the market price of input 1, and w_2 that of input 2. The profit from a production plan $(x_1, x_2; y_1, y_2)$, if technologically feasible, is therefore equal to $p_1y_1 + p_2y_2 - w_1x_1 - w_2x_2$. Mimic the proof of Ineq. (4) in Section 4 to derive an inequality analogous to Ineq. (4).

7. A firm uses a single kind of input to produce several kinds of outputs. If x is equal to the quantity of input the firm employs, then its revenue is equal to $R(x)$ for some function R, and its profit is equal to $R(x) - wx$ if w is equal to the market price of the input. The market price of the input is a parameter given to the firm, but the prices of the outputs need not be so. Mimic the proof of Ineq. (4) in Section 4 to derive an inequality analogous to Ineq. (4).